STEP Mathematics

Sixth Term Examination Papers **9475**
MATHEMATICS 3 Morning
Monday 16 June 2025 Time: 3 hours

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully.

Do **NOT** open this question paper until you are told that you may do so.

Read and follow the additional instructions on the front of the answer booklet.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.

Each question is marked out of 20.

You may answer as many questions as you choose. You are advised to concentrate on no more than six questions. Little credit will be given for fragmentary answers.

All your answers will be marked.

Crossed out work will NOT be marked.

Your final mark will be based on the six questions for which you gain the highest marks.

There is NO Mathematical Formulae Booklet.

Calculators are NOT permitted.

Bilingual dictionaries are NOT permitted.

Wait to be told you may begin before turning this page.

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Section A: Pure Mathematics

1 *You need not consider the convergence of the improper integrals in this question.*

For $p, q > 0$, define

$$\mathrm{b}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x.$$

(i) Show that $\mathrm{b}(p,q) = \mathrm{b}(q,p)$.

(ii) Show that $\mathrm{b}(p+1,q) = \mathrm{b}(p,q) - \mathrm{b}(p,q+1)$ and hence that $\mathrm{b}(p+1,p) = \frac{1}{2}\,\mathrm{b}(p,p)$.

(iii) Show that

$$\mathrm{b}(p,q) = 2\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,\mathrm{d}\theta.$$

Hence show that $\mathrm{b}(p,p) = \dfrac{1}{2^{2p-1}}\,\mathrm{b}\big(p,\frac{1}{2}\big)$.

(iv) Show that

$$\mathrm{b}(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}t.$$

(v) Evaluate

$$\int_0^\infty \frac{t^{\frac{3}{2}}}{(1+t)^6}\,\mathrm{d}t.$$

2 Let $\mathrm{f}(x) = 7 - 2|x|$.

A sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = a$ and $u_n = \mathrm{f}(u_{n-1})$ for $n > 0$.

(i) **(a)** Sketch, on the same axes, the graphs with equations $y = \mathrm{f}(x)$ and $y = \mathrm{f}(\mathrm{f}(x))$.

(b) Find all solutions of the equation $\mathrm{f}(\mathrm{f}(x)) = x$.

(c) Find the values of a for which the sequence $u_0, u_1, u_2, \ldots$ has period 2.

(d) Show that, if $a = \frac{28}{5}$, then the sequence $u_2, u_3, u_4, \ldots$ has period 2, but neither u_0 or u_1 is equal to either of u_2 or u_3.

(ii) **(a)** Sketch, on the same axes, the graphs with equations $y = \mathrm{f}(x)$ and $y = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$.

(b) Consider the sequence $u_0, u_1, u_2, \ldots$ in the cases $a = 1$ and $a = -\frac{7}{9}$. Hence find all the solutions of the equation $\mathrm{f}(\mathrm{f}(\mathrm{f}(x))) = x$.

(c) Find a value of a such that the sequence $u_3, u_4, u_5, \ldots$ has period 3, but where none of u_0, u_1 or u_2 is equal to any of u_3, u_4 or u_5.

3 Let $\mathrm{f}(x)$ be defined and positive for $x > 0$.

Let a and b be real numbers with $0 < a < b$ and define the points $A = (a, \mathrm{f}(a))$ and $B = (b, -\mathrm{f}(b))$.

Let $X = (m, 0)$ be the point of intersection of line AB with the x-axis.

(i) Find an expression for m in terms of a, b, $\mathrm{f}(a)$ and $\mathrm{f}(b)$.

(ii) Show that, if $\mathrm{f}(x) = \sqrt{x}$, then $m = \sqrt{ab}$.

Find, in terms of n, a function $\mathrm{f}(x)$ such that $m = \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$.

(iii) Let $\mathrm{g}_1(x)$ and $\mathrm{g}_2(x)$ be defined and positive for $x > 0$. Let $m = M_1$ when $\mathrm{f}(x) = \mathrm{g}_1(x)$ and let $m = M_2$ when $\mathrm{f}(x) = \mathrm{g}_2(x)$.

Show that if $\dfrac{\mathrm{g}_1(x)}{\mathrm{g}_2(x)}$ is a decreasing function then $M_1 > M_2$.

