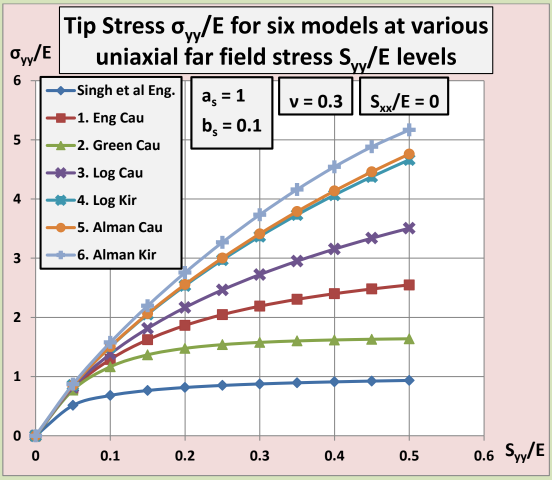
Understanding Non-Linear Strain-Displacement Relations In The Elliptical Co-ordinate System

(with applications to holes in a thin plate)



Rajen Merchant

Introductory Sample

(not for sale)

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(with applications to holes in a thin plate)

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Dedication

This book is dedicated to

my wife

Rekha

Preface

The second monograph is also specially written for students and researchers in mechanical and/or civil engineering, solid mechanics, materials science, physics, and mathematics. Professors teaching this subject will find it very useful.

The first monograph (titled ***Understanding the Elastic Stress Field Around an Elliptical Hole in a Thin Plate (in the deformed configuration)- ISBN 978-1-48359-262-6***) utilized stress equilibrium conditions to develop expressions for stress components which contain a parameter α_e. This second monograph does not deal with stress equilibrium conditions. It deals with strain-displacement relations as well as stress-strain relations and shows how to evaluate α_e for linear elastic materials.

I thank Prof. Rajendra Dubey for introducing me to the basic concepts of current configuration and the use of elliptical co-ordinate system. Prof. Mahesh Pandey provided initial financial support for Phase 1 of this research. Prof. G. Glinka often set aside time out of his busy schedule whenever I requested help. I am indeed thankful to Prof. Katerina Papoulia for providing partial financial support for Phase 1 of this research as well as pointing out difficulties in the original model. However, most of this Phase 2 research was developed after 2013. Dr. Amin Eshraghi devoted (and continues to devote) a lot of time verifying my work. Dr. Anup Sahoo also spent a considerable amount of time reviewing my work and providing me encouragement when I needed it most. His guidance was useful since he had worked on similar models.

The University of Waterloo provided some financial support while I was registered as a graduate student. Prof. Rashmi Desai was always available for any difficulties I faced. I am thankful to Prof. Amit Singh for a detailed review of this second book and Prof. Jit Sharma for writing the Foreword. Prof. Bhalchandra Puranik's guidance was very helpful.

I sincerely thank Rekha (my wife) for logistical support and being supportive when I was spending hours at a time analyzing various scenarios. My children Reena and Himanshu helped a lot in the production of this monograph and related video presentations. My special thanks to the publisher bookbaby.com for taking care of all details related to printing and distribution of this monograph.

I bear full responsibility for the content and presentation including any errors and omissions in this work. However, I and the publisher assume no responsibility for damages resulting from the use of the information contained herein. I will greatly appreciate any comments and suggestions. I can be reached at 1-416-725-0909 or rajen.merchant@gmail.com.

I appeal to you not to copy any part of the full version of the monograph. Monographs are usually not sold in large numbers. Also, monographs that involve complicated mathematical expressions are not easy to typeset. I have tried my best to keep the cost of each book to an affordable level so that as many students (and researchers) can buy, read, understand, and apply these concepts in their work.

Rajen Merchant

Toronto, August 2019

Foreword

It gives me great pleasure to write this foreword for Mr. Rajen Merchant's second book on the stress-strain analysis of an elliptical hole (representing a defect in a continuum) in an infinite thin plate. The first book focused exclusively on the evaluation of stress field based upon the stress equilibrium conditions. The Airy stress functions were developed in the first book so as to satisfy the field equations of equilibrium, expressed in the current (deformed) configuration. The results presented in the first book showed stress concentrations that are orders of magnitude higher in certain conditions once the defect changes shape. This second book incorporates non-linear strain-displacement relations as well as a constitutive (stress-strain) relation for isotropic linear elastic materials. The most innovative aspect of these two books is Rajen's use of the elliptical coordinate system, which enables a clearer evaluation of the parametric trends of the resulting stress terms.

