Optimal CAT 2.2 Manual

Optimal CAT is an easy-to-use *production-grade* Computer Adaptive Testing (CAT) engine. It's both for psychometricians to run simulations and for IT engineers to deploy to production.

It builds upon the shadow-test approach. It leverages state-of-the-art Mathematical Programming solvers (FICO Xpress and/or Gurobi) to deliver *optimal* tests to each student (hence the name Optimal CAT). It guarantees that test blueprints and constraints are always met, and measurement accuracy is maximized.

The engine is built as a microservice (stateless REST API interface) and can be easily integrated with many test delivery systems.

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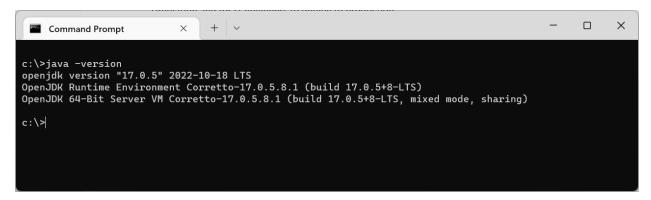
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Optimal CAT License

	Use	Limitation and Support				
Optimal-CAT Free Version	Personal use and research	Users need to install solver separately				
	use	(FICO Xpress or Gurobi) before using				
available on <u>www.optimal-</u>		Optimal-CAT				
<u>cat.com</u>						
		Current Optimal-CAT version expires:				
		10/31/2025 (download a new version				
		to renew)				
		Max load 10 tests				
		Limited technical support				
Optimal-CAT Licensed Version	Commercial use	Solver and license are embedded in				
shared with commercial users		Optimal-CAT as OEM components (optimized for performance and				
through private links		deployment)				
		No max load or time limits				
		Support both Windows and Linux platform				
		Free training, consultation and full technical support				
		Please contact us for trial and pricing (solver license included)				

Prerequisites

To use the Optimal CAT engine, you need to have **Java 17 or above** installed on your computer. If you are not sure which Java to use, you could try Amazon Corretto (https://aws.amazon.com/corretto). Once you have installed Java, make sure you check the version.



The Optimal CAT engine also uses **Microsoft Excel** (or Apache OpenOffice) as the editing tool for CAT test configurations.

* Optimal-CAT Free Version Users: need to install solver software (Fico Xpress or Gurobi) before using Optimal-CAT. If you currently do not have a valid solver license, you could try FICO Xpress Community License or Gurobi Academics License (if you are an academic user).

FICO Xpress Community License (needs to install **version 9.5** or above) https://www.fico.com/en/fico-xpress-community-license

Gurobi (needs to install **version 12.0.1**) https://www.gurobi.com/

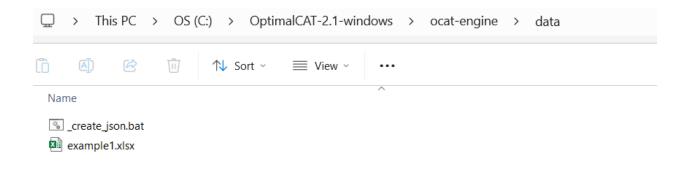
Getting Started Guide (Windows version, 30 minutes)

Download and unzip Optimal-CAT from www.optimal-cat.com. In the unzipped folder, you should see the following sub-folders.

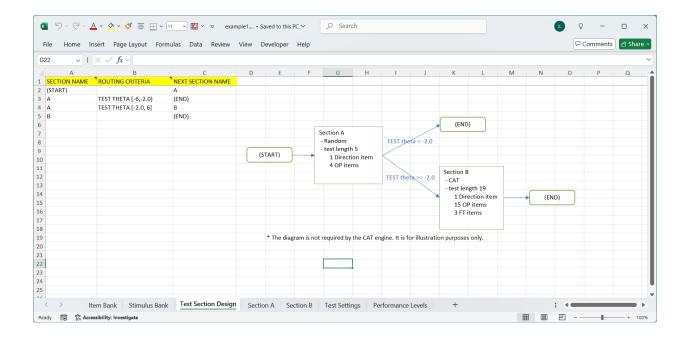
- API_examples
 cotal-engine
 cotal-loadtest
 cotal-simulator
 - API_examples: API examples in Java, Python and R ¹
 - ocat-engine: Optimal CAT engine
 - ocat-loadtest: Optimal CAT load test tool for IT engineers
 - ocat-simulator: Optimal CAT simulator for psychometricians

1. Test configuration

First, go to the "ocat-engine" folder, and then the "data" folder. There is one test configuration Excel file (example1.xlsx) in the folder. You could have multiple test configuration Excel files, each for one test.



