HSC exam solutions

2026 4U Mathematics Extension 2 HSC exam solutions

Multiple choice

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
A	B	D	D	B	C	C	B	A	C

Question 1

Since $f(x)$ has real coefficients, by the conjugate root theorem $3-i$ is also a root. Let $\alpha = 3-i$. By the product of roots,

$$\begin{aligned}(3+i)(3-i)\beta &= -\frac{-40}{1} \\ 10\beta &= 40 \\ \beta &= 4.\end{aligned}$$

Thus, $\alpha = 3-i$ and $\beta = 4$.

Hence, the answer is A.

Question 2

$$\begin{aligned}\left|-2\underset{\sim}{i}+3\underset{\sim}{j}-6\underset{\sim}{k}\right| &= \sqrt{(-2)^2+3^2+(-6)^2} \\ &= \sqrt{4+9+36} \\ &= \sqrt{49} \\ &= 7\end{aligned}$$

Hence, the answer is B.

Question 3

A sphere of radius 9 centred at $(-1, 3, 0)$ has equation

$$\begin{aligned}(x+1)^2 + (y-3)^2 + z^2 &= 81 \\ x^2 + 2x + 1 + y^2 - 6y + 9 + z^2 &= 81 \\ x^2 + 2x + y^2 - 6y + z^2 &= 71.\end{aligned}$$

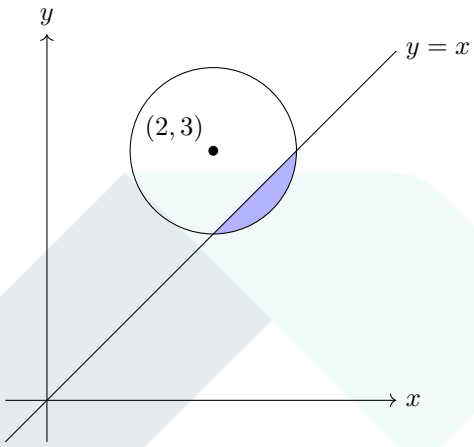
Hence, the answer is D.

Question 4

The inequality $|z-2-3i| \leq 1$ is the disc of radius 1 centred at $2+3i$. Since $3 > 2$, the centre lies above the line $y = x$. Letting $z = x + iy$,

$$\text{Re}(z) \geq \text{Im}(z)$$
$$x \geq y$$

which is the region on or below the line $y = x$. Hence, the region is the part of the disc centred at $(2, 3)$ that lies below the line $y = x$.

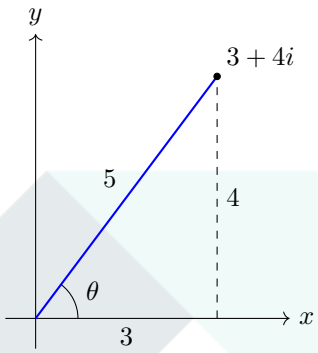


Hence, the answer is D.

Question 5

Using the law of arguments,

$$\begin{aligned}\theta &= \arg(w) - \arg(z) \\ &= \arg\left(\frac{w}{z}\right) \\ &= \arg(3+4i) \\ &= \tan^{-1}\left(\frac{4}{3}\right)\end{aligned}$$



$$\begin{aligned}|3+4i| &= \sqrt{3^2+4^2} \\ &= 5 \\ \sin\theta &= \frac{4}{5}\end{aligned}$$

Hence, the answer is B.

Question 6

To negate the statement, each quantifier is swapped ($\forall$ becomes $\exists$ and $\exists$ becomes $\forall$) and the final condition is negated. The conditions $p > 0$ and $q > p$ attached to each quantifier stay the same. Thus, the negation is

$$\exists p > 0 \text{ such that } q^2 \neq p \quad \forall q > p.$$

Hence, the answer is C.

Question 7

Using partial fractions,

$$\begin{aligned}\frac{7}{(x+1)(x-6)^2} &= \frac{P}{x+1} + \frac{Q}{x-6} + \frac{R}{(x-6)^2} \\ 7 &= P(x-6)^2 + Q(x+1)(x-6) + R(x+1)\end{aligned}$$

When $x = 6$,

$$\begin{aligned}7 &= 7R \\ R &= 1\end{aligned}$$

When $x = -1$,

$$\begin{aligned}7 &= 49P \\ P &= \frac{1}{7}\end{aligned}$$

Equating coefficients of x^2,

$$\begin{aligned}0 &= P + Q \\ Q &= -\frac{1}{7}.\end{aligned}$$

Thus,

$$\begin{aligned}\int \frac{7}{(x+1)(x-6)^2}\,dx &= \int \left(\frac{\frac{1}{7}}{x+1} - \frac{\frac{1}{7}}{x-6} + \frac{1}{(x-6)^2}\right) dx \\ &= \frac{1}{7}\ln|x+1| - \frac{1}{7}\ln|x-6| - \frac{1}{x-6} + K.\end{aligned}$$

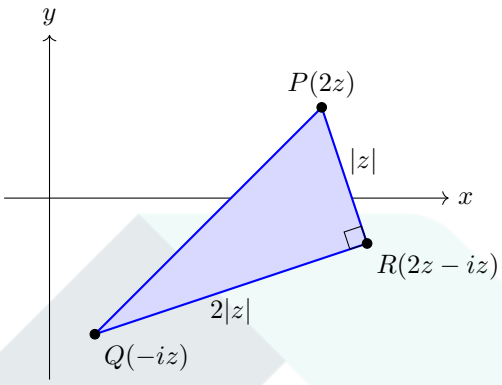
Thus, it is of the form $A\log_e|x+1| + B\log_e|x-6| + \dfrac{C}{x-6}$.

