2026

Mathematical Methods

Question and Answer Book

SAC 2b (Tech-active)

- Reading time is **5 minutes**
- Writing time is **55 minutes**

Approved materials

- One scientific calculator and one approved CAS calculator
- Bound reference
- Writing equipment
- The VCAA formula sheet

Materials supplied

- Question and Answer Book of 15 pages

Instructions

Students are not permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents

SECTION A – Multiple-choice questions

Instructions for Section A
Answer **all** questions by circling the correct answer or clearly indicating it
Choose the response that is **correct** or that **best answers** the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f(x) = xe^{x+2}$. Then $f'(x) = -1$ at

A. $x = -2$

B. $x = -1$

C. $x = 0$

D. $x = 1$

E. $x = 2$

Solution. $f'(x) = e^{x+2} + xe^{x+2}$ by the product rule. Now use CAS to solve for when this equals -1 to get that $x = -2$.

Question 2

The integral

$$\int_{-\pi/2}^{\pi/2} 24x^2 + a\cos(x)\ dx$$

is equal to zero. The value of a is

A. 0

B. π^3

C. $-\pi^2$

D. $-\pi^3$

E. $2\pi^3$

Solution. Use the CAS to evaluate the integral: it is $2a + 2\pi^3$ and so it equals zero when $a = -\pi^3$.

Question 3

The average value of the derivative of $x^2 \log_e(ax^2 + 1)$ (where a is a real number) from $x = -1$ to $x = 1$ is

A. -2

B. 0

C. 1

D. 3

E. 2

Solution. The quick solution is that the integral of the derivative of $x^2 \log_e(ax^2 + 1)$ from $x = -1$ to $x = 1$ is just $1^2 \log_e(a \cdot 1^2 + 1) - (-1)^2$ \log_e(a \cdot (-1)^2 + 1) $= 0$ and so the average value is 0. Alternatively, you can calculate the derivative in the CAS, then the

Question 4

The tangent to the graph of $y = f(x)$ at $x = 2$ has the equation $y = x + 5$. The tangent to the graph of $y = 2f(1 - x)$ at $x = -1$ has equation

A. $y = x + 5$

B. $y = 6 - 2x$

C. $y = 12 - 2x$

D. $y = 8 - 5x$

E. $y = 2x + 10$

Solution. We need to determine the transformations taking us from $y = f(x)$ to $y = 2f(1 - x)$. We want $y' = 2f(1 - x')$, and since $\frac{y'}{2} = f(1 - x')$, we can just take $y = y'/2$, $x = 1 - x'$. Now substitute these into the line $y = x + 5$ to get $y'/2 = 1 - x' + 5$, so $y' = 2(6 - x') = 12 - 2x'$.

Question 5

Let $f(x) = cx^3$. The area bounded between the graphs of $y = f(x)$ and $y = f^{-1}(x)$ is equal to 5. The value of c is

A. 0.10

B. 0.15

C. 0.20

D. 0.25

E. 0.50

Solution. By symmetry in the line $y = x$, this area is equal to twice the area bounded between the graph of $y = f(x)$ and the line $y = x$. That area, by symmetry around the origin, is equal to twice the area bounded between the graph of $y = f(x)$ and the line $y = x$ in the 1st quadrant, i.e. between $x = 0$ and $x = 1/\sqrt{c}$. So solve for

$$\int_0^{1/\sqrt{c}} x - cx^3 \, \mathrm{d}x = \frac{5}{4}$$

to get that $c = \frac{1}{5}$.

Question 6

The function $f: R \to R$ satisfies $f(0) = 1$, and $f'(x) = e^{x^2} - 1$ for all x. At the point $(0,1)$, the graph of $y = f(x)$ has

A. An x-intercept

B. A local maximum

C. A local minimum

D. A point of inflection

E. A discontinuity

Solution. Firstly, $f''(x) = 2xe^{x^2}$, so $f''(0) = 0$. Furthermore, $f'(x)$ is positive for all $x \neq 0$, since $e^{x^2} > e^0 = 1$ for these x. So we have a stationary point of inflection (as the graph is strictly increasing).

