Mathematical Methods
Practice SAC 1a SOLUTIONS

Reading time: TBA
Writing time: TBA

QUESTION AND ANSWER BOOK
Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
6	6	30

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 9 pages
- Working space is provided throughout the book.

Instructions

- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- You may not take anything out of the examination room.

Instructions

Answer **all** questions in the spaces provided.
Where a question asks for a diagram, axes are not provided but **must** be drawn and labelled.
In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.
Appropriate working **must** be shown for all questions regardless of mark allocation.
No marks will be given if more than one answer is completed for any question.

Section 1

Question 1 (3 marks)

(a) (1 mark) Let $y = (x^3 + 6)^3$. Find $\frac{dy}{dx}$.

Solution. Using the chain rule with $u = x^3 + 6$, we get $y = u^3$, so $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 3x^2$. Therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3(x^3 + 6)^2 \cdot 3x^2 = 9x^2(x^3 + 6)^2.$$

(b) (2 marks) Let $f(x) = \dfrac{1}{x^2 + 1}$. Find $f'(-1)$.

Solution. Using the chain rule with $u = x^2 + 1$, we get

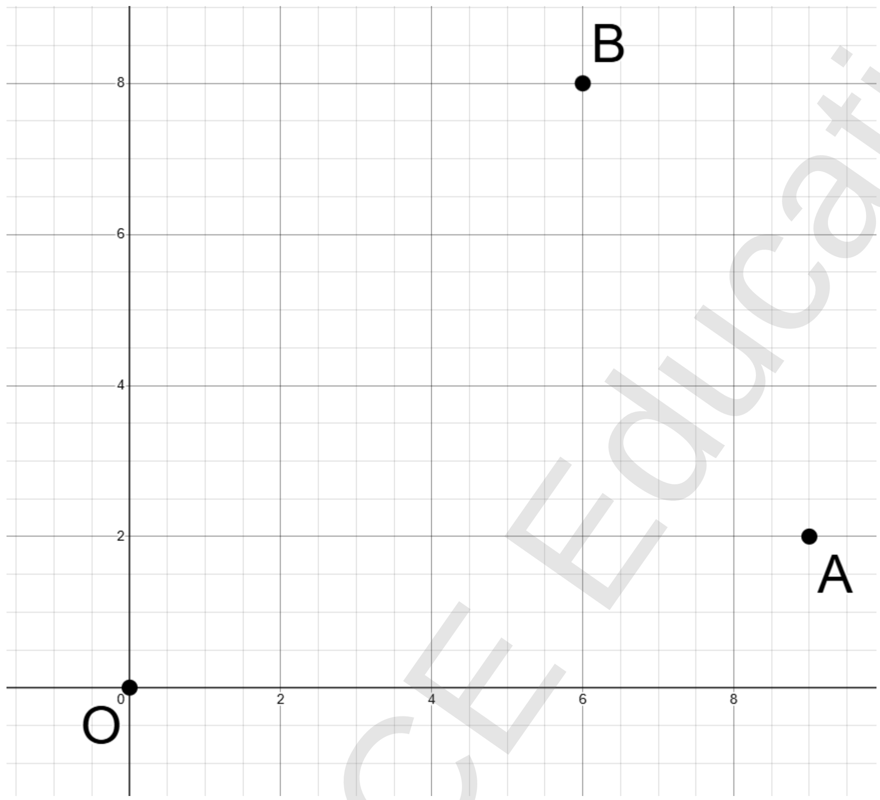
$$f'(x) = -\frac{2x}{(x^2 + 1)^2}$$

because the derivative of $\frac{1}{u}$ is $-\frac{1}{u^2}$ and the derivative of $x^2 + 1$ is $2x$.
Therefore

$$f'(-1) = -\frac{2 \cdot (-1)}{((-1)^2 + 1)^2} = \frac{2}{4} = \frac{1}{2}.$$

Question 2 (5 marks)

Consider the points $A = (9, 2)$ and $B = (6, 8)$, as depicted below.



