HSC exam solutions

2026 3U Mathematics Extension 1 HSC exam solutions

Multiple choice

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
A	D	C	D	C	B	B	D	A	A

Question 1

Using the double angle formula,

$$\begin{aligned}\sin\alpha\cos\alpha &= \frac{1}{2}(2\sin\alpha\cos\alpha)\\ &= \frac{1}{2}\sin 2\alpha.\end{aligned}$$

Since $-1 \leq \sin 2\alpha \leq 1$, the maximum value of $\sin\alpha\cos\alpha$ is $\frac{1}{2}$.

Hence, the answer is A.

Question 2

By the chain rule,

$$\begin{aligned} y &= \sin^{-1}\left(\frac{2x}{3}\right) \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1-\left(\frac{2x}{3}\right)^2}} \cdot \frac{2}{3} \\ &= \frac{2}{3\sqrt{\frac{9-4x^2}{9}}} \\ &= \frac{2}{\sqrt{9-4x^2}}. \end{aligned}$$

Hence, the answer is D.

Question 3

Since $y^2 \geq 0$, the graph of $y^2 = f(x)$ only exists where $f(x) \geq 0$. From the graph, $f(x) = 0$ at $x = -2, 0$ and 5, with $f(x) \geq 0$ for $-2 \leq x \leq 0$ and $x \geq 5$, and $f(x) < 0$ for $0 < x < 5$. For option C,

$$\begin{aligned} f(x) &= x(x+2)(x-5) \\ f(-1) &= (-1)(1)(-6) \\ &= 6 > 0 \\ f(1) &= (1)(3)(-4) \\ &= -12 < 0 \\ f(6) &= (6)(8)(1) \\ &= 48 > 0. \end{aligned}$$

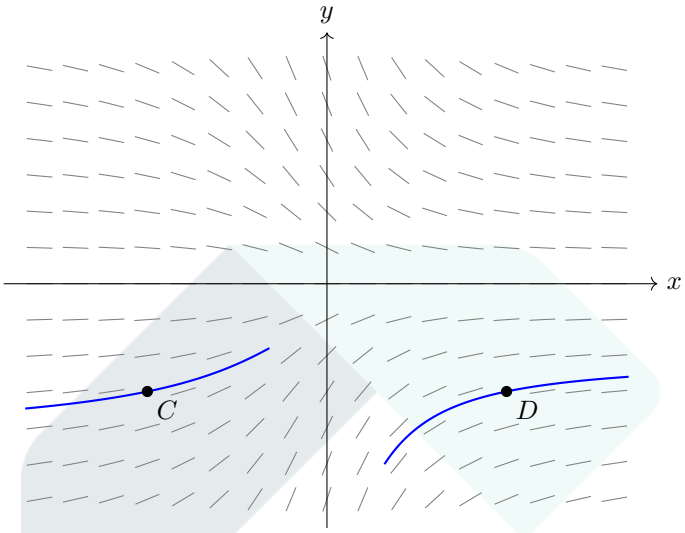
So $f(x) = x(x+2)(x-5)$ has the required roots and signs.

Hence, the answer is C.

Question 4

An increasing function has a positive gradient, so the point must lie below the x-axis, where the line segments slope upwards. This leaves points C and D.

A function is concave down where its gradient decreases as x increases. Moving to the right through point D, the line segments become less steep, so the gradient decreases and the solution curve is concave down. Moving to the right through point C, the line segments become steeper, so the solution curve is concave up.



The solution curve through D is increasing and concave down, while the solution curve through C is increasing and concave up.

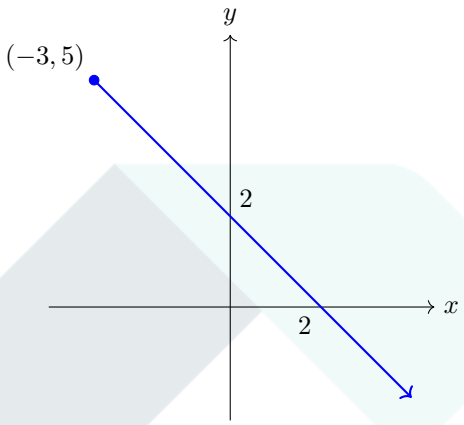
Hence, the answer is D.