Hence, the answer is C.

Question 8

Let the vertices be $P(2z)$, $Q(-iz)$ and $R(2z-iz)$. Consider the sides meeting at R:

$$\begin{aligned}\overrightarrow{RP} &= 2z-(2z-iz)\\ &= iz\\ \overrightarrow{RQ} &= -iz-(2z-iz)\\ &= -2z.\end{aligned}$$



Since iz is z rotated by $\frac{\pi}{2}$, $\overrightarrow{RP} \perp \overrightarrow{RQ}$, so the triangle is right-angled at R. Thus,

$$\begin{aligned}\text{Area} &= \frac{1}{2}|iz||-2z|\\ &= \frac{1}{2}\cdot|z|\cdot 2|z|\\ &= |z|^2.\end{aligned}$$

Hence, the answer is B.

Question 9

Comparing the direction vectors,

$$\begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

so the lines are parallel.

Now check whether $(3, 2, -2)$, which lies on ℓ_2, also lies on ℓ_1:

$$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

$\lambda = 1$ from the first component

$$\begin{pmatrix} 2+1 \\ 3-1 \\ -4+2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}.$$

Since the lines are parallel and share a point, they are the same line. Thus, the lines are intersecting and parallel.

Hence, the answer is A.

Question 10

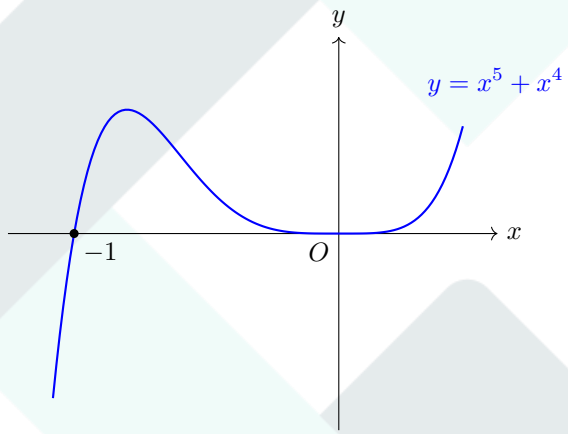
A counter-example must have $f''(0) = 0$ without a change in concavity at $x = 0$. For option C,

$$\begin{aligned} f(x) &= x^5 + x^4 \\ f'(x) &= 5x^4 + 4x^3 \\ f''(x) &= 20x^3 + 12x^2 \\ &= 4x^2(5x + 3) \\ f''(0) &= 0. \end{aligned}$$

Testing values either side of $x = 0$,

x	-0.1	0	0.1
$f''(x)$	0.1	0	0.14

For $-\frac{3}{5} < x < 0$ and for $x > 0$, both $4x^2 > 0$ and $5x + 3 > 0$, so $f''(x) > 0$ on both sides of $x = 0$. The concavity does not change, so there is no point of inflection at $x = 0$. Thus, $f(x) = x^5 + x^4$ is a counter-example.



Hence, the answer is C.

Question 11

(a) **1 mark**

If a is odd then a^3 is odd.

(b) (i) **1 mark**

$$\begin{aligned} 3z + w &= 3(3 + 2i) + (-1 - 5i) \\ &= 9 + 6i - 1 - 5i \\ &= 8 + i \end{aligned}$$

(ii) **1 mark**

$$\begin{aligned} 2\bar{z}w &= 2(3-2i)(-1-5i) \\ &= 2\left(-3-15i+2i+10i^2\right) \\ &= 2(-13-13i) \\ &= -26-26i \end{aligned}$$

(c) **2 marks**

$$\overrightarrow{OP_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OP_2} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix}$$

$$\left|\overrightarrow{OP_1}\right| = 1$$

$$\left|\overrightarrow{OP_2}\right| = \sqrt{\frac{1}{10} + \frac{9}{10}}$$

$$= 1$$

$$\cos\theta = \frac{\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}}{\left|\overrightarrow{OP_1}\right| \left|\overrightarrow{OP_2}\right|}$$

$$= \frac{1}{\sqrt{10}}$$

$$\sec^2\theta = 10$$

$$\tan^2\theta = \sec^2\theta - 1$$

$$= 9$$

(d) **2 marks**

Let $(x+iy)^2 = 11+60i$, where $x, y \in \mathbb{R}$.

$$\begin{aligned} x^2 - y^2 + 2xyi &= 11 + 60i \\ x^2 - y^2 &= 11 \text{ by equating real parts} \\ 2xy &= 60 \text{ by equating imaginary parts} \\ xy &= 30 \\ x = 6, y = 5 &\text{ or } x = -6, y = -5 \text{ by inspection} \end{aligned}$$

Hence, the square roots of $11+60i$ are $6+5i$ and $-6-5i$.

