Calculus Differentiator Questions - Solutions

Note: All of the below questions are calculator active

Q1) Feodor is trying to construct a new roller coaster. His initial design for one of the sections follows the graph of $f: [0, 2] \to R, f(x) = x^3 - ax^2 + 2x + b$ (where a, b are real numbers).

a) **If Feodor wants the track to be flat at the right-hand end, show that $a = \frac{7}{2}$.**

$f'(x) = 3x^2 - 2ax + 2$

Sub $f'(2) = 0$, solve for a

$0 = 12 - 4a + 2$
$4a = 14$
$a = \frac{7}{2}$

b) **Suppose that $a = \frac{7}{2}$ and the track stops right at the ground (which is taken to be $y = 0$). What is the value of b?**

$f(x) = x^3 - \frac{7}{2}x^2 + 2x + b$

Sub $f(2) = 0$, solve for b

$0 = 8 - 14 + 4 + b$

$b = 2$

c) **For his choice of b (and taking $a = \frac{7}{2}$), what are the coordinates of the highest point of the track?**

$f'(x) = 3x^2 - 7x + 2$

Solve $f'(x) = 0$

$x = \frac{1}{3}, x = 2$

We already know that $x = 2$ is the endpoint, so the local maximum must be at $x = \frac{1}{3}$.

$f\left(\frac{1}{3}\right) = \frac{125}{54}$

Highest point is $(\frac{1}{3}, \frac{125}{54})$.

Feodor wants to create a supporting beam which is normal to the track at some point P and also meets the ground at $x = 0$.

d) Find the possible coordinates of point P, to two decimal places.

Equation of the normal to the curve at $x = c$ (using 'Normal Line' function on CAS):

$y = -\frac{1}{f'(c)}(x - c) + f(c)$

Sub $(0,0)$

$0 = \frac{c}{f'(c)} + f(c)$, solve for c

$c = 0.3655 \ldots, c = 1.3847 \ldots$ (as $0 \le c \le 2$)

$f(0.3655 \ldots) = 2.3122 \ldots, f(1.3847 \ldots) = 0.7135 \ldots$

P is $(0.37, 2.31)$ or $(1.38, 0.71)$

e) Find the length of the supporting beam which meets the track at the highest possible point, to two decimal places.

This is the beam to $c = 0.365 \ldots$

Find distance between $(0.3655 \ldots, 2.3122 \ldots)$ and $(0,0)$:

$d = \sqrt{(0.3655 - 0)^2 + (2.3122 - 0)^2}$

$d = 2.34$

This proposal unfortunately failed council guidelines, which require that the slope of the track is never less than -2 or more than 2.

f) Find the largest possible value of a which meets council guidelines.

$f'(x) = 3x^2 - 2ax + 2$ (slope of curve)

Needs to be within range for $0 \le x \le 2$

Check endpoints:

$f'(0) = 2$ (within range)

$f'(2) = 14 - 4a$

$-2 \le 14 - 4a \le 2$

$3 \le a \le 4$

Check turning point:

$x = \frac{2a}{2 \times 3} = \frac{a}{3}$

Note: Since the turning point is at $x = \frac{a}{3}$, this is between $x = 0$ and $x = 2$, which means that the minimum occurs at $x = \frac{a}{3}$ and the y-coordinate needs to be at least -2 (then because $f(x)$ is a positive quadratic, the track will satisfy the conditions).

$f'\left(\frac{a}{3}\right) = 3\left(\frac{a}{3}\right)^2 - 2a\left(\frac{a}{3}\right) + 2 = -\frac{a^2}{3} + 2$

$-\frac{a^2}{3} + 2 \geq -2$

$a^2 \leq 12$

$a = 2\sqrt{3}$ is the largest value of a

Suppose that Feodor chooses the value of a found in part f). He wants to make the remainder of the track follow the graph of $g: [2, 5] \to g(x) = cx^2 - 5cx$, such that the two sections connect smoothly.

g) Find the values of b and c. Give your answers in the form $r + s\sqrt{t}$, where r, s, t are integers and t is as small as possible.

$f(x) = x^3 - 2\sqrt{3}x^2 + 2x + b$

$f'(x) = 3x^2 - 4\sqrt{3}x + 2$

$g(x) = cx^2 - 5cx$

$g'(x) = 2cx - 5c$

Solve $f(2) = g(2)$ and $f'(2) = g'(2)$ simultaneously on CAS

$b = 72 - 40\sqrt{3}, c = -14 + 8\sqrt{3}$

Q2) An underground collection tank has burst under the street. Water flows out of the tank for some time before the water flow ceases. The rate of water coming out, in litres per minute, can be modelled by the function $R(t) = 100t(log_e(t) - 3)^2$.

a) What is the implied domain of $R(t)$?

Solve $R(t) = 0$ for t

$t = e^3$ (when water flow ceases)

Implied domain is $(0, e^3]$

Note: $log_e(t)$ is undefined at $t = 0$, so 0 is not included in the domain.

b) How much water was in the tank (to the nearest litre)?

Since $R(t)$ is the rate of loss of water (in L/min), the area under the graph represents the total volume of water lost.

$\int_0^{e^3} R(t)\,dt = 10086 \text{ L}$

c) At what point was the water leaking out of the tank the fastest?

Solve $R'(t) = 0$ for t

$t = e, t = e^3$

Since $t = e$ is a local maximum, this is when water was leaking out the fastest.

d) What was the average rate of water loss from when the tank burst to this point (to the nearest litre/minute)?

Need to find average value of $R(t)$ for the implied domain.

$Average\ value = \frac{1}{e^3-0}\int_0^{e^3} R(t)\,dt = 502 \text{ L/min}$

e) Consider now the same situation, but now there is an emergency team which comes to try and stop the leak. If their goal is to stop the leak before 70% of the water has been lost, what is the latest time at which they can plug the tank (in minutes, to two decimal places)?

70% of volume:

$V = 0.70 \times \int_0^{e^3} R(t)\,dt = 7060 \text{ L}$

Solve $\int_0^{T} R(t)\,dt = 7060$ for T

$T = 7.71 \text{ min}$