CAMBRIDGE

Sixth Term Examination Paper (STEP)

Mathematics 3 (9475)

2025

Examiners’ report and **mark scheme**

STEP Mathematics 3 mark scheme

Question		Answer	Marks	Guidance
1	(i)	$b(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$. The substitution $u = 1-x$ leads to	M1	For identifying suitable substitution and making some progress with it.
		$b(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_1^0 -(1-u)^{p-1}u^{q-1}du$		For the M1 it is sufficient to see either the limit swap or the presence of factor of (-1).
		$= \int_0^1 (1-u)^{p-1}u^{q-1}du = b(q,p)$.	A1	Fully correct working leading to result (AG)
			[2]	
	(ii)	$\int_0^1 x^p(1-x)^{q-1}dx + \int_0^1 x^{p-1}(1-x)^q dx$		
		$= \int_0^1 x^{p-1}(1-x)^{q-1}(x+(1-x))dx + \int_0^1 x^{p-1}(1-x)^{q-1}dx$	M1	Method involving common factor in two integrands

Question	Answer	Marks	Guidance
		A1	Fully correct working and conclusion.
	$b(p+1,q)+b(p,q+1)=b(p,q)$.		
	Setting $p = q$ then gives $b(p+1,p)+b(p,p+1)=b(p,p)$.	**B1**	Need both parts of this argument (namely setting $p = q$ and use of $b(p,q) = b(q,p)$)
	Since $b(p+1,p)=b(p,p+1)$ by part (i) this gives $b(p+1,p)=\frac{1}{2}b(p,p)$		
			The use of $b(p,q)=b(q,p)$ can be implied by presence of $2b(p+1,p)$ term.
		[3]	
(iii)	Starting with the original $b(p,q)$, the substitution $x=\sin^2\theta$ leads to	**M1**	For identifying suitable substitution and making some progress with it.
	$b(p,q)=\int_0^{\pi/2}(\sin\theta)^{2(p-1)}\,(\cos\theta)^{2(q-1)}2\sin\theta\cos\theta\,d\theta=$		
	$2\int_0^{\pi/2}(\sin\theta)^{2p-1}\,(\cos\theta)^{2q-1}d\theta$.	**A1**	Fully correct working leading to result (AG)

Question	Answer	Marks	Guidance
	This gives that $b(p,p) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2p-1}\,d\theta$ $= 2\int_0^{\pi/2} \frac{(\sin 2\theta)^{2p-1}}{2^{2p-1}}\,d\theta$	M1	For setting $q = p$ and use of $2\sin\theta\cos\theta = \sin 2\theta$.
	The substitution $t = 2\theta$ then gives $b(p,p) = 2\times\frac{1}{2}\int_0^{\pi} \frac{(\sin t)^{2p-1}}{2^{2p-1}}\,dt$	M1	For identifying suitable substitution and carrying it out correctly. Allow one incorrect limit for M1. Also can give M1 if fully correct but with no working (since substitution is particularly straightforward).
*	by the symmetry of integrand about $\frac{\pi}{2}$. $= 2\int_0^{\pi/2} \frac{(\sin t)^{2p-1}}{2^{2p-1}}\,dt =$	B1	Use of symmetry of sin to deal with a factor of 2. Need to see word "symmetry" (or equivalent e.g. diagram or identity) and change of upper limit to $\frac{\pi}{2}$.

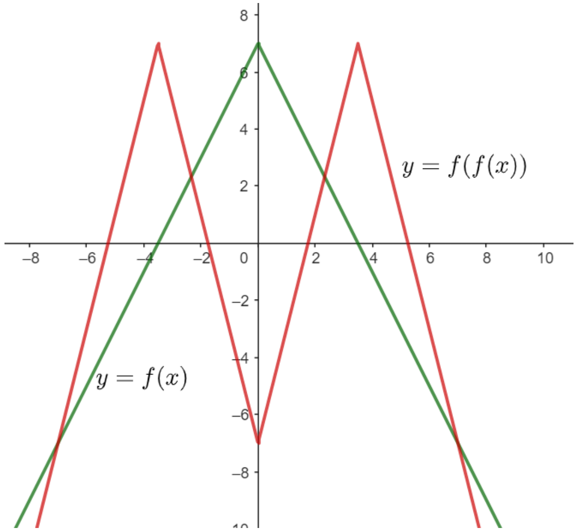
Question	Answer	Marks	Guidance
**	$2\int_0^{\pi/2} \frac{(\sin t)^{2p-1}(\cos t)^{2\times\frac{1}{2}-1}}{2^{2p-1}}dt = \frac{b(p,\frac{1}{2})}{2^{2p-1}}$	A1	Bring together for fully correct result (given), noting use of $(\cos t)^0 = 1$ Writing out $b\left(p,\frac{1}{2}\right)$ independently, with the absence of the $\cos t$ term, is to be considered equivalent to the above.
		[6]	
	Alternative for last two marks of (iv)		
*	$b(p,p) = 2\times\frac{1}{2}\int_0^{\pi}\frac{(\sin t)^{2p-1}}{2^{2p-1}}dt = 4\int_0^{\frac{\pi}{4}}\frac{(\sin 2\theta)^{2p-2}}{2^{2p-1}}\frac{4\sin 2\theta\cos 2\theta d\theta}{4\cos 2\theta}$		
	$= \frac{1}{2^{2p-1}}\int_0^1 \frac{u^{p-1}}{(1-u)^{\frac{1}{2}}}du$	M1	Substitution $u = \sin^2 2\theta$
**	$= \frac{1}{2^{2p-1}}b\left(p,\frac{1}{2}\right)$	A1	

Question		Answer	Marks	Guidance
	(iv)	Using the substitution $t = \frac{x}{1-x}$ (or equivalently $x = \frac{t}{1+t}$)	M1	For identifying suitable substitution(s). Other possible substitutions: $t = \tan^2(\theta)$ $x = 1/(1+t)$ $x = t/(t-1)$ Approaches involving a series of two substitutions $t = \tan(\theta)$ then $u = t^2$ $x = \frac{1}{t}$ then $u = t - 1$ $x = (t-1)/t$ then $u = 1 + t$ $u = \ln(1/x)$ then $t = e^u - 1$ $x = \tanh^2(\theta)$ then $u = \sinh^2(\theta)$
		$b(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx =$ $\int_0^\infty \left(\frac{t}{1+t}\right)^{p-1}\left(1-\frac{t}{1+t}\right)^{q-1}\frac{1}{(1+t)^2}dt$	M1	For working in method of substitution

Question		Answer	Marks	Guidance
		$= \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,dt$	**A1** **[3]**	Fully correct working leading to result (AG). (Note that if arrives at same integral but with power of $q - 1$ in numerator need to quote (i) to switch that to $p - 1$ and gain this mark.)
	(v)	By part (iv) the integral given is $b\left(\frac{5}{2}, \frac{7}{2}\right)$	**B1**	Use of previous answers. May attempt direct integration of this see ALT below.
		By part (i) this is equal to $b\left(\frac{7}{2}, \frac{5}{2}\right)$.	**B1**	Use of previous answers
		and then by part(ii) equal to $\frac{1}{2} b\left(\frac{5}{2}, \frac{5}{2}\right)$. By part (iii) this is equal to $\frac{1}{2^5} b\left(\frac{5}{2}, \frac{1}{2}\right)$	**B1** **B1**	Use of previous answers This can be given if seen, without link to previous step.

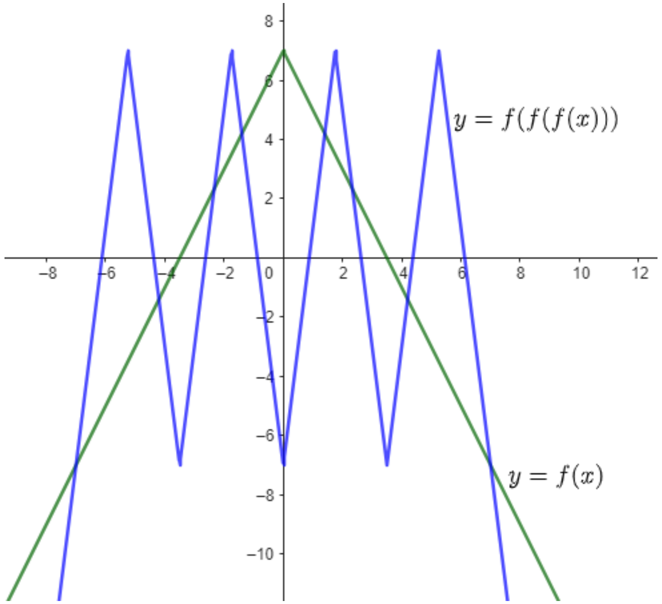
Question		Answer	Marks	Guidance
		$\frac{1}{2^4}\int_0^{\pi/2}(\sin\theta)^4\,d\theta = \frac{1}{2^6}\int_0^{\pi/2}(1-\cos 2\theta)^2 d\theta =$	M1	Need to have got to an expression with multiple angles only, using the double angle formulae.
		$= \frac{1}{2^6}\int_0^{\pi/2} 1 - 2\cos 2\theta + \frac{\cos 4\theta + 1}{2} d\theta$		
		$= \frac{1}{64}\left[\frac{3\theta}{2} - \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{2}} =$		
		$\frac{3\pi}{256}$	A1	Fully correct answer
			[6]	

Question		Answer	Marks	Guidance
		Alternative (v)		
		By part (iv) the integral given is $b\left(\frac{5}{2},\frac{7}{2}\right)$.	**B1**	
		Using the result in (iii) this is		
		$2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\, d\theta$	**M1**	For stating switching to this form of $b(p,q)$
		de Moivre, with $z=\cos\theta+i\sin\theta$	**M1**	For use of de Moivre in this way $(z+z^{-1})$ etc.
		$2^{10}\sin^4\theta\cos^6\theta=(z-z^{-1})^4(z+z^{-1})^6$		
		$=(z^8-4z^4+6-4z^{-4}+z^{-8})(z^2+2+z^{-2})=$		
		$z^{10}+z^{-10}+2(z^8+z^{-8})-3(z^6+z^{-6})-8(z^4+z^{-4})+2(z^2+z^{-2})+12$		
		$=2\cos 10\theta+4\cos 8\theta-6\cos 6\theta-16\cos 4\theta+2\cos 2\theta+12$		
		Therefore		
		$2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta\, d\theta=$	**M1**	For arriving at an easily integrable expression.
		$\frac{1}{256}\int_0^{\frac{\pi}{2}}\cos 10\theta+2\cos 8\theta-3\cos 6\theta-8\cos 4\theta+\cos 2\theta+6\, d\theta=$		
		$\frac{1}{256}\left[\frac{\sin(10\theta)}{10}+\frac{\sin 8\theta}{4}-\frac{\sin 6\theta}{2}-2\sin 4\theta+\frac{\sin 2\theta}{2}+6\theta\right]_0^{\frac{\pi}{2}}=\frac{3\pi}{256}$	**M1** **A1**	

Question			Answer	Marks	Guidance
2	**(i)**	**(a)**		**G1**	Two straight lines for y = f(x), roughly correct gradients, meeting on y-axis
				G1	Four straight lines for y = f(f(x)) with roughly correct gradients (twice as steep than those for y = f(x))
				G1	Symmetry between (0,7) on y = f(x) and (0,-7) on y = f(f(x)) 'Peaks' on both graphs have same y-coordinates (y = 7).
				G1	Intersections at $(\pm 7, -7)$ correspond to minimum point on y-axis.
				[4]	
		(b)	Using graph, need to find the intersections of $y = x$ with each of $y = 4x - 7, y = -4x - 7, y = 4x + 21, y = -4x + 21.$	**M1**	Method which involves solving for intersections of the relevant lines, guided by graph, (or equivalent such as attempting to solve $7 - 2\|7 - 2\|x\|\| = x$ algebraically). For attempts to solve using quadratics, award M1 for getting

Question	Answer	Marks	Guidance
			any one quadratic: $15x^2+126x+147=0$ or $15x^2-98x+147=0$ $5x^2+14x-147=0$ or $15x^2-14x-49=0$
	This gives values of $\frac{7}{3},\frac{-7}{5},-7,\frac{21}{5}$	**A2**	Award **A1** for 3 correct solutions or if any incorrect values are included.
		[3]	
(c)	For this we need a such that $f(a)\neq a$ but $f\big(f(a)\big)=a$.	**M1**	For property of a point with period 2.
	Candidates are solutions of $f\big(f(x)\big)=x$, from above, namely: $\frac{7}{3},\frac{-7}{5},-7,\frac{21}{5}$.		
	We have $f\left(\frac{7}{3}\right)=\frac{7}{3}, f(-7)=-7$ and $f\left(\frac{-7}{5}\right)=\frac{21}{5}, f\left(\frac{21}{5}\right)=-\frac{7}{5}$. Therefore sequence has period 2 only when $a=\frac{-7}{5},\frac{21}{5}$.	**A1FT**	Checking leading to correct values (could be via graph) Follow through, their answer to (i)(b) with 7/3 and –7 removed.
		[2]	

Question	Answer	Marks	Guidance
(c)	**Alternative: Some candidates may consider a sequence with period 1 to also have period 2. In this case marks as follows**		
	A sequence which has period 1 also has period 2.	M1	
	Such a are $\frac{7}{3}, \frac{-7}{5}, -7, \frac{21}{5}$, from answer to (b)	A1	
		[2]	
(d)	With $a = \frac{28}{5}$ we get $u_0 = \frac{28}{5}, u_1 = \frac{-21}{5}, u_2 = \frac{-7}{5}$, and (by part (c)) the sequence is periodic from there.	M1	For calculation of values necessary for checking and comment that sequence is periodic.
	$u_3 = \frac{21}{5}, (u_4 = \frac{-7}{5}, u_5 = \frac{21}{5}$ and will then repeat $-\frac{7}{5}, \frac{21}{5} - \frac{7}{5}, \frac{21}{5}, ..)$ Neither of u_0, u_1 is equal to either of u_2, u_3.	A1	Value of u_3 and check that other condition is satisfied.
		[2]	

Question	Answer	Marks	Guidance
(ii) (a)		G1 G1 G1 [3]	8 straight lines for $y = f(f(f(x)))$, looking roughly correct (position, gradients etc) 4 peaks for $y = f(f(f(x))$ at same y-coordinate (y = 7) as the peak of $y = f(x)$. Three mins with same negative y-coordinate (y = -7) and the intersections at ± 7 at this y-coordinate.
(b)	(We can see from the graph, by adding $y = x$, that there are a total of 8 solutions.) Any solution of $f(x) = x$ is also a solution of $f(f(f(x))) = x$. From earlier, these are $\frac{7}{3}, -7$.	B1 FT	Identifying both of these values . Follow through from their solutions to f(x)=x identified in part (i)(c).