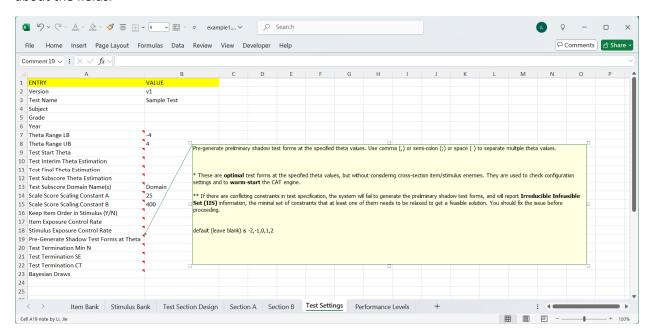
¹ No programming is needed for psychometricians to set up CAT tests and run simulations. These API examples are provided to assist IT engineers to integrate the Optimal-CAT engine into your testing delivery systems. You may also use these examples to build your own simulation client.



A quick overview of the sheets in test configuration files:

- **Item Bank** sheet contains item info. Yellow highlighted columns are required by the system. You could add any other columns you need, and you could use them later in Constraint definition.
 - o The engine supports the following IRT models: 1PL, 2PL, 3PL, GPCM and GRM.
- Stimulus Bank sheet contains stimulus info.
- **Test Section Design** sheet contains test section routing info. You can have many test sections and define routing criteria from one section to another. This allows building many test configurations like Multi-Stage Testing (MST). There are two default dummy sections (START) and (END) to designate the beginning and the end of a test.
- Section A, Section B ... (or simply A, B...) for each section defined in Test Section Design sheet, there should be a corresponding sheet for the section, in which you define things like test format for the section. The engine supports the following test formats:
 - CAT item/unit level CAT using Maximum Fisher Information
 - CAT_BM item/unit level CAT using b-param matching
 - o Random linear test, item order is random
 - Linear linear test, item order specified by user
 - o LOFT linear on-the-fly test, this is a special case of CAT with no test form refresh
- Test Settings sheet contains test level parameters, e.g., theta range, scale score constants, etc.
- Performance Levels sheet contains cut scores and performance levels definition.

The **cell comments** (a pop-up message when moving over the cell with a red triangle) provide useful tips about the fields.

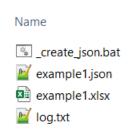


Once you have test specifications defined in Excel configuration files, the next step is to run **_create_json.bat**.

* Optimal-CAT Free Version Users: Before you run _create_json.bat, you need to go to "ocat-engine" folder, make sure the "solver" setting in the application.properties file (1 – Xpress, 2 – Gurobi) matches the solver installed on your computer.

```
X
    application.properties
                                                                                                                                                          .
                                                                                                                                                                 (g)
File Edit View
server.port = 8080
cat.jsonfolder = data
solver = 1
# solver : 1 - Xpress, 2 - Gurobi
# Other settings and their default values:
# n.digits = 6
                                      # output digits
                                     # solver run time limit (in seconds) before termination
# MIP (Mixed-Integer Programming) relative gap before declaring optimality
# timelimit = 1
# mipgap = 0.005
```

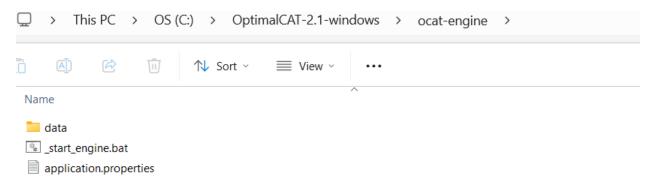
After running _create_json.bat, you should see a log file (log.txt) and a JSON file (example1.json) for each test. Check the screen output and the log file (log.txt) carefully to make sure no error is reported. Otherwise, you should correct errors and then re-run.



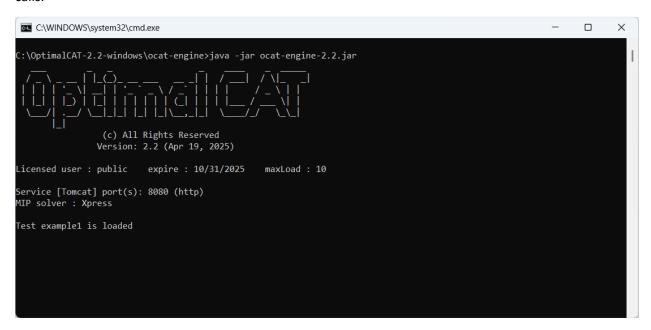
Notice: the CAT engine reads only JSON files, not test configuration Excel files.