(e) **2 marks**

$$
\begin{aligned}
z &= re^{i\theta} \\
\overline{z} &= re^{-i\theta} \\
z\overline{z} &= re^{i\theta} \cdot re^{-i\theta} \\
&= r^2 e^0 \\
&= r^2 \\
|z| &= \left|re^{i\theta}\right| \\
&= r \\
\therefore z\overline{z} &= |z|^2 \text{ as required.}
\end{aligned}
$$

(f) **3 marks**

Completing the square,

$$\begin{aligned} 6x - x^2 &= 9 - \left(x^2 - 6x + 9\right) \\ &= 9 - (x-3)^2. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\frac{3}{2}}^{3} \frac{1}{\sqrt{6x - x^2}}\, dx &= \int_{\frac{3}{2}}^{3} \frac{1}{\sqrt{9 - (x-3)^2}}\, dx \\ &= \left[\sin^{-1}\left(\frac{x-3}{3}\right)\right]_{\frac{3}{2}}^{3} \\ &= \sin^{-1}(0) - \sin^{-1}\left(-\frac{1}{2}\right) \\ &= 0 - \left(-\frac{\pi}{6}\right) \\ &= \frac{\pi}{6} \end{aligned}$$

(g) (i) **1 mark**

$$\begin{aligned} |z| &= \sqrt{a^2 + a^2} \\ &= \sqrt{2}a \text{ since } a > 0 \\ \left|z^5\right| &= |z|^5 \\ &= \left(\sqrt{2}a\right)^5 \\ &= 4\sqrt{2}a^5 \end{aligned}$$

(ii) **2 marks**

Since $a > 0$, $z = a + ai$ lies in the first quadrant on the line $y = x$.

$$\begin{aligned}
\arg(z) &= \frac{\pi}{4} \\
\arg\left(z^5\right) &= 5 \cdot \frac{\pi}{4} \\
&= \frac{5\pi}{4} \\
\operatorname{Arg}\left(z^5\right) &= \frac{5\pi}{4} - 2\pi \\
&= -\frac{3\pi}{4}
\end{aligned}$$

Question 12

(a) (i) **3 marks**

$$\begin{aligned}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^4 &= \frac{\left(e^{i\theta}-e^{-i\theta}\right)^4}{(2i)^4}\\ &= \frac{1}{16}\left(e^{4i\theta}-4e^{3i\theta}e^{-i\theta}+6e^{2i\theta}e^{-2i\theta}-4e^{i\theta}e^{-3i\theta}+e^{-4i\theta}\right)\\ &= \frac{1}{16}\left(e^{4i\theta}-4e^{2i\theta}+6-4e^{-2i\theta}+e^{-4i\theta}\right)\\ &= \frac{1}{16}\left[\left(e^{4i\theta}+e^{-4i\theta}\right)-4\left(e^{2i\theta}+e^{-2i\theta}\right)+6\right]\\ &= \frac{1}{16}\left(2\cos 4\theta-8\cos 2\theta+6\right)\\ &= \frac{1}{8}\left(\cos 4\theta-4\cos 2\theta+3\right) \text{ as required.}\end{aligned}$$

(ii) **2 marks**

Since $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$,

$$\begin{aligned}
\frac{e^{i\theta} - e^{-i\theta}}{2i} &= \frac{2i\sin\theta}{2i} \\
&= \sin\theta \\
\sin^4\theta &= \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3) \text{ from part (i)} \\
\int \sin^4\theta \, d\theta &= \frac{1}{8}\int (\cos 4\theta - 4\cos 2\theta + 3) \, d\theta \\
&= \frac{1}{8}\left(\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta\right) + C \\
&= \frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{3\theta}{8} + C
\end{aligned}$$

(b) **2 marks**

Since the polynomial has real coefficients, by the conjugate root theorem $1-6i$ and $3+4i$ are also zeroes.

$$\begin{aligned}(x-(1+6i))\,(x-(1-6i)) &= (x-1)^2-(6i)^2\\ &= x^2-2x+1+36\\ &= x^2-2x+37\\ (x-(3-4i))\,(x-(3+4i)) &= (x-3)^2-(4i)^2\\ &= x^2-6x+9+16\\ &= x^2-6x+25\end{aligned}$$

Hence, a suitable polynomial is

$$P(x)=(x-2)\left(x^2-2x+37\right)\left(x^2-6x+25\right).$$

(c) **2 marks**

Since a, b and c are positive, apply the given inequality to each factor:

$$\begin{aligned} a + 7b &\geq 2\sqrt{7ab} \\ b + 2c &\geq 2\sqrt{2bc} \\ c + 14a &\geq 2\sqrt{14ca}. \end{aligned}$$

Since both sides of each inequality are positive, they can be multiplied together:

$$\begin{aligned} (a + 7b)(b + 2c)(c + 14a) &\geq 8\sqrt{7ab \cdot 2bc \cdot 14ca} \\ &= 8\sqrt{196a^2b^2c^2} \\ &= 8 \cdot 14abc \\ &= 112abc \text{ as required.} \end{aligned}$$

(d) **3 marks**

From the graph, $\ddot{x}$ has period 4 and amplitude π^2, with $\ddot{x} = \pi^2$ when $t = 0$.

$$\begin{aligned} n &= \frac{2\pi}{4} \\ &= \frac{\pi}{2} \\ \ddot{x} &= \pi^2 \cos\left(\frac{\pi t}{2}\right) \\ \dot{x} &= 2\pi \sin\left(\frac{\pi t}{2}\right) + C_1 \end{aligned}$$

When $t = 0$, $\dot{x} = 0$ since the particle is initially at rest. Thus,

$$\begin{aligned} 0 &= 0 + C_1 \\ C_1 &= 0 \\ x &= -4\cos\left(\frac{\pi t}{2}\right) + C_2 \end{aligned}$$

When $t = 0$, $x = -2$. Thus,

$$\begin{aligned} -2 &= -4 + C_2 \\ C_2 &= 2 \\ x &= 2 - 4\cos\left(\frac{\pi t}{2}\right) \end{aligned}$$

(e) **3 marks**

$$\begin{aligned}\int \frac{e^x + e^{2x}}{1 + e^{2x}}\, dx &= \int \left(\frac{e^x}{1 + (e^x)^2} + \frac{e^{2x}}{1 + e^{2x}} \right) dx \\ &= \int \frac{e^x}{1 + (e^x)^2}\, dx + \frac{1}{2} \int \frac{2e^{2x}}{1 + e^{2x}}\, dx \\ &= \tan^{-1}(e^x) + \frac{1}{2} \ln\left(1 + e^{2x}\right) + C\end{aligned}$$

Question 13

(a) **2 marks**

From the pattern, the k^{th} line is $4k = (k+1)^2 - (k-1)^2$. Let k be a positive integer. Then,

$$\begin{aligned}(k+1)^2 - (k-1)^2 &= \left(k^2 + 2k + 1\right) - \left(k^2 - 2k + 1\right) \\ &= 4k.\end{aligned}$$

Since $k+1$ and $k-1$ are integers, every positive multiple of 4 can be written as a difference of squares.