Question	Answer	Marks	Guidance
	We have that $f(f(f(1)) = 1$. We also that $f(f\left(f(\frac{-7}{9})\right) = \frac{-7}{9}$. Therefore $x = 1$ $x = -\frac{7}{9}$ are both solutions of $f(f(f(x)) = x$.	M1	For checking these two values
	The other solutions must be other values in the 3-cycles which include $1, -\frac{7}{9}$, namely $f(\frac{-7}{9}), f\left(f(\frac{-7}{9})\right), f(1), f(f(1))$.	E1	For commenting that the other values in the cycles will also be solutions.
	These values are $\frac{49}{9}, \frac{-35}{9}, 5, -3$.	A1	Identifying all four of these values
		[4]	
(c)	u_3 has to be one of $1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3$.	B1FT	Follow through using of their answers to (ii)(b) as a value for u_3.
	So we are looking for solutions, x, of $f(f(f(x)) = 1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3$. which are not 'values above' and so that, when taking them as u_0, neither of u_1, u_2 are included in the values above. The full list of values is $$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{11}{2}, \pm\frac{13}{2}, \pm\frac{7}{18}, \pm\frac{35}{18}, \pm\frac{49}{18}, \pm\frac{77}{18}, \pm\frac{91}{18}, \pm\frac{119}{18}$$	B1	Any one will do
		[2]	

Question	Answer	Marks	Guidance
(c)	**Alternative: Some candidates may consider a sequence with period 1 to also have period 3. In this case marks as follows, this must be consistent with their approach to 2(i)(c)**		
	u_3 has to be one of $1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3, -7, \frac{7}{3}$	B1	
	So we are looking for solutions, x, of $f(f(f(x)) = 1, -\frac{7}{9}, \frac{49}{9}, \frac{-35}{9}, 5, -3, -7, \frac{7}{3}$ which are not 'values above' and so that, when taking them as u_0, neither of u_1, u_2 are included in the values above. The full list of such values is $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{11}{2}, \pm\frac{13}{2}, \pm\frac{7}{18}, \pm\frac{35}{18},$ $\pm\frac{49}{18}, \pm\frac{77}{18}, \pm\frac{91}{18}, \pm\frac{119}{18}, \pm\frac{7}{2}, \pm\frac{7}{6}, \pm\frac{35}{6}$	B1	Any one will do
		[2]	

Question		Answer	Marks	Guidance
3	(i)	The equation of the line AB is $\left(y - f(a)\right) = \left(\frac{-(f(a)+f(b))}{b-a}\right)(x-a)$.	M1	Finding equation of line and
		Substituting $y = 0, x = m$ gives $-f(a) = \left(\frac{-(f(a)+f(b))}{b-a}\right)(m-a)$ leading to	M1	Sub $y = 0$, $x = m$. No need to set these in a single step.
		$\frac{f(a)(b-a)}{f(a)+f(b)} + a = m \Leftrightarrow m = \frac{f(a)(b-a)+a\left(f(a)+f(b)\right)}{f(a)+f(b)} \Leftrightarrow$ $m = \frac{f(a)b + af(b)}{f(a)+f(b)}$.	A1	Fully correct working
			[3]	

Question	Answer	Marks	Guidance
(ii)	Setting $f(x) = \sqrt{x}$ gives $m = \frac{b\sqrt{a}+a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$. Noting that $b\sqrt{a} + a\sqrt{b} = (\sqrt{a} + \sqrt{b})\sqrt{ab}$, we see that $m = \sqrt{ab}$.	**B1**	Use of correct relationship that leads to result. Or use difference of squares. No FT as AG.
*	We need to find a function $f(x)$ such that $m = \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{f(a)b+f(b)a}{f(a)+f(b)}$ $\Leftrightarrow f(a)a^n b + f(a)b^{n+1} + f(b)a^{n+1} + f(b)ab^n$ $= f(a)a^{n+1} + f(a)b^{n+1} + f(b)a^{n+1} + f(b)b^{n+1}$	**M1**	
**	$\Leftrightarrow\ [b - a] \cdot (f(a)a^n - f(b)b^n)\ =\ 0$ $\Leftrightarrow\ f(a)a^n = f(b)b^n$	**A1**	Equate the two fractions and at least get to the second line of calculation for the M1.
***	Finding any function $f(x)$ that satisfies the above e.g. $f(x) = \frac{1}{x^n}$ or $f(x) = \frac{a^n b^n}{x^n}$. *(Alternatively, if the candidate recognises the need for switching a and b and/or writes the function of a form similar to $f(x) = (a + b - x)^n$ then the E1 is awarded.)*	**E1**	Award for giving a function that satisfies relationship above.
		[4]	

Question	Answer	Marks	Guidance
	Alternatively, for last three marks in (ii), candidates could		
***	state an appropriate function $f(x)$.	**E1**	
	and then go on to show that it meets the requirements in the question that $f(x)$ is defined and positive for $x > 0$, and that		
*;**	$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{f(a)b+f(b)a}{f(a)+f(b)}$.	**M1A1**	
(iii)	We have $M_1 = \frac{g_1(a)b+ag_1(b)}{g_1(a)+g_1(b)}$ and $M_2 = \frac{g_2(a)b+ag_2(b)}{g_2(a)+g_2(b)}$.		
	So $M_1 > M_2 \Leftrightarrow \frac{g_1(a)b+ag_1(b)}{g_1(a)+g_1(b)} > \frac{g_2(a)b+ag_2(b)}{g_2(a)+g_2(b)}$ $\Leftrightarrow \big(g_1(a)b + ag_1(b)\big)\big(g_2(a) + g_2(b)\big) > \big(g_2(a)b + ag_2(b)\big)\big(g_1(a) + g_1(b)\big)$ $\Leftrightarrow bg_1(a)g_2(b) + ag_1(b)g_2(a) > bg_2(a)g_1(b) + ag_2(b)g_1(a)$ $\Leftrightarrow b\big(g_1(a)g_2(b) - g_2(a)g_1(b)\big) + a(g_1(b)g_2(a) - g_2(b)g_1(a)) > 0$ $\Leftrightarrow (b-a)\big(g_1(a)g_2(b) - g_2(a)g_1(b)\big) > 0$	**M1**	For simplifying $M_1 > M_2$ to "useable form", need at least one line of simplifying
	We know that $\frac{g_1(x)}{g_2(x)}$ is a decreasing function so that, since $b > a$, $\frac{g_1(b)}{g_2(b)} < \frac{g_1(a)}{g_2(a)}$ giving $g_1(b)g_2(a) < g_1(a)g_2(b)$ and $0 < g_1(a)g_2(b) - g_1(b)g_2(a)$	**M1**	Correct use of decreasing function

Question	Answer	Marks	Guidance
	Therefore, since $b > a$ we have $(b-a)\big(g_1(a)g_2(b) - g_2(a)g_1(b)\big) > 0$ and so $M_1 > M_2$ by the working above.	**A1**	Fully correct explanation.
	Take $g_1(x) = 1$. By formula in (i) $M_1 = \frac{a+b}{2}$. As seen earlier, taking $g_2(x) = \sqrt{x}$, gives $M_2 = \sqrt{ab}$. Alternatively taking $g_1(x) = \sqrt{x}$ and $g_2(x) = x$ is also valid for proving the right side of the inequality first.	**M1**	Correct choice of g_1 and g_2.
	Since $\frac{g_1(x)}{g_2(x)} = \frac{1}{\sqrt{x}}$ is a decreasing function the result just established gives us that $\frac{a+b}{2} > \sqrt{ab}$. Same can be used to prove $\sqrt{ab} > \frac{2ab}{a+b}$.	**A1**	Decreasing function statement required
	Then $\frac{a+b}{2} > \sqrt{ab} \Leftrightarrow 1 > \frac{2\sqrt{ab}}{a+b} \Leftrightarrow \sqrt{ab} > \frac{2ab}{a+b}$ where the last iff holds by multiplying both sides of the previous inequality by $\sqrt{ab} > 0$. Similarly, for $\frac{a+b}{2} > \sqrt{ab}$.	**B1**	Correct working of first inequality. This could also be given by taking a direct approach to other inequality with correct choice of g_1, g_2
		[6]	
(iv)	$f(a) = p(c-a)^3$ and $-f(b) = p(c-b)^3$.	**M1**	Equations for $f(a), f(b)$
	Dividing one by the other we get $\frac{(c-a)^3}{(b-c)^3} = \frac{f(a)}{f(b)}$ giving $\frac{c-a}{b-c} = \frac{f(a)^{\frac{1}{3}}}{f(b)^{\frac{1}{3}}}$	**A1**	Fully correct working

Question	Answer	Marks	Guidance
	We then have $f(b)^{\frac{1}{3}}c - f(b)^{\frac{1}{3}}a = f(a)^{\frac{1}{3}}b - f(a)^{\frac{1}{3}}c \Leftrightarrow (f(b)^{\frac{1}{3}} + f(a)^{\frac{1}{3}})c = f(a)^{\frac{1}{3}}b + f(b)^{\frac{1}{3}}a$ So we have $c = \frac{f(a)^{\frac{1}{3}}b+f(b)^{\frac{1}{3}}a}{f(b)^{\frac{1}{3}}+f(a)^{\frac{1}{3}}}$	**B1**	For correct c. FT if e.g. in M1 missed a minus sign or MR (as method is valid and c not given in Q3)
*	Now $c < m \Leftrightarrow \frac{f(a)^{\frac{1}{3}}b+f(b)^{\frac{1}{3}}a}{f(b)^{\frac{1}{3}}+f(a)^{\frac{1}{3}}} < \frac{f(a)b+af(b)}{f(a)+f(b)} \Leftrightarrow$ $\left(f(a)^{\frac{1}{3}}b + f(b)^{\frac{1}{3}}a\right)\left(f(a) + f(b)\right) < \left(f(b)^{\frac{1}{3}} + f(a)^{\frac{1}{3}}\right)\left(f(a)b + af(b)\right) \Leftrightarrow$ $f(a)^{\frac{4}{3}}b + f(a)^{\frac{1}{3}}bf(b) + f(b)^{\frac{1}{3}}af(a) + f(b)^{\frac{4}{3}}a$ $< f(b)^{\frac{1}{3}}f(a)b + f(b)^{\frac{4}{3}}a + f(a)^{\frac{4}{3}}b + f(a)^{\frac{1}{3}}f(b)a$ $\Leftrightarrow f(a)^{\frac{1}{3}}bf(b) + f(b)^{\frac{1}{3}}af(a) < f(b)^{\frac{1}{3}}f(a)b + f(a)^{\frac{1}{3}}f(b)a$ $\Leftrightarrow (b-a)\left(f(a)^{\frac{1}{3}}f(b) - f(b)^{\frac{1}{3}}f(a)\right) < 0 \Leftrightarrow f(a)^{\frac{1}{3}}f(b) - f(b)^{\frac{1}{3}}f(a) < 0$ $\Leftrightarrow \frac{f(a)^{\frac{1}{3}}}{f(b)^{\frac{1}{3}}} < \frac{f(a)}{f(b)} \Leftrightarrow f(b)^{\frac{2}{3}} < f(a)^{\frac{2}{3}}$	**M1**	For working and at least one line of simplifying

Question	Answer	Marks	Guidance
**	So $c < m \Leftrightarrow f(b)^{\frac{2}{3}} < f(a)^{\frac{2}{3}}$	**A1**	Equivalence of inequalities.
***	If x and y are positive, then $x < y \Leftrightarrow x^{\frac{1}{3}} < y^{\frac{1}{3}} \Leftrightarrow x^{\frac{2}{3}} < y^{\frac{2}{3}}$. Applying this, since $f(x)$ is decreasing and positive, so is $f(x)^{\frac{2}{3}}$. *(Note that using derivates here is not appropriate as the function may not be differentiable, however, if the method carries along in the right spirit the M1 is awarded but not the A1 for accuracy of the argument.)*	**M1**	Statement, with justification, about $f(x)^{\frac{2}{3}}$ being decreasing, FT their inequality c < m. Also award when the candidate states that any decreasing function raised to a positive power is also decreasing – note the generality of the statement.
****	Therefore $f(b)^{\frac{2}{3}} < f(a)^{\frac{2}{3}}$ giving that $c < m$.	**A1**	CSO
		[7]	

Question	Answer	Marks	Guidance
	Alternative for last four marks in (iv)		
*	Bringing c and m in the form such as $c = ka + (1-k)b$ and $m = qa + (1-q)b$, where $k = \frac{f(b)^{\frac{1}{3}}}{f(b)^{\frac{1}{3}}+f(a)^{\frac{1}{3}}} = \frac{1}{1+\left(\frac{f(a)}{f(b)}\right)^{\frac{1}{3}}}$ and $q = \frac{f(b)}{f(b)+f(a)} = \frac{1}{1+\frac{f(a)}{f(b)}}$.	M1	For bringing the expression for c and m in a form that can be compared and splitting the expressions into a form weight_1*a + weight_2*b.
**	Hence as both k and q positive and $a < b$, we need to prove that $k > q$ and hence $(1-k) < (1-q)$ to show that $c < m$. *(Note that as both k and q are in $(0,1)$ we may think of c and m as expectation values weighting the values of a and b and hence the expression which has "less a and more b " in it will be bigger.)*	A1	For setting up correct, simplified inequalities that can be used to find $c < m$.
***	As $f(x)$ is positive and decreasing we have $\frac{f(a)}{f(b)} > 1$ and hence $\frac{f(a)}{f(b)} > \left(\frac{f(a)}{f(b)}\right)^{\frac{1}{3}} > 1$. Using this one finds $k > q$ and $(1-k) < (1-q)$. *(Note that using derivates here is not appropriate as the function may not be differentiable, however, if the method carries, assuming differentiability, is correct, the M1 is awarded but not the A1 for accuracy of the argument.)*	M1	Statement, with justification, of the form for $x > 1$ one finds $x > x^{\frac{1}{3}} > 1$, or similar, that can be used to show how k and q can be related.
****	Therefore as $k > q$ and $(1-k) < (1-q)$ one finds $m > c$ as $a < b$.	A1	Note that showing $\frac{f(a)}{f(b)} > 1$ here is important.

