2026

Mathematical Methods

Question and Answer Book

SAC 2a (Tech-Free)

- Reading time is **5 minutes**
- Writing time is **45 minutes**

Approved materials

- Writing equipment (students are **not** permitted to bring any technology or notes of any kind)
- The VCAA formula sheet

Materials supplied

- Question and Answer Book of 8 pages

Instructions

Students are not permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents

Instructions

Answer **all** questions in the spaces provided.

Write your responses in English.

In all questions where a numerical answer is required, an **exact** value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (3 marks)

a. Let $y = -\log_e(\cos(x))$. Find $\frac{dy}{dx}$. (1 mark)

Solution. We apply the chain rule with $u = \cos(x)$ (hence $y = -\log_e(u)$.) Since $\frac{dy}{du} = -\frac{1}{u}$ and $\frac{du}{dx} = -\sin(x)$, by the chain rule $\frac{dy}{dx} = -\frac{-\sin(x)}{\cos(x)} = \tan(x)$.

b. Let $f(x) = e^{-3x}\sin(x)$.

Find $f'\left(\frac{\pi}{3}\right)$. (2 marks)

Solution. First we apply the product rule to obtain $f'(x)$. We get

$$f'(x) = -3e^{-3x}\sin(x) + e^{-3x}\cos(x).$$

Therefore $f'\left(\frac{\pi}{3}\right) = -3e^{-\pi} \cdot \frac{\sqrt{3}}{2} + e^{-\pi} \cdot \frac{1}{2} = \frac{1-3\sqrt{3}}{2}e^{-\pi}$.

Question 2 (3 marks)

Consider the polynomial $P(x) = 2x^3 - x^4$.

a. Show that $y = P(x)$ has exactly two $x - intercepts$. (1 mark)

Solution. Factorise out x^3 to get $P(x) = x^3(2 - x)$. Hence if $y = P(x) = 0$, then $x = 0$ or $2 - x = 0$, i.e. $x = 0$ or $x = 2$. So the x-intercepts are 0,0 and $(2,0)$.

b. Let the x-intercepts of $y = P(x)$ be a and b, with $a < b$.

Find the average value of $P(x)$ on $[a, b]$. (2 marks)

Solution. The average value of $P(x)$ over $[a, b] = [0,2]$ is defined by the expression

$$\frac{1}{2-0}\int_0^2 P\,(x)\,\mathrm{d}x$$

An anti-derivative of $P(x)$ is given by $\frac{1}{2}x^4 - \frac{1}{5}x^5$. Therefore we get

$$\int_0^2 P\,(x)\,\mathrm{d}x = \left(\frac{1}{2}\cdot 2^4 - \frac{1}{5}\cdot 2^5\right) - (0-0) = \frac{8}{5}$$

and so the desired average value is $\frac{1}{2}\cdot\frac{8}{5} = \frac{4}{5}$.

Question 3 (4 marks)

Let $g: R\backslash\{-2\} \to R$ be a function such that $g(0) = 1$ and

$$g'(x) = \frac{-3}{(x+2)^4}.$$

a. What is the equation of the normal to the graph of $y = g(x)$ at $x = 0$? (2 marks)

Solution. The slope of the tangent is $g'(0) = \frac{-3}{(0+2)^4} = \frac{-3}{16}$. Using the rule $m_1 m_2 = -1$, we get that the slope of the normal is therefore $\frac{16}{3}$. Hence the line takes the form $y = \frac{16}{3}x + c$ for some number c. Since $g(0) = 1$, $1 = \frac{16}{3}\cdot 0 + c$ and so $c = 1$. So the normal has equation $y = \frac{16}{3}x + 1$.

b. Find $g(1)$. (2 marks)

Solution. We know that $g(1) - g(0) = \int_0^1 g\,'(x)\,\mathrm{d}x$. An anti-derivative of $g'(x)$ is given by $(x+2)^{-3}$. Hence $g(1) - g(0) = (1+2)^{-3} - (0+2)^{-3} = \frac{1}{27} - \frac{1}{8} = -\frac{19}{216}$. Since $g(0) = 1$, we get that $g(1) = 1 - \frac{19}{216} = \frac{197}{216}$.