Students and researchers from a wide range of disciplines from mathematics, physics, and material science to solid mechanics and engineering would find these two books extremely useful for analyzing a wide range of problems, such as glass panel fractures (in car windscreens, for example), tunnelling in soft ground and hard rock, reservoir geomechanics problems associated with carbon dioxide sequestration, etc. In the case of tunnelling and reservoir geomechanics problems, it is not clear how well the finite element and boundary element models capture the deformed state and the risk of localized rupture and subsequent progression of failure; the solutions contained in these two books will be of tremendous help in this regard. Last, but not least, this is a standard problem in steel connections (perforated members bolted to each other) where establishing the strength of the critical crack / tear path is an important design issue. The equations described in these two books could be used to analyze and design plates containing multiple holes via superposition, thereby deriving critical values for the spacing of openings to preclude tear-off prior to global yielding of the steel plate overall.

I commend Rajen for his very detailed and innovative analysis of this age-old problem in classical theory of elasticity using a more general elliptical coordinate system of which the polar coordinate system is a special case. I have no doubt that these two books will be well received and extensively used by the scientific community.

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Introduction

Stress at the tip of a central elliptical hole in a thin plate (very long in other two dimensions) has not been understood well because of difficulties in determining displacement and deformation gradients when an elliptical shape deforms to form another elliptical shape. It is indeed convenient to use elliptical coordinate system while analysing elliptical holes. However, the grid in an elliptical coordinate system is not fixed; it can shift when an elliptical shape deforms to form another elliptical shape.

In this second book, it is shown that when displacement caused by such a shift is included in the total displacement, it leads us to correct expressions for displacement and deformation gradients and consequently to a much improved analytical solution.

In the first book (titled ***Understanding the Elastic Stress Field Around an Elliptical Hole in a Thin Plate (in the deformed configuration)- ISBN 978-1-48359-262-6***), only stress analysis was carried out in the deformed configuration. We did not involve strain-displacement relations, stress-strain relations, and material properties. Hence, the complete solution could not be attained. That is why expressions for stress components presented at the end of Chapter 2 in the first book contained a parameter α_e. How to determine the value of α_e is shown in this second book.

Not to involve strain-displacement relations, stress-strain relations, and material properties in the first book was intentional. By doing so, the dichotomy present in the total solution can be emphasized and the sole impact of geometrical non-linearity can be readily interpreted. Furthermore, since strain-displacement relations, stress-strain relations, and material properties are not considered in the first book, the analysis and the results presented in the first book are applicable to all materials.

The first book does not deal with deformation. It shows that, even without dealing with deformation (small or large), the stress field depends on the final configuration. Furthermore, there are special features (e.g. the zero stress point, localized nature of the disturbance, tip stress depends on the shape and not the size of the hole) of such a stress field which are presented in Chapter 3 onward in the first book.

Since the first book does not deal with deformation, it can be a strong reference book for a beginning course on stress analysis where deformation is not introduced (or discussed).

The first book shows how to exploit features of the elliptical co-ordinate system and the complex analytic functions to obtain expressions for the stress field that contain only one parameter α_e (in comparison with only one parameter in polar co-ordinate system i.e. radius r). It illustrates how to apply the boundary conditions in the deformed configuration to study the sole impact of geometric non-linearity.

In this second book, we study non-linearity in strain-displacement relations. However, we consider only the linear portion of stress-strain relations. We assume Young's modulus, E, and Poisson's ratio, ν, are constant. Consequently, the analysis and the results presented in this second book are applicable to only linear elastic materials.

Thus, the second book addresses mathematical aspects of only kinematics of this topic and subsequent solutions. Historical perspective and references to any other articles or books are not included. Details of analysis related to stress equilibrium are also not included here. Historical perspective, references, and detailed stress analysis are presented in the first book. However, results derived from the first book which focussed on analysis involving stress equilibrium are used here.