Question			Answer	Marks	Guidance
4	(i)		(x_1, y_1) on $G_1 \Rightarrow \frac{x_1^2}{9} + \frac{y_1^2}{4} = 1.$	**B1**	For statement about (x_1, y_1) being on G_1
		*	Then $(x_2, y_2) = \left(\frac{x_1 - y_1}{\sqrt{2}}, \frac{x_1 + y_1}{\sqrt{2}}\right)$ and, using the inverse matrix, $(x_1, y_1) = \left(\frac{x_2 + y_2}{\sqrt{2}}, \frac{-x_2 + y_2}{\sqrt{2}}\right)$	**M1**	For getting (x_1, y_1) in terms of (x_2, y_2) Allow for $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$
		**	Therefore $$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$$ $$\Rightarrow \frac{\left(\frac{x_2}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}\right)^2}{9} + \frac{\left(-\frac{x_2}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}\right)^2}{4} = 1$$ So that (x_2, y_2) is on G_2.	**A1**	Fully correct working to show that if (x_1, y_1) on G_1 then (x_2, y_2) is on G_2 Allow **BOD** for **A1** if **B0** If **B0** earlier then can allow **B1** here if they state something like "since (x_1, y_1) is on G_1 this is true" If the implication is shown in the "wrong direction" then allow **SC**

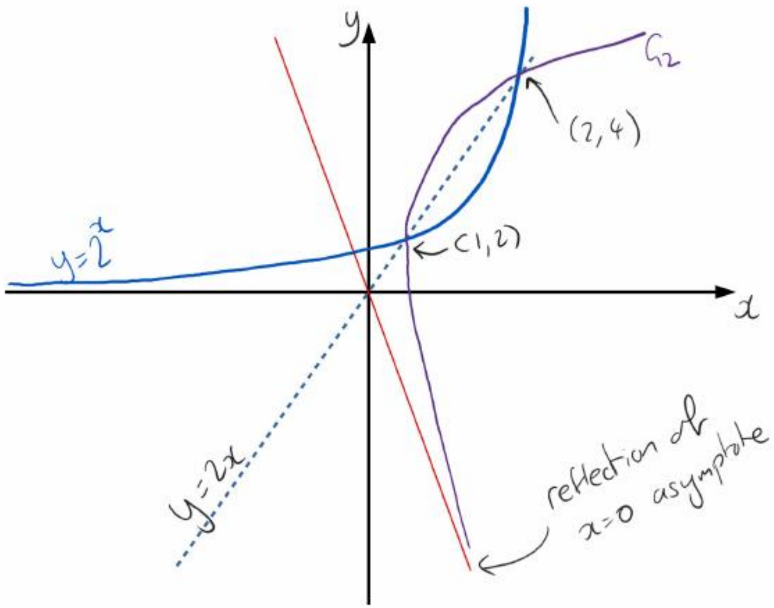
Question			Answer	Marks	Guidance
					B2 for using (x_1, y_1) on G_1 and (x_2, y_2) on G_2 to obtain given relationship for (x_2, y_2) in terms of (x_1, y_1). If they convincingly argue that the implication can be reversed then allow **SC B3**

ALT		*	We have $$\frac{\left(\frac{x_2}{\sqrt{2}}+\frac{y_2}{\sqrt{2}}\right)^2}{9}+\frac{\left(-\frac{x_2}{\sqrt{2}}+\frac{y_2}{\sqrt{2}}\right)^2}{4}$$ $$=\frac{\left(\frac{x_1-y_1}{2}+\frac{x_1+y_1}{2}\right)^2}{9}+\frac{\left(\frac{y_1-x_1}{2}+\frac{x_1+y_1}{2}\right)^2}{4}$$	**M1**	For taking G_2 equation and replacing (x_2, y_2) with expressions in (x_1, y_1)
		**	$$=\frac{(x_1)^2}{9}+\frac{(y_1)^2}{4}$$ $$=1$$	**A1**	For completing argument
			We know that the relationship $\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}$ means that the point (x_2, y_2) is an anticlockwise rotation of (x_1, y_1) by 45° about the origin. Therefore, every point on G_2 is a rotation of a point on G_1 and so G_2 is an anticlockwise rotation of G_1 about the origin through 45 degrees.	**B1**	Don't need to see "every point" Identifying matrix as relevant rotation matrix (no working needed) **and** stating conclusion for the graphs. If general rotation matrix seen must have cos / sin in correct places.
				[4]	

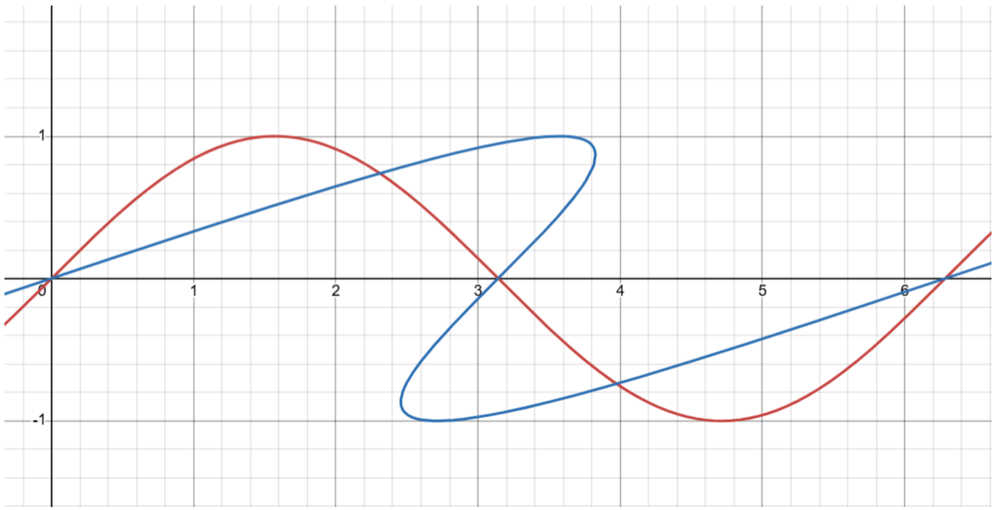
	(ii)	**(a)**	Need points (x, y) such that $\begin{pmatrix}-0.6 & 0.8\\ 0.8 & 0.6\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$	**M1**	Setting up method for finding LOIP If Invariant lines need to see one more step of working before **M1**
			$\Leftrightarrow 0.8y = 1.6x \Leftrightarrow y = 2x.$	**A1**	Might see set up for invariant lines. Can have the **M1** but for **A1** need to see incorrect lines rejected If using general reflection matrix allow: **M1:** Setting up a quadratic equation in $t = \tan\theta$ **A1:** Correctly solving quadratic equation to obtain $y = 2x$ only (i.e. must reject one solution of quadratic equation)
				[2]	
		(b) *	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}-0.6 & 0.8\\ 0.8 & 0.6\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix}$	**M1**	Relationship between (x_1, y_1) and (x_2, y_2)

		**	If (x_1, y_1) is on $y = 2^x$ then we have: $$y_1 = 2^{x_1}$$ $$0.8x_2 + 0.6y_2 = 2^{-0.6x_2+0.8y_2}$$ and so (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$	**A1**	Do not need to see "if and only if" stated M1 A1 for convincingly establishing relationship between the two graphs Must show that (x_1, y_1) on $y = 2^x$ implies that (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$

ALT		*	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$	**M1**	
		**	Consider $$0.8x_2 + 0.6y_2 = 2^{-0.6x_2+0.8y_2}$$ This is equivalent to: $$0.8(-0.6\,x_1 + 0.8y_1) + 0.6(0.8x_1 + 0.6y_1) = 2^{-0.6(-0.6\,x_1+0.8y_1)+0.8(0.8x_1+0.6y_1)}$$ i.e. $y = 2^x$ And so if (x_1, y_1) is on $y = 2^x$ then (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$	**A1**	M1 A1 for convincingly establishing relationship between the two graphs Must show that (x_2, y_2) is on $0.8x + 0.6y = 2^{-0.6x+0.8y}$ implies (x_1, y_1) on $y = 2^x$

			Therefore $y = 2^x$ is a reflection of $0.8x + 0.6y = 2^{-0.6x+0.8y}$ in $y = 2x$.  	**B3**	B1: $y = 2^x$ and what looks like a reflection in an oblique line with positive gradient (reflection line might not appear) Award even if looks like has been reflected in $y = x$ B1: Roughly correct asymptote as $y \to -\infty$ for G_2 shown and used. No need to see a line for asymptote but if shown must be through origin and looks like a reflection of x-axis B1: Two points of intersection shown between G_1 and G_2. Not necessary to find the coordinates but if labelled then they must be correct. Follow through for a maximum of **B2** possible if equation of line of reflection is wrong (as long as line of reflection is oblique). Must be using a reflection between the graphs (not rotation etc)
				[5]	

	(iii)	*	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.	**M1**	Relationship between (x_1, y_1) and (x_2, y_2) Could see for inverse relationship $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ Allow **M1 BOD** for sight of one of the two correct matrices even if points are backwards
		**	If (x_1, y_1) is on $y = \sin x$ then we have: $y_1 = \sin x_1$ $y_2 = \sin(x_2 - 2y_2)$. And so (x_2, y_2) is on $y = \sin(x - 2y)$	**A1**	Do not need to see "if and only if" If **M1** awarded for inverse relationship then this part must be carefully explained. Must show that (x_1, y_1) is on $y = \sin x$ implies that (x_2, y_2) is on $y = \sin(x - 2y)$

ALT		*	Suppose (x_1, y_1) and (x_2, y_2) are such that $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.		
		**	Then consider $y_2 = \sin(x_2 - 2y_2)$ This is equivalent to $y_1 = \sin(x_1 + 2y_1 - 2y_2)$ i.e. $y_1 = \sin x_1$ And so if (x_1, y_1) is on $y = \sin x$ then (x_2, y_2) is on $y = \sin(x - 2y)$		Must show that (x_2, y_2) is on $y = \sin(x - 2y)$ implies that (x_1, y_1) is on $y = \sin x$
			 Note: only need range $[0, 2\pi]$	**B2**	Generosity needed. $y = \sin(x\text{-}2y)$ needs to be between -1 and 1 (or at least intended to be!), includes some x with multiple y, some vertical tangents in roughly the right places. **B1** for smooth, continuous graph with correct x intercepts and looks like it had been pushed to the right, and at some point it must "escape" $y = \sin x$. **SC B1 for shear in wrong direction, must be between ±1 and have correct *x* intercepts**

			Differentiating $y = \sin(x - 2y)$ with respect to x gives $\frac{dy}{dx} = \left(1 - 2\frac{dy}{dx}\right)\cos(x - 2y)$ Therefore $\frac{dy}{dx}(1 + 2\cos(x - 2y)) = \cos(x - 2y) \Rightarrow \frac{dy}{dx} = \frac{\cos(x-2y)}{1+2\cos(x-2y)}$	**M1**	Recognisable attempt at differentiation using implicit and chain rules. Might be incorrect. Do not need to rearrange to $\frac{dy}{dx} =$ for the M mark
		*	So $\frac{dy}{dx} = 0 \Leftrightarrow x - 2y = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.	**M1**	Using $\frac{dy}{dx} = 0$ to get an equation in x and y Might see $2n\pi \pm \frac{\pi}{2}$ instead of two values
		**	When $x - 2y = \frac{\pi}{2}, y = \sin(x - 2y) = 1$ and $x = \frac{\pi}{2} + 2$ When $x - 2y = \frac{3\pi}{2}, y = \sin(x - 2y) = -1$ and $x = \frac{3\pi}{2} - 2$ Therefore points where $\frac{dy}{dx} = 0$ have form $\left(\frac{\pi}{2} + 2, 1\right)$ and $\left(\frac{3\pi}{2} - 2, -1\right)$.	**A1**	CAO Note: ONLY need the points within the given range

