CAMBRIDGE

Sixth Term Examination Paper (STEP)

Mathematics 3 (9475)

2025

Examiners' report and mark scheme

STEP Mathematics 3 examiners' report

Paper 3 overview

The majority of candidates focused solely on the pure questions, with questions 1, 2 and 8 the most popular. The statistics questions were more popular than the mechanics questions in this exam series.

Candidates who did well on this paper generally:	**Candidates who did less well on this paper generally:**
• were careful to explain and justify the steps in their arguments, explaining what they had done rather than expecting the examiner to infer what had been done from disjointed groups of calculations • paid close attention to what was required by the questions • made fewer unnecessary mistakes with calculations • thought carefully about how to present rigorous arguments involving trig functions and their inverse functions, especially in relation to domain considerations • understood that questions set on the STEP papers require sufficient justification to earn full credit • knew the difference between 'positive' and 'non-negative' • attempted all parts of a question, picking up marks for later parts even when they had not necessarily attempted or completed previous parts.	• did not pay attention to 'Hence' instructions: this means that you must use the previous part • presented explanations that were not precise enough (e.g. in Question 3 describing the transformations but not in the context of the graphs or in Question 8 not explaining use of trigonometric relationships sufficiently well) • made additional assumptions, e.g. that a function was differentiable when this had not been given • tried to present if and only if arguments in a single argument when dealing with each direction separately would have been more appropriate and safer (note that this is not always the case; in general candidates need to consider what is the most appropriate presentation of an if and only if argument) • tried to carry out too many steps in one go, resulting in them not justifying the key steps sufficiently • did not take sufficient care with graphs/curve sketching.

Section A: Pure Mathematics overview

Most candidates focused on the pure section, but generally the more successful candidates selected questions to answer rather than attempting each question in the order given on the paper.

Question 1

1 *You need not consider the convergence of the improper integrals in this question.*

For $p, q > 0$, define

$$\mathrm{b}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x.$$

(i) Show that $\mathrm{b}(p,q) = \mathrm{b}(q,p)$.

(ii) Show that $\mathrm{b}(p+1,q) = \mathrm{b}(p,q) - \mathrm{b}(p,q+1)$ and hence that $\mathrm{b}(p+1,p) = \frac{1}{2}\,\mathrm{b}(p,p)$.

(iii) Show that

$$\mathrm{b}(p,q) = 2\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,\mathrm{d}\theta.$$

Hence show that $\mathrm{b}(p,p) = \dfrac{1}{2^{2p-1}}\,\mathrm{b}\big(p,\frac{1}{2}\big)$.

(iv) Show that

$$\mathrm{b}(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}t.$$

(v) Evaluate

$$\int_0^\infty \frac{t^{\frac{3}{2}}}{(1+t)^6}\,\mathrm{d}t.$$

This question was the most popular question in terms of the number of attempts, and it was generally well done. Some candidates spent significant time attempting methods involving integration by parts in the early parts of this question which did not work.

In part (iii) the most common method in the 'Hence show that...' involved using the substitution $u = 2\theta$ at some point. After this the integral looks like what is required in the given answer but with the limits 0 to π rather than 0 to $\frac{\pi}{2}$. Candidates needed to point out that the symmetry in the integral enabled the limits to be changed back to 0 to $\frac{\pi}{2}$ with the appearance of a factor of 2.

Part (iv) was generally well done by those who got that far.

In part (v) some marks were given for piecing together the earlier results to get to an integral that is relatively easy to calculate. Some candidates did not provide sufficient justification to be awarded full credit. A good number of candidates got to a final correct value of the integral. Some candidates had success with alternative methods involving more difficult integration having not made so much use of the properties of b to simplify the calculation.

Question 2

2 Let $f(x) = 7 - 2|x|$.

A sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = a$ and $u_n = f(u_{n-1})$ for $n > 0$.

(i) **(a)** Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(x))$.

(b) Find all solutions of the equation $f(f(x)) = x$.

(c) Find the values of a for which the sequence $u_0, u_1, u_2, \ldots$ has period 2.

(d) Show that, if $a = \frac{28}{5}$, then the sequence $u_2, u_3, u_4, \ldots$ has period 2, but neither u_0 or u_1 is equal to either of u_2 or u_3.

