Mathematical Methods
Practice SAC 1b SOLUTIONS

Reading time: TBA
Writing time: TBA

QUESTION AND ANSWER BOOK
Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	10	10	10
B	3	3	35

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 11 pages
- Working space is provided throughout the book.

Instructions

- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- You may not take anything out of the examination room.

Instructions

Answer **all** questions in the spaces provided.
Where a question asks for a diagram, axes are not provided but **must** be drawn and labelled.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
Appropriate working **must** be shown for all questions regardless of mark allocation.
No marks will be given if more than one answer is completed for any question.

Section A - Multiple-choice questions

Question 1

What are the amplitude and period of $f(x) = -2\sin(6\pi x + 3)$?

A. Amplitude -2, period $1/3$

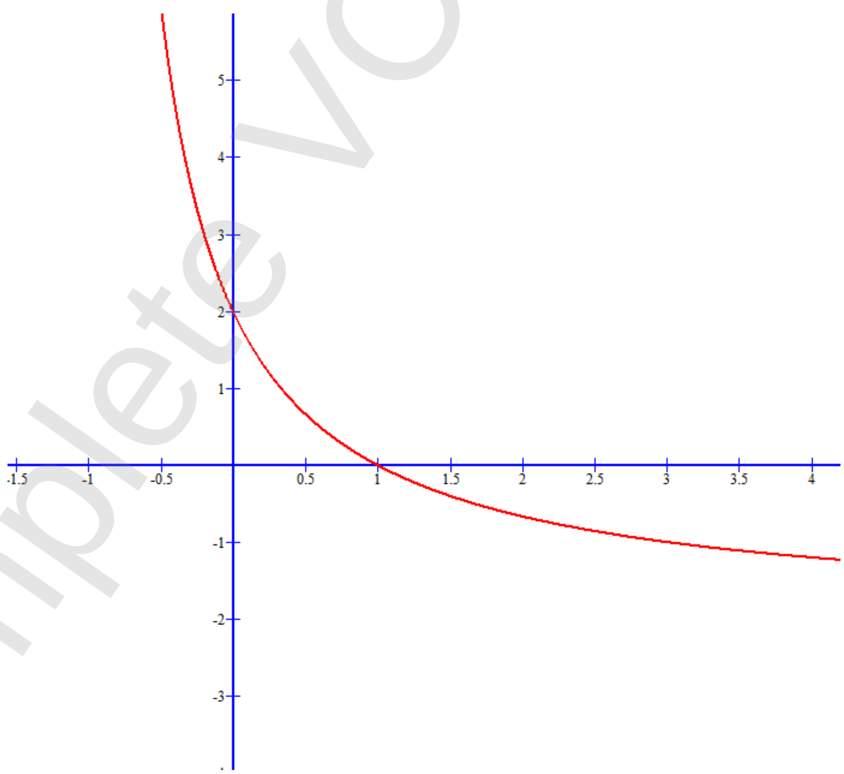
B. **Amplitude 2, period 1/3**

C. Amplitude -2, period $1/6$

D. Amplitude 2, period $1/6$

Question 2

The graph below is for $y = \frac{A}{x+1} + B$. The values of A and B are:



A. $A = 2,\ B = -1$

B. $A = 2,\ B = 0$

C. $A = 4,\ B = -2$

D. $A = 4,\ B = 2$

Question 3

Consider the function $f : [0, 4\pi]\backslash\{\pi, 3\pi\} \to \mathbb{R}$, $f(x) = \tan(x/2)^2$. Which of the following statements is false?