ALT		*	For previous two marks only $$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}\frac{\pi}{2}\\1\end{pmatrix}=\begin{pmatrix}\frac{\pi}{2}+2\\1\end{pmatrix}$$	**B1**	For correct point found via transformation of maximum of sin x
		**	$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}\frac{3\pi}{2}\\-1\end{pmatrix}=\begin{pmatrix}\frac{3\pi}{2}-2\\-1\end{pmatrix}$$	**B1**	For second correct point found via transformation **NOTE:** you can either award via **M1 A1** scheme or via **B1 B1** scheme.
			$y=\sin(2x-y)$ has infinite gradient when $\cos(x-2y)=-\frac{1}{2}\Leftrightarrow x-2y=\frac{2\pi}{3}$ or $\frac{4\pi}{3}$	**M1**	Using $\frac{dy}{dx}=\infty$ to get an equation in x and y Might see more general cases
			When $x-2y=\frac{2\pi}{3}$, $y=\sin(x-2y)=\frac{\sqrt{3}}{2}$, so $x=\sqrt{3}+\frac{2\pi}{3}$ When $x-2y=\frac{4\pi}{3}$, $y=\sin(x-2y)=-\frac{\sqrt{3}}{2}$, so $x=-\sqrt{3}+\frac{4\pi}{3}$. So we have vertical tangent at points of form $\left(-\sqrt{3}+\frac{4\pi}{3},-\frac{\sqrt{3}}{2}\right)$ and $\left(\sqrt{3}+\frac{2\pi}{3},\frac{\sqrt{3}}{2}\right)$	**A1**	CAO
				[9]	

Question		Answer	Marks	Guidance
5	(i) *	Let $\overrightarrow{OL} = \delta\boldsymbol{a}$ for some δ. Then $\overrightarrow{OX} = \overrightarrow{OL} + \overrightarrow{LX} = \overrightarrow{OL} + \lambda\overrightarrow{LP} = \delta\mathbf{a} + \lambda((\alpha - \delta)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c})$ for some λ.	M1	Equation for $\overrightarrow{OX}$ Sight of correct line equation in **a, b, c** gets M1
	**	Also $\overrightarrow{OX} = \overrightarrow{OC} + \mu\overrightarrow{CB} = \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})$ for some μ.	M1	Second equation for $\overrightarrow{OX}$ Could be $\mathbf{b} + \mu(\mathbf{c} - \mathbf{b})$, $(1 - \mu)\mathbf{b} + \mu\mathbf{c}$, $(1 - \mu)\mathbf{c} + \mu\mathbf{b}$
ALT	*	**First two marks** can be awarded as: Let $\overrightarrow{OL} = \delta\boldsymbol{a}$ for some δ. Then $\overrightarrow{LX} = \lambda\overrightarrow{LP} = \lambda(\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} - \delta\boldsymbol{a})$ for some λ.	M1	Can only award first two marks via first scheme or this scheme. i.e. cannot award one **M1** from each scheme.
	**	Also $\overrightarrow{LX} = \overrightarrow{LO} + \overrightarrow{OX} = \mathbf{c} + \mu(\mathbf{b} - \mathbf{c}) - \delta\boldsymbol{a}$ for some μ	M1	Sight of correct line equation in **a, b, c** gets **M1**

Question		Answer	Marks	Guidance
		Equating coefficients gives the equations $\delta + \lambda(\alpha - \delta) = 0$, $\lambda\beta = \mu$ and $\lambda\gamma = 1 - \mu$. Adding the last two together leads to $\lambda = \frac{1}{\beta+\gamma}$ and then also $\mu = \frac{\beta}{\beta+\gamma}$.	**M1**	Attempt to solve simultaneous equations found by equating coefficients. Enough to have attempted just one of λ or μ in terms of β and γ
		So we have $\overrightarrow{OX} = \mathbf{c} + \frac{\beta}{\beta+\gamma}(\mathbf{b} - \mathbf{c}) = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$.	**A1**	For correct $\overrightarrow{OX}$ aef
		From the first of the three equations earlier $\delta = \frac{\lambda\alpha}{\lambda-1}$	**M1**	For valid method to attempt to find $\overrightarrow{OL}$
		$= \frac{\frac{\alpha}{\beta+\gamma}}{\frac{1}{\beta+\gamma}-1} = \frac{\alpha}{1-\beta-\gamma} = \frac{\alpha}{k+\alpha}$ as required.	**A1**	Fully correct **(AG)**

ALT		**All 6 marks in (i)** $\overrightarrow{OX} = [\overrightarrow{OC} + \mu\overrightarrow{CB}] = \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})$ for some μ.	**M1**	Equation for $\overrightarrow{OX}$ any of $\mathbf{b} + \mu(\mathbf{c} - \mathbf{b})$, $(1-\mu)\mathbf{b} + \mu\mathbf{c}$, $(1-\mu)\mathbf{c} + \mu\mathbf{b}$
		Let $\overrightarrow{OL} = \delta\boldsymbol{a}$ for some δ. $\overrightarrow{LX} = -\delta\mathbf{a} + \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})$ $\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OL} + \lambda\overrightarrow{LX} = \delta\boldsymbol{a} + \lambda[-\delta\mathbf{a} + \mathbf{c} + \mu(\mathbf{b} - \mathbf{c})]$	**M1**	Equation for $\overrightarrow{OP}$ Correct equation for $\overrightarrow{OP}$ implies previous method mark as well
		Equating coefficients gives: $\alpha = \delta(1-\lambda)$ $\beta = \lambda\mu$ $\gamma = \lambda(1-\mu)$ Adding the last two gives $\lambda = \beta + \gamma$ and so $\mu = \frac{\beta}{\lambda} = \frac{\beta}{\beta+\gamma}$	**M1**	Attempt to solve simultaneous equations found by equating coefficients. Enough to have attempted just one of λ or μ in terms of β and γ

		So we have $\overrightarrow{OX} = \mathbf{c} + \frac{\beta}{\beta+\gamma}(\mathbf{b} - \mathbf{c}) = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$.	**A1**	For correct $\overrightarrow{OX}$ aef
		From the first of the three equations earlier $\delta = \frac{\alpha}{1-\lambda}$	**M1**	For valid method to attempt to find $\overrightarrow{OL}$
		$= \frac{\alpha}{1-(\beta+\gamma)} = \frac{\alpha}{1-\beta-\gamma} = \frac{\alpha}{k+\alpha}$ as required.	**A1**	Fully correct **(AG)**
			[6]	

	(ii) *	LMN is horizontal if and only if the vectors $\overrightarrow{LM}$ and $\overrightarrow{LN}$ are both linear combinations of the horizontal vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$	**E1**	Characterisation of "*LMN* is horizontal" aef
	**	Suppose $\overrightarrow{LM}$ is a linear combination of the horizontal vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$. Then $\overrightarrow{LM} = r(\mathbf{b}-\mathbf{a}\,) + s(\mathbf{c}-\mathbf{a})$ for some real numbers r, s. Also $\overrightarrow{LM} = \overrightarrow{LO} + \overrightarrow{OM} = -\frac{\alpha}{k+\alpha}\mathbf{a} + \frac{\beta}{k+\beta}\mathbf{b}$.	**M1**	Setting up equations for either $\overrightarrow{LM}$ or $\overrightarrow{LN}$
	***	Equating coefficients gives $s = 0$ and then $r = \frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$ so that $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$.	**M1**	Equating coefficients
ALT	*	**First 3 marks** *L* lies on *OA*, *M* on *OB* and *N* on *OC* so *LMN* will be parallel to *ABC* if and only if the ratios *OL* : *OA*; *OM* : *OB* and *ON* :*OC* are the same	**E1**	Must mention **all three** ratios for this mark. Withhold if explanation not convincing
	**	Therefore we need $\frac{\alpha}{k+\alpha}$: 1 and $\frac{\beta}{k+\beta}$:1 to be the same	**M1**	Only 2 ratios needed here
	***	Therefore $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$	**M1**	

ALT	*	**First 3 marks** L lies on OA, M on OB and N on OC so LMN will be parallel to ABC if and only if the line LM is parallel to AB and the line LN is parallel to AC	**E1**	Need **two** different lines considered here for E1. Withhold if explanation not convincing
	**	$$\overrightarrow{LM} = \overrightarrow{LO} + \overrightarrow{OM} = \frac{\beta}{k+\beta}\mathbf{b} - \frac{\alpha}{k+\alpha}\mathbf{a} = \lambda\overrightarrow{AB}$$ And $\lambda\overrightarrow{AB} = \lambda(\mathbf{b} - \mathbf{a})$	**M1**	
	***	Therefore $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$	**M1**	
	*	$$\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$$ $$\alpha(k+\beta) = \beta(k+\alpha)$$ $$k\alpha = k\beta$$ And since $k \neq 0$ we have $\alpha = \beta$.	**A1FT**	Deducing $\alpha = \beta$ from $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta}$. Must have the previous M mark, but **M0 M1 A1FT** is OK Need to see $k \neq 0$ or other convincing argument for this mark

	**	A symmetrical argument with $\overrightarrow{LN} = \overrightarrow{LO} + \overrightarrow{ON} = -\frac{\alpha}{k+\alpha}\mathbf{a} + \frac{\gamma}{k+\gamma}\mathbf{c}$ shows that $\alpha = \gamma$ so that $\alpha = \beta = \gamma$ [and vector equation of the line *OP* can be written as $\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$]	**A1FT**	BOD "By symmetry we also have $\alpha = \gamma$" Allow if **A0** for previous mark through lack of $k \neq 0$ Deducing by symmetry $\alpha = \beta = \gamma$ enough for **A1** here **M0 M1 A0 A1 FT** is fine
ALT	*	**Previous two marks** $\mathrm{f}(x) = \frac{x}{k+x}$ has derivative $\mathrm{f}'(x) = \frac{k}{(k+x)^2}$ so as k > 0 this is a strictly increasing function	**A1FT**	For showing that this function is strictly increasing
	**	Therefore if $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta} = \frac{\gamma}{k+\gamma}$ then we must have $\alpha = \beta = \gamma$	**A1FT**	Allow even if did not earn previous A mark though lack of k > 0

		The equation of the plane ABC is $s = \mathbf{a} + x(\mathbf{b} - \mathbf{a}) + y(\mathbf{c} - \mathbf{a}) = (1 - x - y)\mathbf{a} + x\mathbf{b} + y\mathbf{c}$. The point of intersection of OP with the plane ABC is when all these coefficients are equal, i.e. $x = y = 1 - x - y$ (which has solution $x = y = \frac{1}{3}$) therefore has position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$	**B1**	Solving for intersection of OP and ABC to show given result for $\overrightarrow{OG}$ Award for convincing argument that OP in the form as $\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$ intersects ABC at G BOD mention of centroid
		Conversely if OP intersects with the plane ABC at the point G with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ then we must have that $\alpha = \beta = \gamma$.	**M1**	Deducing $\alpha = \beta = \gamma$ in converse argument
	*	This means that $\overrightarrow{LM} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\beta}{k+\beta}\boldsymbol{b} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\alpha}{k+\alpha}\boldsymbol{b} = \frac{\alpha}{k+\alpha}(\boldsymbol{b} - \boldsymbol{a})$.	**M1**	For $\overrightarrow{LM}$ as a multiple of $\mathbf{b} - \mathbf{a}$.
	**	Similarly, $\overrightarrow{LN}$ is a multiple of $\boldsymbol{c} - \boldsymbol{a}$.	**M1**	For $\overrightarrow{LN}$ is a multiple of $\mathbf{c} - \mathbf{a}$.
	***	and so LMN is horizontal	**A1**	Conclusion

ALT	*	**Previous 3 marks** This means that $\overrightarrow{OL} = \frac{\alpha}{k+\alpha}\boldsymbol{a}$, $\overrightarrow{OM} = \frac{\alpha}{k+\alpha}\mathbf{b}$ and $\overrightarrow{ON} = \frac{\alpha}{k+\alpha}\mathbf{c}$	**M1**	Showing that the coefficients of the three vectors are the same
	**	So the ratio of the lengths *OL*:*OA*, *OL*:*OA* and *OL*:*OA* are all equal	**M1**	Attempt to show that proportions of distance travelled along OA etc are the same
	***	Therefore *LMN* is parallel to *ABC* and so *LMN* is horizontal	**A1**	Withhold if argument not convincing. Must be clear that *L* lies on line *OA*

		We have $\overrightarrow{OX} = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$, when $\beta = \gamma$, this gives $\overrightarrow{OX} = \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{b}$,.	**M1**	Calculating $\overrightarrow{OX}$
		so X is the midpoint of BC.	**A1**	Allow M1 A1 for just stating X is the midpoint of BC (or equivalent for Y/Z)
		Similarly, Y is the midpoint of AC and Z is the midpoint of AB.	**B1**	
			[13]	
	(iii)	It's on the interior of the tetrahedron $OABC$.	**B1**	
			[1]	
		Additional Alternative for (ii)		
	(ii)	Plane ABC has normal $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ and so plane LMN will have to have a normal parallel to this	**E1**	Characterisation of "LMN is horizontal" aef
		LMN normal is given by $\overrightarrow{LM} \times \overrightarrow{LN} = \left(\frac{\beta}{k+\beta}\mathbf{b} - \frac{\alpha}{k+\alpha}\mathbf{a}\right) \times \left(\frac{\gamma}{k+\gamma}\mathbf{c} - \frac{\alpha}{k+\alpha}\mathbf{a}\right)$	**M1**	Attempt to find normal for LMN