(a) (1 mark) Determine the equation of line AB.

Solution. The slope of the line is $\frac{2-8}{9-6} = -2$. So the line has equation $y = -2x + c$ for some c. It passes through $(x, y) = (6, 8)$, so $8 = -2 \cdot 6 + c = c - 12$, so $c = 20$. Hence the equation of the line is $y = -2x + 20$.

(b) (1 mark) Determine the equation of the line through O perpendicular to AB.

Solution. The line has slope $\frac{-1}{-2} = \frac{1}{2}$. It passes through the origin, so it has equation $y = x/2$.

(c) (1 mark) On the graph above, mark the point P on line AB such that OP is perpendicular to AB, labelled with coordinates.

Solution. Intersecting $y = x/2$ with $y = -2x+20$ gives $x/2 = -2x+20$, so $5x/2 = 20$, so $x = 8$. Then $y = 8/2 = 4$. So $P = (8, 4)$. (Hopefully you can label that on the graph!)

(d) (2 marks) What is the area of $\triangle OAB$?

Solution. We use the formula $A = \frac{1}{2}bh$, where we take b to be the base AB and h to be the height OP.
The length of AB is $\sqrt{(9-6)^2 + (2-8)^2} = \sqrt{9+36} = 3\sqrt{5}$. The length of OP is $\sqrt{(8-0)^2 + (4-0)^2} = \sqrt{64+16} = 4\sqrt{5}$. Therefore the area is $\frac{1}{2} \cdot 3\sqrt{5} \cdot 4\sqrt{5} = 30$ square units.

Question 3 (6 marks)

Consider the polynomial $p(x) = x^3 + 3x^2 - 4$.

(a) (1 mark) Verify that $(x-1)$ is a factor of $p(x)$.

Solution. By the factor theorem, if $p(1) = 0$, then $(x-1)$ is a factor of $p(x)$. But we see that $p(1) = 1^3 + 3 \times 1^2 - 4 = 1 + 3 - 4 = 0$.

(b) (1 mark) Hence or otherwise, factorise $p(x)$.

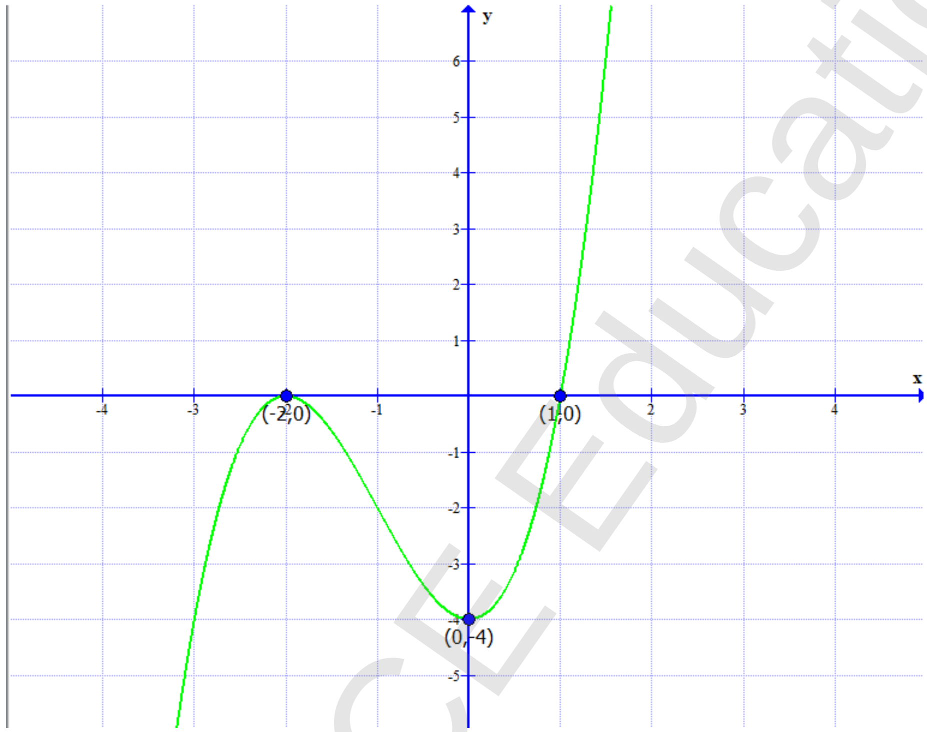
Solution. Divide $p(x)$ by $(x-1)$ to get $x^2 + 4x + 4$, which equals $(x+2)^2$. Therefore $p(x) = (x-1)(x+2)^2$.

