CAMBRIDGE

Sixth Term Examination Paper (STEP)

Mathematics 2 (9470)

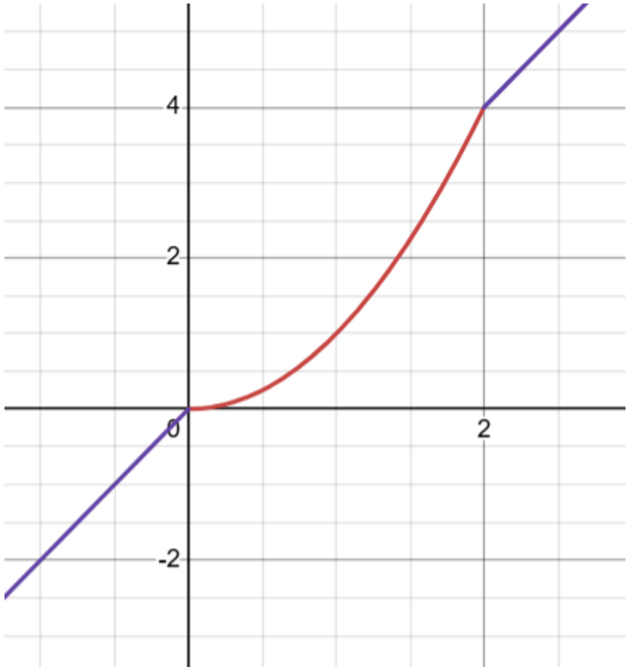
2025

Examiners' report and **mark scheme**

CAMBRIDGE

STEP Mathematics 2 mark scheme

Question		Answer	Marks	Guidance
1	i	$\text{Min}(x^2, 2x) = \begin{matrix} x^2 & 0 \le x \le 2 \\ 2x & \text{otherwise} \end{matrix}$	B1	

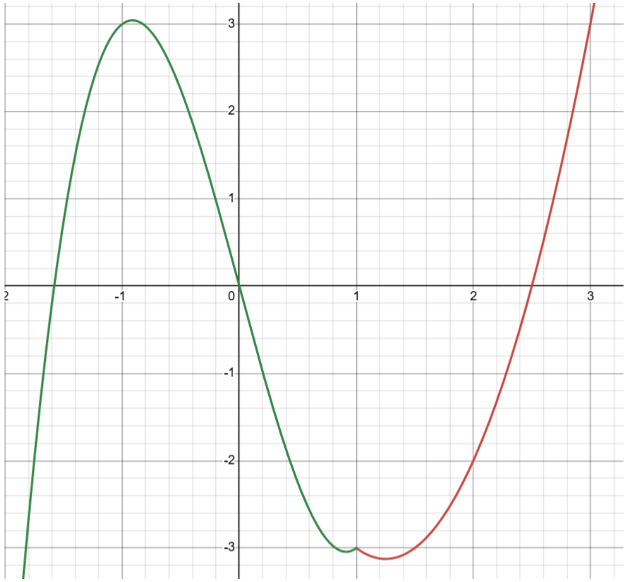
Question		Answer	Marks	Guidance
			G1 G1 [3]	Two straight line sections part of the same line through the origin. Quadratic portion correct and (2,4) indicated.

Question		Answer	Marks	Guidance
	ii	For $0 \leq x \leq 2$, $2x^2 = 5x - 3$ $(x-1)(2x-3) = 0$		
		$x = 1, x = \frac{3}{2}$	**B1**	Solutions must be seen as part of final answer.
		Otherwise, $4x = 5x - 3$		
		$x = 3$	**B1**	Solution must be seen as part of final answer.
		$x = 1, x = \frac{3}{2}, x = 3$	**E1**	Must be clear that there has been a check for at least one of the solutions to see if it is valid. Can be awarded for a correct judgement from an incorrect solution.
			[3]	
	iii	$\text{Min}(x^3, 4x) = x^3$ when $x^3 \leq 4x$		
		$x(x-2)(x+2) \leq 0$		
		$x \leq -2$ or $0 \leq x \leq 2$	**B1**	Award this mark if the equations are solved and then each solution is checked by evaluating the Min function at that point. If checking after solving then all **four** possible equations need to be considered (i.e. $x^2 + 4x = mx$ needs to be included as well).
		For $x \leq -2$: $x^3 + 2x = mx$		

Question		Answer	Marks	Guidance
		If $m = 2$, this is $x^3 = 0$, so no solution within this range. If $m = 6$, this is $x^3 - 4x = 0$, the only solution within the range is $x = -2$. For $-2 < x < 0$: $$2x + 4x = mx$$ If $m = 2$, this is $4x = 0$, so no solution within this range. If $m = 6$, this is true for all values within the range. For $0 \leq x \leq 2$: $$x^3 + x^2 = mx$$ If $m = 2$, this is $x(x + 2)(x - 1) = 0$, so the only solutions within the range are $x = 0$ and $x = 1$. If $m = 6$, this is $x(x + 3)(x - 2) = 0$, so the only solution within the ranges are $x = 0$ and $x = 2$.		

Question		Answer	Marks	Guidance
		For $x > 2$: $$2x + 4x = mx$$ If $m = 2$, this is $4x = 0$, so no solution within this range. If $m = 6$, this is true for all values within the range.	**B1**	Equation stated correctly for all four cases either for both values of *m* or written in terms of *m*.
		$m = 2$: $x = 0, x = 1$	**B1**	
		$m = 6$: $-2 \leq x \leq 0$,	**B1**	
		$x \geq 2$	**B1**	
			[5]	
	iv	$$x^2 \leq x^3$$ $$x^3 - x^2 \geq 0$$		
		$$\text{Min}(x^2, x^3) = \begin{matrix} x^3 & x \leq 1 \\ x^2 & x > 1 \end{matrix}$$	**B1**	

Question		Answer	Marks	Guidance
		If $y = 2x^3 - 5x$, $\frac{dy}{dx} = 6x^2 - 5$, so $y = 2x^3 - 5x$ has a positive gradient at the point $(1, -3)$.		
		If $y = 2x^2 - 5x$, $\frac{dy}{dx} = 4x - 5$, so $y = 2x^2 - 5x$ has a negative gradient at the point $(1, -3)$.		
		Therefore $y = 2Min(x^2, x^3) - 5x$ has a local maximum at $(-1, 3)$.	**E1**	
		Stationary points where $x < 1$:		
		$6x^2 - 5 = 0$, so $x = \pm\sqrt{\frac{5}{6}}$,		
		both of which satisfy $x < 1$.		
		Points are $\left(\pm\sqrt{\frac{5}{6}}, \mp\frac{10}{3}\sqrt{\frac{5}{6}}\right)$	**B1**	
		Stationary points where $x > 1$:		
		$4x - 5 = 0$, so $x = \frac{5}{4}$,		
		which satisfies $x > 1$.	**E1**	Confirmation that all values are valid.

Question		Answer	Marks	Guidance
		Point is $\left(\frac{5}{4}, -\frac{25}{8}\right)$	B1	
		$\left(\frac{10}{3}\sqrt{\frac{5}{6}}\right)^2 = \frac{250}{27} = \frac{1250}{135}$ and $\left(\frac{25}{8}\right)^2 = \frac{625}{64} = \frac{1250}{128}$		
		Therefore $-\frac{10}{3}\sqrt{\frac{5}{6}} > -\frac{25}{8}$ and $\left(\sqrt{\frac{5}{6}}, -\frac{10}{3}\sqrt{\frac{5}{6}}\right)$ is higher than $\left(\frac{5}{4}, -\frac{25}{8}\right)$		
			G1	Graph through origin and correct shape for $x < 1$. Ignore slight inaccuracies in the vicinity of $(1, -3)$. The stationary points should be similar distances above/below the x -axis (allow no more than a ratio of 2 between the distances).
			G1	Correct shape for $x > 1$. Ignore slight inaccuracies in the vicinity of $(1, -3)$
			G1	$\left(-\sqrt{\frac{5}{2}}, 0\right)$ and $\left(\frac{5}{2}, 0\right)$ and correct behaviour at $(1, -3)$Correct relative positions for the two local minima but only if some justification is seen. Only awarded if the coordinates were calculated correctly.
			G1	
			[9]	

Question			Answer	Marks	Guidance
2	i	a	$\lvert z+b\rvert$ and a are both real, so z^2 must be real.	B1	Also award if equating imaginary parts of a *correct* equation. If they have squared the equation, do not award for finding $4xy(x^2-y^2-a)=0$
			If $z=x+iy$, then $z^2=x^2-y^2+2ixy$	M1	If using arguments, M1 for stating possible values for the argument of z^2. Also award for stating that $z^2 = \pm k$ for a *positive* real number k .
			$2xy=0$		
			$x=0$ (z is purely imaginary) or $y=0$ (z is purely real)	E1	Both cases explained. Allow follow through if a small algebraic error beforehand still allows the same analysis to be done. If using the alternative method, award for saying that if $z^2=-k$ then $z=\pm\sqrt{k}i$, and if $z^2=k$ then $z=\pm\sqrt{k}$.
				[3]	
		b	Suppose $z=iy$ for some real value of y:		
			$-y^2+\sqrt{\left(\frac{5}{2}\right)^2+y^2}=\frac{7}{2}$	M1	Can award with BOD if numbers are incorrect.
			$\frac{25}{4}+y^2=\frac{49}{4}+7y^2+y^4$	A1	For writing down the correct quartic equation (brackets expanded, does not have to have terms grouped).
			$y^4+6y^2+6=0$		

Question			Answer	Marks	Guidance
			$y^2 = -\frac{6\pm\sqrt{6^2-4(6)}}{2}$ Both values of y^2 are negative, so there are no real solutions for y. (Therefore there are no purely imaginary roots.)	E1 [3]	For establishing that the quartic equation does not have real roots and understanding what this implies for solutions to the original problem (this may be implicit). If the equation has an algebraic error, only award if analysed using methods that would be correct for the correct equation (computing roots and realising they are negative or noting that everything on the LHS is positive).