END OF SECTION A

TURN OVER

SECTION B

Instructions for Section B
Answer **all** questions in the spaces provided.
Write your responses in English.
In all questions where a numerical answer is required, an **exact** value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (12 marks)

Lucy is trying to build a roller coaster. Her initial plan starts with a section of the graph of the function $f: R \to R, f(x) = x^3 - 5x^2 + 7x$.

a. Her plan is for the segment to start at $x = 0$ and end at a local minimum $x = a$ of the graph of $y = f(x)$. Show that the segment ends at the point $\left(\frac{7}{3}, \frac{49}{27}\right)$. (2 marks)

Solution. First, $f'(x) = 3x^2 - 10x + 7$. This factorises as $(3x - 7)(x - 1)$, so it is zero when $x = 7/3$ and when $x = 1$. But, drawing a sign diagram, $f'(x) > 0$ for $x < 1$ and $x > 7/3$ and $f'(x) < 0$ for $1 < x < 7/3$, so $x = 1$ is a local maximum and $x = 7/3$ is the local minimum in question (hence $a = 7/3$). Since $f(7/3) = 49/27$, the segment ends at $\left(\frac{7}{3}, \frac{49}{27}\right)$.

b. What is the steepest negative slope on the roller coaster? (2 marks)

Solution. We want to find the minimum of $f'(x)$ between $x = 0$ and $x = 7/3$. We can differentiate again to get $f''(x) = 6x - 10$, so this occurs at $x = 10/6 = 5/3$. Now $f'(5/3) = -4/3$.

c. Lucy intends to build some supporting beams for the roller coaster segment. For one of them, she selects the normal to the graph of $y = f(x)$ at $x = 2$, drawn from the point $(2,2)$ to the point at which it intersects the roller coaster segment again.

i. Find the equation of this normal. (1 mark)

Solution. Use the normalLine function in the CAS. It is $y = x$.

ii. Find (to two decimal places) the coordinates of the point at which the beam meets the roller coaster segment again. (1 mark)

Solution. Solve $f(x) = x$ to get $(0,0)$.

iii. How long (to two decimal places) is the beam? (1 mark)

Solution. The length is $2\sqrt{2} = 2.83$.

d. She decides to include some space for advertising material in the design. For this purpose, she reserves the space **above** the supporting beam in part c., between the beam and the track. To two decimal places, what is the area of this advertising space? (1 mark)

Solution. The area is

$$\int_0^2 f(x) - x \, dx = \frac{8}{3} = 2.67.$$

e. For the second segment of the roller coaster, she decides to add another drop. To do this, she decides to make the second segment the graph of $y = g(x)$, where $g: \left[\frac{7}{3}, \frac{49}{27}\right] \to R$ is a cubic. She wants the following two conditions to hold:

- The graphs of $y = f(x)$ and $y = g(x)$ are joined smoothly at $x = \frac{7}{3}$ so that the ride is not bumpy;
- The graph of $y = g(x)$ joins smoothly to the x-axis at $x = 4$ so that the ride can end there.

i. By considering the derivative of $g(x)$, show that

$$g(x) = a\left(x^3 - \frac{19}{2}x^2 + 28x\right) + b$$

for some real numbers a and b. (2 marks)

Solution. We know that $g'(7/3) = g'(4) = 0$ by the two conditions. It follows that $g'(x) = a(x-4)(3x-7)$ for some real number a, i.e. $a(3x^2 - 19x + 28)$. Taking the anti-derivative, we know that $g(x) = a\left(x^3 - \frac{19}{2}x^2 + 28x\right) + b$ for some real numbers a and b.

ii. To two decimal places, what are the values of a and b? (2 marks)

Solution. We know that $g(7/3) = 49/27$ and $g(4) = 0$. Use the CAS to solve this system of equations: it is $a = 98/125$, $b = -2352/125$. That is, $a = 0.78$, $b = -18.82$.

Question 2 (8 marks)

A farmer is draining his rainwater tank because maintenance is required. After t hours, the amount of water that has left the tank is modelled by $V(t)$, where

$$\frac{\mathrm{d}V}{\mathrm{d}t} = e^{-t}(2-t)^2$$

and V is measured in millions of litres. He keeps the tap open until all of the water has been emptied out of the tank and water stops flowing out, at which point the model ends.

a. What is the implied domain of $V(t)$? (1 mark)

Solution. We are looking for the first time where $\frac{dV}{dt} = 0$, since it is always non-negative. This is clearly at $t = 2$, and so the domain is $t \in [0,2]$.

b. How much water is in the tank? (1 mark)

Solution. The amount of water in the tank is the total change in the volume:

$$\int_0^a e^{-t}\,(2-t)^2\,\mathrm{d}t = 2 - 2/e^2$$

(in millions of litres).

c. In litres per hour, how quickly does water exit the rainwater tank when he first opens the tap? (1 mark)

Solution. This is $\frac{dV}{dt}$ at $t = 0$, which is $4e^0 = 4$, i.e. 4 million litres/hour.

d. To the nearest minute, find the time at which the water is draining from the tank at one hundred thousand litres per hour. (1 mark)

Solution. Solve $\frac{dV}{dt} = 1/10$ to get $t = 1.372\ldots$, i.e. after 82 minutes.

e. To the nearest minute, how long does it take for half of the water to have emptied from the tank? (2 marks)