2. Start CAT engine

To start the CAT engine, go to "ocat-engine" folder and run _start_engine.bat.



You should then see a command window like below. It shows that the engine is started and listening on port 8080, MIP solver is Xpress or Gurobi (depending on the settings in *application.properties* file) and test example1 is loaded. Leave the command window there. It's now ready to process incoming API calls.



The CAT engine has the following public REST API endpoints (see Appendix for details).

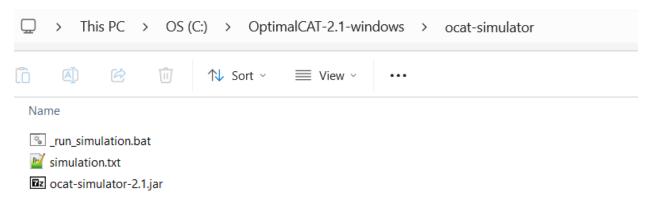
- HTTP GET request. It returns basic info of the engine and the loaded tests.
- /{testID} HTTP GET request. It returns the test JSON file.
- /IEC/{testID} HTTP GET request. It returns item/stimulus exposure counts.
- /cat HTTP POST request. It returns the next items to administer and final scores.
- /rescore HTTP POST request. It is for rescoring.

For GET request endpoints, you could use a browser to access them, for example,

For POST request endpoints (/cat and /rescore), they are for IT engineers to integrate with test delivery systems for test administration and scoring.

3. Run simulation (for Psychometricians)

Now the CAT engine is up-running, you could start simulation. Go to "ocat-simulator" folder



The *simulation.txt* file defines the parameters to run the simulation.

```
C:\OptimalCAT-2.1-windows\ocat-simulator\simulation.txt - Notepad++
                                                                                                                                                                                                                                                                                                            X
File Edit Search View Encoding Language Settings Tools Macro Run Plugins Window 2
             # Comments start with #
             testID = example1
             # Generate response files (optional)
# response.1 = student1.csv 100 FIXED(-2,-1,0,1,2)
# response.2 = student2.csv 100 NORMAL(0,1)
# response.3 = student3.csv 100 UNIFORM(-3,3)
                                                                                                                                                   # 100 students each at the fixed points -2, ..., 2, save to student1.csv
# 100 students from Normal(0,1) distribution, save to student2.csv
# 100 students from Uniform(-3,3) distribution, save to student3.csv
              # Simulator runs in multiple threads: thread.1, thread.2, ... (must start from thread.1)
# thread.1 = student1.csv
# thread.2 = 100 FIXED(-2,-1,0,1,2)
# thread.3 = 100 NORMAL(0,1)
# thread.4 = 100 UNIFORM(-3,3)
  18
19
20
21
22
23
24
25
26
27
28
29
30
             thread.1 = 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2) thread.2 = 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2) thread.3 = 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2) thread.4 = 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2) thread.5 = 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2)
                                                                                                         # simulation (fast mode) only runs on localhost
# random.seed = 0 is no random seed
              # engine.url = localhost:8080
              # random.seed = 12345
                                                                                                                            length: 1,146 lines: 32
                                                                                                                                                                                   Ln:1 Col:1 Pos:1
                                                                                                                                                                                                                                                        Windows (CR LF) UTF-8
Normal text file
```