A. The graph of $y = f(x)$ has an asymptote at $x = \pi$

B. There is no inverse function for f

C. **The function f is strictly increasing on $[0, \pi) \cup [2\pi, 3\pi)$**

D. f is periodic

Question 4

The minimum value of the function $f : [0, 2] \to \mathbb{R}$, $f(x) = 2x^3 - 6x^2 + 5x$ occurs at

A. $x = 0$

B. $x = 1$

C. $x = 1 - \sqrt{6}/6$

D. $x = 1 + \sqrt{6}/6$

Question 5

Functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given such that $f(0) = 2$, $g(0) = 2$, $f'(0) = 6$ and $g'(0) = 4$. The derivative of $f(x)/g(x)$, evaluated at $x = 0$, is

A. 0

B. 1

C. 2

D. 3

Question 6

Consider the following system of equations, where $k \in \mathbb{R}$ is a parameter.

$$\begin{aligned} kx + y &= 3 \\ 6x + (k + 1)y &= 9 \end{aligned}$$

The values of k for which this system has infinitely many solutions are

A. $k = -3$

B. $k \in \mathbb{R}\backslash\{-3, 2\}$

C. $k = 2$

D. $k \in \{-3, 2\}$

Question 7

For how many real numbers a in the interval $[-1, 1]$ does the equation $(2\sin(x)-1)(\cos(x)-a)$ have two solutions in the interval $0 \leq x \leq \pi$?

A. 1

B. 2

C. 3

D. 4

Question 8

A function f has domain $[-1, 5]$. After applying a dilation of factor a from the y-axis and a translation by b units in the positive direction of the x-axis to the graph of $y = f(x)$, the result has a domain of $[-1, 8]$. The value of b is

A. 0

B. 3

C. 3/2

D. 1/2

Question 9

To two decimal places, the positive real number a for which the graphs of $y = \sqrt[3]{x+a}$ and $y = x^3 - a$ are tangent is

A. 0.37

B. 0.38

C. 0.39

D. 0.40

Question 10

Let $r < 0$ be a real number. The value(s) of c for which the function $f : (1, \infty) \to \mathbb{R}$, $f(x) = \log_e(x^r + c)$ is well-defined are:

A. $c \in [0, \infty)$

B. $c \in [1, \infty)$

C. $c \in \mathbb{R}$

D. $c \in (-\infty, -1]$

Section B

Question 1 (13 marks)

An alien spy has come to Earth, and just so happened to land in Australia. It needs to learn English in order to blend in; unfortunately, the only word it bothered to learn before coming to Earth was "hello".

The number of English words it knows d days after arriving in Australia can be modelled as $W(d) = 1 + cd^n$, where c and n are parameters that depend on how good the alien is at learning words.

(a) (2 marks) By day 4 the alien has only managed to pick up "goodbye", but by day 20 the alien knows exactly one word which starts with each letter of the alphabet! (That's 26 words.) Show that $n = 2$ and $c = 1/16$.

Solution. Substituting gives $1 = c \cdot 4^n$ and $25 = c \cdot 20^n$, so $25 = 5^n$, i.e. $n = 2$; then $c = 1/4^2 = 1/16$.

(b) (1 mark) Online estimates suggest that a vocabulary of 8000 words is sufficient to read the newspaper, a key skill for any spy who wants to understand Earth-dwelling society. To the nearest day, how long will this take the alien?

Solution. Solve $1 + \frac{d^2}{16} = 8000$ to get $d = 358$ (to the nearest integer). So 358 days.

(c) (1 mark) On day X, the alien's computer sends a congratulatory notification, remarking that the alien has managed to learn an average of 2 words per day since landing in Australia! What is the value of X?

Solution. Solve $2 = \left(\frac{d^2}{16}\right)/d$ to get $d = 32$. So $X = 32$.

(d) (1 mark) At the one-year mark (assume 365 days), what is the instantaneous rate at which the alien is learning words?

Solution. The derivative of $W(d)$ is $d/8$. So $365/8$ words per day.

The alien decides to get some practice by going out and conversing with its neighbours, but quickly finds that this is quite difficult due to a lack of confidence. It goes back home and finds out that a scientifically-backed measure of language confidence is the base 2 logarithm of the number of words known, i.e. $\log_2(W(d))$.

(e) (2 marks) To two decimal places, find the number t such that from the point of time t in days, it takes 60 days to improve the alien's confidence level by 1.