Question			Answer	Marks	Guidance
		c	Suppose $z = x$ for some real value of x:		
			$x^2 + \sqrt{\left(x + \frac{7}{2}\right)^2} = \frac{5}{2}$	M1	For writing down the equation correctly having used that $y = 0$. If sticking with z variable, then enough to have some indication that the modulus is of a real number. Can award with BOD if the numbers are incorrect.
			$x^2 + \left\lvert x + \frac{7}{2} \right\rvert = \frac{5}{2}$		
			If $x \geq -\frac{7}{2}$:	M1	Identification of two cases and the conditions on x for each. Can also award if the conditions are not stated, but the candidate has checked that the solutions fall into the correct range or solve the relevant equations.
			$x^2 + x + 1 = 0$ Discriminant = -3, so no roots.	E1	For correctly analysing the quadratic equation. May be awarded if the equation is wrong but the difficulty is unchanged (i.e. still a quadratic equation) and the analysis is correct. Do not award if there is no evidence of having recognised that this is one of two cases.
			If $x < -\frac{7}{2}$: $x^2 - x - 6 = 0$ $(x - 3)(x + 2) = 0$		
			Roots are 3 and -2, neither of which is valid.	E1	Same as above.
			(Therefore there are no purely real roots.)		
				[4]	

Question			Answer	Marks	Guidance
		d	Suppose $z = iy$ for some real value of y: $-y^2 + \sqrt{b^2 + y^2} = \frac{1}{2}$ $y^2 + b^2 = \frac{1}{4} + y^2 + y^4$		
			$y^4 - \left(b^2 - \frac{1}{4}\right) = 0$	M1	For writing down the quartic equation; OR for identifying values which can be used to show existence of a root via the IVT (if it is otherwise clear that this is the method they are using).
			There will be a positive value for y^2 if $b^2 > \frac{1}{4}$, which is satisfied if $\frac{1}{2} < b < \frac{3}{4}$, so there must be at least one purely imaginary root.	E1	For establishing that the correct equation has at least one real root, using any method (using the intermediate value theorem is included). If the equation is incorrect, only award marks if analysis is practically unchanged compared to the correct equation.
			Suppose $z = x$ for some real value of x: $x^2 + \sqrt{(x+b)^2} = \frac{1}{2}$		
			$x^2 + \|x + b\| = \frac{1}{2}$	M1	For writing down the quadratic equation. Allow for line above also. OR for identifying values which can be used to show existence of a root via the IVT (if it is otherwise clear that this is the method they are using).
			If $x \geq -b$:		
			$x^2 + x + b - \frac{1}{2} = 0$	A1	For correctly analysing the absolute value above OR substituting in above values correctly.

Question			Answer	Marks	Guidance
			Roots are $\frac{-1\pm\sqrt{3-4b}}{2}$ $\frac{-1+\sqrt{3-4b}}{2} > -\frac{1}{2} > -b$ if $\frac{1}{2} < b < \frac{3}{4}$, so there must be at least one purely real root.	E1 **[5]**	For establishing that at least one real root exists, using any method. If the equation is incorrect, only award marks if analysis is practically unchanged compared to the correct equation. If they have not split into cases based on the absolute value, they must check that this is a solution to the original equation.

Question			Answer	Marks	Guidance
	ii		$\lvert z+2\rvert^2$ and 4 are both real, so z^3 must be real.		
			$z = x + iy$, $3x^2y - y^3 = 0$. OR: $z = k\omega^n$, k is real, ω is a cube root of unity and n is either 0, 1 or 2.	B1	Mark awarded for recognising that z^3 is real and the algebraic form of z or the relation between x and y that is implied. Can be awarded if one of the cases (e.g. n = 2) is missing.
			$n = 0$: $k^3 + k^2 + 4k = 0$ $k(k^2 + k + 4) = 0$	M1	Write down the cubic equation (in one variable) for k (or x or y) in the purely real (0) case. If working with z , then they need to give some indication that they are treating z as real.
			$k = 0$ is the only real solution as the discriminant of the quadratic factor is -15.	A1	For analysing the equation correctly in this case and finding 0 as the only real solution.
			$n \neq 0$ $k^3 + k^2 - 2k = 0$ $k(k-1)(k+2) = 0$	M1	Write down the cubic equation for k (or x or y) in the complex case. Can be a FT for similar method if the case is slightly wrong, or for an expansion with an algebraic error.
			$k = 0, 1, -2$	A1	For analysing the quadratic correctly in this case and finding all remaining solutions. Award for finding the correct x values in the case $y = 3x^2$. More needed if the case is just $y = \sqrt{3}x$ to demonstrate they have found all four solutions.

Question			Answer	Marks	Guidance
			Roots are $0,\ \omega,\ \omega^2, -2\omega, -2\omega^2$	**[5]**	

Question			Answer	Marks	Guidance
3	i		$\frac{dy}{dx} = \frac{(1-\ln x)}{x^2}$ Stationary point when $\frac{(1-\ln x)}{x^2} = 0 \Rightarrow x = \mathrm{e}$ Vertical asymptote $x = 0$, curve crosses the x-axis at $(1,0)$ and nothing to the left of the y-axis. Horizontal asymptote $y = 0$ and maximum at $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$.	B1 G1 G1 [3]	G1 for (1,0), increasing on LHS and asymptote at $x = 0$ G1 for $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$, decreasing on RHS and asymptote
	ii	a	$\mathrm{e} < 3 < \pi$ From the graph: $\frac{\ln 3}{3} > \frac{\ln \pi}{\pi}$ $\pi \ln 3 > 3 \ln \pi$ $3^{\pi} > \pi^3$	M1 A1 [2]	Need to see some reference to $\mathrm{e} < 3 < \pi$, but this could be on their graph A1=CSO; algebra must be seen somewhere. Either a conclusion must be present, or they must have made clear that it was sufficient to show the M1 line.

Question			Answer	Marks	Guidance
		b	$\sqrt{5} < \frac{9}{4} < \mathrm{e}$	E1	Must acknowledge both inequalities somehow (can be on graph or reference to being in the increasing range); no formal justification required for either.
			From the graph:		
			$\frac{\ln\sqrt{5}}{\sqrt{5}} < \frac{\ln\frac{9}{4}}{\frac{9}{4}}$	M1	
			$\frac{9}{4}\ln\sqrt{5} < \sqrt{5}\ln\frac{9}{4}$		
			$\sqrt{5}^{\frac{9}{4}} < \left(\frac{9}{4}\right)^{\sqrt{5}}$	A1	A1=CSO; algebra can be skipped if they did it correctly in ii)a); conclusion of some form still needed.
				[3]	
	iii		$1 < x < 2 < \mathrm{e}$, so $\frac{\ln x}{x} < \frac{\ln 2}{2}$		
			$\mathrm{e} < 3 < x+2 < 4$, so $\frac{\ln 4}{4} < \frac{\ln(x+2)}{x+2}$ $$\frac{\ln 4}{4} < \frac{\ln(x+2)}{x+2}$$	M1	Both inequalities found; must see some acknowledgement of these inequalities using decreasing/increasing ranges on graph ($x < 2,\ x+2 > 3$ is enough, need not see e)

Question			Answer	Marks	Guidance
			$\frac{\ln 4}{4} = \frac{2\ln 2}{4} = \frac{\ln 2}{2}$, so $\frac{\ln x}{x} < \frac{\ln(x+2)}{x+2}$	**M1**	Dependent on first M1
			$(x+2)\ln x < x\ln(x+2)$		
			$x^{x+2} < (x+2)^x$	**A1**	Algebra can be skipped if done correctly in earlier parts.
				[3]	
			Alternative (iii)		
			Correctly drawn graphs of $\frac{\ln x}{x}$ and $\frac{\ln(x+2)}{x+2}$	**M1**	
			Showing intersection of the two graphs is at $x = 2$	**M1**	
			Deducing the inequality $(x+2)^x > x^{(x+2)}$	**A1**	With fully correct working only.

Question			Answer	Marks	Guidance
`	iv		$9^{\sqrt{2}} = (3^2)^{\sqrt{2}} = 3^{2\sqrt{2}}$ $(\sqrt{2})^9 = \left((\sqrt{2})^3\right)^3 = (2\sqrt{2})^3$ So, the two inequalities are equivalent.	E1	Must either manipulate both sides or make reference to equivalence (i.e. cannot just have "⇒" all the way down). For the first line here a jump is ok. Must see working for the second.
			Since $e^2 < 8$, $e < 2\sqrt{2} < 3$	B1	Must see both inequalities (at least implicitly), must show $e < 2\sqrt{2}$ rigorously if trying to use decimal expansion; otherwise may simply be stated.
			$e < 2\sqrt{2} < 3$		
			$\frac{\ln 2\sqrt{2}}{2\sqrt{2}} > \frac{\ln 3}{3}$	M1	First line of reasoning here can be implicit from a present $e < 2\sqrt{2}$ or by reference to/showing understanding in previous parts
			$(2\sqrt{2})^3 > 3^{2\sqrt{2}}$		
			So $(\sqrt{2})^9 > 9^{\sqrt{2}}$	A1	Algebra can be skipped if done correctly in an earlier part.