Question 5

Adding the two equations eliminates t:

$$\begin{aligned} x + y &= \left(t^2 - 3\right) + \left(5 - t^2\right) \\ &= 2 \\ y &= 2 - x. \end{aligned}$$

Since $t^2 \geq 0$, $x = t^2 - 3 \geq -3$ and $y = 5 - t^2 \leq 5$. So the graph is the part of the line $y = 2 - x$ starting at $(-3, 5)$ and continuing to the right.



Hence, the answer is C.

Question 6

Since g is the inverse of f, $f(g(x)) = x$. Differentiating both sides with respect to x,

$$\begin{aligned} f'(g(x))\, g'(x) &= 1 \\ g'(x) &= \frac{1}{f'(g(x))}. \end{aligned}$$

Now $f'(x) = 3x^2 - 7$ and $g(-6) = 2$. Thus,

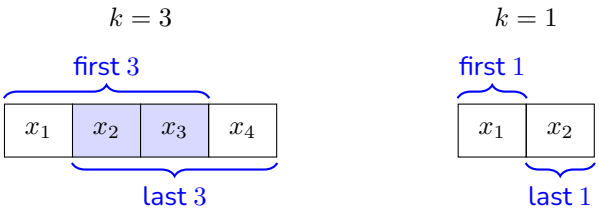
$$\begin{aligned} g'(-6) &= \frac{1}{f'(2)} \\ &= \frac{1}{3(2)^2 - 7} \\ &= \frac{1}{5}. \end{aligned}$$

Hence, the answer is B.

Question 7

$S(1)$ is true, since a single number is equal to itself.

The inductive step takes a list of $k+1$ numbers and applies $S(k)$ to two subsets of numbers: the first k numbers and the last k numbers. Each subset has all its numbers equal. When $k \geq 2$, the two subsets share at least one number, so every number in the list equals that shared number. This makes "if $S(2)$ then $S(3)$" and "if $S(3)$ then $S(4)$" true.



When $k = 1$, the subsets of x_1 and x_2 are meant to illustrate that the set $\{x_1, x_2\}$ has all numbers equal. However, these subsets share no number so $S(1)$ gives no link between x_1 and x_2. Thus "if $S(1)$ then $S(2)$" is false.

Hence, the answer is B.

Question 8

From the position vector,

$$\begin{aligned} x &= b - \sqrt{a}\cos t \\ \cos t &= \frac{b-x}{\sqrt{a}} \\ y &= a - \frac{1}{b}\sin t \\ \sin t &= b(a-y). \end{aligned}$$

Using $\cos^2 t + \sin^2 t = 1$,

$$\frac{(x-b)^2}{a} + b^2(y-a)^2 = 1.$$

For the path to be a circle, the coefficients of $(x-b)^2$ and $(y-a)^2$ must be equal:

$$\begin{aligned} \frac{1}{a} &= b^2 \\ ab^2 &= 1. \end{aligned}$$

Hence, the answer is D.

Question 9

The largest selection with no matching pair contains the odd sock and one sock from each of the n pairs, which is $n+1$ socks. Any selection of $n+2$ socks must contain a matching pair.

Since 6 socks do not guarantee a pair but 7 socks do,

$$\begin{aligned} n+1 &= 6 \\ n &= 5. \end{aligned}$$

Hence, the answer is A.

Question 10

There are $\binom{46}{10}$ equally likely sets of 10 drawn balls. For the player to win, the 10 drawn balls must include all 6 of the player's numbers. The remaining 4 drawn balls can be any of the other 40 balls, which can happen in $\binom{40}{4}$ ways. Thus,

$$\begin{aligned} P(\text{win}) &= \frac{\binom{6}{6}\binom{40}{4}}{\binom{46}{10}} \\ &= \frac{\binom{40}{4}}{\binom{46}{10}}. \end{aligned}$$

Hence, the answer is A.