Run *run simulation.bat* to start the simulation. Once it's completed, you should see the output below.

```
C:\\OptimalCAT-2.1-windows\ocat-simulator>java -jar ocat-simulator-2.1.jar simulation.txt

engine.url = http://localhost:8080
testID = example1

thread.2 : 02/15/25 14:37:18 --- sent to server : 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2)
thread.4 : 02/15/25 14:37:18 --- sent to server : 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2)
thread.3 : 02/15/25 14:37:18 --- sent to server : 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2)
thread.5 : 02/15/25 14:37:18 --- sent to server : 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2)
thread.1 : 02/15/25 14:37:18 --- sent to server : 100 FIXED(-2,-1.5,-1,-0.5,0,0.5,1,1.5,2)
Run Statistics: 1.0 minutes, 4500 students

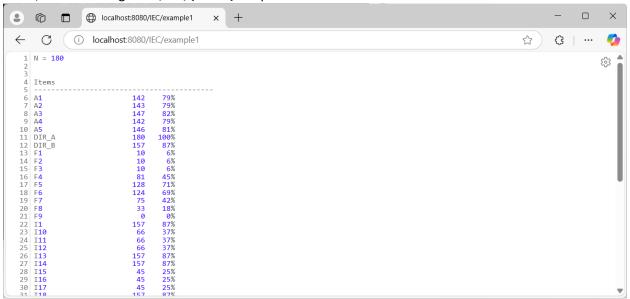
Saving files ...
Saving files ...
Saving files ... done

C:\OptimalCAT-2.1-windows\ocat-simulator>pause
Press any key to continue . . . _
```

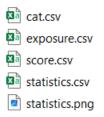
The simulation is run on the server side (fast mode, to save the API traffic between client and server). While the simulation is running, you could see its progress in the Server window.

```
C:\WINDOWS\system32\cmd.exe
 :\OptimalCAT-2.1-windows\ocat-engine>java -jar ocat-engine-2.1.jar
                   (c) All Rights Reserved
                  Version: 2.1 (Feb 15, 2025)
Licensed user : public
                             expire : 10/31/2025
                                                       maxLoad : 10
Service [Tomcat] port(s): 8080 (http)
Test example1 is loaded
[example1] thread.5 : 02/15/25 14:37:18 --- started
[example1] thread.2 : 02/15/25 14:37:18 --- started
example1] thread.4 : 02/15/25 14:37:18 --- started
[example1] thread.3 : 02/15/25 14:37:18 --- started
[example1] thread.1 : 02/15/25 14:37:18 --- started
[example1] thread.1 : 02/15/25 14:37:24 --- 100/900
                                                                62 ms/test
[example1] thread.3 : 02/15/25 14:37:24 --- 100/900
[example1] thread.2 : 02/15/25 14:37:25 --- 100/900
                                                                64 ms/test
                                                                65 ms/test
[example1] thread.5 : 02/15/25 14:37:25 --- 100/900
                                                                65 ms/test
[example1] thread.4 : 02/15/25 14:37:25 --- 100/900
                                                                66 ms/test
[example1] thread.1 : 02/15/25 14:37:30 --- 200/900
                                                                56 ms/test
```

While the simulation is running, you could also check the item/stimulus exposure counts and turned-off items/stimuli through the /IEC/{testID} endpoint.

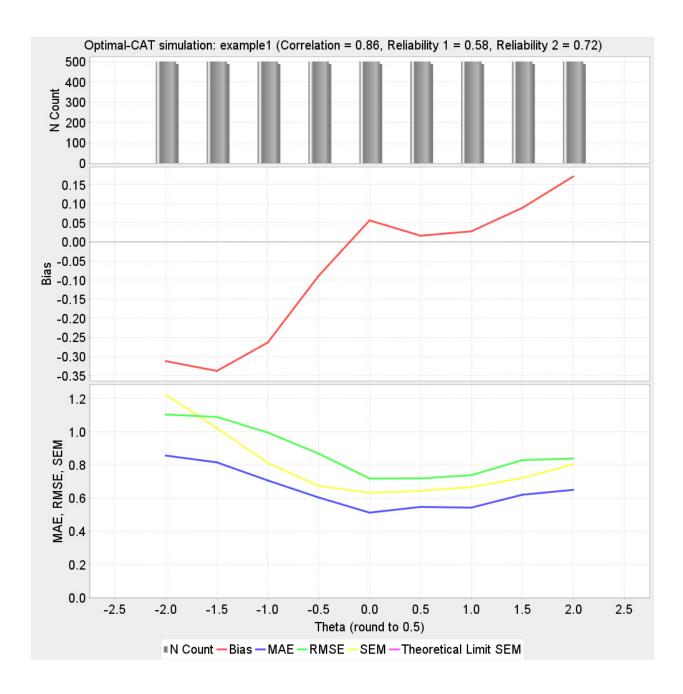


After the simulation run is completed, you should see results in a newly created folder named as {testid}_{date}_{time}. There are five files in the folder:



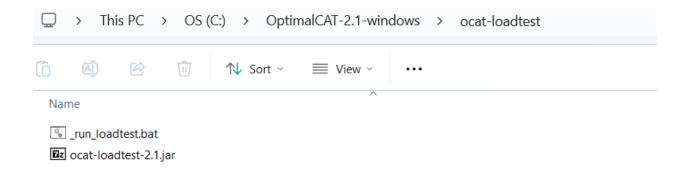
- cat.csv this file contains detailed test session information, item by item.
- exposure.csv this file contains exposure counts for both items and stimuli.
- score.csv this file contains final scores and sub-scores.
- statistics.csv this file contains the following measurement statistics
 - N Count
 - Bias
 - MAE Mean Absolute Error
 - o RMSE Root Mean Square Error
 - SEM (or Bayesian_SD) Standard Error of Measurement (or Standard Deviation in Bayesian CAT)
 - Theoretical Limit SEM it is calculated by aggregating all the pre-generated shadow test forms. It is shown only when the CAT engine detects that all students see all the test sections defined in the configuration Excel file; otherwise, the Theoretical Limit SEM is not shown (you must manually aggregate only the test sections that the student routes through to find out).

- O Correlation of true theta (θ_i) and estimated theta $(\hat{\theta}_i)$
- o Reliability 1 defined as $\rho^2 = \frac{Var(\theta_j)}{Var(\widehat{\theta}_j)}$
- $\qquad \text{Reliability 2 defined as } \widehat{\rho^2} = 1 \frac{E(CSEM_{\widehat{\theta}_j}^2)}{Var(\widehat{\theta}_j)}, \text{ where } CSEM_{\widehat{\theta}_j} \text{ is the Conditional Standard Error of Measurement (CSEM) at the estimated ability } \widehat{\theta}_j$
- Statistics.png chart using the numbers in statistics.csv



4. Run load test (for IT engineers)

To get an idea of the actual response time of engine API calls in real operations, a load test tool is provided in "ocat-loadtest" folder.



You could change the command line arguments to fit your load test needs.

```
C:\OptimalCAT-2.1-windows\ocat-loadtest\_run_loadtest.bat - Notepad++
                                                                                        ×
] 🚽 🗎 🖺 🖺 🧸 🥱 🚵 🚜 🐚 🖍 🕽 🗲 🗀 🥌 💘 🤏 📭 🖫 🚍 🖺 🌃 👂 📾 🐠 🗨 🗎 🖼
☐ _run_loadtest.bat 🗵
     REM ===
      REM ocat-loadtest-2.1.jar command line arguments:
     REM [1] engine.url
     REM [2] testID
     REM [3] # of threads
     REM [4] # of students per thread
      REM ===
  8
 11
12
     java -jar ocat-loadtest-2.1.jar localhost example1 5 1000
 13
      pause
 14
 15
 16
                                                                              UTF-8
Batch file
               length: 358 lines: 18
                                      Ln:1 Col:1 Pos:1
                                                                   Windows (CR LF)
```

Run _run_loadtest.bat to start a load test. You should see a window like below.

```
C:\WINDOWS\system32\cmd.exe
                                                                                                                   \Box
                                                                                                                          ×
:\OptimalCAT-2.1-windows\ocat-loadtest>REM ==================
 :\OptimalCAT-2.1-windows\ocat-loadtest>REM ocat-loadtest-2.1.jar command line arguments:
 :\OptimalCAT-2.1-windows\ocat-loadtest>REM [1] engine.url
:\OptimalCAT-2.1-windows\ocat-loadtest>REM [2] testID
:\OptimalCAT-2.1-windows\ocat-loadtest>REM [3] # of threads
 :\OptimalCAT-2.1-windows\ocat-loadtest>REM [4] # of students per thread
 :\OptimalCAT-2.1-windows\ocat-loadtest>java -jar ocat-loadtest-2.1.jar localhost example1 5 1000
engine.url = http://localhost:8080
testID = example1
threads = 5
students/thread = 1000
[example1] thread.1 : 02/15/25 15:07:59 --- started
example1] thread.2 : 02/15/25 15:07:59 --- started
[example1] thread.5 : 02/15/25 15:07:59 --- started
[example1] thread.3 : 02/15/25 15:07:59 --- started
[example1] thread.4 : 02/15/25 15:07:59 --- started
example1] thread.4 : 02/15/25 15:08:30 --- 100/1000 students, 1606 API calls, ~19.4 ms/call (client side)
example1] thread.3 : 02/15/25 15:08:30 --- 100/1000 students, 1606 API calls, ~19.6 ms/call (client side)
example1] thread.2 : 02/15/25 15:08:30 --- 100/1000 students, 1625 API calls, ~19.5 ms/call (client side)
[example1] thread.1 : 02/15/25 15:08:31 --- 100/1000 students, 1663 API calls, ~19.6 ms/call (client side)
example1] thread.5 : 02/15/25 15:08:34 --- 100/1000 students, 1796 API calls, ~19.5 ms/call (client side)
example1] thread.3 : 02/15/25 15:09:02 --- 200/1000 students, 3250 API calls, ~19.6 ms/call (client side)
[example1] thread.4 : 02/15/25 15:09:04 --- 200/1000 students, 3326 API calls, ~19.9 ms/call (client side)
[example1] thread.1 : 02/15/25 15:09:05 --- 200/1000 students, 3364 API calls, ~19.6 ms/call (client side)
example1] thread.2 : 02/15/25 15:09:05 --- 200/1000 students, 3402 API calls, ~19.6 ms/call (client side)
example1] thread.5 : 02/15/25 15:09:08 --- 200/1000 students, 3516 API calls, ~19.6 ms/call (client side)
```