(ii) **(a)** Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(f(x)))$.

(b) Consider the sequence $u_0, u_1, u_2, \ldots$ in the cases $a = 1$ and $a = -\frac{7}{9}$. Hence find all the solutions of the equation $f(f(f(x))) = x$.

(c) Find a value of a such that the sequence $u_3, u_4, u_5, \ldots$ has period 3, but where none of u_0, u_1 or u_2 is equal to any of u_3, u_4 or u_5.

In terms of attempts, this was the third most popular question. Most candidates who attempted this question were able to make good progress with many of the parts. Candidates were generally able to sketch the graph of $y = \mathrm{f}(x)$, but sketches of $y = \mathrm{f}(\mathrm{f}(x))$ often had some features missing or incorrect. Many candidates opted to work out the equation for each of the straight-line segments before sketching the graph and, while this generally resulted in the correct overall shape, important points such as the vertical positioning of the points where the two graphs cross were often incorrect. A number of candidates would have benefitted from making their sketches larger. A small number of candidates did not sketch the two graphs on the same set of axes, which meant that some of the marks for this part of the question were not accessible.

Part (c) was not answered well, with many candidates simply restating their solutions to part (b) without considering the fact that some of the solutions would lead to sequences with a period of 1. Allowance was made here for those candidates who stated that a solution with period 1 also has period 2 and listed all their solutions to (b).

In part (d) candidates successfully calculated the terms of the sequence and many identified the connection with the previous parts to explain that the remainder of the sequence would have a period of 2. A small number of candidates only checked that $u_0 \neq u_2$ and $u_1 \neq u_3$ and therefore did not fully answer this part of the question.

Those who had made good sketches for the graphs in part (i) (a) generally made good attempts at the sketches in part (ii) (a), although similar issues were encountered with the positioning of the intersections of the two graphs.

Almost all candidates who attempted part (ii) (b) were able to calculate the sequences starting with the two given values, although many did not realise that all three values that appeared in each solution would also be solutions. The question was posed using 'Hence' and so this approach was required. Some candidates simply stated their set of solutions without providing any explanation. While some candidates commented on the fact that there must be eight solutions based on their sketch, a significant number of candidates did not realise that the two period 1 values that lead to a constant sequence would also be solutions.

Question 3

3 Let $\mathrm{f}(x)$ be defined and positive for $x > 0$.

Let a and b be real numbers with $0 < a < b$ and define the points $A = (a, \mathrm{f}(a))$ and $B = (b, -\mathrm{f}(b))$.

Let $X = (m, 0)$ be the point of intersection of line AB with the x-axis.

(i) Find an expression for m in terms of a, b, $\mathrm{f}(a)$ and $\mathrm{f}(b)$.

(ii) Show that, if $\mathrm{f}(x) = \sqrt{x}$, then $m = \sqrt{ab}$.

Find, in terms of n, a function $\mathrm{f}(x)$ such that $m = \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$.

(iii) Let $\mathrm{g}_1(x)$ and $\mathrm{g}_2(x)$ be defined and positive for $x > 0$. Let $m = M_1$ when $\mathrm{f}(x) = \mathrm{g}_1(x)$ and let $m = M_2$ when $\mathrm{f}(x) = \mathrm{g}_2(x)$.

Show that if $\dfrac{\mathrm{g}_1(x)}{\mathrm{g}_2(x)}$ is a decreasing function then $M_1 > M_2$.

Hence show that

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}.$$

(iv) Let p and c be chosen so that the curve $y = p(c - x)^3$ passes through both A and B. Show that

$$\frac{c-a}{b-c} = \left(\frac{\mathrm{f}(a)}{\mathrm{f}(b)}\right)^{\frac{1}{3}}$$

and hence determine c in terms of a, b, $\mathrm{f}(a)$ and $\mathrm{f}(b)$.

Show that if f is a decreasing function, then $c < m$.

Part (i) was generally well done. However, some candidates simply stated the result without showing sufficient working and full credit could not be given. Sign errors were another common pitfall and usually meant that the accuracy mark could not be awarded.