		Expanding gives the normal as: $$n_1 = \mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{a}$$ $$n_2 = \frac{\beta}{k+\beta}\frac{\gamma}{k+\gamma}\,\mathbf{b}\times\mathbf{c} - \frac{\alpha}{k+\alpha}\frac{\gamma}{k+\gamma}\mathbf{a}\times\mathbf{c} - \frac{\alpha}{k+\alpha}\frac{\beta}{k+\beta}\mathbf{b}\times\mathbf{a}$$ Since normal are parallel we have $\frac{\beta}{k+\beta}\frac{\gamma}{k+\gamma} = \frac{\alpha}{k+\alpha}\frac{\gamma}{k+\gamma}$	**M1**	Equating coefficients
		$$\frac{\alpha\gamma}{(k+\alpha)(k+\gamma)} = \frac{\beta\gamma}{(k+\beta)(k+\gamma)}$$ $$\alpha(k+\beta) = \beta(k+\alpha)$$ $$k\alpha = k\beta$$ And since $k \neq 0$ we have $\alpha = \beta$.	**A1**	Deducing $\alpha = \beta$ Need to see $k \neq 0$ or other convincing argument for this mark
		A symmetrical argument with $\frac{\beta}{k+\beta}\frac{\gamma}{k+\gamma} = \frac{\alpha}{k+\alpha}\frac{\beta}{k+\beta}$ shows that $\alpha = \gamma$ so that $\alpha = \beta = \gamma$ [and vector equation of the line OP can be written as $\lambda(\mathbf{a}+\mathbf{b}+\mathbf{c})$]	**A1**	BOD "By symmetry we also have $\alpha = \gamma$" Independent to previous mark Deducing by symmetry $\alpha = \beta = \gamma$ enough for **A1** here

	*	The equation of the plane ABC is $s = \mathbf{a} + x(\mathbf{b} - \mathbf{a}) + y(\mathbf{c} - \mathbf{a}) = (1 - x - y)\mathbf{a} + x\mathbf{b} + y\mathbf{c}$. The point of intersection of OP with the plane ABC is when all these coefficients are equal, i.e. $x = y = 1 - x - y$ (which has solution $x = y = \frac{1}{3}$) therefore has position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$	**B1**	Solving for intersection of OP and ABC to show given result for $\overrightarrow{OG}$ Award for convincing argument that OP in the form as $\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})$ intersects ABC at G
ALT for prev B1	*	Equation of plane is given by **r.n** = **a.n** $[\lambda(\mathbf{a} + \mathbf{b} + \mathbf{c})] \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a})$ $\lambda[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ $3\lambda[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ $\lambda = \frac{1}{3}$ Therefore the point of intersection of OP with the plane ABC has position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$	**B1**	
		Conversely if OP intersects with the plane ABC at the point G with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ then we must have that $\alpha = \beta = \gamma$.	**M1**	Deducing $\alpha = \beta = \gamma$ in converse argument
		This means that $\overrightarrow{LM} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\beta}{k+\beta}\boldsymbol{b} = -\frac{\alpha}{k+\alpha}\boldsymbol{a} + \frac{\alpha}{k+\alpha}\boldsymbol{b} = \frac{\alpha}{k+\alpha}(\boldsymbol{b} - \boldsymbol{a})$.	**M1**	For $\overrightarrow{LM}$ as a multiple of **b** – **a**.

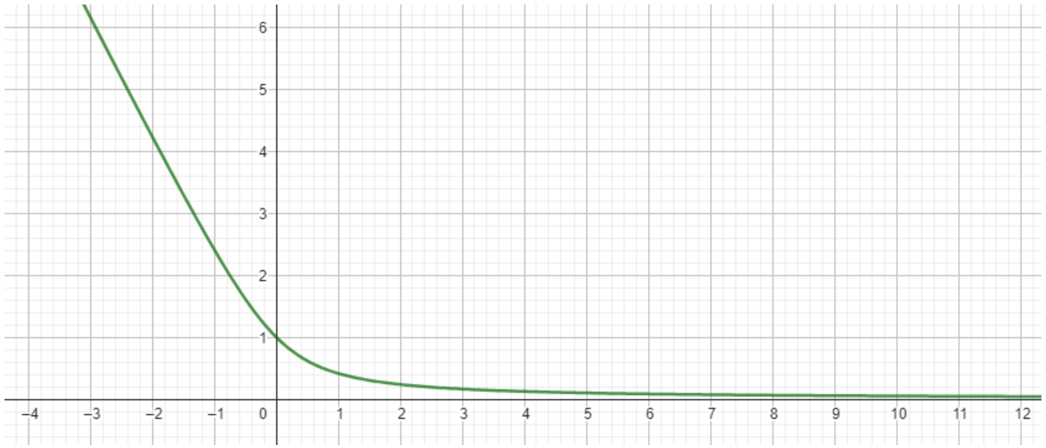
		Similarly, $\overrightarrow{LN}$ is a multiple of $\boldsymbol{c} - \boldsymbol{a}$.	**M1**	For $\overrightarrow{LN}$ is a multiple of $\mathbf{c} - \mathbf{a}$.
		and so *LMN* is horizontal as the normal to the plane *LMN* is parallel to the normal to plane *ABC*	**A1**	Conclusion
		We have $\overrightarrow{OX} = \frac{\gamma}{\beta+\gamma}\mathbf{c} + \frac{\beta}{\beta+\gamma}\mathbf{b}$, when $\beta = \gamma$, this gives $\overrightarrow{OX} = \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{b}$,.	**M1**	Calculating $\overrightarrow{OX}$
		so X is the midpoint of *BC.*	**A1**	
		Similarly, *Y* is the midpoint of *AC* and *Z* is the midpoint of *AB*.	**B1**	
			[13]	

Question		Answer	Marks	Guidance
6	(i)	If a, b, c are real and non-zero then we would have $a^2 + b^2 + c^2 > 0$ since each of $a^2, b^2, c^2 > 0$.	B1	
		We have that $2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 0$.	M1	For identity
		Therefore $ab + bc + ca = 0$.	A1	
	*	A polynomial with roots a, b, c is $z^3 - (a + b + c)z^2 + (ab + bc + ca)z - abc = 0$. In this case, this is $z^3 - abc = 0$. Each of a, b, c is a root of this so $a^3 = abc, b^3 = abc, c^3 = abc$.	M1	Common values of cubes
	**	Therefore $\lvert a\rvert = \lvert b\rvert = \lvert c\rvert = \sqrt[3]{\lvert abc\rvert}$.	A1	
			[5]	

Question	Answer	Marks	Guidance
	Alternative for last two marks of (i)		
*	$a^2 = a(-b-c) = bc, b^2 = ca, c^2 = ab$	**M1**	Or equivalent expression. May appeal to symmetry. One expression of this form only is insufficient for M1,
**	So e.g. $$\frac{\lvert a\rvert^2}{\lvert b\rvert} = \frac{\lvert b\rvert^2}{\lvert a\rvert} \Longrightarrow \lvert a\rvert = \lvert b\rvert = \lvert c\rvert$$	**A1**	Convincingly arriving at the given statement by rearranging. Must conclude about all three magnitudes being equal, not just 2 of 3.
(ii)	Suppose $a+b+c=0$ and $a^3+b^3+c^3=0$.		
	Since $a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2)-ab^2-ac^2-ba^2-bc^2-ca^2-cb^2$	**M1**	For identity
	$0 = ab^2+ac^2+ba^2+bc^2+ca^2+cb^2$	**M1**	
	$= ab(b+a)+bc(c+b)+ac(a+c) = -abc-abc-abc = -3abc.$	**M1**	Use of $a+b+c=0$
	This means that $abc=0$,	**A1**	

Question		Answer	Marks	Guidance
		which cannot be the case as all of a, b, c are non-zero.	**A1** **[5]**	
		Alternative for (ii)		
		$c = -(a+b)$, so $(a+b)^3 = a^3 + b^3$	**M1**	Using $a+b+c=0$ to simplify this
		$a^3 + b^3 + 3ab^2 + 3ab^2 = a^3 + b^3$ $ab(a+b) = 0$	**M1**	Expanding, $(a+b)^3$ expanded correctly (condone sign error $-(a+b)^3 = a^3 + b^3$) Factorised, = 0
		so $abc = 0$	**A1**	Correctly justified
		So one of $a, b, c = 0$, contradiction	**A1**	
	(iii)	$a + b = -(c+d)$. Therefore $(a+b)^3 = -(c+d)^3$ giving $a^3 + b^3 + 3ab(a+b) = -\left(c^3 + d^3 + 3cd(c+d)\right)$	**M1** **M1**	Use of this

Question		Answer	Marks	Guidance
		Using $a^3 + b^3 = -c^3 - d^3$ and $a + b = -(c + d)$ gives $(a + b)(ab - cd) = 0$.	A1	For $(a + b)(ab - cd) = 0$.
		If $a = -b$ then a and b have the same modulus and we are done.	M1	For this case
		Else $ab = cd$. The roots of the equation $x^2 - (a + b) + ac = 0$ are a and b. But this equation is the same as $x^2 + (c + d)x + cd = 0$ which has roots $-c$ and $-d$	M1	
		Therefore a is either $-c$ or $-d$ and we are done.	A1	
			[6]	
	(iv)	We need $a + b = -2$ and $a^3 + b^3 = -20$.	M1	
		Since $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ we must also have that $a^2 - ab + b^2 = 10$.	M1	
		Substituting $b = -2 - a$ into this gives $a^2 + a(a + 2) + (a + 2)^2 = 10$ or $3a^2 + 6a - 6 = 0$	M1	
		Solving $a^2 + 2a - 2 = 0$ give $a = -1 \pm \sqrt{3}$ which leads to $b = -1 \mp \sqrt{3}$	A1	Correct pair
			[4]	

Question			Answer	Marks	Guidance
7	(i)	*	$\frac{\sqrt{x^2+1}}{\lvert x\rvert}\sqrt{\frac{x^2+1}{x^2}} = \sqrt{1+\frac{1}{x^2}}$.	M1	Rearrange to expression with a 'small' term that can be expanded.
		**	Using the binomial expansion, where $\lvert x\rvert$ is large:		
			$\frac{\sqrt{x^2+1}}{\lvert x\rvert} = \sqrt{\frac{x^2+1}{x^2}} = \sqrt{1+\frac{1}{x^2}} \approx 1+\frac{1}{2}\times\frac{1}{x^2} = 1+\frac{1}{2\lvert x\rvert^2}$.		
			Multiplying both sides by $\lvert x\rvert$ gives the required result.	A1	
				G1	For generally correct (allow 'inflection points' but must have no minima/maxima) shape $x \geq -C$, $C \geq 0$, correct y-intercept, and $y \to 0$ as $x \to +\infty$
				G1	Looks like a straight line as $x \to -\infty$. (No need to explicitly say it looks like $y \approx -2x$, but if the asymptotic behaviour is

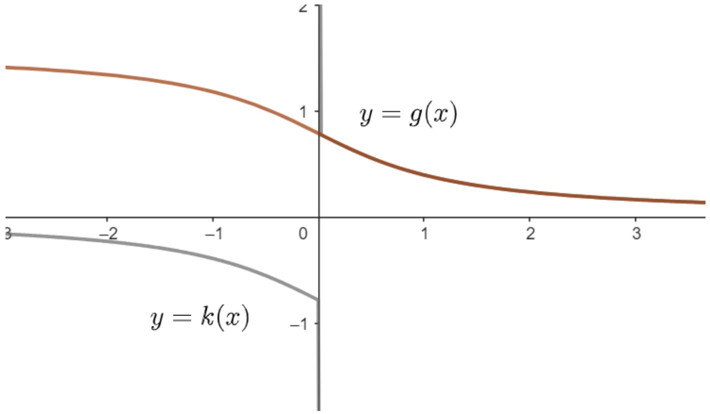
Question			Answer	Marks	Guidance
				[4]	explicitly incorrect, or written down wrong in working, say $y \approx -x$, then G0.)
			Alternative for first two marks		
		*	Note that $\sqrt{x^2+1} - \lvert x\rvert = \frac{1}{\sqrt{x^2+1}+\lvert x\rvert}$. Thus, $\sqrt{x^2+1} = \lvert x\rvert + \frac{1}{\sqrt{x^2+1}+\lvert x\rvert}$.	**M1**	
		**	For large $\lvert x\rvert$, we see $\sqrt{x^2+1} \approx \sqrt{x^2} = \lvert x\rvert$, and so $\sqrt{x^2+1} \approx \lvert x\rvert + \frac{1}{2\lvert x\rvert}$.	**A1**	
	(ii)	**(a)**	We have $f(x)f(-x) = \left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right) = 1$. For $0 < A, B < \frac{\pi}{2}$, we have $\tan A \tan B = 1 \Rightarrow \cos A \cos B - \sin A \sin B = 0 \Rightarrow \cos(\mathrm{A}+\mathrm{B}) = 0 \Rightarrow A + B = \frac{\pi}{2}$.	**M1**	Simplifying $f(x)f(-x)$.
			Since $g(x) \in (0, \frac{\pi}{2})$ for all x, $f(x)f(-x) = \tan g(x) \tan g(-x) = 1$ implies $g(x) + g(-x) = \frac{\pi}{2}$.	**A1** **[2]**	Correct use of suitable trig identity.