Question			Answer	Marks	Guidance
				[4]	
			Alternative (iv)		
			$9^{\sqrt{2}} = (3^2)^{\sqrt{2}} = 3^{2\sqrt{2}}$ $\left(\sqrt{2}\right)^9 = \left(\left(\sqrt{2}\right)^3\right)^3 = \left(2\sqrt{2}\right)^3$ So, the two inequalities are equivalent.	**E1**	Must either manipulate both sides or make reference to equivalence (i.e. cannot just have "⇒" all the way down). For the first line here a jump is ok. Must see working for the second.
			Consider $y = \frac{ln\,x}{\sqrt{x}}$ and prove its maximum is at $x = e^2$	**M1**	
			Show by any valid method that the inequality at hand is equivalent to comparing $ln\,9\,/3$ and $ln\,8\,/2\sqrt{2}$	**M1**	
			Use $e^2 < 8 < 9$ to deduce that $ln\,9\,/3 < ln\,8\,/2\sqrt{2}$ and hence that $\sqrt{2}^9$ is the larger value	**A1**	
				[4]	

Question			Answer	Marks	Guidance
	v		$8^{\sqrt[3]{3}} = (2^3)^{\sqrt[3]{3}} = 2^{3\sqrt[3]{3}}$	B1	
			$\left(\sqrt[3]{3}\right)^8 = \left(\left(\sqrt[3]{3}\right)^4\right)^2 = \left(3\sqrt[3]{3}\right)^2$	B1	Enough to state both for the marks.
			$\left(3\sqrt[3]{3}\right)^3 = 81 > 64 = 4^3$, so $e < 4 < 3\sqrt[3]{3}$ $e < 4 < 3\sqrt[3]{3}$	M1	Don't need to see 4 > e at this point (clear enough and seen similar enough times). Must see some working.
			$\frac{\ln 4}{4} > \frac{\ln 3\sqrt[3]{3}}{3\sqrt[3]{3}}$	M1	
			$\frac{\ln 2}{2} > \frac{\ln 3\sqrt[3]{3}}{3\sqrt[3]{3}}$		
			$2^{3\sqrt[3]{3}} > \left(3\sqrt[3]{3}\right)^2$		
			$8^{\sqrt[3]{3}} > \left(\sqrt[3]{3}\right)^8$	A1	Must correctly and clearly compare the inequalities (not ok for logic to go in the wrong direction) for full marks. Again, algebra can be skipped if seen elsewhere.
				[5]	

Question			Answer	Marks	Guidance
			Alternative (v)		
			$8^{\sqrt[3]{3}} = (2^3)^{\sqrt[3]{3}} = 2^{3\sqrt[3]{3}}$	**B1**	
			$\left(\sqrt[3]{3}\right)^8 = \left(\left(\sqrt[3]{3}\right)^4\right)^2 = \left(3\sqrt[3]{3}\right)^2$	**B1**	
			$\left(3\sqrt[3]{3}\right)^3 = 81 > 64 = 4^3$, so $e < 4 < 3\sqrt[3]{3}$	**M1**	
			Consider $y = 2^t - t^2, \frac{dy}{dt} = (\ln 2)2^t - 2t > 0$ for $t \geq 4$ (by any valid method)	**M1**	Any valid proof that y increasing for $t \geq 4$
			$y(4) = 0$ and hence $y\left(3\sqrt[3]{3}\right) > 0$, whence $8^{\sqrt[3]{3}} > \left(\sqrt[3]{3}\right)^8$	**A1**	Any valid way to finish off
				[5]	

Question			Answer	Marks	Guidance
4	i		Let $\delta = x - \lfloor x \rfloor$ $x + n = \lfloor x \rfloor + n + \delta$ $\lfloor x \rfloor + n$ is an integer and $\delta < 1$, so $\lfloor x + n \rfloor = \lfloor x \rfloor + n$	M1 E1 [2]	Any attempt to show that $\lfloor x \rfloor + n$ satisfies the definition of $\lfloor x + n \rfloor$, either as given in the question or by using $\delta \in [0,1)$ (or any attempt to consider a 'decimal part' or similar, wrt $x + n$). Give BOD for $\delta > 0$. Correct explanation with $\delta < 1$ (can be lenient with stating that '$\lfloor x \rfloor + n$ is an integer'); or 'it is the largest possible integer $\leq x + n$'.
	ii	a	$f_n\left(x + \frac{1}{n}\right) - f_n(x)$ $= (\lfloor x + 1 \rfloor - \lfloor x \rfloor) - (\lfloor nx \rfloor - \lfloor nx + 1 \rfloor)$ $= 1 - 1 = 0$	M1 A1 [2]	Any attempt to compare $f_n\left(x + \frac{1}{n}\right)$ and $f_n(x)$, expanding $f_n\left(x + \frac{1}{n}\right)$ is enough. Must clearly show application of (i); it is enough to see the terms that cancel in brackets as shown.
		b	If $0 \leq t < \frac{1}{n}$, then $t + \frac{k}{n} < 1$ for all integers $0 \leq k \leq n - 1$, so $\left\lfloor t + \frac{k}{n} \right\rfloor = 0$ $nt < 1$, so $\lfloor nt \rfloor = 0$	E1 E1	This can be implicit if $t + \frac{n-1}{n} < 1$ is stated. If the inequalities are stated without $\left\lfloor t + \frac{k}{n} \right\rfloor = 0$ or $\lfloor nt \rfloor = 0$, and the answer is not clear enough to give BOD (i.e. $f_n(t) = 0 + \cdots + 0 - \ 0$ is sufficient for BOD and

Question			Answer	Marks	Guidance
			Therefore $f_n(t) = 0$ for $0 \leq t < \frac{1}{n}$.	**E1** **[3]**	the final E mark) then give the corresponding marks for the inequalities and withhold the final E mark.

Question			Answer	Marks	Guidance
		c	By repeated application of (a):		
			$f_n\left(y+\frac{k}{n}\right)=f_n(y)$	M1	Any attempt to 'iterate' (ii)(a) to some $k \in \mathbb{Z}$ other than 0,1.
			For any value of x, there is an integer k and a value $0 \le \delta < \frac{1}{n}$ such that $x=\delta+\frac{k}{n}$		
			Therefore $f_n(x)=0$	E1	Any argument explicitly restricting to $x \ge 0$ gets E0. Claiming that $f_n\left(t+\frac{k}{n}\right)=f_n(t)$ for <u>integer</u> k, while only showing 'positive' iteration is sufficient for E1. Any response that does not specify $k \in \mathbb{Z}$, and it cannot be inferred from their argument gets E0.
				[2]	
	iii	a	Let $x=2k+\delta$, where k is an integer and $0 \le \delta < 2$		
			$\left\lfloor\frac{x}{2}\right\rfloor=k$		
			$\left\lfloor\frac{x+1}{2}\right\rfloor= \begin{matrix} k & 0 \le \delta < 1 \\ k+1 & 1 \le \delta < 2 \end{matrix}$	M1	Identification of the two cases.
			$\lfloor x\rfloor= \begin{matrix} 2k & 0 \le \delta < 1 \\ 2k+1 & 1 \le \delta < 2 \end{matrix}$	A1	Either case completed correctly.

Question			Answer	Marks	Guidance
			In both cases, $\left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{x+1}{2}\right\rfloor = \lfloor x\rfloor$	**E1**	Complete solution.
				[3]	
			Alternative (iii)(a)		
			Attempt to use part (ii)(c), $f_n(x) \equiv 0$.	**M1**	
			Used $n = 2$: $\lfloor x\rfloor + \left\lfloor x + \frac{1}{2}\right\rfloor = \lfloor 2x\rfloor$ or equivalent.	**A1**	
			$x \longmapsto \frac{x}{2}$ or equivalent, correct solution.	**E1**	
				[3]	

Question			Answer	Marks	Guidance
		b	Let $g_n(x) = \left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \ldots + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor$	B1	
			$g_n(x) = g_{n-1}(x) + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor$		
			$g_n(x) + \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$ $= g_{n-1}(x) + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor + \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$	M1	
			Let $y = \frac{x+2^n}{2^n}$		
			$\frac{y}{2} = \frac{x+2^n}{2^{n+1}}$ and $\frac{y+1}{2} = \frac{x+2^{n+1}}{2^{n+1}}$		
			$g_n(x) + \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$ $= g_{n-1}(x) + \left\lfloor \frac{x+2^n}{2^n} \right\rfloor$	A1	
			$g_0(x) + \left\lfloor \frac{x+2}{2} \right\rfloor = \lfloor x+1 \rfloor$	M1	
			Therefore $g_n(x) = \lfloor x+1 \rfloor - \left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor$	A1	

Question			Answer	Marks	Guidance
			For $x \geq 0$ and n sufficiently large, $\left\lfloor \frac{x+2^{n+1}}{2^{n+1}} \right\rfloor = \left\lfloor \frac{x}{2^{n+1}} + 1 \right\rfloor = 1$		
			Therefore $g_n(x) = \lfloor x+1 \rfloor - 1$ for large values of n	**E1**	

Question			Answer	Marks	Guidance
			For $x < 0$ and n sufficiently large,		
			$\left\lfloor \frac{x + 2^{n+1}}{2^{n+1}} \right\rfloor = \left\lfloor \frac{x}{2^{n+1}} + 1 \right\rfloor = 0$		
			Therefore $g_n(x) = \lfloor x + 1 \rfloor$ for large values of n	E1	
			$\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{x + 2^n}{2^{n+1}} \right\rfloor + \cdots$ $= \begin{matrix} \lfloor x + 1 \rfloor - 1 & x \geq 0 \\ \lfloor x + 1 \rfloor & x < 0 \end{matrix}$	A1	
				[8]	
			Alternative (iii)(b)		
			Let $g_n(x) = \sum_{k=0}^{n} \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor$	B1	This mark can be implicit, i.e. $\sum_{k=0}^{\infty} \left\lfloor \frac{x+2^k}{2^{k+1}} \right\rfloor = \lim_{n \to \infty} \left(\lfloor x \rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor \right)$ gets the mark for considering the limit of the partial sum; any statement that $x/2^n$ goes to zero gets B1 for consideration of 'term at infinity' (even if incorrectly stated as $\left\lfloor \frac{x}{2^n} \right\rfloor$ goes to zero, since this loses the final three marks anyway).
			$\left\lfloor \frac{x + 2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{x}{2^{k+1}} + \frac{1}{2} \right\rfloor =$	M1	Attempt to manipulate kth term using (iii)(a).
			$= \left\lfloor \frac{x}{2^k} \right\rfloor - \left\lfloor \frac{x}{2^{k+1}} \right\rfloor$, using (iii)(a).	A1	Correct simplification. (Instead of kth term, exhibiting multiple terms is enough.)