Part (ii) was frequently attempted, though for many candidates it marked the end of their attempt. A common mistake was overlooking the fact that the function $\mathrm{f}(x)$ was defined to be positive for $x > 0$, leading to marks being unavailable. Another frequent issue was providing insufficient justification – some candidates simply stated a function without explaining their reasoning or showing it had the required properties, which prevented them from earning full marks.

Part (iii) was less frequently attempted, especially the first subpart. Some candidates assumed that the given functions were differentiable and attempted to provide arguments involving derivatives which did not gain credit. The second subpart was generally handled better, though again, a lack of justification was common. A few candidates also attempted alternative methods not involving the previous part, thus ignoring 'hence' in the question.

Part (iv) was relatively well done by those who attempted it. The first subpart was accessible to most candidates. The second subpart was more challenging and required careful attention, especially when working with inequalities and avoiding unwarranted assumptions of equality.

Question 4

4 **(i)** x_2 and y_2 are defined in terms of x_1 and y_1 by the equation

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

G_1 is the graph with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and G_2 is the graph with equation

$$\frac{\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{9} + \frac{\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{4} = 1.$$

Show that, if (x_1, y_1) is a point on G_1, then (x_2, y_2) is a point on G_2.

Show that G_2 is an anti-clockwise rotation of G_1 through $45°$ about the origin.

(ii) **(a)** The matrix

$$\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$$

represents a reflection. Find the line of invariant points of this matrix.

(b) Sketch, on the same axes, the graphs with equations

$$y = 2^x \quad \text{and} \quad 0.8x + 0.6y = 2^{-0.6x+0.8y}.$$

(iii) Sketch, on the same axes, for $0 \leqslant x \leqslant 2\pi$, the graphs with equations

$$y = \sin x \quad \text{and} \quad y = \sin(x - 2y).$$

You should determine the exact co-ordinates of the points on the graph with equation $y = \sin(x - 2y)$ where the tangent is horizontal and those where it is vertical.

This question was one of the less popular pure questions but still had a good number of attempts. In the question, candidates are led through an example of how to apply the mathematics they know to a new context and then are expected to apply what they have learnt to other problems.

For part (i) most candidates knew how to approach the problem but showed insufficient detail in their working for full credit, usually by not making the link (x_1, y_1) on $G_1 \Rightarrow \frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$ when trying to show that (x_2, y_2) is on G_2.

A handful did not realise the significance of the indexing on the points and instead tried to show that the equations of the two curves were equivalent.

Almost all candidates recognised that the given matrix was a rotation matrix, but some did not make the link between this and the relationship between the points on the curves clear.

In part (ii) (a) it became evident that a number of candidates do not know the difference between a line of invariant points and an invariant line (in the second case points can move under the transformation but must stay on the same line). This meant some candidates did a lot more working than was necessary and often ended up with an extra answer, meaning that they could not gain full credit for this part.

A few candidates used the general form of a reflection matrix in the line $y = \tan\theta$ and a t substitution to find the required line. This method also required candidates to reject one solution, which was usually done by those taking this route.

In part (ii) (b) only the most successful candidates showed convincingly that if (x_1, y_1) was on the graph of $y = 2^x$ then $(x_2, y_2) = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ was on the other graph, but most realised that the two graphs were reflections of each other and so could make an attempt at the sketch. The most common mistakes here were assuming that the second graph was asymptotic to the *y* axis and not showing the two graphs intersecting twice.

Attempts were variable for part (iii). A lot of candidates found a matrix connecting points on the two curves, but often had the relationship the 'wrong way around' with $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ rather than the correct version $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

Many candidates could differentiate the implicit equation $y = \sin(x - 2y)$ successfully and some successfully went on to find the points where the tangent was horizontal and vertical. Some who had found the correct transformation matrix could use this to find the points with horizontal tangents but struggled to use a similar argument convincingly for the vertical tangents.

Some candidates successfully set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$ to get $x - 2y = \frac{\pi}{2}$ or $x - 2y = \frac{2\pi}{3}$ but were then uncertain how they could use this to find the coordinates of the relevant points.

Many of the candidates who found the points with horizontal or vertical tangents could 'join the dots' to complete the sketch, but some joined them in the wrong order. Many candidates laboured under the misunderstanding that it is not possible for an implicit function to be one-to-many valued which caused a variety of different mistakes.