Question 4 (6 marks)

Consider the function $f(x) = \sqrt{x} + \sqrt{4 - x}$.

a. What is the implied domain of $f(x)$? (1 mark)

Solution. Firstly, for $\sqrt{x}$ to be defined we need $x \geq 0$. Secondly, for $\sqrt{4-x}$ to be defined we need $4 - x \geq 0$, i.e. $x \geq 4$. Hence the implied domain is $0 \leq x \leq 4$, i.e. $x \in [0,4]$.

b. Find the maximum of $f(x)$. (3 marks)

Solution. Calculate that $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{4-x}}$. Therefore, $f'(x) = 0$ means that $\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4-x}}$, and so $\sqrt{x} = \sqrt{4-x}$, so $x = 4 - x$, so $x = 2$; if $x < 2$, then $\sqrt{x} < \sqrt{4-x}$, so $f'(x) > 0$, and if $x > 2$, then $\sqrt{x} > \sqrt{4-x}$, so $f'(x) < 0$. Hence $x = 2$ is a maximum for $f(x)$ (rather than a minimum), and $f(2) = \sqrt{2} + \sqrt{4-2} = 2\sqrt{2}$.

c. Using the trapezium rule with four trapezia, estimate

$$\int_0^4 f(x)\, \mathrm{d}x.$$

Solution. The trapezium rule gives the approximation

$$\int_0^4 f(x)\, \mathrm{d}x \approx \frac{4-0}{2 \cdot 4}\big(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\big).$$

We can compute that $f(0) = f(4) = 2$, $f(1) = f(3) = 1 + \sqrt{3}$ and $f(2) = 2\sqrt{2}$. Therefore, the approximation is

$$\frac{1}{2}\big(2 + 2 + 2\sqrt{3} + 4\sqrt{2} + 2 + 2\sqrt{3} + 2\big) = 4 + 2\sqrt{3} + 2\sqrt{2}.$$

Question 5 (7 marks)

Consider the function $f: [-\pi, \pi] \to R$, $f(x) = a\cos(x) + b$ (where a, b are real numbers).

a. It is known that f has a point of inflection at $\left(\frac{\pi}{2}, 2\right)$ and that the y-intercept of f is $(0,4)$.

   i. Show that $(0,4)$ is a stationary point of y. (1 mark)

   Solution. $f'(x) = -a\sin(x)$, so $f'(0) = -a\sin(0) = 0$. Therefore $(0,4)$ is a stationary point of the graph.

   ii. Explain why it is a local maximum of $y = f(x)$. (1 mark)