Hence show that

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}.$$

(iv) Let p and c be chosen so that the curve $y = p(c - x)^3$ passes through both A and B. Show that

$$\frac{c-a}{b-c} = \left(\frac{\mathrm{f}(a)}{\mathrm{f}(b)}\right)^{\frac{1}{3}}$$

and hence determine c in terms of a, b, $\mathrm{f}(a)$ and $\mathrm{f}(b)$.

Show that if f is a decreasing function, then $c < m$.

4 **(i)** x_2 and y_2 are defined in terms of x_1 and y_1 by the equation

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

G_1 is the graph with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and G_2 is the graph with equation

$$\frac{\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{9} + \frac{\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{4} = 1.$$

Show that, if (x_1, y_1) is a point on G_1, then (x_2, y_2) is a point on G_2.

Show that G_2 is an anti-clockwise rotation of G_1 through $45°$ about the origin.

(ii) **(a)** The matrix

$$\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$$

represents a reflection. Find the line of invariant points of this matrix.

(b) Sketch, on the same axes, the graphs with equations

$$y = 2^x \quad \text{and} \quad 0.8x + 0.6y = 2^{-0.6x+0.8y}.$$

(iii) Sketch, on the same axes, for $0 \leqslant x \leqslant 2\pi$, the graphs with equations

$$y = \sin x \quad \text{and} \quad y = \sin(x - 2y).$$

You should determine the exact co-ordinates of the points on the graph with equation $y = \sin(x - 2y)$ where the tangent is horizontal and those where it is vertical.

5 Three points, A, B and C, lie in a horizontal plane, but are not collinear. The point O lies above the plane.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

P is a point with $\overrightarrow{OP} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$, where α, β and γ are all positive and $\alpha + \beta + \gamma < 1$.

Let $k = 1 - (\alpha + \beta + \gamma)$.

(i) The point L is on OA, the point X is on BC and LX passes through P.

Determine $\overrightarrow{OX}$ in terms of β, γ, $\mathbf{b}$ and $\mathbf{c}$ and show that $\overrightarrow{OL} = \dfrac{\alpha}{k + \alpha}\mathbf{a}$.

(ii) Let M and Y be the unique pair of points on OB and CA respectively such that MY passes through P, and let N and Z be the unique pair of points on OC and AB respectively such that NZ passes through P.

Show that the plane LMN is also horizontal if and only if OP intersects plane ABC at the point G, where $\overrightarrow{OG} \equiv \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Where do points X, Y and Z lie in this case?

(iii) State what the condition $\alpha + \beta + \gamma < 1$ tells you about the position of P relative to the tetrahedron $OABC$.

6 **(i)** Let a, b and c be three non-zero complex numbers with the properties $a + b + c = 0$ and $a^2 + b^2 + c^2 = 0$.

Show that a, b and c cannot all be real.

Show further that a, b and c all have the same modulus.

(ii) Show that it is not possible to find three non-zero complex numbers a, b and c with the properties $a + b + c = 0$ and $a^3 + b^3 + c^3 = 0$.

(iii) Show that if any four non-zero complex numbers a, b, c and d have the properties $a + b + c + d = 0$ and $a^3 + b^3 + c^3 + d^3 = 0$, then at least two of them must have the same modulus.

(iv) Show, by taking $c = 1$, $d = -2$ and $e = 3$ that it is possible to find five real numbers a, b, c, d and e with distinct magnitudes and with the properties $a + b + c + d + e = 0$ and $a^3 + b^3 + c^3 + d^3 + e^3 = 0$.

7 Let $\mathrm{f}(x) = \sqrt{x^2+1} - x$.

(i) Using a binomial series, or otherwise, show that, for large $|x|$, $\sqrt{x^2+1} \approx |x| + \dfrac{1}{2|x|}$.

Sketch the graph $y = \mathrm{f}(x)$.

(ii) Let $\mathrm{g}(x) = \tan^{-1}\mathrm{f}(x)$ and, for $x \neq 0$, let $\mathrm{k}(x) = \frac{1}{2}\tan^{-1}\dfrac{1}{x}$.

(a) Show that $\mathrm{g}(x) + \mathrm{g}(-x) = \frac{1}{2}\pi$.