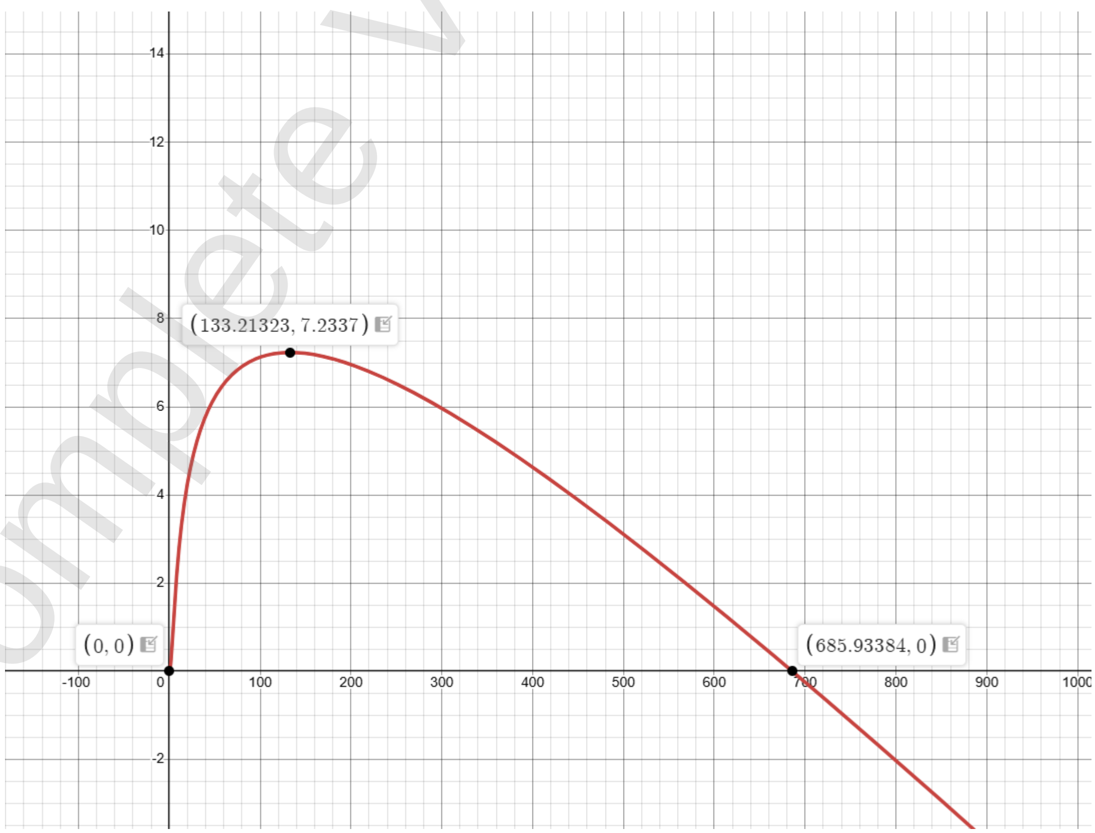
Solution. Use the CAS to solve $\log_2(W(t+60)) - \log_2(W(t)) = 1$. The answer is approximately 144.758..., so 144.76.

Unfortunately, the bad experience was so demotivating that the alien slowed down in language learning ability. The alien's backup team's modelling suggests that this resulted in a change to $W(d)$, such that it is better modelled by the function $W_1(d) = e^{-bd/1000}W(d)$, for some $b > 0$.

(f) (2 marks) The initial estimate is $b = 15$. What is the largest the alien's vocabulary can possibly get (to the nearest word)?

Solution. Now the confidence is $\log_2\left(e^{-15d/1000}W(d)\right)$. Using the CAS, the maximum occurs at $d = 133.21...$ with $7.2337....$ This equals a vocabulary of $2^{7.2337...} = 151$ words (to the nearest whole number).

(g) (2 marks) Draw a graph of the alien's **confidence** as a function of d if $b = 15$, labelling all axis intercepts and the turning point.