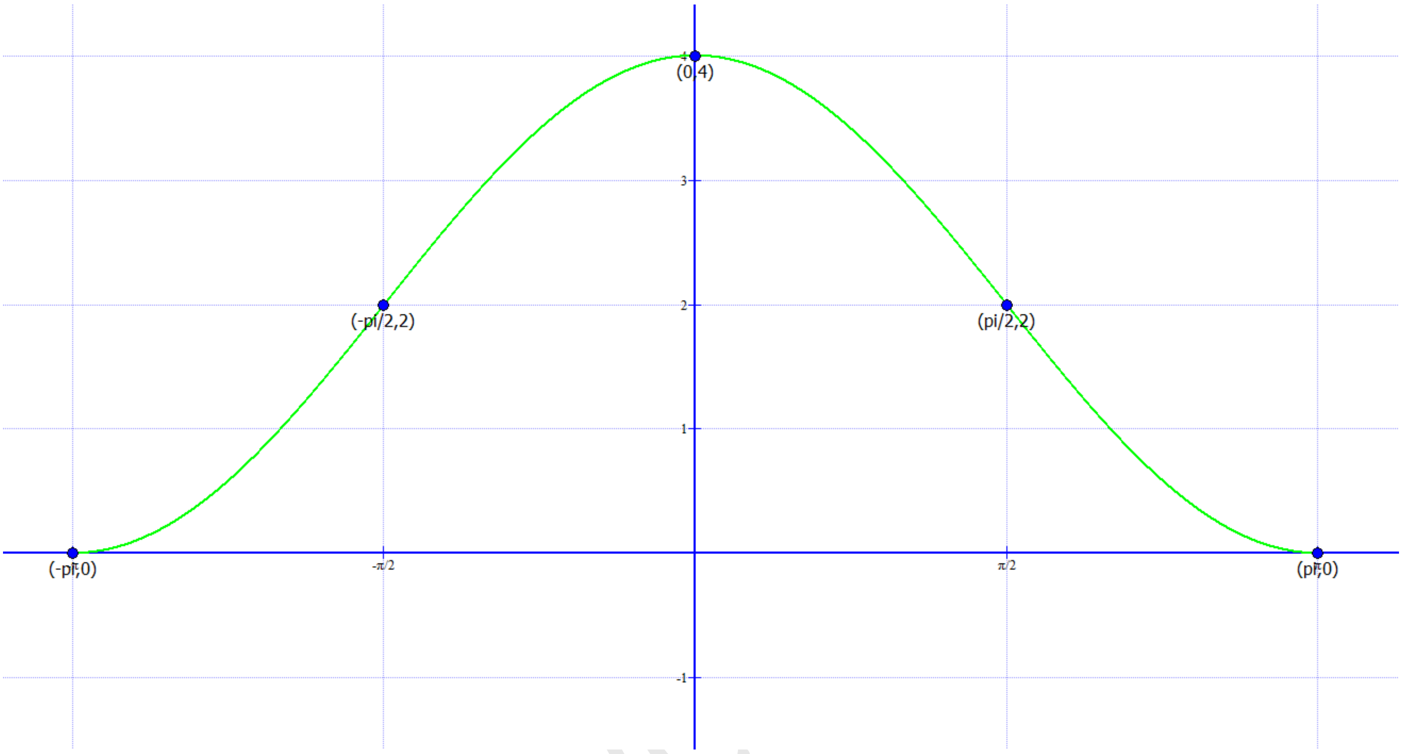
   Solution. On a sine graph, the local maxima and minima occur above and below the points of inflection respectively. Since $4 > 2$, this is above the point of inflection, so it is a local maximum rather than a local minimum.

b. What are the values of a and b? (2 marks)

Solution. Since the points of inflection occur at $y = b$, we know that $b = 2$. Therefore $a\cos(0) + 2 = 4$ (substitute in the point 0,4), and so $a = 2$.

c. Sketch $y = f(x)$ on the axes below. Label all axis intercepts, turning points and points of inflection with their coordinates. (3 marks)

Solution.



Question 6 (7 marks)

Given a real number $a > 1$, consider the function $f: R \to R$, $f(x) = xa^x$.

a. Show that $y = x$ is tangent to the graph of $y = f(x)$ at $x = 0$. (2 marks)

Solution. $f'(x) = a^x + x\log_e(a) \cdot a^x$ by the product rule. So $f'(0) = a^0 + 0 \cdot \log_e(a) \cdot a^0 = 1$ and therefore the tangent has slope 1. Since $f(0) = 0 \cdot a^0 = 0$, the tangent is thus $y = x$.

b. Show that the tangent at $x = 1$ to the graph of $y = f(x)$ has a negative y −intercept. (3 marks)

Solution. Firstly, referring to the above, $f'(1) = a + \log_e(a)\, a$. Also, $f(1) = a$. So the tangent has equation $y = a(\log_e(a) + 1)x + c$ for some number c. Letting $x = 1$ and $y = a$, we get $a = a(\log_e(a) + 1) + c$ and so $c = -a\log_e(a)$. Therefore the tangent has equation $y = a(\log_e(a) + 1)x - a\log_e(a)$ and so the y-intercept is $(0, -a\log_e(a))$. Since $a > 1$, $\log_e(a) > 0$ and so $-a\log_e(a) < 0$ is negative.

c. Over what interval is f strictly increasing? (2 marks)

Solution. We need $f'(x) > 0$. This means that $a^x(1 + x\log_e(a)) > 0$. Since $a^x > 0$ for all real numbers x, we need $1 + x\log_e(a) > 0$, i.e. $x > -1/\log_e(a)$. So the interval is

$$\left(-\frac{1}{\log_e(a)}, \infty\right).$$

END OF QUESTION AND ANSWER BOOK