STEP Mathematics

# Sixth Term Examination Papers — 9470
## MATHEMATICS 2 — Morning
**Wednesday 11 June 2025** — Time: 3 hours

Additional Material: Answer Booklet

*4606618491*

**INSTRUCTIONS TO CANDIDATES**

Read this page carefully.

Do **NOT** open this question paper until you are told that you may do so.

Read and follow the additional instructions on the front of the answer booklet.

**INFORMATION FOR CANDIDATES**

There are 12 questions in this paper.

Each question is marked out of 20.

You may answer as many questions as you choose. You are advised to concentrate on no more than six questions. Little credit will be given for fragmentary answers.

All your answers will be marked.

Crossed out work will NOT be marked.

Your final mark will be based on the six questions for which you gain the highest marks.

**There is NO Mathematical Formulae Booklet.**

**Calculators are NOT permitted.**

**Bilingual dictionaries are NOT permitted.**

**Wait to be told you may begin before turning this page.**



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# Section A: Pure Mathematics

**1** The function Min is defined as

$$\text{Min}(a, b) = \begin{cases} a & \text{if } a \leqslant b \\ b & \text{if } a > b\,. \end{cases}$$

**(i)** Sketch the graph $y = \text{Min}(x^2, 2x)$.

**(ii)** Solve the equation $2\,\text{Min}(x^2, 2x) = 5x - 3$.

**(iii)** Solve the equation $\text{Min}(x^2, 2x) + \text{Min}(x^3, 4x) = mx$ in the cases $m = 2$ and $m = 6$.

**(iv)** Show that $(1, -3)$ is a local maximum point on the curve $y = 2\,\text{Min}(x^2, x^3) - 5x$ and find the other three local maxima and minima on this curve.

Sketch the curve.

**2** **(i)** **(a)** Show that if the complex number $z$ satisfies the equation

$$z^2 + |z+b| = a,$$

where $a$ and $b$ are real numbers, then $z$ must be either purely real or purely imaginary.

**(b)** Show that the equation

$$z^2 + \left|z + \tfrac{5}{2}\right| = \tfrac{7}{2}$$

has no purely imaginary roots.

**(c)** Show that the equation

$$z^2 + \left|z + \tfrac{7}{2}\right| = \tfrac{5}{2}$$

has no purely real roots.

**(d)** Show that, when $\frac{1}{2} < b < \frac{3}{4}$, the equation

$$z^2 + |z+b| = \tfrac{1}{2}$$

will have at least one purely imaginary root and at least one purely real root.

**(ii)** Solve the equation

$$z^3 + |z+2|^2 = 4.$$

**3** **(i)** Sketch a graph of $y = \dfrac{\ln x}{x}$ for $x > 0$.

**(ii)** Use your graph to show the following.

**(a)** $3^\pi > \pi^3$

**(b)** $\left(\frac{9}{4}\right)^{\sqrt{5}} > \sqrt{5}^{\,\frac{9}{4}}$

**(iii)** Given that $1 < x < 2$, decide, with justification, which is the larger of $x^{x+2}$ or $(x+2)^x$.

**(iv)** Show that the inequalities $9^{\sqrt{2}} > \sqrt{2}^9$ and $3^{2\sqrt{2}} > \left(2\sqrt{2}\right)^3$ are equivalent. Given that $e^2 < 8$, decide, with justification, which is the larger of $9^{\sqrt{2}}$ and $\sqrt{2}^9$.

**(v)** Decide, with justification, which is the larger of $8^{\sqrt[3]{3}}$ and $\sqrt[3]{3}^{\,8}$.

**4** Let $\lfloor x \rfloor$ denote the largest integer that satisfies $\lfloor x \rfloor \leqslant x$.
For example, if $x = -4.2$, then $\lfloor x \rfloor = -5$.