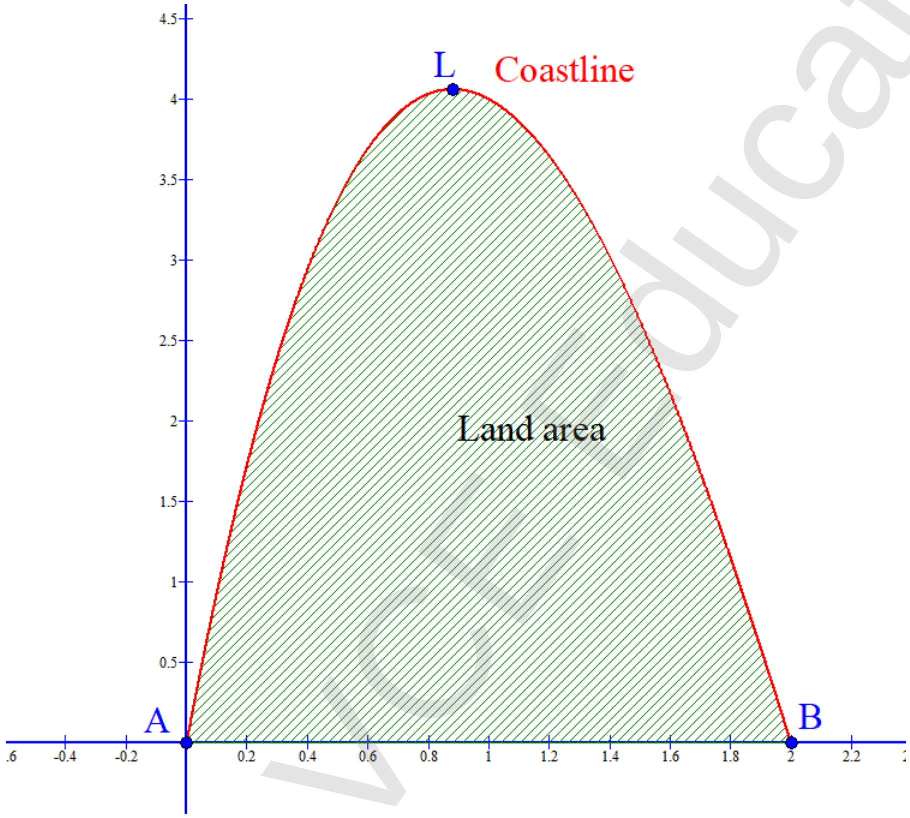
(h) (2 marks) To two decimal places, find the smallest value of b such that the alien will never reach level 13 confidence, which its computer tells it is a good point to try reading a newspaper.

Solution. The confidence can be rewritten as $\log_2(f(x)) - \frac{b\log_2(e)}{1000}x$. We want the maximum value of this to be 13, i.e. $\log_2(f(x)) - \frac{b\log_2(e)}{1000}x \leq 13$ with one value of x that satisfies this. We can rearrange this to $b \geq (\log_2(f(x)) - 13) \cdot \frac{1000}{x\log_2(e)}$. So we need to find the maximum value of this function, which turns out to be 2.03228... at $x = 984.09876....$ Therefore the answer is $b = 2.03$.

Question 2 (9 marks)

Councillors are debating over how best to zone a certain block of land next to the coastline.

The coastline snakes along the plane, and its path (neighbouring the land under consideration) can be modelled by the function $f : [0, 2] \to \mathbb{R}$, $f(x) = x(x-2)(x-5)$, as below. The northernmost point of the coastline has a lighthouse, at point L.



(a) (1 mark) What are the coordinates of L?

Solution. L is the turning point of $y = f(x)$, which is approximately $(0.88, 4.06)$ (CAS).

One proposal is to designate a triangular area on the western side as a commercial area for shops and restaurants, with vertices $(0, 0)$, $(a, 0)$ and $(a, f(a))$ for some $0 < a < 2$, and retain a triangular area on the eastern side as parks and nature reserves, with vertices $(a, 0)$, $(a, f(a))$ and $(2, 0)$.

(b) i. (1 mark) Write the rule $D(a)$ for the area of the western triangular development.

Solution. The area is half times base times height. The base is from 0 to a, and the height is from 0 to $f(a)$, so the area is $\frac{1}{2}af(a) = \frac{1}{2}a^2(a-2)(a-5)$.

ii. (2 marks) One councillor who works in the restaurant business suggests choosing a to maximise the area of the western development. To two decimal places, what is this maximum possible area?