Question 11

(a) **2 marks**

Since $9x^2 - 1 = (3x - 1)(3x + 1)$,

$$P(x) = (3x - 1)(3x + 1)Q(x) + 3x + 7$$

for some polynomial $Q(x)$. By the remainder theorem, the remainder when $P(x)$ is divided by $3x - 1$ is

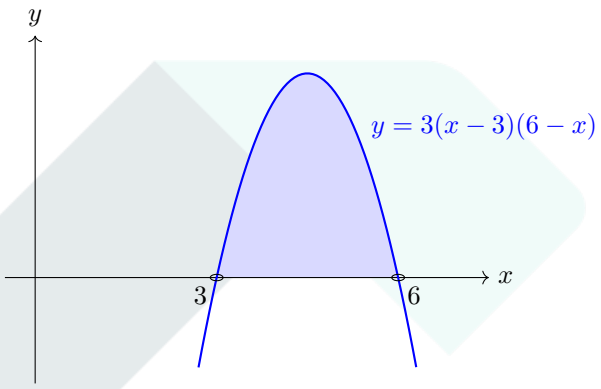
$$\begin{aligned} P\left(\frac{1}{3}\right) &= 0 \cdot \left(3 \cdot \frac{1}{3} + 1\right) Q\left(\frac{1}{3}\right) + 3 \cdot \frac{1}{3} + 7 \\ &= 8. \end{aligned}$$

(b) **3 marks**

Since $x \neq 3$, multiply both sides by $(x-3)^2 > 0$:

$$\begin{aligned} \frac{x+6}{x-3} \cdot (x-3)^2 &> 4(x-3)^2 \\ (x+6)(x-3) &> 4(x-3)^2 \\ (x+6)(x-3) - 4(x-3)^2 &> 0 \\ (x-3)\left[(x+6) - 4(x-3)\right] &> 0 \\ (x-3)(18-3x) &> 0 \\ 3(x-3)(6-x) &> 0. \end{aligned}$$

The parabola $y = 3(x-3)(6-x)$ is concave down with roots $x = 3$ and $x = 6$, so it is positive between its roots.



Hence, $3 < x < 6$.

(c) **2 marks**

The domain of $\sin^{-1} x$ is $-1 \leq x \leq 1$ and its range is $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$. For the domain,

$$\begin{aligned} -1 \leq \frac{x}{3} &\leq 1 \\ -3 \leq x &\leq 3. \end{aligned}$$

For the range,

$$\begin{aligned} -\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x}{3}\right) &\leq \frac{\pi}{2} \\ -\pi \leq 2\sin^{-1}\left(\frac{x}{3}\right) &\leq \pi. \end{aligned}$$

Hence, the domain is $-3 \leq x \leq 3$ and the range is $-\pi \leq y \leq \pi$.

(d) **2 marks**

Since the vectors are parallel, one is a scalar multiple of the other:

$$\begin{pmatrix} m \\ -8 \end{pmatrix} = \lambda \begin{pmatrix} m-2 \\ 1 \end{pmatrix}.$$

Equating the second components gives $\lambda = -8$. Equating the first components,

$$\begin{aligned} m &= -8(m-2) \\ m &= -8m + 16 \\ 9m &= 16 \\ m &= \frac{16}{9}. \end{aligned}$$

(e) **3 marks**

Factorising by grouping,

$$\begin{aligned} 2\cos^3 x - 2\cos^2 x - \cos x + 1 &= 0 \\ 2\cos^2 x\,(\cos x - 1) - (\cos x - 1) &= 0 \\ (\cos x - 1)\left(2\cos^2 x - 1\right) &= 0. \end{aligned}$$

Case 1: $\cos x = 1$

$$x = 0$$

Case 2: $2\cos^2 x - 1 = 0$

$$\begin{aligned} \cos^2 x &= \frac{1}{2} \\ \cos x &= \pm\frac{1}{\sqrt{2}}. \end{aligned}$$

For $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $\cos x \geq 0$, so $\cos x = \frac{1}{\sqrt{2}}$ and $x = \pm\frac{\pi}{4}$.