(b) Show that $\mathrm{k}(x) + \mathrm{k}(-x) = 0$.

(c) Show that $\tan\mathrm{k}(x) = \tan\mathrm{g}(x)$ for $x > 0$.

(d) Sketch the graphs $y = \mathrm{g}(x)$ and $y = \mathrm{k}(x)$ on the same axes.

(e) Evaluate $\displaystyle\int_0^1 \mathrm{k}(x)\,\mathrm{d}x$ and hence write down the value of $\displaystyle\int_{-1}^0 \mathrm{g}(x)\,\mathrm{d}x$.

8 **(i)** Show that

$$z^{m+1} - \frac{1}{z^{m+1}} = \left(z - \frac{1}{z}\right)\left(z^m + \frac{1}{z^m}\right) + \left(z^{m-1} - \frac{1}{z^{m-1}}\right).$$

Hence prove by induction that, for $n \geqslant 1$,

$$z^{2n} - \frac{1}{z^{2n}} = \left(z - \frac{1}{z}\right)\sum_{r=1}^{n}\left(z^{2r-1} + \frac{1}{z^{2r-1}}\right).$$

Find similarly $z^{2n} - \dfrac{1}{z^{2n}}$ as a product of $\left(z + \dfrac{1}{z}\right)$ and a sum.

(ii) **(a)** By choosing $z = \mathrm{e}^{\mathrm{i}\theta}$, show that

$$\sin 2n\theta = 2\sin\theta\sum_{r=1}^{n}\cos(2r-1)\theta\,.$$

(b) Use this result, with $n = 2$, to show that $\cos\frac{2}{5}\pi = \cos\frac{1}{5}\pi - \frac{1}{2}$.

(c) Use this result, with $n = 7$, to show that $\cos\frac{2}{15}\pi + \cos\frac{4}{15}\pi + \cos\frac{8}{15}\pi + \cos\frac{16}{15}\pi = \frac{1}{2}$.

(iii) Show that $\sin\frac{1}{14}\pi - \sin\frac{3}{14}\pi + \sin\frac{5}{14}\pi = \frac{1}{2}$.

Section B: Mechanics

9 In this question, $n \geqslant 2$.

(i) A solid, of uniform density, is formed by rotating through 360° about the y-axis the region bounded by the part of the curve $r^{n-1}y = r^n - x^n$ with $0 \leqslant x \leqslant r$, and the x- and y-axes.

Show that the y-coordinate of the centre of mass of this solid is $\dfrac{nr}{2(n+1)}$.

(ii) Show that the normal to the curve $r^{n-1}y = r^n - x^n$ at the point $\big(rp, r(1-p^n)\big)$, where $0 < p \leqslant 1$, meets the y-axis at $(0, Y)$, where $Y = r\left(1 - p^n - \dfrac{1}{np^{n-2}}\right)$.

In the case $n = 4$, show that the greatest value of Y is $\frac{1}{4}r$.

(iii) A solid is formed by rotating through 360° about the y-axis the region bounded by the curves $r^3y = r^4 - x^4$ and $ry = -(r^2 - x^2)$, both for $0 \leqslant x \leqslant r$.

A and B are the points $(0, -r)$ and $(0, r)$, respectively, on the surface of the solid.

Show that the solid can rest in equilibrium on a horizontal surface with the vector $\overrightarrow{AB}$ at three different, non-zero, angles to the upward vertical. You should not attempt to find these angles.

10 A plank AB of length L initially lies horizontally at rest along the x-axis on a flat surface, with A at the origin.

Point C on the plank is such that AC has length sL, where $0 < s < 1$.

End A is then raised vertically along the y-axis so that its height above the horizontal surface at time t is $\mathrm{h}(t)$, while end B remains in contact with the flat surface and on the x-axis.

The function $\mathrm{h}(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2\mathrm{h}}{\mathrm{d}t^2} = -\omega^2\mathrm{h}, \quad \text{with } \mathrm{h}(0) = 0 \text{ and } \frac{\mathrm{dh}}{\mathrm{d}t} = \omega L \text{ at } t = 0,$$

where ω is a positive constant.

A particle P of mass m remains in contact with the plank at point C.