Solution. Find the maximum of $D(a)$ using the CAS. It is at $a = 5/4$ and it is approximately 2.20.

(c) (2 marks) A less money-driven councillor says it would be better if the eastern area was at least as big as the western area, to preserve nature for the residents of the area. For which values of a would this occur?

Solution. The triangles have the same height, so the eastern area needs a bigger base than the western area. This means that $a \leq 1$, half of the whole width $[0, 2]$. Alternatively, write down a rule for the eastern area as $\frac{1}{2}(2-a)f(a)$, and see that $\frac{1}{2}(2-a)f(a) \geq \frac{1}{2}af(a)$ implies $2 - a \geq a$ implies $a \leq 1$.

(d) (1 mark) A third councillor notes that there is a lighthouse at the northernmost point anyway, and it would be visually appealing if the lighthouse was on the border between the western and eastern sides, because then there could be a nice walking trail in straight lines from A to L to B. To two decimal places, what is the length of this trail?

Solution. This is the distance from A to L plus the distance from L to B, i.e. $\sqrt{0.88037\ldots^2 + 4.06067\ldots^2} + \sqrt{(2 - 0.88037\ldots)^2 + 4.06067\ldots^2} = 8.367...$, so 8.37. Make sure not to prematurely round.

(e) (2 marks) The watchperson at the lighthouse notice a ship in trouble at $(0, 3)$. To get to the ship the fastest, the coast guard needs to determine the closest point along the coastline to the ship. To two decimal places, what are the coordinates of this point?

Solution. Use the calculator to minimise the distance from $(0, 3)$ to a point $(x, f(x))$, which by the distance formula is $\sqrt{x^2 + (f(x) - 3)^2}$. It is minimised at $x = 0.3951\ldots$, which gives $(x, f(x)) = (0.40, 2.92)$ to two decimal places.

Question 3 (13 marks)
Consider the function $f : [0, 2\pi] \to \mathbb{R}$, $f(x) = x + \sin(x)$.

(a) (2 marks) Show that f is an increasing function.

Solution. See that $f'(x) = 1 + \cos(x)$, which is always at least 0, because $-1 \leq \cos(x) \leq 1$, so $0 \leq f'(x) \leq 2$. Hence f is increasing.

(b) (3 marks) Suppose that the tangent at $x = a$ to the graph of $y = f(x)$ passes through the origin. Show that $\tan(a) = a$.

Solution. The coordinates of $(a, f(a))$ are $(a, a + \sin(a))$. The slope of the tangent at $x = a$ to the graph of $y = f(x)$ is $f'(a) = 1 + \cos(a)$. So this means that

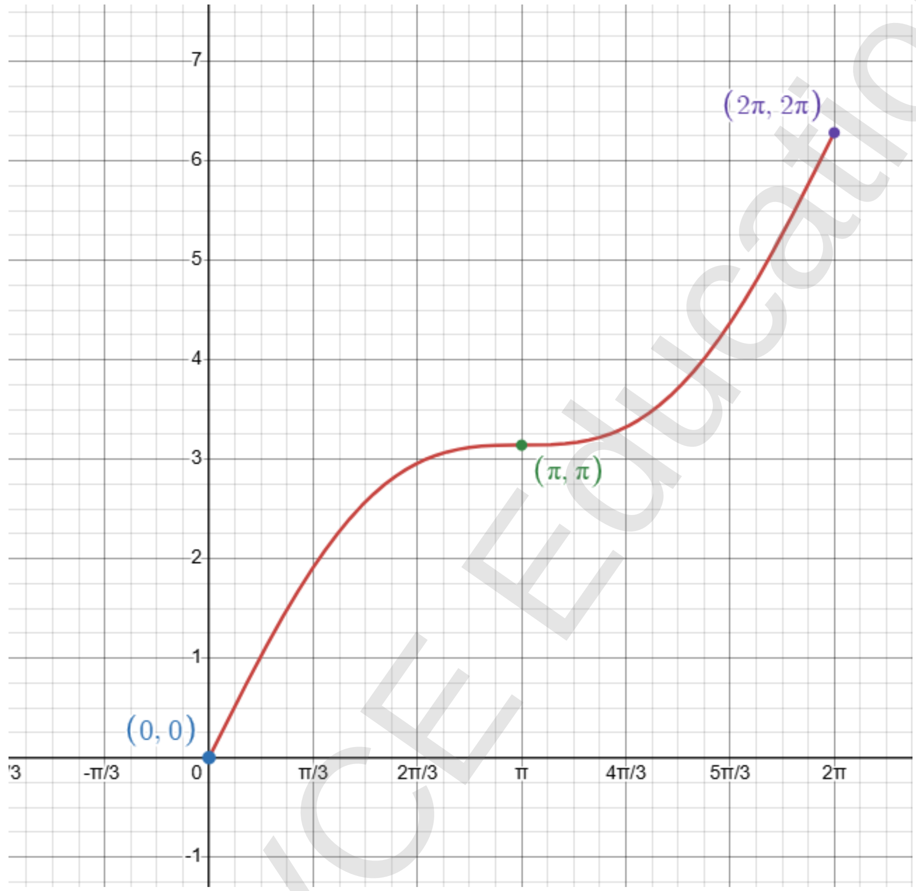
$$\frac{(a + \sin(a)) - 0}{a - 0} = 1 + \cos(a)$$