Hence, $x = -\frac{\pi}{4}$, 0 or $\frac{\pi}{4}$.

(f) **3 marks**

Let $u = x^2$.

$$du = 2x\ dx$$
$$x\ dx = \frac{1}{2}\ du$$

When $x = 0$, $u = 0$. When $x = \frac{1}{\sqrt{2}}$, $u = \frac{1}{2}$. Thus,

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\ dx &= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-(x^2)^2}} \cdot x\ dx \\
&= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\ du \\
&= \frac{1}{2}\left[\sin^{-1} u\right]_0^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right) \\
&= \frac{1}{2}\cdot\frac{\pi}{6} \\
&= \frac{\pi}{12}
\end{aligned}$$

Question 12

(a) **2 marks**

Let X be the number of people with blood type B+. Then $X \sim \text{Bin}(20, 0.12)$.

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{20}{0}(0.12)^0(0.88)^{20} - \binom{20}{1}(0.12)^1(0.88)^{19} \\ &= 1 - (0.88)^{20} - 20(0.12)(0.88)^{19} \\ &\approx 0.711 \end{aligned}$$

(b) **3 marks**

We proceed by mathematical induction.

Base case: $n = 1$

$$1\left(1^2 + 2\right) = 3,$$

which is divisible by 3.

Assumption: $n = k$, for some integer $k \geq 1$

$$k\left(k^2 + 2\right) = 3M, \text{ where } M \text{ is an integer}$$

Inductive step: $n = k + 1$

RTP: $(k+1)\left[(k+1)^2 + 2\right]$ is divisible by 3

$$\begin{aligned}(k+1)\left[(k+1)^2 + 2\right] &= (k+1)\left(k^2 + 2k + 3\right) \\ &= k^3 + 3k^2 + 5k + 3 \\ &= \left(k^3 + 2k\right) + 3k^2 + 3k + 3 \\ &= k\left(k^2 + 2\right) + 3\left(k^2 + k + 1\right) \\ &= 3M + 3\left(k^2 + k + 1\right) \text{ by assumption} \\ &= 3\left(M + k^2 + k + 1\right).\end{aligned}$$

Since $M + k^2 + k + 1$ is an integer, $(k+1)\left[(k+1)^2 + 2\right]$ is divisible by 3.

Hence, by mathematical induction, $n\left(n^2 + 2\right)$ is divisible by 3 for all integers $n \geq 1$.

(c) **2 marks**

$$\begin{aligned}
|\underset{\sim}{a}-2\underset{\sim}{b}|^2 &= (\underset{\sim}{a}-2\underset{\sim}{b})\cdot(\underset{\sim}{a}-2\underset{\sim}{b}) \\
&= \underset{\sim}{a}\cdot\underset{\sim}{a}-4\,\underset{\sim}{a}\cdot\underset{\sim}{b}+4\,\underset{\sim}{b}\cdot\underset{\sim}{b} \\
&= |\underset{\sim}{a}|^2-4\,\underset{\sim}{a}\cdot\underset{\sim}{b}+4\,|\underset{\sim}{b}|^2 \\
&= 2^2-4(3)+4\left(3^2\right) \\
&= 28 \\
|\underset{\sim}{a}-2\underset{\sim}{b}| &= 2\sqrt{7}
\end{aligned}$$

(d) **2 marks**

Let $\alpha = \tan^{-1}(5x)$ and $\beta = \tan^{-1}(3x)$, so $\tan\alpha = 5x$ and $\tan\beta = 3x$. Taking the tangent of both sides,

$$\begin{aligned}
\tan(\alpha - \beta) &= \frac{1}{4} \\
\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} &= \frac{1}{4} \\
\frac{5x - 3x}{1 + 15x^2} &= \frac{1}{4} \\
8x &= 1 + 15x^2 \\
15x^2 - 8x + 1 &= 0 \\
(5x - 1)(3x - 1) &= 0 \\
x &= \frac{1}{5} \text{ or } x = \frac{1}{3}.
\end{aligned}$$

For both values $x > 0$, so $5x > 3x > 0$ and $0 < \alpha - \beta < \frac{\pi}{2}$. Since $\tan(\alpha - \beta) = \frac{1}{4}$ with $\alpha - \beta$ in this interval, $\alpha - \beta = \tan^{-1}\left(\frac{1}{4}\right)$. Hence, $x = \frac{1}{5}$ or $x = \frac{1}{3}$.