(b) **2 marks**

Simple harmonic motion with centre c and amplitude a satisfies $v^2 = n^2\left(a^2 - (x-c)^2\right)$. Completing the square,

$$\begin{aligned} v^2 &= -2x^2 + 8x + 8 \\ &= -2\left(x^2 - 4x - 4\right) \\ &= -2\left[(x-2)^2 - 8\right] \\ &= 2\left(8 - (x-2)^2\right) \\ a^2 &= 8 \\ a &= 2\sqrt{2}. \end{aligned}$$

In one period, the particle travels from one extreme to the other and back, so the distance travelled is

$$\begin{aligned} 4a &= 4 \cdot 2\sqrt{2} \\ &= 8\sqrt{2} \text{ m}. \end{aligned}$$

(c) **3 marks**

Let $t = \tan\frac{\theta}{2}$.

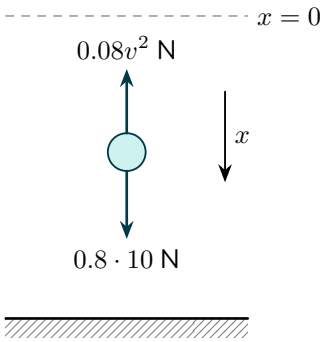
$$\begin{aligned} dt &= \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta \\ &= \frac{1}{2}\left(1+t^2\right)\,d\theta \\ d\theta &= \frac{2}{1+t^2}\,dt \\ \cos\theta &= \frac{1-t^2}{1+t^2} \\ \sin\theta &= \frac{2t}{1+t^2} \end{aligned}$$

Thus,

$$\begin{aligned} \int \frac{d\theta}{5+3\cos\theta-2\sin\theta} &= \int \frac{1}{5+\frac{3(1-t^2)}{1+t^2}-\frac{4t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt \\ &= \int \frac{2}{5\left(1+t^2\right)+3\left(1-t^2\right)-4t}\,dt \\ &= \int \frac{2}{2t^2-4t+8}\,dt \\ &= \int \frac{1}{t^2-2t+4}\,dt \\ &= \int \frac{1}{(t-1)^2+3}\,dt \\ &= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t-1}{\sqrt{3}}\right)+C \\ &= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{\theta}{2}-1}{\sqrt{3}}\right)+C \end{aligned}$$

(d) **3 marks**

On the way down, take downwards as positive, with the origin at the maximum height.



$$\begin{aligned}
0.8\ddot{x} &= 0.8 \cdot 10 - 0.08v^2 \\
\ddot{x} &= 10 - 0.1v^2 \\
v\frac{dv}{dx} &= \frac{100 - v^2}{10} \\
\frac{dx}{dv} &= \frac{10v}{100 - v^2} \\
x &= -5 \ln \left|100 - v^2\right| + C
\end{aligned}$$

When $x = 0$, $v = 0$. Thus,

$$\begin{aligned}
0 &= -5 \ln 100 + C \\
C &= 5 \ln 100 \\
x &= 5 \ln \left(\frac{100}{100 - v^2}\right)
\end{aligned}$$

The stone returns to the point of projection after falling $5 \ln \left(\frac{676}{100}\right)$ m. Thus,

$$\begin{aligned}
5 \ln \left(\frac{676}{100}\right) &= 5 \ln \left(\frac{100}{100 - v^2}\right) \\
\frac{676}{100} &= \frac{100}{100 - v^2} \\
100 - v^2 &= \frac{10000}{676} \\
v^2 &= \frac{67600 - 10000}{676} \\
&= \frac{57600}{676} \\
v &= \frac{240}{26} \\
&= \frac{120}{13} \\
&\approx 9.23 \text{ ms}^{-1}
\end{aligned}$$

(e) (i) **2 marks**

We proceed by contradiction. Suppose that $\sqrt{6}$ is rational. Thus, $\sqrt{6} = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$ and are coprime.

$$\begin{aligned} 6 &= \frac{p^2}{q^2} \\ p^2 &= 6q^2 \\ &= 2\left(3q^2\right) \end{aligned}$$

So p^2 is even. Since the square of an odd number is odd, p is even. Let $p = 2m$, where $m \in \mathbb{Z}^+$.

$$\begin{aligned} (2m)^2 &= 6q^2 \\ 4m^2 &= 6q^2 \\ 3q^2 &= 2m^2 \end{aligned}$$

So $3q^2$ is even. Since 3 is odd, q^2 is even, and hence q is even. Thus, p and q are both even, which contradicts p and q being coprime. Hence, $\sqrt{6}$ is irrational.

(ii) **2 marks**

We proceed by contradiction. Suppose that $\sqrt{3}-\sqrt{2}=r$, where r is rational.

$$\begin{aligned}\left(\sqrt{3}-\sqrt{2}\right)^2 &= r^2 \\ 3-2\sqrt{6}+2 &= r^2 \\ \sqrt{6} &= \frac{5-r^2}{2}\end{aligned}$$

Since r is rational, $\dfrac{5-r^2}{2}$ is rational, so $\sqrt{6}$ is rational. This contradicts part (i). Hence, $\sqrt{3}-\sqrt{2}$ is irrational.