(c) (2 marks) Find the coordinates of the turning points of $p(x)$.

Solution. Set $p'(x)$ equal to zero. We have $p'(x) = 3x^2 + 6x = 3x(x+2)$, so $x = 0$ or $x = -2$. $p(0) = -4$ and $p(-2) = 0$, so the turning points are $(0, -4)$ and $(-2, 0)$.

(d) (2 marks) Graph $y = p(x)$ on the axes below, labelling all turning points and axial intercepts with their coordinates.

Solution.



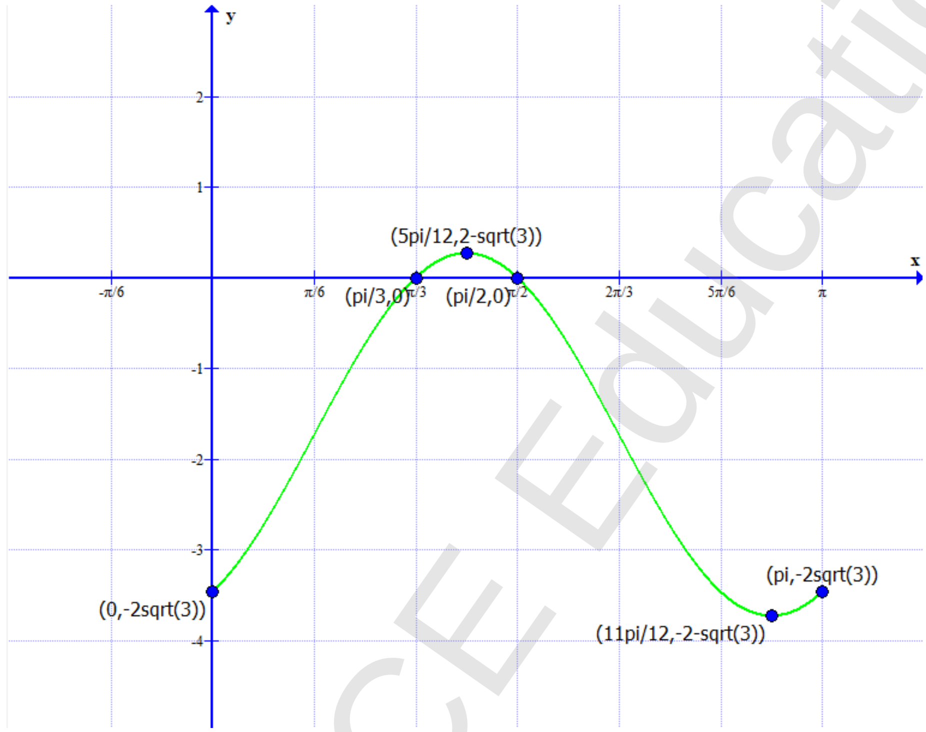
Question 4 (5 marks)

(a) (2 marks) Solve $\sin\left(2x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ in the range $x \in [0, \pi]$.

Solution. Using a general solution, we know that $2x - \frac{\pi}{3} = 2n\pi + \pi/3$ or $(2n+1)\pi - \pi/3$ for some integer n. This means that $x = n\pi + \pi/3$ or $n\pi + \pi/2$ for some integer n. The only solutions in the interval $[0, \pi]$ are thus $x = \pi/3$ and $x = \pi/2$.

(b) (3 marks) Graph $y = 2\sin\left(2x - \frac{\pi}{3}\right) - \sqrt{3}$ for $x \in [0, \pi]$ on the axes below, labelling endpoints, axial intercepts and turning points with their coordinates.