Alternatively, test each solution in the original equation with a calculator:

$$\tan^{-1}\left(\frac{1}{4}\right) \approx 0.2450$$

When $x = \frac{1}{5}$,

$$\tan^{-1}(1) - \tan^{-1}\left(\frac{3}{5}\right) \approx 0.2450$$

When $x = \frac{1}{3}$,

$$\tan^{-1}\left(\frac{5}{3}\right) - \tan^{-1}(1) \approx 0.2450$$

(e) **3 marks**

Using the dot product,

$$\begin{pmatrix} m \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \left| \begin{pmatrix} m \\ 2 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right| \cos \frac{\pi}{4}$$

$$m + 6 = \sqrt{m^2 + 4} \cdot \sqrt{10} \cdot \frac{1}{\sqrt{2}}$$

$$m + 6 = \sqrt{5\,(m^2 + 4)}.$$

The right-hand side is positive, so $m > -6$. Squaring both sides,

$$\begin{aligned} (m+6)^2 &= 5\left(m^2+4\right) \\ m^2 + 12m + 36 &= 5m^2 + 20 \\ 4m^2 - 12m - 16 &= 0 \\ m^2 - 3m - 4 &= 0 \\ (m-4)(m+1) &= 0 \\ m = 4 \text{ or } m &= -1. \end{aligned}$$

Both values satisfy $m > -6$. Hence, $m = 4$ or $m = -1$.

(f) (i) **1 mark**

Since CD is the perpendicular bisector of AB, E is the midpoint of AB. Thus,

$$\begin{aligned}\overrightarrow{OE} &= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) \\ &= \frac{1}{2}\begin{pmatrix} 3\cos\theta - 4\sin\theta + 3\cos\theta + 4\sin\theta \\ 3\sin\theta + 4\cos\theta + 3\sin\theta - 4\cos\theta \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 6\cos\theta \\ 6\sin\theta \end{pmatrix} \\ &= 3\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ as required.}\end{aligned}$$

(ii) **2 marks**

$$\begin{aligned}
\overrightarrow{AE} &= \overrightarrow{OE} - \overrightarrow{OA} \\
&= \begin{pmatrix} 3\cos\theta - (3\cos\theta - 4\sin\theta) \\ 3\sin\theta - (3\sin\theta + 4\cos\theta) \end{pmatrix} \\
&= \begin{pmatrix} 4\sin\theta \\ -4\cos\theta \end{pmatrix} \\
\left|\overrightarrow{AE}\right|^2 &= 16\sin^2\theta + 16\cos^2\theta \\
&= 16 \\
\left|\overrightarrow{OE}\right| &= 3\sqrt{\cos^2\theta + \sin^2\theta} \\
&= 3
\end{aligned}$$

Since E lies on the radius OC and the radius of the circle is 5,

$$\begin{aligned}
\left|\overrightarrow{CE}\right| &= 5 - 3 \\
&= 2 \\
\left|\overrightarrow{DE}\right| &= 5 + 3 \\
&= 8 \\
\left|\overrightarrow{CE}\right|\left|\overrightarrow{DE}\right| &= 2 \cdot 8 \\
&= 16 \\
&= \left|\overrightarrow{AE}\right|^2 \text{ as required.}
\end{aligned}$$

Question 13

(a) **3 marks**

With $p = \frac{1}{4}$,

$$\begin{aligned}\sigma^2 &= \frac{\frac{1}{4} \cdot \frac{3}{4}}{n} \\ &= \frac{3}{16n} \\ \sigma &= \frac{\sqrt{3}}{4\sqrt{n}}.\end{aligned}$$