At the end of the load test run, it reports the engine response time (average milliseconds per API call) from both server side and client side. Based on our experience, most of the real CAT tests have response time below 100 milliseconds. The larger the item bank and the more complicated the test specifications, the longer the response time.

```
[example1] thread.5 : 02/15/25 15:12:54 --- 900/1000 students, 15252 API calls, ~19.1 ms/call (client side)
[example1] thread.2 : 02/15/25 15:13:18 --- 1000/1000 students, 16516 API calls, ~19.2 ms/call (client side)
[example1] thread.1 : 02/15/25 15:13:23 --- 1000/1000 students, 16801 API calls, ~19.1 ms/call (client side)
[example1] thread.4 : 02/15/25 15:13:24 --- 1000/1000 students, 16801 API calls, ~19.2 ms/call (client side)
[example1] thread.5 : 02/15/25 15:13:25 --- 1000/1000 students, 16877 API calls, ~19.2 ms/call (client side)
[example1] thread.3 : 02/15/25 15:13:27 --- 1000/1000 students, 16991 API calls, ~19.0 ms/call (client side)

Run Statistics: 5.5 minutes, 5000 students, 83986 API calls, ~3.9 ms/call (server side), ~19.5 ms/call (client side)

C:\OptimalCAT-2.1-windows\ocat-loadtest>pause

Press any key to continue . . . __
```

Notice: The load test results heavily depend on the hardware and software it runs on. We recommend using **Linux** servers for operation. Microsoft Windows Home/Professional edition is not the best OS to host web services. The CPU utilization varies a lot from time to time due to various background processes in Windows. Also, running on Linux is a lot faster.

A typical load test scenario: IT engineers set up Optimal CAT engine on a Linux server, and then set up 10 or 20 computers as clients each running 10 or 20 threads to stress test the CAT engine and see how many servers they would need for operational tests.

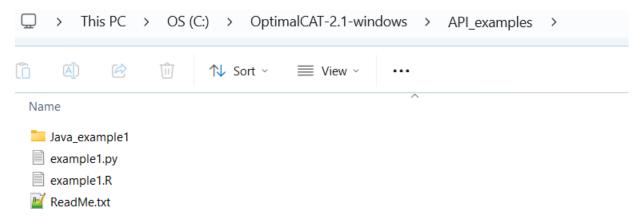
For most assessment programs, we believe one or two powerful servers should be enough. For very large testing programs with high concurrency requirements, we suggest creating a server pool and putting a load balancer in front, e.g., using <u>Load Balancer - Amazon Elastic Load Balancer (ELB) - AWS</u>.

Appendix I. Optimal CAT engine REST API

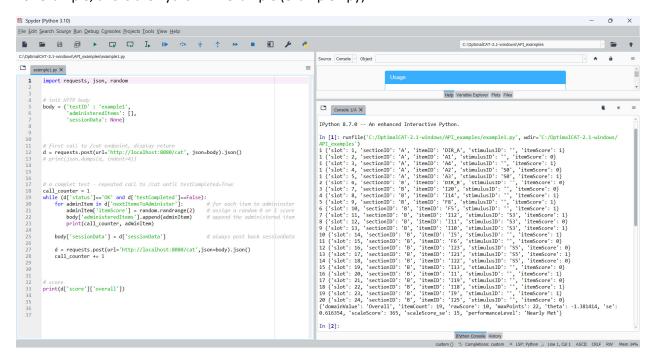
The CAT engine has the following public REST API endpoints:

- HTTP GET request. It returns basic info of the engine and the loaded tests.
- /{testID} HTTP GET request. It returns the test JSON file.
- /IEC/{testID} HTTP GET request. It returns item/stimulus exposure counts.
- /cat HTTP POST request. It returns the next items to administer and final scores.
- /rescore HTTP POST request. It is for rescoring.