Question 14

(a) **4 marks**

Let T be the transmitter at $(2, 2, 51)$. Then,

$$\begin{aligned} \underset{\sim}{r}(t) - \overrightarrow{OT} &= \begin{pmatrix} 5 - \frac{t}{4} \\ 4 \\ 51 + \frac{t}{4} \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 51 \end{pmatrix} \\ &= \begin{pmatrix} 3 - \frac{t}{4} \\ 2 \\ \frac{t}{4} \end{pmatrix}. \end{aligned}$$

The helicopter is 3 m from the transmitter when

$$\begin{aligned} \left(3 - \frac{t}{4}\right)^2 + 2^2 + \left(\frac{t}{4}\right)^2 &= 9 \\ 9 - \frac{3t}{2} + \frac{t^2}{16} + 4 + \frac{t^2}{16} &= 9 \\ \frac{t^2}{8} - \frac{3t}{2} + 4 &= 0 \\ t^2 - 12t + 32 &= 0 \\ (t-4)(t-8) &= 0 \\ t = 4 \text{ or } t &= 8. \end{aligned}$$

The helicopter first comes within 3 m when $t = 4$. At this time,

$$\begin{aligned} \underset{\sim}{r}(4) &= \begin{pmatrix} 4 \\ 4 \\ 52 \end{pmatrix} \\ \dot{\underset{\sim}{r}}(t) &= \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix}. \end{aligned}$$

So the helicopter loses power at a height of 52 m with an upwards vertical velocity of $\frac{1}{4}$ ms^{-1}. Let τ be the time in seconds after it loses power. Vertically,

$$\begin{aligned} \ddot{z} &= -10 \\ \dot{z} &= -10\tau + \frac{1}{4} \\ z &= -5\tau^2 + \frac{\tau}{4} + 52. \end{aligned}$$

The helicopter reaches the ground when $z = 0$:

$$\begin{aligned} -5\tau^2 + \frac{\tau}{4} + 52 &= 0 \\ 20\tau^2 - \tau - 208 &= 0 \\ \tau &= \frac{1 \pm \sqrt{1 + 4 \cdot 20 \cdot 208}}{40} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1 \pm \sqrt{16641}}{40} \\
&= \frac{1 \pm 129}{40} \\
&= \frac{13}{4} \text{ since } \tau > 0.
\end{aligned}
$$

Hence, the helicopter takes 3.25 seconds to reach the ground.

(b) **3 marks**

$$\begin{aligned} V &= \pi \int_0^3 y^2 \, dx \\ &= \pi \int_0^3 x^2 e^x \, dx \end{aligned}$$

Integrating by parts,

$$\begin{aligned} u &= x^2 \\ u' &= 2x \\ v' &= e^x \\ v &= e^x \\ \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx. \end{aligned}$$

Integrating by parts again,

$$\begin{aligned} u &= x \\ u' &= 1 \\ v' &= e^x \\ v &= e^x \\ \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x + C. \end{aligned}$$

Thus,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2\left(x e^x - e^x\right) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= e^x \left(x^2 - 2x + 2\right) + C. \end{aligned}$$

Hence,

$$\begin{aligned} V &= \pi \left[e^x \left(x^2 - 2x + 2\right)\right]_0^3 \\ &= \pi \left[e^3(9 - 6 + 2) - e^0(0 - 0 + 2)\right] \\ &= \pi \left(5e^3 - 2\right) \text{ units}^3 \end{aligned}$$

(c) (i) **1 mark**

$$\begin{aligned}\overrightarrow{BA} &= \begin{pmatrix}4\\5\\2\end{pmatrix} - \begin{pmatrix}3b\\4b\\4b\end{pmatrix} \\ &= \begin{pmatrix}4-3b\\5-4b\\2-4b\end{pmatrix}\end{aligned}$$

Since $\overrightarrow{BA}$ is perpendicular to ℓ,

$$\begin{pmatrix}4-3b\\5-4b\\2-4b\end{pmatrix} \cdot \begin{pmatrix}6\\2\\-3\end{pmatrix} = 0$$

$$\begin{aligned}6(4-3b) + 2(5-4b) - 3(2-4b) &= 0 \\ 24 - 18b + 10 - 8b - 6 + 12b &= 0 \\ 28 - 14b &= 0 \\ b &= 2 \text{ as required.}\end{aligned}$$

(ii) **2 marks**

Since $\angle BAC = \frac{\pi}{2}$ and $\angle ACB = \frac{\pi}{4}$, $\angle ABC = \frac{\pi}{4}$. So triangle ABC is isosceles with $AC = AB$.

$$\begin{aligned}\overrightarrow{BA} &= \begin{pmatrix} 4-6 \\ 5-8 \\ 2-8 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix} \\ \left|\overrightarrow{BA}\right| &= \sqrt{4+9+36} \\ &= 7 \\ \left|\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}\right| &= \sqrt{36+4+9} \\ &= 7\end{aligned}$$

Since the direction vector of ℓ has length 7, C is one direction vector from A along ℓ, in either direction:

$$\begin{aligned}\overrightarrow{OC} &= \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 10 \\ 7 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}.\end{aligned}$$