Question	Answer	Marks	Guidance
	Alternative(ii)		
(ii) (a)	Using the tangent addition formula, we have $\tan\big(g(x)+g(-x)\big)=\frac{\tan(g(x))+\tan(g(-x))}{1-\tan(g(x))\tan(g(-x))}$. The numerator is equal to $f(x)+f(-x)=2\,(x^2+1)^{1/2}>0.$ But the denominator is equal to $1-\left((x^2+1)^{\frac{1}{2}}-x\right)\left((x^2+1)^{\frac{1}{2}}+x\right)=0.$ Therefore, $\tan\big(g(x)+g(-x)\big)=\frac{a}{b}$ with $a>0, b=0$.	**M1**	Any attempt at showing the denominator is 0 i.e. evaluating $f(x)f(-x)$. Allow the use of the 'arctan addition formula' for this M1.
	We also have $f(x)>0 \Rightarrow 0<\ g(x)<\frac{\pi}{2}$, and similarly $0<g(-x)<\frac{\pi}{2}$. Hence, we have $0<\ g(x)+g(-x)<\pi$, and therefore $g(x)+g(-x)=\frac{\pi}{2}$.	**A1**	Correct expression for $\tan\,(g(x)+g(-x))$ and correct use of the range of $g(x)+g(-x)$. A0 for no justification. Allow "$\tan\big(g(x)+g(-x)\big)=\infty$" $\Rightarrow g(x)+g(-x)=(2n+1)\frac{\pi}{2}$. If using the arctan addition formula, allow 'the argument of arctan is infinite, so it must be

Question		Answer	Marks	Guidance
				$(2n+1)\frac{\pi}{2}$ for some n' or equivalent.
		Second alternative(ii)		
(ii)	(a)	$f(x)f(-x) = (\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x) = 1$ For $u > 0$, we have $\tan^{-1} u + \tan^{-1}\frac{1}{u} = \frac{\pi}{2}$,	M1	Any attempt to show $f(-x) = f(x)^{-1}$ May be stated without proof
		And so, since $f(x) > 0$, $g(x) + g(-x) = \frac{\pi}{2}$	A1	Must state that $f(x) > 0$ if using the identity. If identity is "proven" by drawing a suitable right-angled triangle and summing the angles, $f(x) > 0$ need not be explicitly stated and A1 can be given
	(b)	$k(x) + k(-x) = \frac{1}{2}\tan^{-1}\frac{1}{x} + \frac{1}{2}\tan^{-1}\left(-\frac{1}{x}\right) = \frac{1}{2}\tan^{-1}\frac{1}{x} - \frac{1}{2}\tan^{-1}\left(\frac{1}{x}\right) = 0$	E1 [1]	Must see at least this line of working. If differentiating or using trig identities, must be fully correct.

Question	Answer	Marks	Guidance
(c)	For $x > 0$, $\tan 2k(x) = x^{-1}$.	M1	
	Using the tan double angle formula, $\frac{2\tan k(x)}{1-(\tan k(x))^2} = \tan 2k(x) = x^{-1}$, we then have a quadratic for $k(x)$: $(\tan k(x))^2 + 2x\tan k(x) - 1 = 0$, so that $\tan k(x) = -x \pm \sqrt{1+x^2}$.	M1	Use of double angle formula.
	For $x > 0$, $\tan k(x) > 0$, so $\tan k(x) = \sqrt{1+x^2} - x = f(x) = \tan(\tan^{-1} f(x)) = \tan g(x)$.	M1	Attempt to solve for general solution to quadratic. (Completing the square is enough.)
		A1	Correct justification.
		[4]	
	Alternative for (ii)(c)		
(ii) (c)	$\tan k(x) = \frac{\sin\frac{1}{2}\tan^{-1}\frac{1}{x}}{\cos\frac{1}{2}\tan^{-1}\frac{1}{x}}$ $= \frac{\sin\tan^{-1}\frac{1}{x}}{\cos\tan^{-1}\frac{1}{x}+1}$	M1	Use of double (or half) angle formulae

Question	Answer	Marks	Guidance
	$= \dfrac{\dfrac{1}{\sqrt{x^2+1}}}{\dfrac{x}{\sqrt{x^2+1}}+1}$ $= \dfrac{1}{\sqrt{x^2+1}+x}$	M1	Attempt to evaluate $\sin\tan^{-1}\frac{1}{x}$ or $\cos\tan^{-1}\frac{1}{x}$ e.g. using a right-angled triangle
	$= \sqrt{x^2+1} - x$	M1	Rationalisation of fraction.
	$= \tan g(x)$	A1	Fully correct argument; withhold if half-angle formulae are used without justification of signs, i.e. $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

Question	Answer	Marks	Guidance
(d)		G1	Generally correct shape (as in guidance for (i)) for $g(x)$, correct asymptotes.
		G1	$k(x) = g(x)$ for $x > 0$.
		G1	Correct k(x) when $x < 0$, must be $g(x)$ translated by $\frac{\pi}{2}$, i.e. $g(x)$ for $x < 0$ must also be correct. (Using $g(x) = k(x) + \frac{\pi}{2}$ for $x < 0$.) (Special case: if $k(x)$ correct but sketch gets 0 marks as above for incorrect $g(x)$, give G1 G0 G0.)
		[3]	

Question			Answer	Marks	Guidance
		(e)	$\int_0^1 k(x)dx = \frac{1}{2}\int_0^1 \tan^{-1}\frac{1}{x}dx = \frac{1}{2}\left(\left[x\tan^{-1}\frac{1}{x}\right]_0^1 + \int_0^1 \frac{x}{x^2+1}dx\right)$	M1	Use of integration by parts
			$= \frac{1}{2}\left(\frac{\pi}{4} + \left[\frac{1}{2}\ln(x^2+1)\right]_0^1\right)$	M1	Integrating using log.
			$= \frac{\pi + 2\ln 2}{8}$	A1	Fully correct
			We have that $k(x) = g(x)$ for $x > 0$. From (ii)(a), we have, for $x > 0$, $g(-x) = \frac{\pi}{2} - k(x)$.	M1	Use of (ii)(a) with $k(x) = g(x)$ for $x > 0$, or use of sketch in (d).
			Integrating, we get $\int_{-1}^0 g(x)dx = \int_0^1 g(-x)\,dx = \frac{\pi}{2} - \int_0^1 k(x)\,dx$	A1	Correct integral expression, dependent only on previous M.
			$= \frac{\pi}{2} - \frac{\pi + 2\ln 2}{8} = \frac{3\pi - 2\ln 2}{8}$	A1	Allow FT for $\int_0^1 k(x)\,dx$.
				[6]	Since the question says 'write down the value', $\int_{-1}^0 g(x)dx = \frac{\pi}{2} - I$

Question			Answer	Marks	Guidance
					where I is the computed (possibly incorrect) value for $\int_0^1 k(x)\,dx$, gets M1 BOD A1 A1 (FT).

Question		Answer	Marks	Guidance
8	(i)	$\left(z-\frac{1}{z}\right)\left(z^m+\frac{1}{z^m}\right)+\left(z^{m-1}-\frac{1}{z^{m-1}}\right)=z^{m+1}+\frac{1}{z^{m-1}}-z^{m-1}-\frac{1}{z^{m+1}}+z^{m-1}-\frac{1}{z^{m-1}}$	B1	Correct either side and correct six terms.
		$=z^{m+1}-\frac{1}{z^{m+1}}$		
		When $n=1$ the LHS is $z^2-\frac{1}{z^2}$. The RHS is $\left(z-\frac{1}{z}\right)\left(z+\frac{1}{z}\right)$ and these are equal.	M1	Base case
		We then have		
		$z^{2k+2}-\frac{1}{z^{2k+2}}=\left(z-\frac{1}{z}\right)\left(z^{2k+1}+\frac{1}{z^{2k+1}}\right)+\left(z^{2k}-\frac{1}{z^{2k}}\right)$		
		By induction this equals		
		$\left(z-\frac{1}{z}\right)\left(z^{2k+1}+\frac{1}{z^{2k+1}}\right)+\left(z-\frac{1}{z}\right)\sum_{r=1}^{k}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)$		
		$=\left(z-\frac{1}{z}\right)\left[z^{2k+1}+\frac{1}{z^{2k+1}}++\sum_{r=1}^{k}\left(z^{2r-1}+\frac{1}{z^{2r-1}}\right)\right]$		

Question	Answer	Marks	Guidance
	$= \left(z - \frac{1}{z}\right)\left[\sum_{r=1}^{k+1}\left(z^{2r-1} + \frac{1}{z^{2r-1}}\right)\right]$	**M1**	Induction hypothesis (seen or implied)
	So the result is true by induction.	**A1**	Fully correct working
	For other result, first, it can be seen that $\left(z + \frac{1}{z}\right)\left(z^m - \frac{1}{z^m}\right) + \left(\frac{1}{z^{m-1}} - z^{m-1}\right) = z^{m+1} - \frac{1}{z^{m+1}}$	**M1**	
	Base case is easy to check. Then we have $z^{2k+2} - \frac{1}{z^{2k+2}} = \left(z + \frac{1}{z}\right)\left(z^{2k+1} - \frac{1}{z^{2k+1}}\right) + \left(z^{2k} - \frac{1}{z^{2k}}\right)$ By induction this is		

Question	Answer	Marks	Guidance
	$\left(z+\frac{1}{z}\right)\left(z^{2k+1}-\frac{1}{z^{2k+1}}\right)+\left(z+\frac{1}{z}\right)\sum_{r=1}^{k}(-1)^{r+k}\left(z^{2r-1}-\frac{1}{z^{2r-1}}\right)$ $=\left(z+\frac{1}{z}\right)\left[\sum_{r=1}^{k+1}(-1)^{r+k+1}\left(z^{2r-1}-\frac{1}{z^{2r-1}}\right)\right]$	A1 **[6]**	Check that the alternating sign is handled correctly
(ii) **(a)**	Taking $z=e^{i\theta}$ in the first identity in (i) gives $2i\sin 2n\theta = 2i\sin\theta\sum_{r=1}^{n}2\cos(2r-1)\theta$ $\sin(2n\theta)=2\sin(\theta)\sum_{r=1}^{n}\cos\big((2r-1)\theta\big)$	M1 A1 **[2]**	
(b)	Taking $n=2$ and $\theta=\frac{\pi}{5}$ in this gives $\sin\frac{4\pi}{5}=2\sin\frac{\pi}{5}(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5})$. Since $\sin\frac{4\pi}{5}=\sin\frac{\pi}{5}$ this gives $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$	M1 M1	 Implied by disappearance of factor involving sin from each

Question	Answer	Marks	Guidance
			side as the only change.
	and the result follows because $\cos\frac{2\pi}{5} = -\cos\frac{3\pi}{5}$	**A1**	Need to state (AG so strict here). Need to give cos justification.
		[3]	
(c)	Taking $n = 7$ and $\theta = \frac{\pi}{15}$ gives	**M1**	Using $\theta = \frac{2\pi}{15}$ can also work – award M1 here but check other values as subsequent working is more involved
	$\sin\frac{14\pi}{15} = 2\sin\frac{\pi}{15}(\cos\frac{\pi}{15} + \cos\frac{3\pi}{15} + \cos\frac{5\pi}{15} + \cos\frac{7\pi}{15} + \cos\frac{9\pi}{15} + \cos\frac{11\pi}{15} + \cos\frac{13\pi}{15})$		
	So, since $\sin\frac{14\pi}{15} = \sin\frac{\pi}{15}$,	**M1**	Implied by disappearance of factor involving sin from each

Question	Answer	Marks	Guidance
	$\cos\frac{\pi}{15}+\cos\frac{3\pi}{15}+\cos\frac{5\pi}{15}+\cos\frac{7\pi}{15}+\cos\frac{9\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=\frac{1}{2}$		side as the only change.
	Since $\cos\frac{3\pi}{15}=\cos\frac{6\pi}{15}+\frac{1}{2}$ from previous answer and $\cos\frac{5\pi}{15}=\frac{1}{2}$,	**M1**	Needs to be clear.
	$\cos\frac{\pi}{15}+\frac{1}{2}+\cos\frac{6\pi}{15}+\frac{1}{2}+\cos\frac{7\pi}{15}+\cos\frac{9\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=\frac{1}{2}$		
	Now $\cos\frac{6\pi}{15}+\cos\frac{9\pi}{15}=0$ so that $\cos\frac{\pi}{15}+\cos\frac{7\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=-\frac{1}{2}$	**M1**	Needs to be clear.
	So $-\cos\frac{16\pi}{15}-\cos\frac{8\pi}{15}-\cos\frac{4\pi}{15}-\cos\frac{2\pi}{15}=-\frac{1}{2}$ giving the result	**A1**	Need to explain use of identities.
		[5]	

Question			Answer	Marks	Guidance
	(iii)		Taking $z = e^{i\theta}$ in the other identity: $$z^{2n} - \frac{1}{z^{2n}} = \left(z + \frac{1}{z}\right)\sum_{r=1}^{n}(-1)^{r+n}\left(z^{2r-1} - \frac{1}{z^{2r-1}}\right)$$ Gives $2i\sin 2n\theta \;=\; 2\,\cos\theta\,(2i\sin(2n-1)\,\theta - 2i\sin(2n-3)\,\theta + \cdots + (-1)^{n+1}\,2i\sin\theta\,)$	**M1**	Note FT allowed for three M marks but their incorrect identity must be sufficiently similar.
			So $\sin 2n\theta \;=\; 2\,\cos\theta\,(\sin(2n-1)\,\theta - \sin(2n-3)\,\theta + \cdots + (-1)^{n+1}\sin\theta\,)$	**M1**	
			Now taking $n = 3$ and $\theta = \frac{\pi}{14}$ gives $\sin\frac{6\pi}{14} = 2\cos\frac{\pi}{14}(\sin\frac{\pi}{14} - \sin\frac{3\pi}{14} + \sin\frac{5\pi}{14})$	**M1**	Taking $n = 3$ and $\theta = \frac{\pi}{7}$ can also work.
			Since $\sin\frac{6\pi}{14} = \cos\frac{\pi}{14}$ this gives the result.	**A1**	
				[4]	