Question 5

5 Three points, A, B and C, lie in a horizontal plane, but are not collinear. The point O lies above the plane.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

P is a point with $\overrightarrow{OP} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$, where α, β and γ are all positive and $\alpha + \beta + \gamma < 1$.

Let $k = 1 - (\alpha + \beta + \gamma)$.

(i) The point L is on OA, the point X is on BC and LX passes through P.

Determine $\overrightarrow{OX}$ in terms of β, γ, $\mathbf{b}$ and $\mathbf{c}$ and show that $\overrightarrow{OL} = \dfrac{\alpha}{k+\alpha}\mathbf{a}$.

(ii) Let M and Y be the unique pair of points on OB and CA respectively such that MY passes through P, and let N and Z be the unique pair of points on OC and AB respectively such that NZ passes through P.

Show that the plane LMN is also horizontal if and only if OP intersects plane ABC at the point G, where $\overrightarrow{OG} \equiv \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Where do points X, Y and Z lie in this case?

(iii) State what the condition $\alpha + \beta + \gamma < 1$ tells you about the position of P relative to the tetrahedron $OABC$.

This question was the least popular pure question by a large margin, and of the attempts made less than half were 'substantial' attempts.

As is often the case with vector questions, a carefully drawn diagram can be very helpful in selecting an appropriate method for solving the question and the most successful candidates made good use of this.

Various methods were used in part (i), but mostly these involved finding vector equations of relevant lines and manipulating these to show the required results. Candidates should be aware that questions on the STEP papers need enough justification to fully support their solutions. Many candidates lost accuracy marks through their argument not being convincing enough or lacking some details.

Part (ii) had some very good solutions, but many candidates found it difficult to understand what it means for LMN to be horizontal. A clear diagram here would have helped candidates to find a solution method.

Some candidates tried to do both directions of the 'if and only if' in one go. They usually did not gain full marks here, either because they did not link one pair of statements with an if and only if symbol or because they did not appreciate that one step needed a different approach for each direction of implication. It is always 'safer' to approach each direction of implication separately.

The most common issues were not using $k \neq 0$ when justifying $\frac{\alpha}{k+\alpha} = \frac{\beta}{k+\beta} \Rightarrow \alpha = \beta$, or for not convincingly explaining why $\alpha = \beta = \gamma$ mean that LMN was horizontal.

Part (iii) required a one-line answer, and some candidates who had taken the time to read the whole question successfully answered this part even if they had not answered the previous parts. Some candidates confused 'positive' with 'non-negative' and stated that point P could be inside or on the faces of the tetrahedron.

Question 6

6 **(i)** Let a, b and c be three non-zero complex numbers with the properties $a+b+c=0$ and $a^2+b^2+c^2=0$.

Show that a, b and c cannot all be real.

Show further that a, b and c all have the same modulus.

(ii) Show that it is not possible to find three non-zero complex numbers a, b and c with the properties $a+b+c=0$ and $a^3+b^3+c^3=0$.

(iii) Show that if any four non-zero complex numbers a, b, c and d have the properties $a+b+c+d=0$ and $a^3+b^3+c^3+d^3=0$, then at least two of them must have the same modulus.

(iv) Show, by taking $c=1$, $d=-2$ and $e=3$ that it is possible to find five real numbers a, b, c, d and e with distinct magnitudes and with the properties $a+b+c+d+e=0$ and $a^3+b^3+c^3+d^3+e^3=0$.

Question 6 proved to be quite challenging for many candidates, with a significant number scoring fewer than 5 marks and only attempting part (i) or part (i) and part (ii).

In part (i), the first mark was easily earned for strict inequalities or stating the only solution is the zero solution but was not earned if it was stated that 'the square of any real is positive' rather than any non-zero real. For the rest of the question, it was very common for candidates to attempt to consider each of a, b, c in the form $x + y\text{i}$, and then substitute in to obtain four equations in six variables. Those that tried this invariably made no progress. While it is possible to answer the question using real and imaginary parts, it requires far more work and so no credit was awarded for just writing down these four equations. Those who left the algebra in terms of a, b, c or used the roots of a quadratic tended to answer this part well.

Part (ii) also saw attempts to split a, b, c into real and imaginary parts. This saw no further progress, or credit. The most common way that this question was answered was by writing down an identity relating the sum of cubes to the cube of the sum. This identity could be written in several equivalent forms, and saw many errors in the coefficients and signs, for which candidates were penalised accuracy marks.