(d) (i) **2 marks**

While the particle moves in the positive direction, the resistance acts in the negative direction. By Newton's Second Law,

$$\begin{aligned}
m\ddot{x} &= -\left(mk + mv^2\right) \\
\frac{dv}{dt} &= -\left(k + v^2\right) \\
\frac{dt}{dv} &= -\frac{1}{k + v^2} \\
t &= -\int_u^v \frac{1}{k + w^2}\, dw \\
&= \int_v^u \frac{1}{k + w^2}\, dw \\
&= \left[\frac{1}{\sqrt{k}} \tan^{-1}\left(\frac{w}{\sqrt{k}}\right)\right]_v^u \\
&= \frac{1}{\sqrt{k}}\left(\tan^{-1}\frac{u}{\sqrt{k}} - \tan^{-1}\frac{v}{\sqrt{k}}\right) \text{ as required.}
\end{aligned}$$

(ii) **3 marks**

Rearranging part (i) for v,

$$\begin{aligned}\sqrt{k}t &= \tan^{-1}\frac{u}{\sqrt{k}} - \tan^{-1}\frac{v}{\sqrt{k}} \\ \tan^{-1}\frac{v}{\sqrt{k}} &= \tan^{-1}\frac{u}{\sqrt{k}} - \sqrt{k}t\end{aligned}$$

When $t = \dfrac{\pi}{4\sqrt{k}}$,

$$\begin{aligned}\tan^{-1}\frac{v}{\sqrt{k}} &= \tan^{-1}\frac{u}{\sqrt{k}} - \frac{\pi}{4} \\ \frac{v}{\sqrt{k}} &= \tan\left(\tan^{-1}\frac{u}{\sqrt{k}} - \frac{\pi}{4}\right) \\ &= \frac{\frac{u}{\sqrt{k}} - 1}{1 + \frac{u}{\sqrt{k}}} \text{ by the compound angle formula} \\ v &= \frac{\sqrt{k}\left(u - \sqrt{k}\right)}{u + \sqrt{k}}.\end{aligned}$$

Since $u > 0$ and $k > 0$, the denominator $u + \sqrt{k}$ is positive. The particle moves in the negative direction when $v < 0$:

$$\begin{aligned}\frac{\sqrt{k}\left(u - \sqrt{k}\right)}{u + \sqrt{k}} &< 0 \\ u - \sqrt{k} &< 0 \\ u &< \sqrt{k}.\end{aligned}$$

Hence, $0 < u < \sqrt{k}$.

Question 15

(a) **4 marks**

We proceed by mathematical induction.

Base case: $n = 2$

$$\begin{aligned}\text{LHS} &= \frac{1}{3} + \frac{1}{4} \\ &= \frac{7}{12} \\ &= \frac{14}{24} \\ &> \frac{13}{24} \text{ as required.}\end{aligned}$$

Assumption: $n = k$, for some integer $k > 1$

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k-1} + \frac{1}{2k} > \frac{13}{24}$$

Inductive step: $n = k + 1$

RTP: $\dfrac{1}{k+2} + \dfrac{1}{k+3} + \cdots + \dfrac{1}{2k+1} + \dfrac{1}{2k+2} > \dfrac{13}{24}$

$$\begin{aligned}\text{LHS} &= \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right) - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2} \\ &> \frac{13}{24} - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2} \text{ by assumption} \\ &= \frac{13}{24} - \frac{2}{2k+2} + \frac{1}{2k+1} + \frac{1}{2k+2} \\ &= \frac{13}{24} + \frac{1}{2k+1} - \frac{1}{2k+2} \\ &= \frac{13}{24} + \frac{(2k+2) - (2k+1)}{(2k+1)(2k+2)} \\ &= \frac{13}{24} + \frac{1}{(2k+1)(2k+2)} \\ &> \frac{13}{24} \text{ since } k > 0, \text{ as required.}\end{aligned}$$

Hence, by mathematical induction, the statement is true for all integers $n > 1$.

(b) **4 marks**

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 2-1 \\ 0-(-1) \\ 3-2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \ell : \underset{\sim}{r} &= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\end{aligned}$$

Since M lies on the line, $\overrightarrow{OM} = \begin{pmatrix} 1+\lambda \\ -1+\lambda \\ 2+\lambda \end{pmatrix}$ for some λ. Since PQ is perpendicular to the line and M is on PQ,

$$\begin{aligned}\overrightarrow{PM} &= \begin{pmatrix} 1+\lambda-4 \\ -1+\lambda-(-1) \\ 2+\lambda-5 \end{pmatrix} \\ &= \begin{pmatrix} \lambda-3 \\ \lambda \\ \lambda-3 \end{pmatrix} \\ \begin{pmatrix} \lambda-3 \\ \lambda \\ \lambda-3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= 0 \\ 3\lambda - 6 &= 0 \\ \lambda &= 2 \\ \overrightarrow{OM} &= \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}.\end{aligned}$$

Since M is the midpoint of PQ,

$$\begin{aligned}\overrightarrow{OQ} &= 2\overrightarrow{OM} - \overrightarrow{OP} \\ &= \begin{pmatrix} 6 \\ 2 \\ 8 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.\end{aligned}$$

Hence, Q is the point $(2, 3, 3)$.