**(i)** Show that, if $n$ is an integer, then $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

**(ii)** Let $n$ be a positive integer and define function $\mathrm{f}_n$ by

$$\mathrm{f}_n(x) = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \ldots + \left\lfloor x + \frac{n-1}{n} \right\rfloor - \lfloor nx \rfloor$$

**(a)** Show that $\mathrm{f}_n\left(x + \frac{1}{n}\right) = \mathrm{f}_n(x)$.

**(b)** Evaluate $\mathrm{f}_n(t)$ for $0 \leqslant t < \frac{1}{n}$.

**(c)** Hence show that $\mathrm{f}_n(x) \equiv 0$.

**(iii)** **(a)** Show that $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor$.

**(b)** Hence, or otherwise, simplify

$$\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \ldots + \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor + \ldots .$$

**5** *You need not consider the convergence of the improper integrals in this question.*

**(i)** Use the substitution $x = u^{-1}$ to show that

$$\int_0^\infty \frac{\sqrt{x} - 1}{\sqrt{x(x^3+1)}}\,\mathrm{d}x = 0\,.$$

**(ii)** Use the substitution $x = u^{-2}$ to show that

$$\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}x = 2\int_0^\infty \frac{1}{\sqrt{x^6+1}}\,\mathrm{d}x\,.$$

**(iii)** Find, in terms of $p$ and $s$, a value of $r$ for which

$$\int_0^\infty \frac{x^r - 1}{\sqrt{x^s(x^p+1)}}\,\mathrm{d}x = 0,$$

given that $p$ and $s$ are fixed values for which the required integrals converge.

**(iv)** Show that, for any positive value of $k$, it is possible to find values of $p$ and $q$ for which

$$\int_0^\infty \frac{1}{\sqrt{x^p+1}}\,\mathrm{d}x = k\int_0^\infty \frac{1}{\sqrt{x^q+1}}\,\mathrm{d}x\,.$$

**6** **(i)** The circle $x^2 + (y - a)^2 = r^2$ touches the parabola $2ky = x^2$, where $k > 0$, tangentially at two points. Show that $r^2 = k(2a - k)$.

Show further that if $r^2 = k(2a - k)$ and $a > k > 0$, then the circle $x^2 + (y - a)^2 = r^2$ touches the parabola $2ky = x^2$ tangentially at two points.

**(ii)** The lines $y = c \pm x$ are tangents to the circle $x^2 + (y - a)^2 = r^2$. Find $r^2$, and the coordinates of the points of contact, in terms of $a$ and $c$.

**(iii)** $C_1$ and $C_2$ are circles with equations $x^2 + (y - a_1)^2 = r_1^2$ and $x^2 + (y - a_2)^2 = r_2^2$ respectively, where $a_1 \neq a_2$ and $r_1 \neq r_2$.

Each circle touches the parabola $2ky = x^2$ tangentially at two points and the lines $y = c \pm x$ are tangents to both circles.

**(a)** Show that $a_1 + a_2 = 2c + 4k$ and that $a_1^2 + a_2^2 = 2c^2 + 16kc + 12k^2$.

**(b)** The circle $x^2 + (y - d)^2 = p^2$ passes through the four points of tangency of the lines $y = c \pm x$ to the two circles, $C_1$ and $C_2$. Find $d$ and $p^2$ in terms of $k$ and $c$.

**(c)** Show that the circle $x^2 + (y - d)^2 = p^2$ also touches the parabola $2ky = x^2$ tangentially at two points.

**7** The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2x\frac{\mathrm{d}x}{\mathrm{d}t}$$

describes the motion of a particle with position $x(t)$ at time $t$. At $t = 0$, $x = a$, where $a > 0$.

**(i)** Solve the differential equation in the case where $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a^2$ when $t = 0$.

What happens to the particle as $t$ increases from 0?

**(ii)** Solve the differential equation in the case where $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a^2 + p^2$ when $t = 0$, where $p > 0$.

What happens to the particle as $t$ increases from 0?

**(iii)** Solve the differential equation in the case where $\dfrac{\mathrm{d}x}{\mathrm{d}t} = a^2 - q^2$ when $t = 0$, where $q > 0$.