(i) Show that the x-coordinate of P is $sL\cos\omega t$, and find a similar expression for its y-coordinate.

(ii) Find expressions for the x- and y-components of the acceleration of the particle.

(iii) N and F are the upward normal and frictional components, respectively, of the force of the plank on the particle. Show that

$$N = mg(1 - k\sin\omega t)\cos\omega t,$$

and that

$$F = mgsk + N\tan\omega t$$

where $k = \dfrac{L\omega^2}{g}$.

(iv) The coefficient of friction between the particle and the plank is $\tan\alpha$, where α is an acute angle.

Show that the particle will not slip initially, provided $sk < \tan\alpha$.

Show further that, in this case, the particle will slip

- while N is still positive,
- when the plank makes an angle less than α to the horizontal.

Section C: Probability and Statistics

11 **(i)** Let $\lambda > 0$. The independent random variables $X_1, X_2, \ldots, X_n$ all have probability density function

$$\mathrm{f}(t) = \begin{cases} \lambda \mathrm{e}^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

and cumulative distribution function $\mathrm{F}(x)$.

The value of random variable Y is the largest of the values $X_1, X_2, \ldots, X_n$. Show that the cumulative distribution function of Y is given, for $y \geqslant 0$, by

$$\mathrm{G}(y) = \left(1 - \mathrm{e}^{-\lambda y}\right)^n .$$

(ii) The values $\mathrm{L}(\alpha)$ and $\mathrm{U}(\alpha)$, where $0 < \alpha \leqslant \frac{1}{2}$, are such that

$$\mathrm{P}(Y < \mathrm{L}(\alpha)) = \alpha \quad \text{and} \quad \mathrm{P}(Y > \mathrm{U}(\alpha)) = \alpha .$$

Show that

$$\mathrm{L}(\alpha) = -\frac{1}{\lambda} \ln\left(1 - \alpha^{\frac{1}{n}}\right)$$

and write down a similar expression for $\mathrm{U}(\alpha)$.

(iii) Use the approximation $\mathrm{e}^t \approx 1 + t$, for $|t|$ small, to show that, for sufficiently large n,

$$\lambda \mathrm{L}(\alpha) \approx \ln(n) - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right) .$$

(iv) Hence show that the median of Y tends to infinity as n increases, but that the width of the interval $\mathrm{U}(\alpha) - \mathrm{L}(\alpha)$ tends to a value which is independent of n.

(v) You are given that, for $|t|$ small, $\ln(1 + t) \approx t$ and that $\mathrm{e}^3 \approx 20$.

Show that, for sufficiently large n, there is an interval of width approximately $4\lambda^{-1}$ in which Y lies with probability 0.9 .

12 **(i)** Show that, for any functions f and g, and for any $m \geqslant 0$,

$$\sum_{r=1}^{m+1}\left(\mathrm{f}(r)\sum_{s=r-1}^{m}\mathrm{g}(s)\right)=\sum_{s=0}^{m}\left(\mathrm{g}(s)\sum_{r=1}^{s+1}\mathrm{f}(r)\right).$$

(ii) The random variables $X_0, X_1, X_2, \ldots$ are defined as follows

- X_0 takes the value 0 with probability 1;
- X_{n+1} takes the values $0, 1, \ldots, X_n + 1$ with equal probability, for $n = 0, 1, \ldots$.

(a) Write down $\mathrm{E}(X_1)$.

Find $\mathrm{P}(X_2 = 0)$ and $\mathrm{P}(X_2 = 1)$ and show that $\mathrm{P}(X_2 = 2) = \frac{1}{6}$.

Hence calculate $\mathrm{E}(X_2)$.

(b) For $n \geqslant 1$, show that

$$\mathrm{P}(X_n = 0) = \sum_{s=0}^{n-1}\frac{\mathrm{P}(X_{n-1} = s)}{s+2}$$

and find a similar expression for $\mathrm{P}(X_n = r)$, for $r = 1, 2, \ldots, n$.

(c) Hence show that $\mathrm{E}(X_n) = \frac{1}{2}\left(1 + \mathrm{E}(X_{n-1})\right)$.

Find an expression for $\mathrm{E}(X_n)$ in terms of n, for $n = 1, 2, \ldots$.

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