Careful thought about presentation was required before commencing the algebraic manipulation to part (iii) to avoid introducing sign and arithmetic errors. Complicated identities were common and often contained errors. Establishing that $abc + bcd + acd + abd = 0$ was a common approach and led to considering the roots of a quartic.

Part (iv) was generally well answered. Most candidates that attempted it were able to identify a quadratic and solve it. Several candidates that could not solve some of parts (i), (ii), (iii) skipped straight to this part and picked up some marks. This is good general exam practice and STEP candidates should remember that subsequent parts of a question can often still be answered even if an early part seems challenging.

Question 7

7 Let $\mathrm{f}(x) = \sqrt{x^2+1} - x$.

(i) Using a binomial series, or otherwise, show that, for large $|x|$, $\sqrt{x^2+1} \approx |x| + \dfrac{1}{2|x|}$.

Sketch the graph $y = \mathrm{f}(x)$.

(ii) Let $\mathrm{g}(x) = \tan^{-1}\mathrm{f}(x)$ and, for $x \neq 0$, let $\mathrm{k}(x) = \frac{1}{2}\tan^{-1}\dfrac{1}{x}$.

(a) Show that $\mathrm{g}(x) + \mathrm{g}(-x) = \frac{1}{2}\pi$.

(b) Show that $\mathrm{k}(x) + \mathrm{k}(-x) = 0$.

(c) Show that $\tan \mathrm{k}(x) = \tan \mathrm{g}(x)$ for $x > 0$.

(d) Sketch the graphs $y = \mathrm{g}(x)$ and $y = \mathrm{k}(x)$ on the same axes.

(e) Evaluate $\displaystyle\int_0^1 \mathrm{k}(x)\,\mathrm{d}x$ and hence write down the value of $\displaystyle\int_{-1}^0 \mathrm{g}(x)\,\mathrm{d}x$.

In part (i), some candidates tried to expand $\sqrt{1+x^2}$ as a series in increasing powers of x^2, not appreciating that they needed $|x|$ to be small for such an expansion to be valid.

A lot of candidates used the expansion to correctly identify the asymptotes of $\mathrm{f}(x)$.

In part (ii) (a), the most common approach was to consider $\tan\big(\mathrm{g}(x)+\mathrm{g}(-x)\big)$, which, using the tan double angle formula, '$=\infty$'. Only a small number justified their answer by using the positivity of $\mathrm{f}(x)$ to get $0<\mathrm{g}(x)+\mathrm{g}(-x)<\pi$. Many candidates then simply stated the answer, or wrote $\mathrm{g}(x)+\mathrm{g}(-x)=\tan^{-1}\infty$, stating that this is $\frac{\pi}{2}$. A common theme in this question was a lack of consideration of ranges/domains of the trigonometric functions which meant there were marks that were unavailable.

Part (ii) (b) was done well in general, with many candidates knowing that $y=\arctan x$ is an odd function. Some overcomplicated it, using the double angle formula again, and not gaining a mark for justifying the range of $k(x)$, i.e. $\tan\big(k(x)+k(-x)\big)=0 \nRightarrow k(x)+k(-x)=0$ in general.

In part (ii) (c), the most common approach was again to use the tan double angle formula, realising that $\tan 2k(x)=x^{-1}$, and arriving at a quadratic for $\tan k(x)$. Marks were again unavailable for those candidates that either did not attempt to solve the quadratic or did solve for the two roots but then not explaining why $\tan k(x)=\mathrm{f}(x)$ was the correct root to choose.

There were also some nice geometric arguments for (c), drawing a right-angled triangle with angle $2k(x)$, then bisecting the angle and finding the side lengths of the smaller right-angled triangle with angle $k(x)$ to find $\tan k(x)$.

The sketches in part (ii) (d) were good in general, although some candidates' sketches contradicted the relations for $g(x)$,$k(x)$, given in the question, for example sketching $k(x)$ as an even function, or not using $\tan k(x)=\tan g(x)$ for $x>0$.