because the slope of the tangent has to equal the slope of the line joining $(0, 0)$ to $(a, f(a))$. Hence $a + \sin(a) = a + a\cos(a)$, so $\sin(a) = a\cos(a)$, so $\tan(a) = a$.

(c) i. (2 marks) Show that the graph of $y = f(x)$ has a point of inflection.

Solution. We further see that $f''(x) = -\sin(x)$. So $f''(x) = 0$ at $x = 0$, $x = \pi$, $x = 2\pi$ in the domain. We reject $x = 0$ and $x = 2\pi$ as endpoints, but at $x = \pi$, we see by drawing a sign table that $f''(x)$ is negative for $x < \pi$ and positive for $x > \pi$, so there is a point of inflection at $x = \pi$. (It has coordinates (π, π).)

ii. (2 marks) Sketch $y = f(x)$ over its domain, labelling endpoints and the inflection point with their coordinates.



Consider the set of functions $f_c : [0, 2\pi c] \to \mathbb{R}$, where the graph of $y = f_c(x)$ is obtained by applying a dilation of factor $c > 0$ to the graph of $y = f(x)$.

(d) (2 marks) For which values of c do $y = f_c(x)$ and $y = f_c^{-1}(x)$ intersect exactly once? Give your answer to two decimal places.

Solution. $f_c(x)$ is an increasing function, so $y = f_c(x)$ and $y = f_c^{-1}(x)$ intersect on $y = x$. We can see that it always intersects $y = x$ at $x = 0$, so we need to make sure there are no other intersections. So either the graph of $y = f_c(x)$ is always under $y = x$ or always over. The simplest thing to do here is to use the slider. Alternatively, there is a more exact way: if you want $f_c(x) = f(x/c) \geq x$ always, then you want $f(x) \geq cx$ always; similarly if you want $f(x/c) \leq x$ always, then you want $f(x) \leq cx$ always. So we just need to find the minimum and maximum values of $f(x)/x$. Using a calculator, these are 0.7827... and 2. Hence the answer is $c < 0.78$ or $c > 2$.

Consider the set of functions of the form $f_b : [0, 2\pi] \to \mathbb{R}$, $f_b(x) = f(x) + b$, where b is a real number.

(e) (2 marks) For which values of b do the graphs of $y = f_b(x)$ and $y = f_b^{-1}(x)$ have no points of intersection?

Solution. Again, $f_b(x)$ is an increasing function, so $y = f_b(x)$ and $y = f_b^{-1}(x)$ intersect on $y = x$. So we want $f_b(x) = x$ to have no solutions. But this means that $b + \sin(x) = 0$ has no solutions, i.e. $\sin(x) = -b$ has no solutions. This means $-b \in (-\infty, -1) \cup (1, \infty)$, so $b \in (-\infty, -1) \cup (1, \infty)$.