There are examples (written in Java, Python and R) in "API_examples" folder for your reference.



For example, this is the Python API example (example1.py).



GET /

Parameters: (none)

Returns: basic info of Optimal CAT engine and loaded test

GET /{testID}

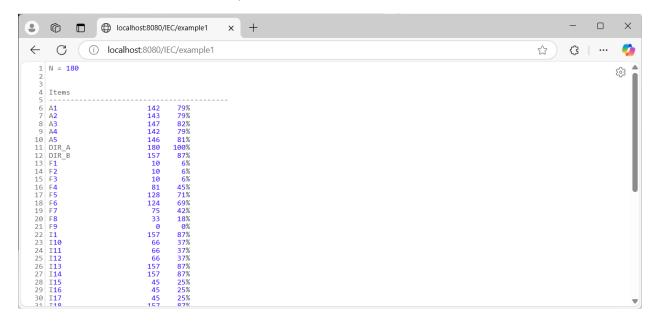
Parameters: testID String

Returns: test JSON file, including item bank, test sections, test settings, etc.

GET IEC/{testID}

Parameters: testID String

Returns: item and stimulus exposure counts



POST/CAT - for test administration (what is the next item?)

Header: Content-Type = application/json

Parameters: (none)

Body: JSON format of the following fields:

testID

• ;	admin	isteredItems	list of administered item information, as follows					
	0	slot	Integer	the slot in the test				
	0	sectionID	String	the section in the test				
	0	itemID	String	the item ID				
	0	stimulusID	String	the item's stimulus ID				
	0	itemScore	Integer	item score. If the item cannot be scored				
				in real time, use -1, and the item will be				
				ignored in theta calculation. If the item				
				is a FT or a non-scorable item,				
				itemScore is ignored by the engine.				

valid testID

String

sessionData

The is the sesstionData returned by /cat call. The CAT engine requires the client to post back sessinData. See below Returns for details of the sessionData.

testStartTheta (optional)
 for overriding the default test start theta

o distribution: String NORMAL, UNIFORM, FIXED

param1 Doubleparam2 Double

for NORMAL distribution, param1 is mean and param2 is standard deviation for UNIFORM distribution, param1 is min value and param2 is max value for FIXED distribution, param1 is the fixed value, param2 is ignored

• avoidItems (optional) list of String avoid item IDs. If the item ID is not in

the item pool, it will be ignored.

Returns: JSON format of the following fields:

•	testID	String	testID
---	--------	--------	--------

• **status** String the status of the request. "**OK**" if

successful, otherwise see the detailed

error message

• **testCompleted** Boolean whether the test has completed

• nextItemsToAdminster a list of the next items to administer

slot Integer the slot in the testsectionID String the section in the test

o itemID String the item ID

o stimulusID String the item's stimulus ID

• score same structure as "score" in /RESCORE (see below)

score is only filled when testCompleted = True

• **sessionData** the session data that the CAT engine requires the client

to always post back

o shadowTest the current shadow test. A list of the following fields

slot Integer the slot in the test sectionID String the section in the test

itemID String the item ID

stimulusID String the item's stimulus ID
lastSlotSeen int the last slot seen
test_theta Double interim test theta
test_se Double interim test theta S.E.
session_theta Double interim section theta

o session_se Double interim session theta S.E.

o bayesian_theta

mean Double mean of theta drawssd Double S.D. of theta draws

vector Array of Double theta draws

POST / RESCORE - for test rescoring

Header: Content-Type = application/json

Parameters: (none)

Body: JSON format

the same structure as in /CAT request body (see above), but sessionData could/should

be set to null, because this endpoint is for rescoring.