As in the first book, we assume there exists a central elliptical hole in a plate whose length (along X axis) and width (along Y axis) are much larger than its thickness (along Z axis) such that we can treat it as a two dimensional (XY) plane stress problem. It has been shown in the first book (page 32) that at the tip of such an elliptical hole with its major axis along X axis, there exists only vertical stress

$\sigma_{yy} = [S_{yy}] \left[1 + \frac{2a_e}{b_e}\right] - [S_{xx}]$, where S_{xx} and S_{yy} are far field applied stresses (in horizontal i.e. X and vertical i.e. Y directions respectively) while a_e and b_e are semi major and semi minor axes of the elliptical hole at the end (not in the beginning), i.e. in the deformed state. Similarly, at the top of an elliptical hole with its minor axis along Y axis there exists only horizontal stress $\sigma_{xx} = [S_{xx}] \left[1 + \frac{2b_e}{a_e}\right] - [S_{yy}]$. Many researchers do not distinguish between the predeformed (starting) and deformed (ending) values for a and b. Many have formulas for tip stress for uniaxial loading only where terms involving S_{xx} are not involved.

In this second book, first we develop expressions for displacement and deformation gradients when an elliptical shape deforms to form another elliptical shape. Subsequently, these developed expressions for displacement and deformation gradients are linked with the above mentioned expressions for stresses (from the first book) through Hooke's law using four different definitions of strains (Green, engineering, logarithmic, and Almansi) to obtain lengths of semi major axis a_e and semi minor axis b_e of the deformed elliptical hole.

REVIEWS AND ENDORSEMENTS

<u>A REVIEW</u>

by

Amit Singh

Assistant Professor

Department of Mechanical Engineering

Indian Institute of Technology, Bombay, India

July 8, 2019

This monograph by Rajen Merchant is a welcome addition to the vast literature around the stress distribution in a thin plate with circular and elliptical holes. Kirsch [1] in 1898 came up with the first analytical solutions for the stress concentration around a circular hole in an infinite plate using Airy function approach. Then Inglis [2] in 1913 used complex potential functions and developed solutions for the stress field around elliptical holes in an infinite plate with the help of elliptical coordinate system (ECS). This led to further growth in literature [3, 4] where many such problems have been solved.

The attempts have also been made to develop closed-form solutions in Cartesian coordinates since solutions in elliptical coordinates are not convenient for application to strength analysis [5]. However, any information about the deformed configuration is not explicit in these relations developed in

Cartesian coordinates, moreover, the relations become cumbersome, therefore, the author likes the ECS and presented the stress field in terms of a shape parameter α_e of the deformed configuration in a monograph published earlier [6].

The present monograph develops relations for displacements and deformation gradients in ECS and uses four different definitions of strain (Engineering, Green, logarithmic and Almansi) as well as Hooke's law as constitutive equation, to obtain lengths of semi major and minor axes, a_e and b_e, respectively, of the deformed elliptical hole. Then, the monograph derives expressions for stress distribution in terms of material parameters, Youngs modulus E and Poissons ratio ν, applied stresses and initial geometry described by a_s and b_s.

The real substance of the whole book has already been presented by the author in the final Chapter 12 where summaries and conclusions of each chapter from 1 to 11 have been provided. The first two chapters are very interesting from the point of view of understanding nuances of the ECS when compared to Cartesian coordinate system (CCS), especially for the beginner. The first chapter tries to find analogies of straight lines x = constant and y = constant in the CCS with ellipses α = constant and hyperbolas β = constant, when the coordinate transformation of a point (x, y) in the CCS to (α, β) in the ECS reveals that $x = c \cosh\alpha \cos\beta$ and $y = c \sinh\alpha \sin\beta$, where $\tanh\alpha = b/a$, $a = c \cosh\alpha$ and $b = c \sinh\alpha$. Thereafter, the effects of changes in c and α upon the nature of elliptical and hyperbolic curves have been explored. Chapter 2 deals with the displacements in the ECS and concludes that displacements at the tip and top of the elliptical curve are horizontal and vertical, respectively, whereas they are not perpendicular to the ellipse at other points on the curve. Furthermore, it has also been shown that any displacement vector can be decomposed into two components, one due to change in c and the other due to change in α from the starting elliptical configuration to the end elliptical configuration. The analogy of a deformation with constant α and changing c with constant passenger coordinates with respect to driver in a moving car is illuminating and helps build the intuition among the readers about the meaning of α, β, and c in the ECS.