Question			Answer	Marks	Guidance
			so $g_n(x) = \sum_{k=0}^{n} \left\lfloor \frac{x}{2^k} \right\rfloor - \sum_{k=0}^{n} \left\lfloor \frac{x}{2^{k+1}} \right\rfloor$		
			$= \sum_{k=0}^{n} \left\lfloor \frac{x}{2^k} \right\rfloor - \sum_{k=1}^{n+1} \left\lfloor \frac{x}{2^k} \right\rfloor$	M1	Attempt to re-index or use telescoping series
			$\therefore g_n(x) = \lfloor x \rfloor - \left\lfloor \frac{x}{2^{n+1}} \right\rfloor$	A1	Can be inside a limit. Cannot be inferred from statement as in second half of B1 guidance above.
			For $x \geq 0$, and n sufficiently large, $\left\lfloor \frac{x}{2^{n+1}} \right\rfloor = 0$, and therefore $g_n(x) = \lfloor x \rfloor$.	E1	
			For $x < 0$, and n sufficiently large, $\left\lfloor \frac{x}{2^{n+1}} \right\rfloor = -1$, and therefore $g_n(x) = \lfloor x \rfloor + 1$.	E1	
			The infinite sum is therefore: $\left\lfloor \frac{x+1}{2} \right\rfloor + \left\lfloor \frac{x+2}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{x+2^n}{2^{n+1}} \right\rfloor + \cdots = \begin{cases} \lfloor x \rfloor & : x \geq 0 \\ \lfloor x \rfloor + 1 & : x < 0 \end{cases}$	A1	
				[8]	

Question		Answer	Marks	Guidance
5	**i**	$\frac{dx}{du} = -u^{-2}$	**M1**	Dx=-du/u^2 fine. Done correctly. Seeing –u^(-2) term in integrand is sufficient. Must include an attempt at substitution. Incorrect differentiation gets M0 May calculate du/dx= -x^(-2) and sub this in for M1, but need integrand in terms of u for A1
		As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$	**B1**	May be implicit in calculation. Must be earned in (i), not from correct attempts in (ii) onward. If not done explicitly, and in integrand there is no minus sign and limits are 0 to inf give B0 NGE. In this case, final two marks can be earned. Give BOD on limits when d(1/u) is used, until they are dealt with fully.
		$\int_0^\infty \frac{\sqrt{x}}{\sqrt{x(x^3+1)}} dx = \int_\infty^0 \frac{\sqrt{u^{-1}}}{\sqrt{u^{-1}(u^{-3}+1)}} (-u^{-2}) du$ $= \int_0^\infty \frac{1}{u^2\sqrt{u}\sqrt{u^{-1}(u^{-3}+1)}} du$	**A1**	Substitution completed correctly. Allow incorrect limits. For correct form of the integrand, may just be the first/second term in the integrand dealt with correctly.
		$= \int_0^\infty \frac{1}{\sqrt{u(u^3+1)}} du$	**A1**	Fully simplified integrand. No errors in calculation up to this point. Needs to be some indication of legitimate simplification of their substitution.
		Therefore: $\int_0^\infty \frac{\sqrt{x}}{\sqrt{x(x^3+1)}} dx = \int_0^\infty \frac{1}{\sqrt{x(x^3+1)}} dx$, so $\int_0^\infty \frac{\sqrt{x}-1}{\sqrt{x(x^3+1)}} dx = 0$	**E1**	Using their transformation, done accurately, to prove the given statement. A complete proof of the statement, including convincing manipulation of the integrals. A0E1 is not possible.

Question		Answer	Marks	Guidance
			[5]	
	ii	$\frac{dx}{du} = -2u^{-3}$	**M1**	Dx=-2du/u^3 fine. Differentiation done correctly. Seeing –2u^(-3) term in integrand is sufficient. Must include an attempt at substitution. If attempting to use (i) to solve (ii) will need 2 substitutions, M1 only after the second substitution is made.
		As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$		
		$\int_0^\infty \frac{1}{\sqrt{x^3+1}} dx = \int_\infty^0 \frac{1}{\sqrt{u^{-6}+1}} (-2u^{-3})\, du$	**A1**	Substitution completed correctly. May have incorrect limits at this stage. For correct form of the integrand.
		$= 2\int_0^\infty \frac{1}{\sqrt{u^6+1}} du$ $\int_0^\infty \frac{1}{\sqrt{x^3+1}} dx = 2\int_0^\infty \frac{1}{\sqrt{x^6+1}} dx$	**A1**	Completely correct simplification, including limits dealt with correctly. Give BOD on limits when d(1/u^2) is used, until they are dealt with fully. Can give BOD for not changing dummy variable back to x. Candidates cannot gain this A mark if they leave out the differential element du while performing the change of variables, but after that step allow with BOD provided no other serious mistakes.
			[3]	

Question		Answer	Marks	Guidance
	iii	Consider $\int_0^\infty \frac{x^k}{\sqrt{x^p+1}}dx$, for fixed values of p and k for which the integral converges:		
		Substitute $x = u^{-1}$:		
		$\frac{dx}{du} = -u^{-2}$	M1	Dx=-du/u^2 fine. Done correctly. Seeing –u^(-2) term in integrand is sufficient. Must include an attempt at substitution. Need power of u in substitution to be -1
		As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$		
		$\int_0^\infty \frac{x^k}{\sqrt{x^p+1}}dx = \int_\infty^0 \frac{u^{-k}}{\sqrt{u^{-p}+1}}(-u^{-2})\,du$	A1	Substitution completed correctly. May have incorrect limits at this stage. For correct form of the integrand.
		$= \int_0^\infty \frac{1}{u^{k+2}\sqrt{u^{-p}+1}}du$		
		$= \int_0^\infty \frac{1}{\sqrt{u^{2k+4-p}+u^{2k+4}}}du$	A1	Correct form of integrand, with correct powers of u collected, including limits dealt with correctly. Numerator and denominator correct. Want to see numerator = 1 (or containing u^{r-1}), denominator containing $\sqrt{u^{p+1}}$. Can also be awarded for a form from which they have correctly extracted the necessary information.
		Therefore: $\int_0^\infty \frac{x^k}{\sqrt{x^p+1}} - \frac{1}{\sqrt{x^{2k+4-p}+x^{2k+4}}}dx = 0$		

Question		Answer	Marks	Guidance
		$\int_0^\infty \frac{x^{k+\frac{s}{2}}}{\sqrt{x^s(x^p+1)}} - \frac{1}{\sqrt{x^{2k+4-p}(x^p+1)}}\,dx = 0$	**M1**	Writing the integrand in an appropriate form to compare powers of the dummy variable and attempting to obtain an equation for *r*. Incorrect justification must score M0.
		Therefore, if $s = 2k + 4 - p$,		
		$\int_0^\infty \frac{x^{k+\frac{s}{2}-1}}{\sqrt{x^s(x^p+1)}}\,dx = 0$		
		$k = \frac{p+s-4}{2}$	**A1**	Correct equation making integral = 0.
		$r = k + \frac{s}{2} = \frac{p}{2} + s - 2$	**A1**	Correct final answer.
			[6]	

Question		Answer	Marks	Guidance
	iv	$\int_0^\infty \frac{1}{\sqrt{x^p+1}}dx$		
		Use the substitution $x = u^{-k}$	M1	$dx = -k\frac{du}{u^{k+1}}$ fine. Seeing $\frac{-k}{u^{k+1}}$ term in integrand is sufficient. Must include an attempt at substitution. Power on u must be $-k$ May award M1 for error in power of u after substitution (e.g. $\frac{-k}{u^{k-1}}$) May use index other than k, and solve entire problem in terms of this arbitrary index. This is acceptable. Withold mark until clear this is the case. Other powers that "will work" are acceptable, e.g. $\frac{-q}{p}$, $\frac{-p}{q}$, $\frac{2}{2-p}$, $\frac{2-q}{2}$ $x = u^k$, $u^{\frac{p}{q}}$ is accepted, with the final answers being $p = 2 - \frac{2}{k}$, $q = 2 - 2k$ This is marked as normal
		$\frac{dx}{du} = -ku^{-k-1}$		
		For any positive value of k: As $x \to 0, u \to \infty$ As $x \to \infty, u \to 0$	E1	Must be clear that it applies for all positive values of k. . There must be a statement somewhere which can be read as being linked to the consideration of the limits, can be generous with how clearly this is stated.
		$\int_0^\infty \frac{1}{\sqrt{x^p+1}}dx = \int_\infty^0 \frac{-ku^{-k-1}}{\sqrt{u^{-pk}+1}}du$	A1	Correct substitution, including limits dealt with correctly.

Question		Answer	Marks	Guidance
		$= k\int_0^\infty \frac{1}{\sqrt{u^{2k+2-pk}+u^{2k+2}}}du$	**M1**	Writing the integrand in an appropriate form to compare powers of the dummy variable and attempting to obtain (or obtaining) an equation for *p* or *q* in terms of *k* by comparing appropriate powers of *u*. Want to see numerator of +-1, limits correct, denominator simplified as given, or factorised. The equation *q* = *pk* alone is not sufficient.
		If $p = \frac{2(k+1)}{k}$, $u^{2k+2-pk} = 1$ and	**A1**	At least one correct explicit equation for *p* or *q* in terms of *k*, unless from obviously incorrect working. If equations are implicit for *p*(*k*), *q*(*k*) there must be two correct equations including *q*/*p* = *k* or equivalent.
		$\int_0^\infty \frac{1}{\sqrt{x^p+1}}dx = k\int_0^\infty \frac{1}{\sqrt{x^q+1}}dx$		
		Where q= $2(k+1)$	**A1**	Both *p* and *q* correctly identified. Might not both be explicitly given in terms of *k* (e.g. *p*(*k*) given, *q* = *kp*). Should be some specification of *p* and *q* in terms of *k*, not just *k*(*p*) etc (e.g. "choose *k* = ..." is A0 NGE). If neither *p* or *q* given explicitly in terms of *k*, then candidate must **convincingly** argue that given equations for *k*, *p*, *q*, can be solved for *p, q* for all *k* > 0.
			[6]	