The student passes when $\hat{p} \geq 0.2$. Converting into z-scores,

$$\begin{aligned}P\left(\hat{p} \geq 0.2\right) &= P\left(Z \geq \frac{0.2 - 0.25}{\sigma}\right) \\ &= P\left(Z \leq \frac{0.05}{\sigma}\right) \text{ by symmetry.}\end{aligned}$$

From the table, $P(Z \leq 2.33) = 0.9901$. For at least a 99.01% chance of passing,

$$\begin{aligned}\frac{0.05}{\sigma} &\geq 2.33 \\ 0.05 \cdot \frac{4\sqrt{n}}{\sqrt{3}} &\geq 2.33 \\ \sqrt{n} &\geq \frac{2.33\sqrt{3}}{0.2} \\ n &\geq \frac{2.33^2 \cdot 3}{0.04} \\ &= 407.1675.\end{aligned}$$

Hence, the smallest value of n is 408.

(b) **2 marks**

$$\begin{aligned}
\text{RHS} &= \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1} \\
&= \frac{m(m-1)}{2} + \frac{n(n-1)}{2} + mn \\
&= \frac{m^2 - m + n^2 - n + 2mn}{2} \\
&= \frac{\left(m^2 + 2mn + n^2\right) - (m+n)}{2} \\
&= \frac{(m+n)^2 - (m+n)}{2} \\
&= \frac{(m+n)(m+n-1)}{2} \\
&= \binom{m+n}{2} \\
&= \text{LHS}
\end{aligned}$$

Alternate Method:

Consider the identity

$$(1+x)^{m+n} = (1+x)^m (1+x)^n.$$

By the binomial theorem, the coefficient of x^2 on the left-hand side is $\binom{m+n}{2}$. On the right-hand side,

$$(1+x)^m (1+x)^n = \left[\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots\right]\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots\right].$$

The x^2 terms come from multiplying the x^0 and x^2 terms, the x^1 and x^1 terms, and the x^2 and x^0 terms. So the coefficient of x^2 is

$$\binom{m}{0}\binom{n}{2} + \binom{m}{1}\binom{n}{1} + \binom{m}{2}\binom{n}{0} = \binom{n}{2} + \binom{m}{1}\binom{n}{1} + \binom{m}{2}.$$

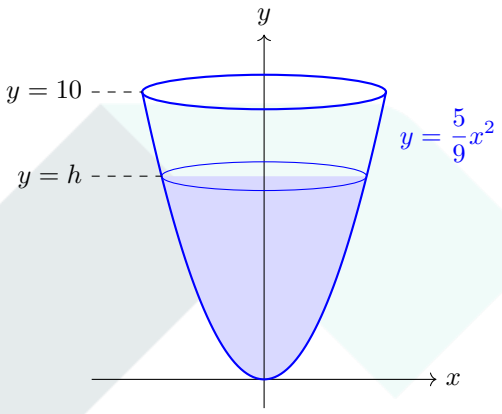
Equating the coefficients of x^2,

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}.$$

(c) **4 marks**

Rearranging $y = \frac{5}{9}x^2$ gives $x^2 = \frac{9y}{5}$. The volume of water when it has height h is

$$\begin{aligned} V &= \pi \int_0^h x^2 \, dy \\ &= \pi \int_0^h \frac{9y}{5} \, dy \\ &= \frac{9\pi}{5} \left[\frac{y^2}{2} \right]_0^h \\ &= \frac{9\pi h^2}{10}. \end{aligned}$$



The capacity of the glass is the volume when $h = 10$:

$$\begin{aligned} V &= \frac{9\pi \left(10^2\right)}{10} \\ &= 90\pi. \end{aligned}$$

At half capacity, $V = 45\pi$:

$$\begin{aligned} \frac{9\pi h^2}{10} &= 45\pi \\ h^2 &= 50 \\ h &= 5\sqrt{2} \text{ since } h > 0. \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dh} \cdot \frac{dh}{dt} \\ 15 &= \frac{9\pi h}{5} \cdot \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{75}{9\pi h} \\ &= \frac{25}{3\pi h}. \end{aligned}$$