Chapter 3 provides a detailed derivation of both α in terms of x, y, c and x, y, β, and β in terms of x, y, c and x, y, α. Chapters 4 and 5 help derive the displacement and deformation gradients for the displacement $u \equiv (u_x, u_y)$ of a point from (x_s, y_s) to (x_e, y_e). Chapter 6 presents Hooke's law for the plane strain/stress and derives strains at the tip and top of the hole in terms of material constants, applied stresses and the ratio $\frac{a_e}{b_e}$. Chapter 7 presents four different definitions of strain. Chapter 8 provides the expressions for a_e and b_e for the deformed configuration when the engineering strain is chosen for Hooke's law showing relationship between the Cauchy stress and the engineering strain. This helps find solutions for stresses in final configuration and both tip and top of the holes have been analyzed in great details. The expressions for stresses were compared with Singh, Glinka and Dubey (1994) [7] and it was found that the results obtained by them are the special cases ($\nu = 1$) of the general solution developed in the monograph. A comparison with small deformation theory has also been provided and it was concluded that stress values obtained by small deformation theory serve as upper bound for the general results.

Chapter 9 develops equations for obtaining a_e and b_e when Green, logarithmic and Almansi strains are used in the analysis. Chapter 10 describes procedures for solving the equations obtained in Chapter 9 for Hooke's law based five different constitutive equations relating (a) the Green strain with the Cauchy stress, (b) the logarithmic strain with the Cauchy stress, (c) the logarithmic strain with the Kirchoff stress, (d) the Almansi strain with the Cauchy stress, and (e) the Almansi strain with the Kirchoff stress. Thus we get six different models, five described in Chapter 10 and one in Chapter 8.

Finally, Chapter 11 compares results for six different models under four loading conditions. It was found that, for the loading case $S_{xx} = 0$, b_e does not depend upon a_s for all six models, i.e., it does not matter if the initial hole is circular, elliptical or a sharp long crack. Similarly, when $S_{yy} = 0$, a_e does not depend upon b_s for all six models. The displacements of points on the contour of the elliptical hole were also compared for all four loading conditions and important conclusions were drawn with regard to expansion and contraction of the semi major and minor axes. Zero stress points and tip stress were also compared. In the last section of the Chapter, limitations of models were considered, where it was first noted that material non-linearity was not discussed at all and secondly, the geometrical non-linearity implicit in strain definitions may not find any solutions for particular loading conditions and/or initial geometry.

Overall, the chapters have been written clearly with enough details and therefore can aid teaching of any undergraduate or graduate level courses without much difficulties. The uncluttered graphs and explicit tables help understand the equations and procedures. There are couple of suggestions though:

- The notion of β appears on page 6 without any introduction. It assumes familiarity with the ECS or the monograph published earlier. We felt that upper portion of the first paragraph on page 7 can be shifted to the beginning of second paragraph of page 6.
- The trial and error procedures introduced in Chapter 10 are not very sophisticated. Some advance methods to solve the non-linear equations of Chapter 9 based on iteration and convergence in different function spaces such as non-linear Gauss - Seidel method and other methods [8, 9] could have been introduced.

Regardless of these minor issues, which can be easily addressed, the monograph is excellent in convincing the reader that working with the ECS and final shape parameters in deformed configuration makes the analytical solutions simpler to understand, which also helps think about the problems in a much clearer way with better intuitions. Its inclusion for both students and teachers of the subject as a nice text cannot be recommended more.

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Author's note : The first suggestion by Prof. Singh is implemented by reshuffling the text on pages 6 and 7. For the second suggestion, a paragraph on top of Chapter 10 has been inserted noting that a discussion on sophisticated computational procedures is beyond the scope of this book. However, if there is sufficient interest, such a discussion in detail may be presented in future as part of the third book on "Applications" where analytical results can be compared with results from FEM codes and/or experimental studies. Suggestion have been made to use analytical results as benchmarks to evaluate FEM codes.