Question			Answer	Marks	Guidance
6	i		Let point of contact be (x_1, y_1):		
			The centre of the circle is $(0, a)$		
			Gradient of radius of circle to contact point $= \frac{y_1 - a}{x_1}$		
			$2k\frac{dy}{dx} = 2x$	M1 A1	M1: differentiating both equations (or considering two relevant gradients). Need to get $\frac{dy}{dx}$ by itself (i.e. need to divide by *x*) - done implicitly or explicitly. A1: correct gradients (can be gradient of a perpendicular).
			Gradient of tangent at contact point $= \frac{x_1}{k}$		
			$\left(\frac{y_1 - a}{x_1}\right)\left(\frac{x_1}{k}\right) = -1$		
			$y_1 - a = -k$	M1	Having an equation that equates two gradients.
			$x_1 = \pm\sqrt{2ka - 2k^2}$ $y_1 = a - k$		
			$2ka - 2k^2 + k^2 = r^2$		

Question			Answer	Marks	Guidance
			$r^2 = k(2a-k)$	**A1**	CSO
			Points of intersection between circle and parabola:		
			$2ky + (y-a)^2 = r^2$	**M1**	Substitution (r^2 and then x or y).
			$y^2 + 2(k-a)y + a^2 - k(2a-k) = 0$		
			$y^2 + 2(k-a)y + a^2 - 2ak + k^2 = 0$		
			$y^2 + 2(k-a)y + (k-a)^2 = 0$		
			$(y+k-a)^2 = 0$		
			$y = a-k$		
			$x^2 = 2k(a-k)$, which is positive since $a > k$ so there are two points of intersection.	**E1**	Showing 2 intersections ($a > k$ stated or equivalent).
			Points of intersection are $\left(\pm\sqrt{2k(a-k)}, a-k\right)$		

Question			Answer	Marks	Guidance
			Gradient of tangent to $2ky = x^2$ at $\left(\pm\sqrt{2k(a-k)}, a-k\right)$ is $\pm\sqrt{\frac{2(a-k)}{k}}$		
			Gradient of radius of circle to $\left(\pm\sqrt{2k(a-k)}, a-k\right)$ is		
			$\frac{-k}{\pm\sqrt{2k(a-k)}} = \mp\frac{k}{\sqrt{2k(a-k)}}$		
			$\pm\sqrt{\frac{2(a-k)}{k}} \times \mp\frac{k}{\sqrt{2k(a-k)}} = -1$, so the circle touches the parabola tangentially at both points.	**E1**	Showing tangential.
				[7]	

Question			Answer	Marks	Answer
	ii		Gradients of tangents are ± 1 so the points of intersection are the lower intersections between the circle and the lines $y = a \pm x$		
			$2(y-a)^2 = r^2$	M1	May be seen graphically. Going towards one coordinate. Need something like equation only in one coordinate in the context of a valid method.
			Lower intersection is at $y = a - \frac{r\sqrt{2}}{2}$		
			Points of intersection are $\left(\pm\frac{r\sqrt{2}}{2}, a - \frac{r\sqrt{2}}{2}\right)$		
			$y - \left(a - \frac{r\sqrt{2}}{2}\right) = \pm\left(x \mp \frac{r\sqrt{2}}{2}\right)$	M1	Substituting back in for other coordinate (quite easy to get). Say something reasonable about there being one y (okay if discriminant not stated to be zero yet).
			$y = \pm x - \frac{r\sqrt{2}}{2} + a - \frac{r\sqrt{2}}{2}$		
			$y = a - r\sqrt{2} \pm x$		
			Therefore, $c = a - r\sqrt{2}$		
			$r = \pm\frac{\sqrt{2}}{2}(a-c)$		
			Points of contact: $\left(\pm\frac{a-c}{2}, \frac{a+c}{2}\right)$	A1	CSO (need to give r^2 or correct value of r).
				[3]	

Question			Answer	Marks	Answer
	iii	a	$r_1^2 = k(2a_1 - k)$ and $r_1 = \frac{\sqrt{2}}{2}(a_1 - c)$		
			$$a_1^2 - 2a_1c + c^2 = 4a_1k - 2k^2$$	M1	Write down a single expression or equality with part i) result and whatever their part ii) result was.
			Similarly. $a_2^2 - 2a_2c + c^2 = 4a_2k - 2k^2$ $a_1^2 - a_2^2 - 2c(a_1 - a_2) = 4(a_1 - a_2)k$ $(a_1 - a_2)(a_1 + a_2 - 2c - 4k) = 0$		
			$a_1 \neq a_2$, so $a_1 + a_2 = 2c + 4k$	A1	Correct expression for $a_1 + a_2$, need to say $a_1 \neq a_2$ or equivalent. (Can be solved via quadratic/Vieta, need to also say $a_1 \neq a_2$ in that case. Take "sum of roots" etc. to imply that roots are distinct.).
			$$a_1^2 + a_2^2 - 2c(a_1 + a_2) + 2c^2 = 4(a_1 + a_2)k - 4k^2$$ $a_1^2 + a_2^2 = (2c + 4k)^2 - 2c^2 - 4k^2$		

Question			Answer	Marks	Answer
			$a_1^2 + a_2^2 = 2c^2 + 16kc + 12k^2$	A1 [3]	Correct expression for $a_1^2 + a_2^2$ (if they get M1, they can use the given expression for $a_1 + a_2$ to get this).
		b	The points $\left(\pm\frac{a_1-c}{2}, \frac{a_1+c}{2}\right)$ and $\left(\pm\frac{a_2-c}{2}, \frac{a_2+c}{2}\right)$ are on the circle. $$\left(\frac{a_1-c}{2}\right)^2 + \left(\frac{a_1+c}{2} - d\right)^2 = p^2$$ $$\left(\frac{a_1-c}{2}\right)^2 + \left(\frac{a_1+c}{2} - d\right)^2 = p^2$$		
			$\left(\frac{a_1-c}{2}\right)^2 - \left(\frac{a_2-c}{2}\right)^2 = \left(\frac{a_1+a_2-2c}{2}\right)\left(\frac{a_1-a_2}{2}\right)$ $= k(a_1 - a_2)$ $\left(\frac{a_1+c}{2} - d\right)^2 - \left(\frac{a_2+c}{2} - d\right)^2$ $= \left(\frac{a_1+a_2+2c}{2} - 2d\right)\left(\frac{a_1-a_2}{2}\right)$ $= (c + k - d)(a_1 - a_2)$	M1	Correctly process difference between the two equations for one corresponding pair of terms. (Substitute in coords to circle equation then get rid of p by subtracting).

Question			Answer	Marks	Answer
			$k(a_1 - a_2) + (c + k - d)(a_1 - a_2) = 0$		
			$d = c + 2k$	**A1**	CSO.
			$\left(\frac{a_1 - c}{2}\right)^2 + \left(\frac{a_1 + c}{2} - d\right)^2 = p^2$ $\left(\frac{a_1 - c}{2}\right)^2 + \left(\frac{a_1 + c}{2}\right)^2 - d(a_1 + c) + d^2 = p^2$ $\frac{a_1^2 + c^2}{2} - d(a_1 + c) + d^2 = p^2$		
			So $\frac{a_1^2 + a_2^2 + 2c^2}{2} - d(a_1 + a_2 + 2c) + 2d^2 = 2p^2$	**M1**	Expanding equation and substituting in d.
			$\frac{2c^2 + 16kc + 12k^2 + 2c^2}{2} - (c + 2k)(4c + 4k) + 2(c + 2k)^2 = 2p^2$	**M1**	Sum completely correct with substitutions made (i.e. expression with p^2,c , k correct but may be long) from their (reasonable) value of d
			$p^2 = 2kc + 3k^2$	**A1**	CSO.
				[5]	

Question			Answer	Marks	Answer
		c	$k(2d-k) = k(2c+4k-k)$		
			$=2kc + 3k^2 = p^2$	**M1**	Reasonable substitution.
			$d > k > 0$, so by part (i) the circle also touches the parabola tangentially at two points.	**E1**	Using part i) of question and simply stating $d > k > 0$ without need to show this inequality.
				[2]	

Question		Answer	Marks	Guidance
7	**i**	$\frac{d}{dt}(x^2) = 2x\frac{dx}{dt}$	**M1**	Reasonable attempt to turn this into 1st order Differential equation
		$\frac{dx}{dt} = x^2 + c$	**A1**	
		At $t = 0$: $a^2 = a^2 + c \Rightarrow c = 0$	**B1**	
		$\int \frac{1}{x^2}dx = \int dt$	**M1**	Separate the variables
		$-\frac{1}{x} = t + c_2$	**A1**	
		At $t = 0$: $\frac{-1}{a} = c_2$ $\frac{-1}{x} = t - \frac{1}{a}$		
		$x = \frac{a}{1-at}$	**A1**	Find c_2 and write a correct expression for x. FT allowed but dependent on M1.

Question		Answer	Marks	Guidance
		As t increases from 0, $1 - at$ decreases from 1, so the particle moves away from the origin.	**E1** **[7]**	Must have some reference to the equation otherwise NGE Must mention behaviour near t=0 not just at the limit $t \to \infty$. FT allowed. In all explanations, if derivatives are used they must be directly applied to the position, x.