(c) **3 marks**

$$\begin{aligned}
|z_{n+1}| &= \left|\frac{3}{4}i\right| |z_n| \\
&= \frac{3}{4} |z_n| \\
|z_n| &= 8\left(\frac{3}{4}\right)^{n-1} \\
z_{n+1} - z_n &= \frac{3}{4}iz_n - z_n \\
&= \left(-1 + \frac{3}{4}i\right) z_n \\
|z_{n+1} - z_n| &= \left|-1 + \frac{3}{4}i\right| |z_n| \\
&= \sqrt{1 + \frac{9}{16}}\, |z_n| \\
&= \frac{5}{4} \cdot 8\left(\frac{3}{4}\right)^{n-1} \\
&= 10\left(\frac{3}{4}\right)^{n-1}
\end{aligned}$$

The lengths form a geometric series with first term 10 and common ratio $\frac{3}{4}$. Since $\left|\frac{3}{4}\right| < 1$, the limiting sum exists:

$$\begin{aligned}
|z_2 - z_1| + |z_3 - z_2| + |z_4 - z_3| + \ldots &= 10 + 10\left(\frac{3}{4}\right) + 10\left(\frac{3}{4}\right)^2 + \ldots \\
&= \frac{10}{1 - \frac{3}{4}} \\
&= 40.
\end{aligned}$$

(d) (i) **2 marks**

Since z_1 is on the circle, $|z_1 - w| = 1$. Thus,

$$\begin{aligned} z_1 - w &= \text{cis}\left(-\frac{2\pi}{3}\right) \\ &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ z_1 &= 2 + i - \frac{1}{2} - \frac{\sqrt{3}}{2}i \\ &= \frac{3}{2} + \left(1 - \frac{\sqrt{3}}{2}\right)i \end{aligned}$$

(ii) **2 marks**

Since a tangent is perpendicular to the radius at the point of contact, the triangle with vertices O, z_2 and w is right-angled at z_2. By Pythagoras' theorem,

$$\begin{aligned} |w| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{5} \\ |z_2|^2 &= |w|^2 - |z_2 - w|^2 \\ &= 5 - 1 \\ |z_2| &= 2. \end{aligned}$$

Let ϕ be the angle between Ow and Oz_2. Then,

$$\begin{aligned} \cos\phi &= \frac{2}{\sqrt{5}} \\ \sin\phi &= \frac{1}{\sqrt{5}}. \end{aligned}$$

From the diagram, z_2 is obtained from w by an anticlockwise rotation through ϕ and a scaling by $\frac{|z_2|}{|w|} = \frac{2}{\sqrt{5}}$:

$$\begin{aligned} z_2 &= \frac{2}{\sqrt{5}} (\cos\phi + i\sin\phi)\, w \\ &= \frac{2}{\sqrt{5}} \left(\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} i \right) (2 + i) \\ &= \frac{2}{5}(2 + i)(2 + i) \\ &= \frac{2}{5}(3 + 4i) \\ &= \frac{6}{5} + \frac{8}{5} i \end{aligned}$$

Question 16

(a) **4 marks**

Let $z = \text{cis}\,\theta$, where $-\pi < \theta \leq \pi$. Since $|z| = 1$, $z^{-1} = \overline{z} = \cos\theta - i\sin\theta$, so $z + z^{-1} = 2\cos\theta$. Thus,

$$\begin{aligned} z^3 + z^2 + z &= z^2\left(z + 1 + z^{-1}\right) \\ &= (1 + 2\cos\theta)z^2 \\ &= (1 + 2\cos\theta)\left(\cos 2\theta + i\sin 2\theta\right) \\ \text{Im}\left(z^3 + z^2 + z\right) &= (1 + 2\cos\theta)\sin 2\theta \\ 0 &= (1 + 2\cos\theta)\sin 2\theta. \end{aligned}$$

Case 1: $\cos\theta = -\dfrac{1}{2}$

$$\begin{aligned} \theta &= \pm\frac{2\pi}{3} \\ z &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{aligned}$$

Case 2: $\sin 2\theta = 0$

$$\begin{aligned} 2\theta &= -\pi, 0, \pi, 2\pi \\ \theta &= -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi \\ z &= -i, 1, i, -1 \end{aligned}$$

Hence, $z = 1, -1, i, -i, -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ or $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$.

Alternate Method:

Let $z = \text{cis}\,\theta$. By De Moivre's Theorem,

$$\begin{aligned} \text{Im}\left(z^3 + z^2 + z\right) &= \sin 3\theta + \sin 2\theta + \sin\theta \\ &= \left(3\sin\theta - 4\sin^3\theta\right) + 2\sin\theta\cos\theta + \sin\theta \\ &= \sin\theta\left(4 - 4\sin^2\theta + 2\cos\theta\right) \\ &= \sin\theta\left(4\cos^2\theta + 2\cos\theta\right) \\ &= 2\sin\theta\cos\theta(2\cos\theta + 1) \\ 0 &= 2\sin\theta\cos\theta(2\cos\theta + 1). \end{aligned}$$

Then $\sin\theta = 0$ gives $z = \pm 1$, $\cos\theta = 0$ gives $z = \pm i$, and $\cos\theta = -\dfrac{1}{2}$ gives $z = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$.

(b) (i) **1 mark**

$$\begin{aligned} I_n - I_{n-1} &= \int_0^1 \frac{1}{(x^2+1)^n}\,dx - \int_0^1 \frac{1}{(x^2+1)^{n-1}}\,dx \\ &= \int_0^1 \left(\frac{1}{(x^2+1)^n} - \frac{x^2+1}{(x^2+1)^n}\right)\,dx \\ &= \int_0^1 \frac{1-(x^2+1)}{(x^2+1)^n}\,dx \\ &= \int_0^1 \frac{-x^2}{(x^2+1)^n}\,dx \text{ as required.} \end{aligned}$$

(ii) **3 marks**

$$\begin{aligned} I_1 &= \int_0^1 \frac{1}{x^2+1}\,dx \\ &= \left[\tan^{-1} x\right]_0^1 \\ &= \frac{\pi}{4} \end{aligned}$$