Solution.



Question 5 (6 marks)

Consider the function $f : [-2, 0] \to \mathbb{R}$, $f(x) = 4^{-x} - 1$.

(a) (2 marks) Define the inverse function $f^{-1}(x)$.

Solution. First we have to find it. Start with $y = 4^{-x} - 1$, swap x and y to get $x = 4^{-y} - 1$, then solve for y: $x + 1 = 4^{-y}$, so $-y = \log_4(x + 1)$, so $y = -\log_4(x + 1)$. Now we need to find the domain. We see that f is decreasing, and $f(-2) = 15$, $f(0) = 0$, so the range of f is $[0, 15]$. But this is the domain of f^{-1}. Therefore the answer is $f^{-1} : [0, 15] \to \mathbb{R}$, $f^{-1}(x) = -\log_4(x + 1)$.

(b) I want to apply a dilation to the graph of $y = f(x)$ so that it passes through the point $(-2, 3)$.

i. (2 marks) State two possible choices of dilation, one from the x-axis and one from the y-axis.

Solution. The dilation from the x-axis maps the point $(-2, f(-2))$ onto $(-2, 3)$, i.e. it dilates $(-2, 15)$ onto $(-2, 3)$. This is a dilation by factor $1/5$ from the x-axis.
The dilation from the y-axis maps the point $(f^{-1}(3), 3)$ onto $(-2, 3)$, i.e. it dilates $(-1, 3)$ onto $(-2, 3)$. This is a dilation of factor 2 from the y-axis.

ii. (2 marks) In each case, find the rule for the resulting graph.

Solution. The first one is $(x, y) \to (x, y/5) = (x', y')$, so we substitute $x = x'$, $y = 5y'$ into $y = 4^{-x} - 1$ to get $5y' = 4^{-x'} - 1$, i.e. a rule of $y = \frac{1}{5}(4^{-x} - 1)$.
The second one is $(x, y) \to (2x, y) = (x', y')$, so we substitute $x = x'/2$, $y = y'$ into $y = 4^{-x} - 1$ to get $y' = 4^{-x'/2} - 1$, i.e. a rule of $y = 4^{-x/2} - 1$ (or equivalent $y = 2^{-x} - 1$.)

Question 6 (5 marks)

(a) (2 marks) Solve the equation $\log_2(4-x)+\log_2(1+x)=2$.

Solution. The left-hand side equals $\log_2((4-x)(1+x))$, so we need $(4-x)(1+x)=2^2=4$. This is just a quadratic: expand the left-hand side to get $4+3x-x^2=4$, so $x^2-3x=0$, so $x=0$ or $x=3$. Both of these lie within the domain of the left-hand side, so they are both solutions.

(b) (3 marks) Find all real numbers c for which the equation $\log_2(c-x)+\log_2(1+x)=2$ has no real solutions.

Solution. We are looking for c such that there is no x in the range $(-1,c)$ for which $\log_2(c-x)+\log_2(1+x)=2$, because these are the values of x for which the left-hand side is defined. In particular, if $c\le -1$ there are immediately no solutions, so now consider $c>-1$.

For these x, this equation is equivalent to $\log_2((c-x)(1+x))=2$, or $(c-x)(1+x)=4$. This is a negative quadratic, and the turning point will lie between -1 and c when $c>-1$, which is the maximum, so we just find the values of $c>-1$ for which the turning point is strictly less than 4.

The turning point has x-coordinate halfway between the two zeroes, $x=-1$ and $x=c$, i.e. $x=\frac{c-1}{2}$. Substituting this in gives a maximum of $\left(c-\frac{c-1}{2}\right)\left(1+\frac{c-1}{2}\right)=\left(\frac{c+1}{2}\right)^2$. So we want $(c+1)^2<16$, i.e. $-4<c+1<4$, i.e. $-5<c<3$. Hence the final answer is $c<3$.

Comment. This is a very hard question.

END OF TEST