Returns: JSON format of the following fields:

• **testID** String testID

domainValue

• status String the status of the request. "OK" if

successful, otherwise see the detailed

"Overall"

error message

• score

o overall overall score details with the following fields

itemcount Integer number of items rawScore Integer raw score maxPoints Integer max possible points theta Double overall theta Double overall theta S.E. se scaleScore Integer overall scale score

String

scaleScore_se Integer overall scale score S.E.

performanceLevel String performance level

subscore list of subcores with the following fields
 domainName String the subscore domain name

domainScore same structure as overall score

bayesian_theta

mean Double mean of theta draws
 sd Double S.D. of theta draws

vector Array of Double theta draws

o scoredItems list of String the list of item IDs used in scoring

o skippedItems list of String the list of item IDs skipped in scoring

Appendix II. Optimal CAT engine IRT models and parameterization

The Optimal CAT engine supports the following IRT models: 1PL, 2PL, 3PL, GPCM and GRM.

The IRT model parameters are defined in the "Item bank" tab sheet in test configuration Excel file.

ITM_ID	STM_ID	ENEMIES	STATUS	MAX POINTS	MODEL	SCALING CONSTANT	А	В	c	D1	D2	D3	D4
DIR_A			DIR										
A1			OP	1	1PL	1		-1.8342					
A2			OP	1	2PL	1	0.4394	-1.4076					
A3			OP	1	3PL	1	0.9085	-1.4063	0.2452				
A4			OP	4	GPCM	1	1.25	-0.8342		0.6032	0.239	-0.2841	-0.5581
A5			OP	4	GRM	1	0.4394	-0.5750		0.7756	0.3673	-0.464	-0.6789

The following notations are used in the formulas below:

- m MAX POINTS
- sc SCALING CONSTANT
- a A parameter (discrimination parameter)
- b − B parameter (difficulty parameter)
- c C parameter (guessing parameter)
- d $[d_1, d_2, ...]$ D parameter vector (threshold parameters for GPCM and GRM)
- θ student ability
- x student score

1PL model

The item response function (the probability of a correct response to a dichotomous item) is

$$p(\theta, x = 1) = \frac{1}{1 + e^{-sc*(\theta - b)}}$$

2PL model

The item response function (the probability of a correct response to a dichotomous item) is

$$p(\theta, x = 1) = \frac{1}{1 + e^{-sc*a*(\theta - b)}}$$

3PL model

The item response function (the probability of a correct response to a dichotomous item) is

$$p(\theta, x = 1) = c + \frac{1 - c}{1 + e^{-sc*a*(\theta - b)}}$$

GPCM model

For the Generalized Partial Credit Model (GPCM, Muraki 1992), there are m+1 possible score points (0, 1, 2, ..., m). The probability of getting a score point of x is

$$p(\theta, x = 0) = \frac{1}{1 + \sum_{r=1}^{m} \exp\left[\sum_{j=1}^{r} sc * a * (\theta - b + d_{j})\right]}$$
$$p(\theta, x = 1, 2, ..., m) = \frac{\exp\left[\sum_{j=1}^{x} sc * a * (\theta - b + d_{j})\right]}{1 + \sum_{r=1}^{m} \exp\left[\sum_{j=1}^{r} sc * a * (\theta - b + d_{j})\right]}$$

GRM model

For the Graded Response Model (GRM, Samejima's 1969), there are m+1 possible score points (0, 1, 2, ..., m). The cumulative probability of getting a score of x or above is

$$p_cum(\theta, x = 0) = 1$$

$$p_cum(\theta, x = 1, 2, ..., m) = \frac{1}{1 + e^{-sc*a*(\theta - b + d_x)}}$$

The probability of getting a score point of x is then

$$p(\theta, x = 0, 1, ..., m - 1) = p_cum(\theta, x) - p_cum(\theta, x + 1)$$

$$p(\theta, x = m) = p_cum(\theta, m)$$

Appendix III. Optimal CAT software libraries and licenses

Optimal-CAT software includes the following software/libraries:

- FICO Xpress, which is proprietary software, needs a valid license for commercial use
- Gurobi, which is proprietary software, needs a valid license for commercial use
- Springframework library, which is licensed under Apache License 2.0.
- Fasterxml jackson library, which is licensed under Apache License 2.0.
- Apache Commons Math library, which is licensed under Apache License 2.0.
- License3j library, which is licensed under Apache License 2.0.
- jfreechart library, which is licensed under GNU Lesser General Public License.
- Pandas library, which is licensed under BSD 3-Clause License.
- Pandasql library, which is licensed under MIT License.
- Openpyxl library, which is licensed under MIT License.