When $h = 5\sqrt{2}$,

$$\frac{dh}{dt} = \frac{25}{3\pi \cdot 5\sqrt{2}}$$

$$
\begin{aligned}
&= \frac{5}{3\sqrt{2}\pi} \\
&= \frac{5\sqrt{2}}{6\pi} \text{ cms}^{-1} \\
&\approx 0.375 \text{ cms}^{-1}.
\end{aligned}
$$

(d) (i) **3 marks**

The plane makes an angle of 45° with the horizontal and passes through O, so it lies along the line $y = x$. The particle hits the plane when its horizontal and vertical components are equal:

$$\begin{aligned} 20t\sin\theta - 5t^2 &= 20t\cos\theta \\ 5t^2 - 20t\sin\theta + 20t\cos\theta &= 0 \\ 5t\left(t - 4\sin\theta + 4\cos\theta\right) &= 0 \\ t &= 4(\sin\theta - \cos\theta) \text{ since } t > 0. \end{aligned}$$

At this time, the horizontal displacement is

$$\begin{aligned} x &= 20 \cdot 4(\sin\theta - \cos\theta)\cos\theta \\ &= 80\cos\theta(\sin\theta - \cos\theta). \end{aligned}$$

Since P lies on $y = x$, the vertical displacement of P equals its horizontal displacement. Thus,

$$\begin{aligned} \ell &= \sqrt{x^2 + x^2} \\ &= \sqrt{2}\,x \\ &= 80\sqrt{2}\cos\theta(\sin\theta - \cos\theta) \text{ as required.} \end{aligned}$$

(ii) **3 marks**

$$\begin{aligned}
\ell &= 80\sqrt{2}\left(\sin\theta\cos\theta - \cos^2\theta\right) \\
\frac{d\ell}{d\theta} &= 80\sqrt{2}\left(\cos^2\theta - \sin^2\theta + 2\cos\theta\sin\theta\right) \\
&= 80\sqrt{2}\left(\cos 2\theta + \sin 2\theta\right)
\end{aligned}$$

Stationary points occur when $\dfrac{d\ell}{d\theta} = 0$:

$$\begin{aligned}
\cos 2\theta + \sin 2\theta &= 0 \\
\tan 2\theta &= -1.
\end{aligned}$$

The particle must be projected above the plane, so $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$, that is $\dfrac{\pi}{2} < 2\theta < \pi$. Thus,

$$\begin{aligned}
2\theta &= \frac{3\pi}{4} \\
\theta &= \frac{3\pi}{8}.
\end{aligned}$$

Testing the nature of the stationary point,

$$\begin{aligned}
\frac{d^2\ell}{d\theta^2} &= 80\sqrt{2}\left(-2\sin 2\theta + 2\cos 2\theta\right) \\
&= 160\sqrt{2}\left(\cos 2\theta - \sin 2\theta\right)
\end{aligned}$$

When $\theta = \dfrac{3\pi}{8}$,

$$\begin{aligned}
\frac{d^2\ell}{d\theta^2} &= 160\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \\
&= -320 \\
&< 0.
\end{aligned}$$

Hence, ℓ is a maximum when $\theta = \dfrac{3\pi}{8}$, that is $\theta = 67.5°$.

Question 14

(a) **2 marks**

Since g is the inverse of f, $f(g(x)) = x$. Differentiating with respect to x,

$$\begin{aligned} f'(g(x))\, g'(x) &= 1 \\ g'(x) &= \frac{1}{f'(g(x))}. \end{aligned}$$

Differentiating again using the chain rule,

$$\begin{aligned} g''(x) &= -\frac{f''(g(x))\, g'(x)}{[f'(g(x))]^2} \\ &= -\frac{f''(g(x))}{[f'(g(x))]^3}. \end{aligned}$$

Since f is increasing, $f'(x) > 0$ for all x, so $[f'(g(x))]^3 > 0$. Since f is concave up, $f''(x) > 0$ for all x, so $f''(g(x)) > 0$. Thus, $g''(x) < 0$, so $g(x)$ is concave down.