Question		Answer	Marks	Guidance
9	(i)	y-coordinate of centre of mass is given by $\frac{A}{B}$ where $A = \pi \int_0^r x^2 y dy$ and $B = \pi \int_0^r x^2 dy$	M1	Correct A and B, including ratio – must be x^2dy not y^2dx
		$r^{n-1}y = r^n - x^n \Leftrightarrow x^2 = r^{\frac{2n-2}{n}}(r-y)^{\frac{2}{n}}$ So $\frac{A}{\pi r^{\frac{2n-2}{n}}} = \int_0^r (r-y)^{\frac{2}{n}} y dy = \left[-\frac{ny}{2+n}(r-y)^{\frac{2+n}{n}}\right]_0^r + \frac{n}{2+n}\int_0^r (r-y)^{\frac{2+n}{n}} dy$ $= 0 + \frac{n}{2+n} \times \left[-\frac{n}{2+2n}(r-y)^{\frac{2+2n}{n}}\right]_0^r = \frac{n^2 r^{\frac{2+2n}{n}}}{(2+n)(2+2n)}$	M1	Correct integral and by parts attempted
		Giving $A = \frac{\pi r^{\frac{2n-2}{n}} n^2 r^{\frac{2+2n}{n}}}{(2+n)(2+2n)} = \frac{\pi r^4 n^2}{(2+n)(2+2n)}$	A1	
		Then $\frac{B}{\pi r^{\frac{2n-2}{n}}} = \int_0^r (r-y)^{\frac{2}{n}} dy = \left[-\frac{n}{2+n}(r-y)^{\frac{2+n}{n}}\right]_0^r = \frac{nr^{\frac{2+n}{n}}}{2+n}$	M1	Integral attempted using a method that works

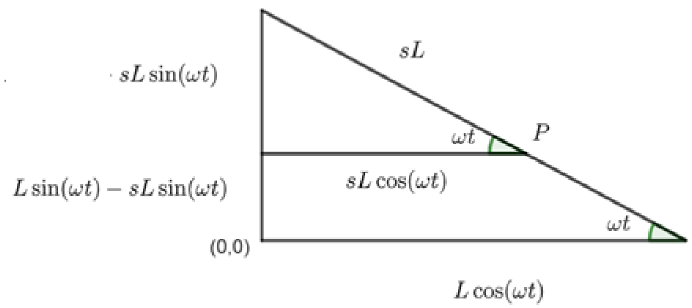
Question	Answer	Marks	Guidance
	So that $B = \frac{nr^{\frac{2+n}{n}}\pi r^{\frac{2n-2}{n}}}{2+n} = \frac{nr^3\pi}{2+n}$	**A1**	
	Then $\frac{A}{B} = \frac{nr}{2(n+1)}$	**A1** **[6]**	Allow FT in the special case where they cancel incorrect minus signs in the ratio, provided they have been penalised already.
	Alternative for (i), using shell formulae.		
(i)	y-coordinate of centre of mass is given by $\frac{A}{B}$ where $A = 2\pi\int_0^r yx \cdot \frac{y}{2}\,dx$ and $B = 2\pi\int_0^r yx\,dx$ (shell formula)	**M1**	Correct A and B, including ratio
	$\frac{A}{\pi r^{2n-2}} = \int_0^r r^{2n}x - 2r^n x^{n+1} + x^{2n+1}\,dy = r^{2n+2}\left[\frac{1}{2} - \frac{2}{n+2} + \frac{1}{2n+2}\right]$	**M1**	Square and integrate term by term

Question	Answer	Marks	Guidance
	$A = \frac{\pi r^4 n^2}{(n+2)(2n+2)}$	A1	
	$B = \pi \int_0^r rx - \frac{x^{n+1}}{r^{n-1}} dx = \pi r^3 \left[\frac{1}{2} - \frac{1}{n+2}\right]$	M1	Substitute in and integrate term by term
	$B = \frac{\pi n r^3}{n+2}$	A1	
	Then $\frac{A}{B} = \frac{nr}{2(n+1)}$	A1	
(ii)	$y = r - \frac{x^n}{r^{n-1}}$ So that $\frac{dy}{dx} = -\frac{nx^{n-1}}{r^{n-1}}$, at $(rp, r(1-p^n)$ we have $\frac{dy}{dx} = -\frac{n(rp)^{n-1}}{r^{n-1}} = -np^{n-1}$	M1	Calculate and try to plug in $x = rp$
	So the equation of the normal is $y - r(1-p^n) = \frac{x-rp}{np^{n-1}}$ Setting $x = 0$ gives $y = \frac{-rp}{np^{n-1}} + r(1-p^n) = r\left(1 - p^n - \frac{1}{np^{n-2}}\right)$	A1	Correctly finding Y

Question	Answer	Marks	Guidance
	Need the max value of this when $n = 4$ as p varies with $0 \leq p < 1$. When $n = 4$ we have $r\left(1 - p^4 - \frac{1}{4p^2}\right)$ Differentiating with respect to p gives $-4rp^3 + \frac{r}{2p^3}$	M1	Differentiating and setting equal to zero
	Setting this equal to zero gives $p^6 = \frac{1}{8}$ so that $p^2 = \frac{1}{2}$	A1	Finding p at the maximum ***and*** getting $Y = \frac{r}{4}$
	Substituting this back into the expression for Y gives $Y = r\left(1 - \frac{1}{4} - \frac{1}{2}\right) = \frac{r}{4}$ (nearby gradient shows this will correspond to a max, derivative decreases as positive p increases).	E1	Need to justify it gives a max, sketch provided for info, not required by question.
		[5]	
(iii)	"Mass" is given by $\frac{nr^3\pi}{2+n}$. So mass of "top part" is $\frac{2r^3\pi}{3}$. Mass of "bottom part" is $\frac{r^3\pi}{2}$.	B1	
	Centre of mass of top part is at $\frac{2r}{5}$, bottom part is at $-\frac{r}{3}$	B1	Needs to be a negative value or "below x-axis"

Question	Answer	Marks	Guidance
	So centre of mass of whole solid is at $$\frac{\left(\frac{2r^3\pi}{3}\times\frac{2r}{5}+\frac{1r^3\pi}{2}\times-\frac{r}{3}\right)}{\frac{2r^3\pi}{3}+\frac{r^3\pi}{2}}$$	**M1**	Attempting to average and get the centre of mass
	$=\frac{6r}{70}$.	**A1**	
	Need normal to meets y-axis there. For top part need $r\left(1-p^4-\frac{1}{4p^2}\right)=\frac{6r}{70}$. Note that RHS is less than maximum value of LHS, so we can expect solutions.	**M1**	
	For this we need $1-p^4-\frac{1}{4p^2}=\frac{6}{70}\Leftrightarrow 128p^2-140p^6-35=0\Leftrightarrow 0=140p^6-128p^2+35$.	**A1**	Must multiply up by p^2
	There are two such positive values of p^2 leading to two possible positive values of p. e.g. it's a cubic in p^2, and when $p=0$ or $p=1$, $140p^6-128p^2+35>0$, but when $p=\frac{1}{\sqrt{2}}$ we have $140p^6-128p^2+35<0$, meaning there are two roots in $0<p<1$.	**E1**	Any justification of this
	For the bottom part need $-r\left(1-p^2-\frac{1}{2}\right)=\frac{6r}{70}$.	**M1**	Must include the minus sign for the method mark

Question	Answer	Marks	Guidance
	For this we need $p^2 + \frac{1}{2} - 1 = \frac{6}{70} \Leftrightarrow 70p^2 - 41 = 0 \Leftrightarrow p = \sqrt{\frac{41}{70}}$.	**A1** **[9]**	Need to make clear the only possible value is the positive root (can just ignore the negative one)

Question		Answer	Marks	Guidance
10	(i)	$\frac{d^2h}{dt^2} = -\omega^2 h$ gives general solution of $h(t) = \alpha e^{i\omega t} + \beta e^{-i\omega t}$ where α, β are complex numbers leading general real-valued solution of $h(t) = A\cos\omega t + B\sin\omega t$.	M1	
		Using $h(0) = 0$ and $\frac{dh}{dt} = \omega L$ gives $A = 0, B = L$ so that $h(t) = L\sin\omega t$	A1	
		 The x coordinate of P is s$L\cos\omega t$.	M1	Needs diagram, or equivalent (AG) - can include mention of similar triangles. Can be implied if both x and y coordinates found correctly.
		The y coordinate of P is $(1-s)L\sin\omega t$.	A1	
			[4]	
	(ii)	$x(t) = sL\cos\omega t \Rightarrow \ddot{x}(t) = -sL\omega^2\cos\omega t$	B1	

Question	Answer	Marks	Guidance
	$y(t) = (1-s)L\sin\omega t \Rightarrow \ddot{x}(t) = (s-1)L\omega^2\sin\omega t$	**B1**	
		[2]	
(iii)	Resolving horizontally, taking right as positive, gives	**M1**	Attempt to resolve horizontally, which must include N, F and an acceleration. Trig functions can be wrong.
	$N\sin\omega t - F\cos\omega t = -msL\omega^2\cos\omega t$ (1)	**A1**	
	Resolving vertically, taking up as positive, gives	**M1**	Attempt to resolve vertically, which must include N, F and an acceleration. Trig functions can be wrong.
	$N\cos\omega t - mg + F\sin\omega t = m(s-1)L\omega^2\sin\omega t$ (2)	**A1**	

Question	Answer	Marks	Guidance
	$(1) \times \sin wt + (2) \times \cos \omega t$ gives $N - mg\cos\omega t = -msL\omega^2\cos\omega t\sin\omega t - m(1-s)L\omega^2\cos\omega t\sin\omega t$	**M1**	Taking a correct combination that would eliminate F, but can be algebra errors. If F was found first, this mark is gained by substituting F into (2) provided they rearrange so that N appears on its own and not multiplied by anything trig.
	So that $N = mg\cos\omega t - mL\omega^2\cos\omega t\sin\omega t = mg\left(\cos\omega t - \frac{L\omega^2}{g}\cos\omega t\sin\omega t\right)$ $= mg\left(1 - \frac{L\omega^2}{g}\sin\omega t\right)\cos\omega t = mg(1 - k\sin\omega t)\cos\omega t$ where $k = \frac{L\omega^2}{g}$.	**A1**	

Question	Answer	Marks	Guidance
	Then (2) $\times \sin wt - (1) \times \cos \omega t$ gives $-m(1-s)Lw^2 \sin^2 \omega t + msL\omega^2 \cos^2 \omega t = F - mg \sin \omega t$	**M1**	Taking the correct combination to eliminate N. Alternatively, divide (1) by cos ωt to get $F = \dots$
	Therefore $F = mg \sin \omega t - mL\omega^2 \sin^2 \omega t + msL\omega^2 = msgk + mg \sin \omega t\,(1 - k \sin \omega t)$ $= mgsk + mg \cos \omega t \tan \omega t\,(1 - k \sin \omega t) = mgsk + N \tan \omega t$	**A1**	
		[8]	

Question	Answer	Marks	Guidance
	Alternative (iii), resolving perpendicular and parallel to slope		
(iii)	Perpendicular (away from plank) component of acceleration is $a_{perp} = -sL\omega^2 \cos\omega t \sin\omega t + (s-1)L\omega^2 \sin\omega t \cos\omega t = -L\omega^2 \sin\omega t \cos\omega t$	**M1** **A1**	Attempts to resolve each component of acceleration found earlier into perp direction – e.g. could have trig functions mixed up but still get M1
	Parallel (down the plank) component of acceleration is $a_{par} = -sL\omega^2 \cos^2\omega t - (s-1)L\omega^2 \sin^2\omega t = -L\omega^2(s - \sin^2\omega t)$	**M1** **A1**	Similar to the above comment
	Resolving perpendicular to the plank gives $N - mg\cos\omega t = -mL\omega^2 \cos\omega t \sin\omega t$ $N = mg\left(1 - \frac{L\omega^2}{g}\sin\omega t\right)\cos\omega t$ which produces the desired result when $k = \frac{L\omega^2}{g}$.	**M1** **A1**	Must have N and gravity component equal to an acceleration for the M1

Question	Answer	Marks	Guidance
	Resolving parallel gives $mg\sin\omega t - F = -mL\omega^2(s - \sin^2\omega t)$ $F = mgsk + mg(1 - k\sin\omega t)\sin\omega t = mgsk + N\tan\omega t$	M1 A1	Must have F and gravity component equal to an acceleration for the M1
(iv)	Since does not slip initially we have $F < \mu N$ when $t = 0$.	M1	
	This is if and only if $mgsk < mg\tan\alpha \Leftrightarrow sk < \tan\alpha$	A1	
	NTS that $F = \mu N$ and $N > 0$ at some time t when the angle the plank makes with the horizontal (hereafter known as the 'angle of the plank') is less than α.		
	Consider $t = \frac{\alpha}{\omega}$.	B1	Identifying $t = \frac{\alpha}{\omega}$.
	When $t = \frac{\alpha}{\omega}$ we have $F = mgsk + N\tan\alpha$, so that $F - \mu N = mgsk + N\tan\alpha - N\tan\alpha = mgsk > 0$	M1	Calculate F at the relevant t and compare it to μN

Question	Answer	Marks	Guidance
	So, the required frictional force is greater than the available frictional force and the particle must have slipped at some earlier time, and therefore at some angle smaller than α to the horizontal.	**A1**	Correctly shown the above and a clear statement showing the particle slips and second bullet point is shown

Question	Answer	Marks	Guidance
	So, if $sk < \tan\alpha$ at time $t = 0$ we have $F < \mu N$ and at time $t = \frac{\alpha}{\omega}$ we have $F > \mu N$. Therefore there is a time t^* with $0 < t^* < \frac{\alpha}{\omega}$ such that $F = \mu N$. We have $0 < \omega t^* < \alpha$ and so $0 < \tan\omega t^* < \tan\alpha$. Earlier equation: $F(t) = mgsk + N(t)\tan\omega t$ Therefore $\frac{F(t) - \mu N(t)}{mg} = \frac{F(t) - \tan\alpha\, N(t)}{mg} = sk + \frac{N(t)(\tan\omega t - \tan\alpha)}{mg}$ And at $t = t^*$ this gives $0 = sk + \frac{N(t^*)(\tan\omega t^* - \tan\alpha)}{mg}$ leading to $N(t^*) = \frac{-skmg}{\tan\omega t^* - \tan\alpha} > 0.$ (Numerator and denominator both negative.)	**E1** **[6]**	Justification of second bullet point in question.