Question		Answer	Marks	Guidance
	ii	At $t = 0$:	B1	
		$a^2 + p^2 = a^2 + c \Rightarrow c = p^2$		
		$\int \frac{1}{x^2+p^2} dx = \int dt$		
		$\frac{1}{p} \arctan\left(\frac{x}{p}\right) = t + c_2$	M1	Reasonable attempt to solve the integral with arctan.
		At $t = 0$:	A1	Correct expression up to constant (x need not be the subject)
		$\frac{1}{p} \arctan\left(\frac{a}{p}\right) = c_2$		
		$x = p\tan\left(pt + \arctan\left(\frac{a}{p}\right)\right)$	A1	Correct expression in terms of x. FT allowed but dependent on M1 and must be a genuinely different type of expression to (i).
		At t increases from 0, x moves away from the origin.	E1	FT allowed, but must be a genuinely different type of expression to (i). Must mention behaviour near t=0 not just at the limit $t \to \infty$. Must have some reference to the equation or a graph otherwise NGE
			[5]	

Question		Answer	Marks	Guidance
	ii	At $t = 0$: $a^2 + p^2 = a^2 + c \Rightarrow c = p^2$	B1	
	iii	At $t = 0$: $a^2 - q^2 = a^2 + c \Rightarrow c = -q^2$	B1	
		$\int \frac{2}{x^2-q^2} dx = \int 2dt$		
		$\frac{2q}{x^2-q^2} = \frac{1}{x-q} - \frac{1}{x+q}$	M1	An attempt at manipulating the fraction
		$\ln \left\vert A\frac{(x-q)}{(x+q)} \right\vert = 2tq$		
		$A\frac{(x-q)}{(x+q)} = e^{2tq}$	M1	Attempt at integrating to $\ln\vert \dots \vert$, must include the absolute value signs

Question		Answer	Marks	Guidance
	ii	At $t = 0$: $a^2 + p^2 = a^2 + c \Rightarrow c = p^2$	B1	
		At $t = 0$: $A\frac{(a-q)}{(a+q)} = 1 \Rightarrow A = \frac{a+q}{a-q}$	M1	Reasonable attempt to find the constant.
		$(a+q)(x-q) = (a-q)e^{2tq}(x+q)$ $x = \frac{q(a+q) + qe^{2tq}(a-q)}{(a+q) - e^{2tq}(a-q)}$ $= -q + \frac{2q(a+q)}{(a+q) - e^{2tq}(a-q)}$	A1	Just for correct expression with x as the subject. Lose this mark if the absolute value signs disappear without explanation or without being absorbed into a constant.
		If $q = a$, then $x = a$, so the particle does not move.	E1	

Question		Answer	Marks	Guidance
	ii	At $t = 0$: $a^2 + p^2 = a^2 + c \Rightarrow c = p^2$	B1	
		If $q < a$, then $(a + q) - e^{2tq}(a - q)$ is decreasing as t increases from 0. Therefore x is increasing and so the particle moves away from the origin.	E1	Must mention behaviour near $t = 0$, not just at the limit $t \to \infty$. FT allowed, but must be a genuinely different type of expression to (i)/(ii).
		If $q > a$, then $(a + q) - e^{2tq}(a - q)$ is increasing as t increases from 0. Therefore x is decreasing and so the particle moves towards the origin.	E1	Must mention behaviour near $t = 0$, not just at the limit $t \to \infty$. FT allowed, but must be a genuinely different type of expression to (i)/(ii).
			[8]	

Question		Answer	Marks	Guidance
8	**i**	The sum of the elements in each subset must be 18.	**B1**	Implied by a correct pair of subsets.
		$\sum_{r=1}^{8} r^2 = \frac{8}{6}(8+1)(2\times 8+1) = 204$	**M1**	Implied by a correct pair of subsets.
		The sum of the squares of the elements in each subset must be 102.	**A1**	Implied by a correct pair of subsets.
		For the subset containing 8, the other three numbers must sum, to 10 and the squares must sum to 38.	**M1**	Choice of one element for a particular set.
		{2,3,5,8} and {1,4,6,7}	**A1**	
			[5]	
	ii	$\sum_{k=1}^{m}(c+b_k)^3 = \sum_{k=1}^{m} c^3 + 3c^2 b_k + 3cb_k^2 + b_k^3$	**M1**	Correct binomial expansion.
		$mc^3 + 3c^2\sum_{k=1}^{m} b_k + 3c\sum_{k=1}^{m} b_k^2 + \sum_{k=1}^{m} b_k^3$	**M1**	Split into separate sums with constants extracted.
		$mc^3 + 3c^2\sum_{k=1}^{m} a_k + 3c\sum_{k=1}^{m} a_k^2 + \sum_{k=1}^{m} b_k^3$	**A1**	Changed terms to refer to the other sequence. Make clear that they are using the properties of balanced subsets.

Question		Answer	Marks	Guidance
		$\sum_{k=1}^{m}(c+a_k)^3$ $$mc^3 + 3c^2\sum_{k=1}^{m} b_k + 3c\sum_{k=1}^{m} b_k^2 + \sum_{k=1}^{m} a_k^3$$ Therefore $$\sum_{k=1}^{m} a_k^3 + \sum_{k=1}^{m}(c+b_k)^3 = \sum_{k=1}^{m}(c+a_k)^3 + \sum_{k=1}^{m} b_k^3$$	E1 A1 **[5]**	Justification, e.g. by "by given result." Fully clear justification.
	iii	Suppose $$\sum_{k=1}^{m} a_k = \sum_{k=1}^{m} b_k$$ $$\sum_{k=1}^{m} a_k^2 = \sum_{k=1}^{m} b_k^2$$ $$\sum_{k=1}^{m} a_k + c = \sum_{k=1}^{m} b_k + c$$	B1	Statement that the sums remain equal if a constant is added to all terms.

Question		Answer	Marks	Guidance
		$\sum_{k=1}^{m}(b_k+c)^2$ $\sum_{k=1}^{m}c^2+2c\sum_{k=1}^{m}b_k+\sum_{k=1}^{m}{b_k}^2$ $\sum_{k=1}^{m}c^2+2c\sum_{k=1}^{m}a_k+\sum_{k=1}^{m}{a_k}^2$	M1	Show that the sums of squares remain equal if a constant is added to all terms.
		$\sum_{k=1}^{m}(a_k+c)^2$	A1	
		Let the sequence a_k be 1, 4, 6, 7 and the sequence b_k be 2, 3, 5, 8.		
		Choosing c=8.	M1	
		The sets:	E1	Explanation that sums and sums of squares are equal.
		$\{1,4,6,7,2+8,3+8,5+8,8+8\}$ and $\{2,3,5,8,1+8,4+8,6+8,7+8\}$ the required properties.	E1	Explanation that the sums of cubes are equal.
		$\{1,4,6,7,10,11,13,16\},\{2,3,5,8,9,12,14,15\}$	A1	
			[7]	

Question		Answer	Marks	Guidance
	iv	$\{1,3,4,5,9,11\}$ and $\{0,2,6,7,8,10\}$ is a balanced partition of the set $\{0,1,2,\dots,11\}$ and each subset contains 6 elements.	B1	If 0 not included, then none of the three marks are available
		Therefore, $\{n+1, n+3, n+4, n+5, n+9, n+11\}$ and $\{n, n+2, n+6, n+7, n+n+8, n+10\}$ is a balanced partition of the set $\{n, n+1, n+2, \dots, n+11\}$	E1	
		$\{(n+1)^2, (n+3)^2, (n+4)^2, (n+5)^2, (n+9)^2, (n+11)^2\}$ and $\{n^2, (n+2)^2, (n+6)^2, (n+7)^2, (n+8)^2, (n+10)^2\}$ is a pair of sets with the required property.	E1 **[3]**	

Question		Answer	Marks	Guidance
9	**i**	GPE = 0 at distance r below A and B. At the point of collision: Let the angle that the string makes with the vertical be θ. $\sin\theta = \frac{\sqrt{2}}{2}$, so $\theta = \frac{\pi}{4}$ Gain in GPE for particle = $mgr\left(1 - \frac{\sqrt{2}}{2}\right)$	**M1**	Or Considering conservation of energy
		For each particle: Initial $KE = \frac{1}{2}mu^2$ Let final speed be v. $mgr\left(1 - \frac{\sqrt{2}}{2}\right) + \frac{1}{2}mv^2 = \frac{1}{2}mu^2$ $v^2 = u^2 - gr\left(2 - \sqrt{2}\right)$	**A1**	Show how the given relationship for v arises.
		Horizontal component of velocity before collision has magnitude $\frac{v\sqrt{2}}{2}$. For collision: Horizontal speed of approach = $v\sqrt{2}$		

Question		Answer	Marks	Guidance
		Horizontal speed of separation = $ev\sqrt{2}$	M1	
		For each particle, horizontal component of speed = $\frac{1}{2}ev\sqrt{2}$	A1	
		Vertical component is unchanged, so = $\frac{1}{2}v\sqrt{2}$	B1	
			[5]	
	ii	Horizontally: $$x = \frac{1}{2}ev\sqrt{2}t$$		
		Vertically: $$y = \frac{1}{2}v\sqrt{2}t - \frac{1}{2}gt^2$$	M1	Set up equations for projectile motion.
		Coordinates of point where particle is located taking the point of suspension as the origin: $$\left(\frac{r\sqrt{2}}{2} - \frac{1}{2}ev\sqrt{2}t, -\frac{r\sqrt{2}}{2} + \frac{1}{2}v\sqrt{2}t - \frac{1}{2}gt^2\right)$$		

Question		Answer	Marks	Guidance
		String becomes taut when: $$\left(\frac{r\sqrt{2}}{2} - \frac{1}{2}ev\sqrt{2}t\right)^2 + \left(-\frac{r\sqrt{2}}{2} + \frac{1}{2}v\sqrt{2}t - \frac{1}{2}gt^2\right)^2 = r^2$$	**M1**	
		$$(r - evt)^2 + \left(-r + vt - \frac{\sqrt{2}}{2}gt^2\right)^2 = 2r^2$$	**A1**	Answer given.
			[3]	

Question		Answer	Marks	Guidance
	iii	$\left(1-\frac{evt}{r}\right)^2+\left(-1+\frac{vt}{r}-\frac{\sqrt{2}}{2r}gt^2\right)^2=2$		
		$(1-ez)^2+\left(-1+z-\frac{v^2t^2}{r^2c}\right)^2=2$		
		$(1-ez)^2+\left(-1+z-\frac{z^2}{c}\right)^2=2$	M1	Substitute the given variables to remove all instances of g and t.
		$1-2ez+e^2z^2+1+z^2+\frac{z^4}{c^2}-2z+\frac{2z^2}{c}-\frac{2z^3}{c}=2$		
		$\frac{z^4}{c^3}-\frac{2z^3}{c^2}+\frac{z^2}{c}(e^2+1)+\frac{2z^2}{c^2}-\frac{2ez}{c}-\frac{2z}{c}=0$	M1	Obviously quartic
		$z\left(\left(\frac{z}{c}\right)^3-2\left(\frac{z}{c}\right)^2+\left(\frac{z}{c}\right)(e^2+1)+\left(\frac{2}{c}\right)\left(\frac{z}{c}\right)-\frac{2}{c}(e+1)\right)=0$		