(b) **3 marks**

Since the curve passes through $\left(\frac{\pi}{4}, -e\right)$, $y \neq 0$. Separating the variables,

$$\begin{aligned}
\cos^2 x \frac{dy}{dx} &= y \\
\frac{1}{y}\frac{dy}{dx} &= \sec^2 x \\
\int \frac{1}{y}\, dy &= \int \sec^2 x\, dx \\
\ln|y| &= \tan x + C \\
|y| &= e^{\tan x + C} \\
y &= Ae^{\tan x}, \text{ where } A = \pm e^C.
\end{aligned}$$

When $x = \frac{\pi}{4}$, $y = -e$. Thus,

$$\begin{aligned}
-e &= Ae^{\tan \frac{\pi}{4}} \\
-e &= Ae \\
A &= -1 \\
y &= -e^{\tan x}
\end{aligned}$$

(c) **3 marks**

Rearranging,

$$\begin{aligned} 3\cos x - 3\sin x + m\sin x &= 5 \\ 3\cos x + (m-3)\sin x &= 5. \end{aligned}$$

Let $3\cos x + (m-3)\sin x = R\cos(x-\alpha)$, where $R > 0$.

$$\begin{aligned} R\cos x\cos\alpha + R\sin x\sin\alpha &= 3\cos x + (m-3)\sin x \\ R\cos\alpha &= 3 \\ R\sin\alpha &= m-3 \\ R^2 &= 3^2 + (m-3)^2 \\ R &= \sqrt{9+(m-3)^2} \end{aligned}$$

Since $-1 \leq \cos(x-\alpha) \leq 1$, the equation $R\cos(x-\alpha) = 5$ has no solutions exactly when $R < 5$:

$$\begin{aligned} \sqrt{9+(m-3)^2} &< 5 \\ 9+(m-3)^2 &< 25 \\ (m-3)^2 &< 16 \\ -4 < m-3 &< 4 \\ -1 < m &< 7. \end{aligned}$$

(d) **4 marks**

Separating the variables and using the given result,

$$\begin{aligned}
\frac{dP}{dt} &= \frac{P(5000-P)}{5000} \\
\frac{5000}{P(5000-P)}\frac{dP}{dt} &= 1 \\
\int \left(\frac{1}{P} + \frac{1}{5000-P}\right) dP &= \int 1 \, dt \\
\ln|P| - \ln|5000-P| &= t + C.
\end{aligned}$$

When $t = 0$, $P = 7000$. Thus,

$$\begin{aligned}
\ln 7000 - \ln|5000-7000| &= C \\
C &= \ln 7000 - \ln 2000 \\
&= \ln \frac{7}{2}.
\end{aligned}$$

When $P = 5100$,

$$\begin{aligned}
\ln 5100 - \ln|5000-5100| &= t + \ln\frac{7}{2} \\
\ln 5100 - \ln 100 &= t + \ln\frac{7}{2} \\
t &= \ln 51 - \ln\frac{7}{2} \\
&= \ln\frac{102}{7} \\
&\approx 2.68.
\end{aligned}$$

Hence, it takes approximately 2.68 years for the population to reach 5100.

(e) **3 marks**

By the product rule,

$$y = x\tan^{-1} x$$
$$\frac{dy}{dx} = \tan^{-1} x + \frac{x}{1+x^2}.$$

When $x = 0$,

$$\frac{dy}{dx} = \tan^{-1} 0 + \frac{0}{1+0}$$
$$= 0,$$

so there is a stationary point at $(0, 0)$.

For $x > 0$, $\tan^{-1} x > 0$ and $\frac{x}{1+x^2} > 0$, so $\frac{dy}{dx} > 0$. For $x < 0$, $\tan^{-1} x < 0$ and $\frac{x}{1+x^2} < 0$, so $\frac{dy}{dx} < 0$.

Thus, $\frac{dy}{dx} \neq 0$ for $x \neq 0$.

Hence, $y = x\tan^{-1} x$ has only one stationary point, at $(0, 0)$.