What happens to the particle as $t$ increases from 0? Give conditions on $a$ and $q$ for the different cases which arise.

**8** If we split a set $S$ of integers into two subsets $A$ and $B$ whose intersection is empty and whose union is the whole of $S$, and such that

- the sum of the elements of $A$ is equal to the sum of the elements of $B$
- and the sum of the squares of the elements of $A$ is equal to the sum of the squares of the elements of $B$,

then we say that we have found a *balanced partition* of $S$ into two subsets.

**(i)** Find a balanced partition of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into two subsets $A$ and $B$, each of size 4.

**(ii)** Given that $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ are sequences with

$$\sum_{k=1}^{m} a_k = \sum_{k=1}^{m} b_k \quad \text{and} \quad \sum_{k=1}^{m} a_k^2 = \sum_{k=1}^{m} b_k^2,$$

show that

$$\sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m} (c + b_k)^3 = \sum_{k=1}^{m} b_k^3 + \sum_{k=1}^{m} (c + a_k)^3$$

for any real number $c$.

**(iii)** Find, with justification, a balanced partition of the set $\{1, 2, 3, \ldots, 16\}$ into two subsets $A$ and $B$, each of size 8, which also has the property that

- the sum of the cubes of the elements of $A$ is equal to the sum of the cubes of the elements of $B$.

**(iv)** You are given that the sets $A = \{1, 3, 4, 5, 9, 11\}$ and $B = \{2, 6, 7, 8, 10\}$ form a balanced partition of the set $\{1, 2, 3, \ldots, 11\}$.

Let $S = \{n^2, (n + 1)^2, (n + 2)^2, \ldots, (n + 11)^2\}$, where $n$ is any positive integer. Find, with justification, two subsets $C$ and $D$ of $S$ whose intersection is empty and whose union is the whole of $S$, and such that

- the sum of the elements of $C$ is equal to the sum of the elements of $D$.

## Section B: Mechanics

**9** Points $A$ and $B$ are at the same height and a distance $\sqrt{2}r$ apart. Two small, spherical particles of equal mass, $P$ and $Q$, are suspended from $A$ and $B$, respectively, by light inextensible strings of length $r$. Each particle individually may move freely around and inside a circle centred at the point of suspension.

The particles are projected simultaneously from points which are a distance $r$ vertically below their points of suspension, directly towards each other and each with speed $u$. When the particles collide, the coefficient of restitution in the collision is $e$.

**(i)** Show that, immediately after the collision, the horizontal component of each particle's velocity has magnitude $\frac{1}{2}ev\sqrt{2}$, where $v^2 = u^2 - gr\left(2-\sqrt{2}\right)$ and write down the vertical component in terms of $v$.

**(ii)** Show that the strings will become taut again at a time $t$ after the collision, where $t$ is a non-zero root of the equation

$$(r - evt)^2 + \left(-r + vt - \tfrac{1}{2}\sqrt{2}gt^2\right)^2 = 2r^2.$$

**(iii)** Show that, in terms of the dimensionless variables

$$z = \frac{vt}{r} \quad \text{and} \quad c = \frac{\sqrt{2}v^2}{rg}$$

this equation becomes

$$\left(\frac{z}{c}\right)^3 - 2\left(\frac{z}{c}\right)^2 + \left(\frac{2}{c} + 1 + e^2\right)\left(\frac{z}{c}\right) - \frac{2}{c}(1+e) = 0.$$

**(iv)** Show that, if this equation has three equal non-zero roots, $e = \frac{1}{3}$ and $v^2 = \frac{9}{2}\sqrt{2}rg$. Explain briefly why, in this case, no energy is lost when the string becomes taut.

**(v)** In the case described in **(iv)**, the particles have speed $U$ when they again reach the points of their motion vertically below their points of suspension. Find $U^2$ in terms of $r$ and $g$.