From part (i) with $n = 2$,

$$\begin{aligned} I_2 - I_1 &= \int_0^1 \frac{-x^2}{(x^2+1)^2}\,dx \\ &= \int_0^1 x \cdot \frac{-x}{(x^2+1)^2}\,dx. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} u &= x \\ u' &= 1 \\ v' &= \frac{-x}{(x^2+1)^2} \\ &= -\frac{1}{2} \cdot 2x\left(x^2+1\right)^{-2} \\ v &= -\frac{1}{2} \cdot \frac{\left(x^2+1\right)^{-1}}{-1} \\ &= \frac{1}{2\left(x^2+1\right)} \\ I_2 - I_1 &= \left[\frac{x}{2\left(x^2+1\right)}\right]_0^1 - \int_0^1 \frac{1}{2\left(x^2+1\right)}\,dx \\ &= \left(\frac{1}{2(1+1)} - 0\right) - \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx \\ &= \frac{1}{4} - \frac{1}{2}I_1 \\ I_2 &= I_1 - \frac{1}{2}I_1 + \frac{1}{4} \\ &= \frac{1}{2}I_1 + \frac{1}{4} \\ &= \frac{1}{2} \cdot \frac{\pi}{4} + \frac{1}{4} \\ &= \frac{\pi}{8} + \frac{1}{4} \end{aligned}$$

(c) (i) **2 marks**

Since A and B lie on the sphere of radius 1, $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right| = 1$, and since $\underset{\sim}{a} \perp \underset{\sim}{b}$, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

$$\begin{aligned}
\left|\underset{\sim}{v}(\theta)\right|^2 &= \underset{\sim}{v}(\theta) \cdot \underset{\sim}{v}(\theta) \\
&= \left(\cos\theta \underset{\sim}{a} + \sin\theta \underset{\sim}{b}\right) \cdot \left(\cos\theta \underset{\sim}{a} + \sin\theta \underset{\sim}{b}\right) \\
&= \cos^2\theta \left(\underset{\sim}{a} \cdot \underset{\sim}{a}\right) + 2\sin\theta\cos\theta \left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right) + \sin^2\theta \left(\underset{\sim}{b} \cdot \underset{\sim}{b}\right) \\
&= \cos^2\theta \left|\underset{\sim}{a}\right|^2 + 0 + \sin^2\theta \left|\underset{\sim}{b}\right|^2 \\
&= \cos^2\theta + \sin^2\theta \\
&= 1 \text{ as required.}
\end{aligned}$$

(ii) **2 marks**

At a point of intersection,

$$\begin{pmatrix} \frac{\cos\alpha}{\sqrt{5}} \\ \frac{2\cos\alpha}{\sqrt{5}} \\ -\sin\alpha \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \frac{\cos\theta}{\sqrt{2}} \\ \frac{\cos\theta}{\sqrt{2}} \end{pmatrix}.$$

Equating the second and third components,

$$\frac{2\cos\alpha}{\sqrt{5}} = -\sin\alpha$$
$$\tan\alpha = -\frac{2}{\sqrt{5}}$$
$$\sin\alpha = -\frac{2}{3}, \cos\alpha = \frac{\sqrt{5}}{3} \text{ or } \sin\alpha = \frac{2}{3}, \cos\alpha = -\frac{\sqrt{5}}{3}.$$

Substituting into C_1,

$$\frac{\sqrt{5}}{3} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$
$$-\frac{\sqrt{5}}{3} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

These lie on C_2 with $\sin\theta = \frac{1}{3}, \cos\theta = \frac{2\sqrt{2}}{3}$ and $\sin\theta = -\frac{1}{3}, \cos\theta = -\frac{2\sqrt{2}}{3}$ respectively. Hence, the points of intersection are $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and $\left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

(iii) **4 marks**

Let $\underset{\sim}{a} = \overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, which is a unit vector. To use part (i), we need a unit vector $\underset{\sim}{b}$ perpendicular to $\underset{\sim}{a}$ in the plane of O, A and P. Projecting $\overrightarrow{OP}$ onto $\overrightarrow{OA}$,

$$\begin{aligned} \overrightarrow{OP} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ \sqrt{2} \\ -1 \end{pmatrix} \\ \text{proj}_{\underset{\sim}{a}} \overrightarrow{OP} &= \frac{\overrightarrow{OP} \cdot \underset{\sim}{a}}{\underset{\sim}{a} \cdot \underset{\sim}{a}} \underset{\sim}{a} \\ &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}. \end{aligned}$$

The component of $\overrightarrow{OP}$ perpendicular to $\underset{\sim}{a}$ is

$$\begin{aligned} \overrightarrow{OP} - \text{proj}_{\underset{\sim}{a}} \overrightarrow{OP} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ \sqrt{2} \\ -1 \end{pmatrix} - \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \\ &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \\ \left| \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right| &= \frac{1}{2\sqrt{3}} \sqrt{3^2 + 0^2 + (-1)^2} \\ &= \frac{\sqrt{10}}{2\sqrt{3}}. \end{aligned}$$

Dividing by its length,

$$\begin{aligned} \underset{\sim}{b} &= \frac{2\sqrt{3}}{\sqrt{10}} \cdot \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}. \end{aligned}$$

Since $\overrightarrow{OP} = \frac{1}{\sqrt{6}} \underset{\sim}{a} + \frac{\sqrt{10}}{2\sqrt{3}} \underset{\sim}{b}$, the point P lies in the plane of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Hence, the circle has vector equation

$$\underset{\sim}{v}(\theta) = \cos\theta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \sin\theta \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \quad 0 \leq \theta < 2\pi.$$