Question		Answer	Marks	Guidance
		$z = 0$ corresponds to the time of the collision, so is not one of the required solutions.	E1	Justify division by z $t \neq 0$ is enough
		$$\left(\frac{z}{c}\right)^3 - 2\left(\frac{z}{c}\right)^2 + \left(\frac{2}{c} + 1 + e^2\right)\left(\frac{z}{c}\right) - \frac{2}{c}(1 + e) = 0$$	A1	Answer given
			[4]	

Question		Answer	Marks	Guidance
	iv	If there are three equal roots then each root must be equal to $\frac{2}{3}$ (since the sum of the roots is the coefficient of z^2).	B1	Can also write $(z-R)^3=0$ and then expand and find $R=\frac{2}{3}$ for this mark.
		$\frac{2}{c}+1+e^2=3\left(\frac{2}{3}\right)^2$		
		$\frac{2}{c}(1+e)=\left(\frac{2}{3}\right)^3$		
		$e=\frac{4c}{27}-1$		
		$\frac{2}{c}+1+\left(\frac{4c}{27}-1\right)^2=\frac{4}{3}$		
		$(1+e)\left(\frac{4}{3}-1-e^2\right)=\frac{8}{27}$		
		$9(1+e)(1-3e^2)=8$		
		$27e^3+27e^2-9e-1=0$	M1	For arranging into a cubic = 0
		$27\left(\frac{1}{3}\right)^3+27\left(\frac{1}{3}\right)^2-9\left(\frac{1}{3}\right)-1=1+3-3-1=0$	M1	For showing $e=\frac{1}{3}$ solves the cubic. Can also show $e=\frac{1}{3}$ and $c=9$ solves the *c*-*e* equations, but in this case M0M1A0
		Therefore $e=\frac{1}{3}$ is a solution.	A1	For correct e and v^2

Question		Answer	Marks	Guidance
		Sum of roots $= -1$ and product of roots $= -1$, so the other two roots are both negative and therefore not possible values for e.	**E1**	(or find the other roots and make the same observations)
		Since the root is repeated, the string must become taut at a point where the direction of motion is tangential to the circle.	**E1**	Ignore subsequent discussion
		.	**[6]**	

Question		Answer	Marks	Guidance
	v	KE immediately after collision: $$\frac{1}{2}m\left(\left(\frac{1}{6}v\sqrt{2}\right)^2+\left(\frac{1}{2}v\sqrt{2}\right)^2\right)=\frac{5}{18}mv^2$$ Since no other energy is lost: $\frac{1}{2}mU^2=\frac{5}{18}mv^2+mgr\left(1-\frac{\sqrt{2}}{2}\right)$ $U^2=\frac{5}{9}v^2+2gr\left(1-\frac{\sqrt{2}}{2}\right)$ $U^2=\frac{5}{2}\sqrt{2}rg+2gr\left(1-\frac{\sqrt{2}}{2}\right)$ $=\left(2+\frac{3\sqrt{2}}{2}\right)rg$	M1 A1 [2]	Must substitute $e=\frac{1}{3}$

Question		Answer	Marks	Guidance
10	**i**	$AM = BM = PM = a$, so A, B and P lie on a circle with centre M.	**E1**	triangles are isosceles, stated or clear from diagram
		Since AB is a diameter of the circle, the angle at P, which is angle APB must be a right angle.	**E1**	
		$BP = 2a\sin\theta$	**B1**	Must be in the final simplified form (which can be given if it is seen later within the calculations)
		$AP = \sqrt{(2a)^2 - (2a\sin\theta)^2} = 2a\cos\theta$	**B1**	
		$\lvert T_A \rvert = \frac{\lambda(2a\cos\theta - l)}{l}$	**B1**	
		$\lvert T_B \rvert = \frac{\lambda(2a\sin\theta - l)}{l}$	**B1**	
		Let $\hat{\boldsymbol{a}}$ be a unit vector in the direction PA. Let $\hat{\boldsymbol{b}}$ be a unit vector in the direction PB.	**M1**	Setting up the pair of vectors.

Question		Answer	Marks	Guidance
		$T_A + T_B = \frac{\lambda(2a\cos\theta - l)}{l}\hat{\boldsymbol{a}} + \frac{\lambda(2a\sin\theta - l)}{l}\hat{\boldsymbol{b}}$	M1	Write the sum of the tensions in vector form (can FT if they previously got incorrect tension expressions) Award equivalent marks if resolving in any two perpendicular directions.
		$= \frac{\lambda}{l}\left(2a\cos\theta\,\hat{\boldsymbol{a}} + \boldsymbol{2}a\sin\theta\,\hat{\boldsymbol{b}}\right) - \lambda\left(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}\right)$	A1	Must be correct and split into the components that include l and those that don't
		$2a\cos\theta\,\hat{\boldsymbol{a}} + \boldsymbol{2}a\sin\theta\,\hat{\boldsymbol{b}} = 2\overrightarrow{\boldsymbol{PM}}$	E1	
		$\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}$ is a vector of length $\sqrt{2}$ in the direction of the bisector of angle APB (since APB is a right angle).	E1	

Question		Answer	Marks	Guidance
		Therefore: $T_A + T_B = \frac{2\lambda}{l}\overrightarrow{\boldsymbol{PM}} - \lambda\sqrt{2}\boldsymbol{v}$		
		where $\boldsymbol{v}$ is a unit vector in the direction of the bisector of angle APB (in the direction towards the ground).	A1	Clear statement that the second bullet point is shown.
		Since $PM = a$, the magnitude of $\frac{2\lambda}{l}\overrightarrow{\boldsymbol{PM}}$ is $\frac{2a\lambda}{l}$	A1	Clear statement that the first bullet point is shown.
			[13]	
	ii	Taking moments about M: $a\cos\theta\, T_B + mg\left(\frac{1}{2}a\cos 2\theta\right) = a\sin\theta\, T_A$	M1	If applying (i): $mg\left(\frac{1}{2}a\cos 2\theta\right) = \lambda\sqrt{2}(a\sin(45° - \theta))$ Require all in terms of a single angle for this mark (should be θ but they may have relabelled). Must have the mass term correct for this method mark but can be errors with other terms
		$\cos\theta\left(\frac{\lambda(2a\sin\theta - l)}{l}\right) + \frac{1}{2}mg\cos 2\theta =$ $\sin\theta\left(\frac{\lambda(2a\cos\theta - l)}{l}\right)$ $-2\lambda\cos\theta + mg\cos 2\theta = -2\lambda\sin\theta$		
		$mg\cos 2\theta = 2\lambda(\cos\theta - \sin\theta)$	A1	

Question		Answer	Marks	Guidance
		Since $\theta < 45°$, $\cos\theta + \sin\theta = \frac{2\lambda}{mg}$	**E1**	
		$\cos\theta + \sin\theta = \sqrt{2}\sin(\theta + 45°)$	**M1**	Quick alt method here: $(\cos\theta + \sin\theta)^2 = 1 + sin\,2\theta$. (M1 earned here)
		Therefore, $1 < \frac{2\lambda}{mg} < \sqrt{2}$		Since $0 < \theta < 45°$ (not essential to state here), then $0 < sin\,2\theta < 1$. $1 < (\cos\theta + \sin\theta)^2 < 2$ Again since $0 < \theta < 45°$, $(\cos\theta + \sin\theta)$ is positive and $1 < (\cos\theta + \sin\theta) < \sqrt{2}$ So that $1 < \frac{2\lambda}{mg} < \sqrt{2}$
		$\frac{1}{2}mg < \lambda < \frac{1}{2}\sqrt{2}mg$	**A1** **[5]**	A1 for fully correct solution either way.

Question		Answer	Marks	Guidance
	iii	Taking moments about P:		
		$mg\left(\frac{1}{2}a\cos 2\theta\right) + Fa\sin 2\theta = Na\cos 2\theta$	**M1**	CSO
		$\frac{1}{2}mg + F\tan 2\theta = N$		
		$N - F\tan 2\theta = \frac{1}{2}mg$	**A1**	
			[2]	

Question			Answer	Marks	Guidance
11	i		$\sum_{r=1}^{\infty} rx^{r-1} = \sum_{s=0}^{\infty}\left(\sum_{r=0}^{\infty} x^{s+r}\right)$	M1	Double sum
			$= \sum_{s=0}^{\infty} \frac{x^s}{1-x}$		
			$= \frac{1}{1-x}\sum_{s=0}^{\infty} x^s$	M1	Use of formula for geometric sum
			$= \frac{1}{(1-x)^2}$	A1	AG
				[3]	
	i (Alt 1)		$\sum_{r=1}^{\infty} x^r = \frac{x}{1-x}$	M1	Use of formula for geometric sum
			Differentiate both sides wrt x:	M1	Differentiating both sides wrt x
			$\sum_{r=1}^{\infty} rx^{r-1} = \frac{1}{1-x} + \frac{x}{(1-x)^2}$		
			$= \frac{1}{(1-x)^2}$	A1 [3]	AG

Question			Answer	Marks	Guidance
	i (Alt 2)		Using the Binomial Theorem: $\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}\frac{(-1)^{2n}(n+1)!\,x^n}{n!}$ $=\sum_{n=0}^{\infty}(n+1)x^n$	M1 M1 A1 **[3]**	Binomial Theorem Simplifying the terms AG
	ii	a	$P(\text{Even, but not 2k})=\frac{k-1}{2k}$ $P(\text{Ali wins})=\sum_{r=0}^{\infty}\left(\frac{k-1}{2k}\right)^r\left(\frac{1}{2k}\right)$ $\left\|\frac{k-1}{2k}\right\|<1$, so $P(\text{Ali wins})=\frac{\left(\frac{1}{2k}\right)}{1-\left(\frac{k-1}{2k}\right)}=\frac{1}{k+1}$	M1 A1 A1 **[3]**	Can be inferred from next line/correct working Can be awarded for P(Ali loses) Don't require sight of $\left\|\frac{k-1}{2k}\right\|<1$