**10** The lower end of a rigid uniform rod of mass $m$ and length $a$ rests at point $M$ on rough horizontal ground. Each of two elastic strings, of natural length $\ell$ and modulus of elasticity $\lambda$, is attached at one end to the top of the rod. Their lower ends are attached to points $A$ and $B$ on the ground, which are a distance $2a$ apart. $M$ is the midpoint of $AB$.

$P$ is the point at the top of the rod and lies in the vertical plane through $AMB$.

Suppose that the rod is in equilibrium with angle $PMB = 2\theta$, where $\theta < 45^\circ$ and $\ell$ is such that both strings are in tension.

**(i)** Show that angle $APB$ is a right angle.

Show that that the force exerted on the rod by the elastic strings can be written as the sum of

- a force of magnitude $\dfrac{2a\lambda}{\ell}$ parallel to the rod
- and a force of magnitude $\sqrt{2}\,\lambda$ acting along the bisector of angle $APB$.

**(ii)** By taking moments about point $M$, or otherwise, show that $\cos\theta + \sin\theta = \dfrac{2\lambda}{mg}$.

Deduce that it is necessary that $\frac{1}{2}mg < \lambda < \frac{1}{2}\sqrt{2}\,mg$.

**(iii)** $N$ and $F$ are the magnitudes of the normal and frictional forces, respectively, exerted on the rod by the ground at $M$.

Show, by taking moments about an appropriate point, or otherwise, that

$$N - F\tan 2\theta = \tfrac{1}{2}mg.$$

## Section C: Probability and Statistics

**11** **(i)** By considering the sum of a geometric series, or otherwise, show that

$$\sum_{r=1}^{\infty} r x^{r-1} = \frac{1}{(1-x)^2} \quad \text{for } |x| < 1.$$

**(ii)** Ali plays a game with a fair $2k$-sided die. He rolls the die until the first $2k$ appears. Ali wins if all the numbers he rolls are even.

**(a)** Find the probability that Ali wins the game.

If Ali wins the game, he earns £1 for each roll, including the final one. If he loses, he earns nothing.

**(b)** Find Ali's expected earnings from playing the game.

**(iii)** Find a simplified expression for

$$1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + \ldots + (n+1)x^n,$$

where $n$ is a positive integer.

**(iv)** Zen plays a different game with a fair $2k$-sided die. She rolls the die until the first $2k$ appears, and wins if the numbers rolled are strictly increasing in size. For example, if $k = 3$, she wins if she rolls 2, 6 or 1, 4, 5, 6, but not if she rolls 1, 4, 2, 6 or 1, 3, 3, 6.

If Zen wins the game, she earns £1 for each roll, including the final one. If she loses, she earns nothing.

Find Zen's expected earnings from playing the game.

**(v)** Using the approximation

$$\left(1 + \frac{1}{n}\right)^n \approx \mathrm{e} \quad \text{for large } n,$$

show that, when $k$ is large, Zen's expected earnings are a little over 35% more than Ali's expected earnings.

**12** Let $X$ be a Poisson random variable with mean $\lambda$ and let $p_r = \mathrm{P}(X = r)$, for $r = 0, 1, 2, \ldots$ . Neither $\lambda$ nor $\lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}$ is an integer.

**(i)** Show, by considering the sequence $d_r \equiv p_r - p_{r-1}$ for $r = 1, 2, \ldots$, that there is a unique integer $m$ such that $\mathrm{P}(X = r) \leqslant \mathrm{P}(X = m)$ for all $r = 0, 1, 2, \ldots$, and that $\lambda - 1 < m < \lambda$.

**(ii)** Show that the minimum value of $d_r$ occurs at $r = k$, where $k$ is such that

$$k < \lambda + \tfrac{1}{2} + \sqrt{\lambda + \tfrac{1}{4}} < k + 1.$$

**(iii)** Show that the condition for the maximum value of $d_r$ to occur at $r = 1$ is

$$1 < \lambda < 2 + \sqrt{2}.$$

**(iv)** In the case $\lambda = 3.36$, sketch a graph of $p_r$ against $r$ for $r = 0, 1, 2, \ldots, 6, 7$.

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