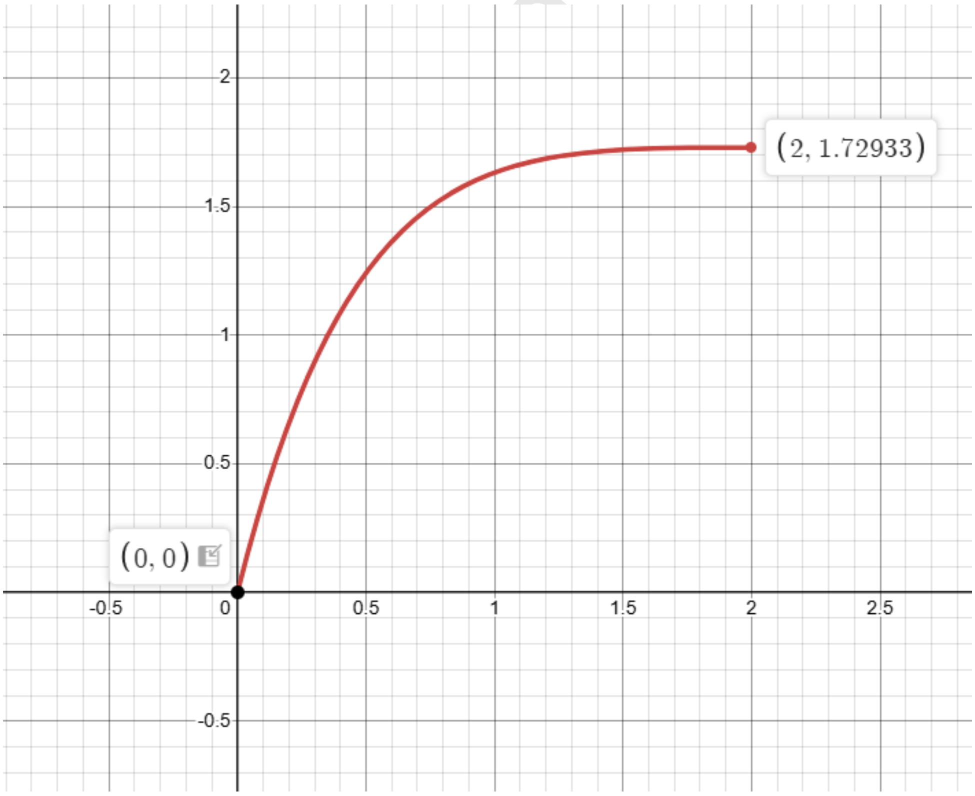
Solution. Solve for the number a such that

$$\int_0^a e^{-t}\,(2-t)^2\,\mathrm{d}t = 1 - 1/e^2.$$

(The right hand side is half of the volume from part b.) We get $a = 0.2855\ldots$, which converts to 17 minutes.

f. Draw a graph of $V(t)$ over its implied domain. Include the axis intercepts with their coordinates. (2 marks)

Solution.



Since $V(0) = 0$, we know that $V(t) = \int_0^t e^{-x}(2 - x)^2 \, dx$.

Question 3 (10 marks)

For each positive integer n, consider the function $f_n(x) = \sqrt{2 - x^{2n}}$.

a. If f_m has an implied domain of $[-\sqrt{2}, \sqrt{2}]$, show that $m = 1$.

Solution. The implied domain is that $2 - x^{2m} \geq 0$, so $x^{2m} \leq 2$, so $-2^{1/2m} \leq x \leq 2^{1/2m}$. This matches $-\sqrt{2} \leq x \leq \sqrt{2}$ when $m = 1$.

b. Show that for any $a \in (-\sqrt{2}, \sqrt{2})$, the normal to the graph of $y = f_1(x)$ at $x = a$ passes through the origin. (3 marks)

Solution. We calculate that $f'_1(x) = \frac{-x}{\sqrt{2-x^2}}$. Therefore the slope of the normal at $x = a$ is $\frac{\sqrt{2-a^2}}{a}$.

Since the coordinate $(a, f_1(a))$ is $(a, \sqrt{2 - a^2})$, the equation of the line is thus $y = \frac{\sqrt{2-a^2}}{a}x + c$ for some c, with $\sqrt{2 - a^2} = \sqrt{2 - a^2} + c$ implies $c = 0$. Therefore the equation is $y = \frac{\sqrt{2-a^2}}{a}x$, which does pass through the origin.

c. Calculate

$$\int_{-\sqrt{2}}^{\sqrt{2}} f_1(x)\, \mathrm{d}x.$$

(1 mark)

Solution. Use the CAS: it is π.

d. Jonathan is trying to approximate the value found in part c.

i. Suppose he uses the trapezium rule with two equal-width trapeziums. What is the resulting estimate? (1 mark)