Question			Answer	Marks	Guidance
	ii	b	$E(Win) = \sum_{r=1}^{\infty} r\left(\frac{k-1}{2k}\right)^{r-1}\left(\frac{1}{2k}\right)$	M1	
			$= \frac{1}{2k} \times \frac{1}{\left(1-\frac{k-1}{2k}\right)^2} = \frac{2k}{(k+1)^2}$	A1	
				[2]	
	iii		$\int \sum_{r=0}^{n}(r+1)\binom{n}{r}x^r\,dx = \sum_{r=0}^{n}\binom{n}{r}x^{r+1} + C$	M1	Termwise integration (sums can be written out in full) Condone missing constant of integration
			$= x\sum_{r=0}^{n}\binom{n}{r}x^r + C$	M1	Factorisation Condone missing constant of integration
			$= x(1+x)^n + C$	A1	Condone missing constant of integration
			$\sum_{r=0}^{n}(r+1)\binom{n}{r}x^r = \frac{d}{dx}(x(1+x)^n)$ $= (1+x)^n + nx(1+x)^{n-1}$	M1	Differentiation of $x(1+x)^n$ wrt x
			$(1+(n+1)x)(1+x)^{n-1}$	A1	

Question			Answer	Marks	Guidance
				[5]	
	iii (Alt 3)		$\sum_{r=0}^{n}\binom{n}{r}x^{r+1}=x\sum_{r=0}^{n}\binom{n}{r}x^{r}$	**M1**	Factorisation (sums can be written out in full)
			$=x(1+x)^n$	**A1**	
			Differentiating wrt x gives: $\sum_{r=0}^{n}(r+1)\binom{n}{r}x^{r}=\frac{d}{dx}[x(1+x)^n]$	**M1**	Termwise differentiation of $\sum_{r=0}^{n}\binom{n}{r}x^{r+1}$ wrt x
			$=(1+x)^n+nx(1+x)^{n-1}$	**M1**	Differentiation of $x(1+x)^n$ wrt x
			$(1+(n+1)x)(1+x)^{n-1}$	**A1**	
				[5]	

Question			Answer	Marks	Guidance
	iv		The number of winning sequences of $(r+1)$ rolls is equal to the number of different combinations of r integers between 1 and $2k-1$ (which are rolled in ascending order, followed by $2k$).	E1	
			$P(\text{Zen wins £(r + 1)}) = \frac{1}{(2k)^{r+1}}\binom{2k-1}{r}$	M1	
			$E(\text{Winnings}) = \sum_{r=0}^{2k-1} \frac{r+1}{(2k)^{r+1}}\binom{2k-1}{r}$		
			Using the result from (iii) with $n = 2k - 1$ and $x = \frac{1}{2k}$:		
			$E(\text{Win}) = \frac{1}{2k}\left(1+\frac{2k}{2k}\right)\left(1+\frac{1}{2k}\right)^{2k-2}$	M1	For using result from (iii) (condone incorrect values of n or x)
			$= \frac{1}{k}\left(1+\frac{1}{2k}\right)^{2k-2}$	A1	
				[4]	

Question			Answer	Marks	Guidance
	v		For large values of k, $E(Z\ win) \approx \dfrac{e}{k\left(1+\frac{1}{2k}\right)^2}$		
			$\dfrac{E(Z\ win)}{E(A\ win)} \approx \dfrac{e}{k\left(1+\frac{1}{2k}\right)^2 \frac{2k}{(k+1)^2}}$	M1	Attempting to compute $\frac{E(Zen's\ Winnings)}{E(Ali's\ Winnnings)}$ and substituting for e
			$= \dfrac{(k+1)^2 e}{2\left(k+\frac{1}{2}\right)^2} \approx \dfrac{e}{2}$	A1	
			Since $\frac{e}{2}$ is slightly more than 1.35, Zen's expected earnings are a little over 35% more than Ali's expected earnings.	E1	Explaining why a factor of $\frac{e}{2}$ corresponds to a percentage increase of a little over 35%
				[3]	

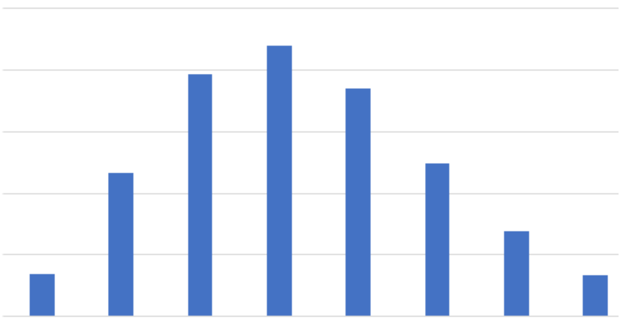
Question		Answer	Marks	Guidance
12	**i**	$d_r = \frac{e^{-\lambda}\lambda^r}{r!} - \frac{e^{-\lambda}\lambda^{r-1}}{(r-1)!}$	**M1**	Must be seen for general r
		$= \frac{e^{-\lambda}\lambda^{r-1}}{r!}(\lambda - r)$	**A1**	Accept correct simplified equivalent forms, e.g. $\frac{e^{-\lambda}\lambda^{r-1}}{(r-1)!}\left(\frac{\lambda}{r} - 1\right)$
		$\frac{e^{-\lambda}\lambda^{r-1}}{r!} > 0$, so If $r < \lambda$, $d_r = p_r - p_{r-1} > 0 \Rightarrow p_r > p_{r-1}$ If $r > \lambda$, $d_r = p_r - p_{r-1} < 0 \Rightarrow p_r < p_{r-1}$	**E1**	Sign of d_r for both cases stated (accept weak inequalities) Must see $\frac{e^{-\lambda}\lambda^{r-1}}{r!} > 0$ OE (unless candidate has written in terms of a probability, e.g. $d_r = P(X = r-1)\left(\frac{\lambda}{r} - 1\right)$, when there is no need to state explicitly that $P(X = r-1) > 0$)

Question		Answer	Marks	Guidance
		Therefore, the maximum value of p_r occurs for the largest integer value of r that is less than λ. Since λ is not an integer, this is equivalent to $\lambda - 1 < r < \lambda$	**E1** **[4]**	AG Must include: • maximum value of p_r occurs for the largest integer value of $r < \lambda$ • λ is not an integer • Conclusion Just using $d_r > 0$ and $d_{r+1} < 0$ to reach the conclusion earns E0 (this only shows that the given answer is necessary, not sufficient)

Question		Answer	Marks	Guidance
	ii	$\frac{d_r}{d_{r-1}} = \frac{\frac{e^{-\lambda}\lambda^{r-1}}{r!}(\lambda - r)}{\frac{e^{-\lambda}\lambda^{r-2}}{(r-1)!}(\lambda - r + 1)}$	**M1**	
		$= \frac{\lambda(\lambda - r)}{r(\lambda - r + 1)}$	**A1**	
		d_r will be positive for $r \leq m$ and negative for $r > m$, so the minimum value of d_r occurs for a value of $r > \lambda$.	**B1**	Statement that arg min is $> \lambda$
		For $r - 1$ in this range, the value of d_r decreases while $\frac{d_r}{d_{r-1}} = \frac{\lambda(\lambda - r)}{r(\lambda - r + 1)} > 1$	**M1**	Can also award for setting equal to 1
		Since $\lambda - r + 1 < 0$: $\lambda(\lambda - r) < r(\lambda - r + 1)$	**E1**	
		$\Rightarrow r^2 - r(1 + 2\lambda) + \lambda^2 < 0$		

Question		Answer	Marks	Guidance
		Critical values are $$\lambda+\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}},\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}$$	**A1**	OE. E.g. if candidate uses $r+1$ in place of r, accept $\lambda-\frac{1}{2}\pm\sqrt{\lambda+\frac{1}{4}}$
		Therefore, the minimum value of d_r occurs at k, the largest integer value of r that is less than $\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}$	**E1**	
		Since $\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}$ is not an integer, this is equivalent to $\mathrm{k}<\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}<\mathrm{k}+1$	**E1**	AG Must include $\lambda+\frac{1}{2}\sqrt{\lambda+\frac{1}{4}}$ is not an integer
			[8]	

Question		Answer	Marks	Guidance
	iii	$d_r \to 0$ as $r \to \infty$, so the maximum value of d_r can only occur at d_1 if $d_1 \geq 0$ $\Leftrightarrow \lambda \geq 1$	B1	Award for strict inequality (although such solutions will lose the final E1)
		The value of d_r increases for $r < \frac{2\lambda + 1 - \sqrt{4\lambda + 1}}{2}$ and then decreases for $\frac{2\lambda + 1 - \sqrt{4\lambda + 1}}{2} < r < \lambda$ after which d_r becomes negative.	E1	
		Therefore the maximum value of d_r occurs at $r = 1$ if $d_2 < d_1$	M1	OE (such as $d_2 - d_1 < 0, \frac{d_1}{d_2} > 1,\ 2 > \lambda + \frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}$ etc.)
		$\frac{\lambda}{2}(\lambda - 2) < \lambda - 1$ $\lambda^2 - 4\lambda + 2 < 0$		

Question		Answer	Marks	Guidance
		$2-\sqrt{2}<r<2+\sqrt{2}$	**A1**	Must include both roots
		But λ is not an integer, so the maximum value of d_r occurs at $r=1$ if: $1<\lambda<2+\sqrt{2}$	**E1**	Must include explanation of why $\lambda = 1$ is excluded.
			[5]	Candidates who instead try to find the condition that the *minimum* value of d_r occurs at $r = 1$ by plugging $r = 1$ into the result of (ii) earn no credit
	iv	$$\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}=3.36+0.5+\sqrt{3.61}=5.76$$	**B1**	Can be awarded for bounding above and below by suitable bounds (to facilitate computing the argmin of d_r)
		The minimum value of d_r occurs at $r=5$		
			**G1** **G1**	Maximum value is for $x = 3$. Largest drop in value occurs from $x = 4$ to $x = 5$. Continuous graphs earn G0 G0
			[3]	