AN ENDORSEMENT

by

Rashmi C. Desai

Professor Emeritus, Department of Physics

University of Toronto, Toronto, Canada

July 12, 2019

The science of elasticity and especially the stress-strain relations are fundamentally important due to their relevance both in Physics and in a large variety of applications. Non-linear relations play very important role in analysis of stresses and associated strains. The first two books by Rajen Merchant in this area are remarkable and outstanding. First one deals with geometrical non-linearity and develops a stress function for elastic stress for a thin plate system with an elliptical hole. The second book describes his derivations and ideas about non-linear strain displacement expressions for the same system.

Rajen uses the elliptical coordinate system since it is much more versatile. This is very challenging; his derivations are impressive and resulting exact expressions are truly new results emerging from his research. He has compared his theory and various approximate models by others, which is also very useful to researchers in this area. This series of books by Rajen Merchant should be in various University libraries, so that they are available to students in Civil Engineering, Physics and other areas where elasticity theory is of use and value. I congratulate Rajen for this novel accomplishment.

AN ENDORSEMENT

by
Amin Eshraghi Ph.D., P.Eng.,
Senior Materials and Structures Engineer
Ottawa, Canada

July 18, 2019

In his first book, Rajen derived closed form solution for the stress field for an elliptical hole in an infinite plate using the stress equilibrium equations. The stress components were expressed in terms of a single parameter α_e. The present book develops expressions for deformation gradients in the elliptical co-ordinate system. These deformation gradients are used to set up both non-linear and linear strain-displacement relations and are used to determine the parameters a_e, semi-major axis and b_e, semi-minor axis of the deformed elliptical hole for linear elastic materials.

These two books provide detailed analysis for the stress and the strain fields of cracks and holes in linear elastic thin plates. The developed expressions in these two books can be used for a wide variety of applications. I commend Rajen for these well written books and I am sure these two books will be strong references for courses where these topics are taught.

AN ENDORSEMENT

by

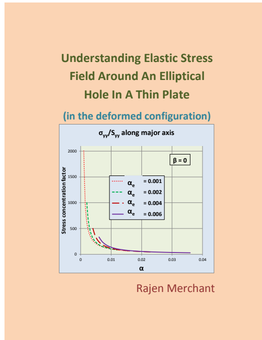
Anup Sahoo Ph.D.,

Senior Engineer/Scientist

Toronto, Canada

July 24, 2019

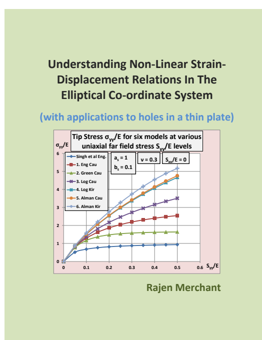
Rajen's first book dealt primarily with geometric non-linearity. This second book deals primarily with non-linearity in strain-displacement relations and non-linearity in definitions of strain. Six different models based on four definitions of strain and two definitions of stress are developed. It is shown that all expressions for all models contain E, Young's Modulus and ν, Poisson's Ratio. Researchers around the world will find these two books very useful while analyzing problems that involve thin sheets made of linear elastic materials. Some libraries have already included the first book in their collection.



ISBN 978-1-48359-262-6

The first monograph can be purchased at

https://store.bookbaby.com/book/Understanding-the-Elastic-Stress-Field-Around-an-Elliptical-Hole-in-a-Thin-Plate



ISBN 978-1-54398-916-8

The second monograph can be purchased at

https://store.bookbaby.com/book/Understanding-Non-Linear-Strain-Displacement-Relations

About the Author

Rajen Merchant studied at Bombay University for one year and then graduated as Bachelor of Technology (a five year course) at the Indian Institute of Technology, Bombay, India. After completing his Master of Science at Carnegie-Mellon University, Pittsburgh, USA, he migrated to Canada. He obtained his Master of Business Administration degree from Queen's University, Kingston, Canada. He has a keen interest in mathematics and its applications. His pursuit for mathematical accuracy has helped him better understand intricacies of the elliptical co-ordinate system. This is his second monograph. The first monograph (titled ***Understanding the Elastic Stress Field Around an Elliptical Hole in a Thin Plate (in the deformed configuration)- ISBN 978-1-48359-262-6***) was published in January 2017. He intends to complete the third book soon. This third book will examine applications of concepts presented in this and the previous monograph.