Solution. The resulting estimate is

$$\frac{\sqrt{2} - -\sqrt{2}}{2 \cdot 2}\left(f_1(-\sqrt{2}) + 2f_1(0) + f_1(\sqrt{2})\right)$$

which is equal to $\frac{\sqrt{2}}{2}\left(0 + 2 \cdot \sqrt{2} + 0\right) = 2$.

ii. Jonathan decides to use two trapeziums, but he is going to try to cheat by making them different widths. Suppose his first trapezium is from $x = -\sqrt{2}$ to $x = a$, and his second from $x = a$ to $x = \sqrt{2}$, where a is some real number. Determine, in terms of a, the resulting estimate. (1 mark)

Solution. Note that the two trapezia are actually triangles, so the resulting estimate is still a triangle, with base $2\sqrt{2}$ and height $f_1(a) = \sqrt{2 - a^2}$. Therefore the resulting estimate is $\frac{1}{2}bh = \sqrt{4 - 2a^2}$.

iii. For which value of a is Jonathan's cheated estimate closest to the true value? (1 mark)

Solution. It is closest when $a = 0$ (the estimate is always at most $2 < \pi$, and equals 2 when $a^2 = 0$.)

e. For each positive integer n, consider the area of the region formed by the graph of $y = f_n(x)$ and the x-axis. What is the largest real number A such that this area is always greater than A, for all positive integer values of n? (1 mark)

Solution. Use the CAS calculator to notice that this area is strictly decreasing as n gets bigger, and it approaches $2\sqrt{2}$ (The region approaches a rectangle with base 2 and height $\sqrt{2}$.)

f. Consider the function g defined by

$$g(x) = \begin{cases} f_n(x) & 0 \le x \le 1 \\ f_n^{-1}(x) & 1 < x \le \sqrt{2} \end{cases}$$

and its graph $y = g(x)$. Its graph, together with the x and y axes, bounds a region.

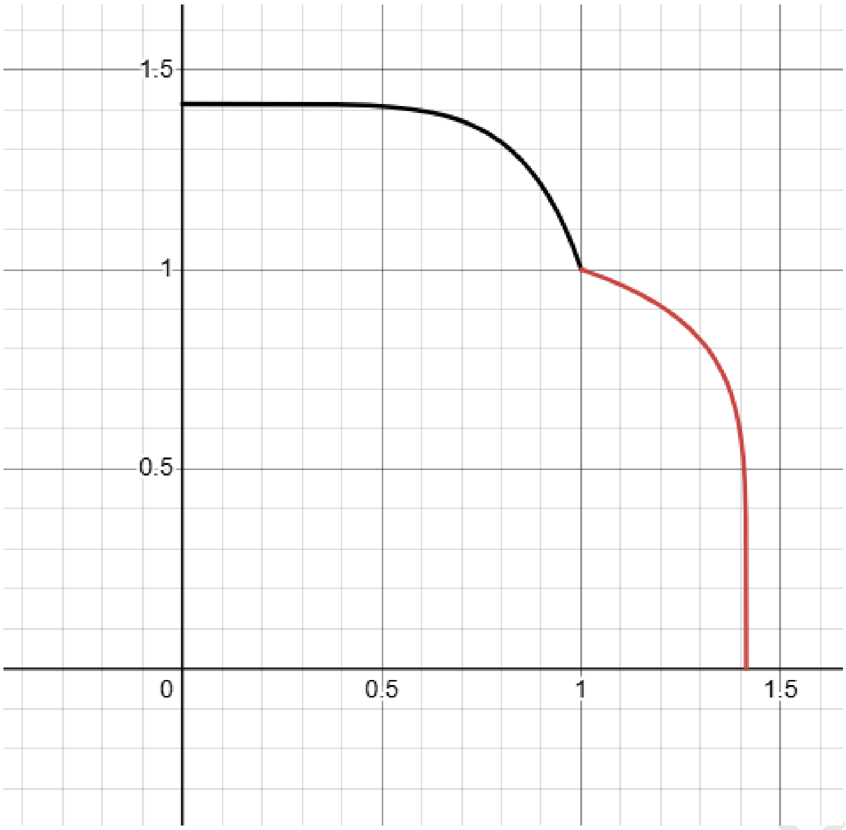
There are real numbers a and b such that the area of this region can be expressed in the form

$$a + b \int_0^1 f_n(x)\, dx$$

regardless of the value of n.

Find the values of a and b. (1 mark)

Solution. Note that $f_n(x)$ and $f_n^{-1}(x)$ intersect at 1,1. Therefore, we can see that the region consists of two copies of the region which defines $\int_0^1 f_n(x)\, dx$, overlapping in a square of area 1. Hence $a = -1$ and $b = 2$. (See next page for picture).



END OF QUESTION AND ANSWER BOOK