Question		Answer	Marks	Guidance
11	**(i)**	$F(x) = \int_0^x \lambda e^{-\lambda t} dt = \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x}$	**B1**	
		$G(y) = (P(X_1) \le y) \times (P(X_2) \le y) \times \ldots \times (P(X_n) \le y)$	**M1**	Must see either the given product or $G(y) = P(X_i \le y \text{ for all } i)$ (OE)
		$= F(y)^n = \left(1 - e^{-\lambda y}\right)^n$	**A1**	**AG**
			[3]	
	(ii)	$P\big(Y < L(\alpha)\big) = \alpha \Leftrightarrow G\big(L(\alpha)\big) = \alpha \Leftrightarrow \left(1 - e^{-\lambda L(\alpha)}\right)^n = \alpha \Leftrightarrow e^{-\lambda L(\alpha)} = 1 - \alpha^{\frac{1}{n}}$	**M1**	For $e^{-\lambda L(\alpha)} = 1 - \alpha^{\frac{1}{n}}$ OE (must take n^{th} root)
		$\Leftrightarrow L(\alpha) = -\frac{1}{\lambda}\ln\left(1 - \alpha^{\frac{1}{n}}\right)$	**A1**	**AG**
		$\mathrm{P}\big(Y > U(\alpha)\big) = \alpha \Leftrightarrow 1 - \mathrm{G}\big(U(\alpha)\big) = \alpha \Leftrightarrow 1 - \left(1 - e^{-\lambda U(\alpha)}\right)^n = \alpha \Leftrightarrow e^{-\lambda U(\alpha)} = 1 - (1-\alpha)^{\frac{1}{n}}$	**M1**	For $e^{-\lambda U(\alpha)} = 1 - (1-\alpha)^{\frac{1}{n}}$ OE (must take n^{th} root)

Question		Answer	Marks	Guidance
		$\Leftrightarrow U(\alpha) = -\frac{1}{\lambda}\ln\left(1-(1-\alpha)^{\frac{1}{n}}\right)$	A1 **[4]**	Correct statement without working earns M1 A1
	(iii)	$\alpha^{\frac{1}{n}} = e^{\frac{\ln\alpha}{n}} \approx 1 + \frac{\ln\alpha}{n}.$	M1	For correct use of given approximation
		This gives $-\lambda L(\alpha) = \ln\left(1-\alpha^{\frac{1}{n}}\right) \approx \ln\left(-\frac{\ln\alpha}{n}\right) = \ln(-\ln\alpha) - \ln n$	M1	Substituting approx for $\alpha^{\frac{1}{n}}$ into expression for $L(\alpha)$ Can be awarded for those who work backwards, provided that it is clearly stated that logic is reversible

Question		Answer	Marks	Guidance
		which gives $\lambda L(\alpha) \approx \ln n - \ln(-\ln \alpha) = \ln n - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right)$	**A1** **[3]**	**AG** A0 if candidates use $=$ instead of $\approx$ either here or in working
	(iv)	Median of Y is $L\left(\frac{1}{2}\right)$ $\approx \frac{1}{\lambda}\left[\ln n - \ln\left(\ln\frac{1}{\alpha}\right)\right]$, which tends to infinity as n increases.	**M1** **A1**	Stating Median $\geq$ $L(\alpha)$ is sufficient. Must relate to the $L(\alpha)$ **AG**

Question		Answer	Marks	Guidance
		$(1-\alpha)^{\frac{1}{n}} = e^{\frac{\ln(1-\alpha)}{n}} \approx 1 + \frac{\ln(1-\alpha)}{n}$.	**M1**	Use of approximation given Mark can be inferred as indicated below, but do not accept alternative methods

Question		Answer	Marks	Guidance
		This gives $\lambda U(\alpha) = \ln\left(1-(1-\alpha)^{\frac{1}{n}}\right) \approx \ln\left(-\frac{\ln(1-\alpha)}{n}\right) = \ln n - \ln(-\ln\{(1-\alpha\})) = \ln n - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right\}\right)\right).$	**A1**	A0 if candidates use = instead of ≈ either here or in working Correct statement without working earns M1 A1 Correct statement except for incorrect use of = earns M1 A0

Question		Answer	Marks	Guidance
		So we have $\lambda\big(\mathrm{U}(\alpha)-\mathrm{L}(\alpha)\big) \approx \ln n - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right)\right\}\right) - \left(\ln n - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right)\right) = \ln\left(\ln\left(\frac{1}{\alpha}\right)\right) - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right)\right\}\right)$		
		In fact $\mathrm{U}(\alpha)-\mathrm{L}(\alpha) \to \frac{1}{\lambda}\left[\ln\left(\ln\left(\frac{1}{\alpha}\right)\right) - \ln\left(\ln\left(\frac{1}{1-\alpha}\right)\right)\right]$ as n increases.	B1 [5]	Failing to note (e.g. via $\to, \approx, \sim$ etc.) that this is an asymptotic formula earns B0
	(v)	By definition of $L(\alpha)$ and $U(\alpha)$, the probability Y is between $L(0.05)$ and $U(0.05)$ is 0.9. For large n, $\mathrm{U}(\alpha)-\mathrm{L}(\alpha) \approx \frac{1}{\lambda}\left(\ln\left(\ln\left(\frac{1}{\alpha}\right)\right) - \ln\left(\ln\left\{\left(\frac{1}{1-\alpha}\right)\right\}\right)\right)$	B1	Must see $\boldsymbol{P}[L(\alpha) < Y < U(\alpha)] = 0.9$ OE

Question		Answer	Marks	Guidance
		Taking $\alpha = 0.05$ in this:		
		$U(\alpha) - L(\alpha) \approx \frac{1}{\lambda}\left(\ln\left(\ln\left(\frac{1}{0.05}\right)\right) - \ln\left(\ln\frac{1}{0.95}\right)\right)$	**M1**	Substituting $\alpha = 0.05$ Can be awarded for substituting later in solutions which work backwards from answer
		$= \frac{1}{\lambda}\left(\ln(\ln(20)) - \ln\left(\ln\frac{20}{19}\right)\right) \approx \frac{1}{\lambda}\left(\ln 3 - \ln\left(ln\frac{20}{19}\right)\right)$	**M1**	Use of $\ln(20) \approx 3$
		$\approx \frac{1}{\lambda}\left(\ln 3 - \ln\frac{1}{19}\right) = \frac{1}{\lambda}\ln 57$	**M1**	Use of $\ln\frac{20}{19} \approx \frac{1}{19}$ or $\ln\frac{19}{20} \approx -\frac{1}{20}$

Question		Answer	Marks	Guidance
		$\approx \frac{1}{\lambda}\left(3 + \ln\frac{57}{20}\right) \approx 4\lambda^{-1}$, since $\frac{57}{20} = 2.85 \approx e$	**A1** **[5]**	**AG** A0 if candidates use $=$ instead of $\approx$ either here or in above working If candidate works backwards from answer, withhold unless it is clearly stated that logic is reversible

Question			Answer	Marks	Guidance
12	**(i)**		$\sum_{r=1}^{m+1}\left(f(r)\sum_{s=r-1}^{m} g(s)\right)$ $= f(1)(g(0)+g(1)+\cdots+g(m))$ $+f(2)(g(1)+g(2)+\cdots+g(m))$ $+f(3)(g(2)+g(3)+\cdots+g(m))$ $\ldots + f(m+1)g(m)$	**M1**	Writing out a representative sample of first sum in full
			$\sum_{s=0}^{m}\left(g(s)\sum_{r=1}^{s+1} f(r)\right)$ $= g(0)f(1)$ $+g(1)(f(1)+f(2))$ $+g(2)(f(1)+f(2)+f(3))$ $\ldots + g(m)(f(1)+f(2)\ldots+f(m+1))$	**M1**	Writing out a representative sample of second sum in full
			$\sum_{r=1}^{m+1}\left(f(r)\sum_{s=r-1}^{m} g(s)\right) = \sum_{s=0}^{m}\left(g(s)\sum_{r=1}^{s+1} f(r)\right)$	**A1**	**AG**
				[3]	

Question			Answer	Marks	Guidance
			Alternative (i), by induction		
		(i)	Proceed by induction. $m = 0$: LHS $= f(1)g(0) =$ RHS Assume true for $m = k$. $m = k+1$: $\sum_{r=1}^{k+2}\left(f(r)\sum_{s=r-1}^{k+1}g(s)\right) = \sum_{r=1}^{k+1}\left(f(r)\sum_{s=r-1}^{k}g(s)\right) + g(k+1)\sum_{r=1}^{k+2}f(r)$	**M1**	Correctly relating the LHS (or RHS) sum for $m = k+1$ to the same sum for $m = k$
			$= \sum_{s=0}^{k}\left(g(s)\sum_{r=1}^{s+1}f(r)\right) + g(k+1)\sum_{r=1}^{k+2}f(r)$	**M1**	Using the induction hypothesis
			$= \sum_{s=0}^{k+1}\left(g(s)\sum_{r=1}^{s+1}f(r)\right)$		
			Thus the claimed identity holds.	**A1**	**AG** Correctly completing the induction

Question			Answer	Marks	Guidance
					(including consideration of the base case)
				[3]	
	(ii)	**(a)**	X_1 is 0 with probability 0.5 and 1 with probability 0.5 so its expected value is 0.5	**B1**	No justification required
			$P(X_2 = 0) = P(X_1 = 0) \times P(X_2 = 0 \mid X_1 = 0) + P(X_1 = 1) \times P(X_2 = 0 \mid X_1 = 1)$	**M1**	Use of Law of Total Probability
			$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{12}$	**A1**	
			$P(X_2 = 1) = P(X_1 = 0) \times P(X_2 = 1 \mid X_1 = 0) + P(X_1 = 1) \times P(X_2 = 1 \mid X_1 = 1)$	**M1**	Use of Law of Total Probability
			$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{12}$	**A1**	Similarly = 2
			$P(X_2 = 2) = P(X_1 = 0) \times P(X_2 = 2 \mid X_1 = 0) + P(X_1 = 1) \times P(X_2 = 2 \mid X_1 = 1)$ $= \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$	**B1** **CSO**	**AG** Any valid justification (e.g. using that

Question			Answer	Marks	Guidance
			Therefore $E(X_2) = 0 \times P(X_2 = 0) + 1 \times P(X_2 = 1) + 2 \times P(X_2 = 2) = \frac{5}{12} + \frac{2}{6} = \frac{9}{12} = \frac{3}{4}$	B1 **[7]**	probabilities sum to 1)
		(b)	Possible values of X_n are $0,1,2,\dots,n$ (so possible values of X_{n-1} are $0,1,2,\dots,n-1$		
			$P(X_n = 0)$ $= P(X_{n-1} = 0) \times P(X_n = 0 \mid X_{n-1} = 0) + P(X_{n-1} = 1) \times P(X_n = 0 \mid X_{n-1} = 1)+\dots$ $P(X_{n-1} = n-1) \times P(X_n = 0 \mid X_{n-1} = n-1)$	M1	Total probability with correct upper limit $[n-1]$ for X_{n-1}
			$= \frac{1}{2} P(X_{n-1} = 0) + \frac{1}{3} P(X_{n-1} = 1) + \dots + \frac{1}{n+1} P(X_{n-1} = n-1) + \dots = \sum_{s=0}^{n-1} \frac{P(X_{n-1} = s)}{s+2}$	A1	AG
			$\mathrm{P}(X_n = r) = \mathrm{P}(X_{n-1} = 0) \times P(X_n = r \mid X_{n-1} = 0) + \mathrm{P}(X_{n-1} = 1) \times P(X_n = r \mid X_{n-1} = 1)+\dots$ $P(X_{n-1} = n-1) \times P(X_n = r \mid X_{n-1} = n-1)$	M1	Total probability with correct upper limit $[n-1]$ for X_{n-1}
			$= \frac{1}{r+1} P(X_{n-1} = r-1) + \frac{1}{r+2} P(X_{n-1} = r) + \dots + \frac{1}{n+1} P(X_{n-1} = n-1)$	M1	Correct lower limit $[r-1]$ for X_{n-1}

Question			Answer	Marks	Guidance
			$= \sum_{s=r-1}^{n-1} \frac{P(X_{n-1}=s)}{s+2}$	**A1** **[5]**	
		(c)	Using the first part of this question, $$E(X_n) = \sum_{r=0}^{n} r \sum_{s=r-1}^{n-1} \frac{P(X_{n-1}=s)}{s+2} = \sum_{s=0}^{n-1} \frac{P(X_{n-1}=s)}{s+2} \sum_{r=1}^{s+1} r$$	**M1**	Correct formula for expectation (FT errors in calculated probabilities in (b)) + use of (a)
			$= \sum_{s=0}^{n-1} \frac{P(X_{n-1}=s)}{s+2} \frac{(s+2)(s+1)}{2}$	**M1**	Sum over r
			$= \sum_{s=0}^{n-1} (X_{n-1}=s) \frac{(s+1)}{2} = \frac{1}{2}\left(\sum_{s=0}^{n-1}(X_{n-1}=s)\, s + \sum_{s=0}^{n-1}(X_{n-1}=s)\right) = \frac{1}{2}(E(X_{n-1})+1)$	**A1**	**AG**
			This recurrence can be solved $x_{n+1} - 0.5x_n = 0$ has solution $x_n = 0.5^n$. Particular solution is $x_n = 1$ giving general solution of $x_n = 1 + A0.5^n$	**M1**	Some evidence of appropriate method for solving the

Question			Answer	Marks	Guidance
			With $E(X_0) = 0,\ E(X_n) = 1 - \left(\frac{1}{2}\right)^n$	**A1** **[5]**	recurrence being applied correctly.