Part (ii) (e) was done well by candidates who attempted it. Some overcomplicated the integral by changing variables, but the majority realised they could integrate by parts directly. For the last part, candidates either used their sketches to find the right area or integrated the relation in part (a) directly using part (c).

Question 8

8 **(i)** Show that

$$z^{m+1} - \frac{1}{z^{m+1}} = \left(z - \frac{1}{z}\right)\left(z^m + \frac{1}{z^m}\right) + \left(z^{m-1} - \frac{1}{z^{m-1}}\right).$$

Hence prove by induction that, for $n \geqslant 1$,

$$z^{2n} - \frac{1}{z^{2n}} = \left(z - \frac{1}{z}\right)\sum_{r=1}^{n}\left(z^{2r-1} + \frac{1}{z^{2r-1}}\right).$$

Find similarly $z^{2n} - \dfrac{1}{z^{2n}}$ as a product of $\left(z + \dfrac{1}{z}\right)$ and a sum.

(ii) **(a)** By choosing $z = \mathrm{e}^{\mathrm{i}\theta}$, show that

$$\sin 2n\theta = 2\sin\theta \sum_{r=1}^{n} \cos(2r-1)\theta\,.$$

(b) Use this result, with $n = 2$, to show that $\cos\frac{2}{5}\pi = \cos\frac{1}{5}\pi - \frac{1}{2}$.

(c) Use this result, with $n = 7$, to show that $\cos\frac{2}{15}\pi + \cos\frac{4}{15}\pi + \cos\frac{8}{15}\pi + \cos\frac{16}{15}\pi = \frac{1}{2}$.

In terms of the number of attempts, this was the second most popular question. The induction in part (i) was generally done very well with a clearly laid out proof. The fifth (method) mark in part (i) was often gained by finding a relevant identity. However, the final mark in part (i) was missed by a large majority of candidates due to not handling the alternating sign in the summation correctly. The summation is quite tricky in this respect, requiring the notation to be set up so that either the final term in the sum is positive, if terms are kept in the same order as for the previous part, or reversing the order of the sum (in which case the first term is positive).

Candidates often missed out on one or both marks in part (ii) (a) due to forgetting factors of 2 and i.

Most candidates that attempted parts (ii) (b) and (ii) (c) gained two method marks for successfully substituting a valid value of θ in both. However, a significant number of attempts at parts (ii) (b) and (ii) (c) did not gain full credit due to insufficient justification when manipulating trigonometric expressions. In general, candidates who stated the trigonometric identities they were using, or which specific terms were equivalent were successful here. Those that did multiple steps at once without justification often missed out on marks because it was not possible to pick out the results they had used. It is very significant here that the answer was given. Candidates should in general attempt to give more details when proving an answer given in the question to show they understand the intermediate steps between the starting point and given answer.

Most candidates did not attempt the part (iii). The successful attempts were from candidates who had given very clear answers to previous parts. There were a few different choices of n and θ that led to the required result.

Section B: Mechanics overview

Of the small number of candidates that attempted the mechanics section, more marks were generally scored on question 10 than on question 9.

Question 9

9 In this question, $n \geqslant 2$.

(i) A solid, of uniform density, is formed by rotating through $360°$ about the y-axis the region bounded by the part of the curve $r^{n-1}y = r^n - x^n$ with $0 \leqslant x \leqslant r$, and the x- and y-axes.

Show that the y-coordinate of the centre of mass of this solid is $\dfrac{nr}{2(n+1)}$.

(ii) Show that the normal to the curve $r^{n-1}y = r^n - x^n$ at the point $\big(rp, r(1-p^n)\big)$, where $0 < p \leqslant 1$, meets the y-axis at $(0, Y)$, where $Y = r\left(1 - p^n - \dfrac{1}{np^{n-2}}\right)$.

In the case $n = 4$, show that the greatest value of Y is $\frac{1}{4}r$.

(iii) A solid is formed by rotating through $360°$ about the y-axis the region bounded by the curves $r^3y = r^4 - x^4$ and $ry = -(r^2 - x^2)$, both for $0 \leqslant x \leqslant r$.

A and B are the points $(0, -r)$ and $(0, r)$, respectively, on the surface of the solid.

Show that the solid can rest in equilibrium on a horizontal surface with the vector $\overrightarrow{AB}$ at three different, non-zero, angles to the upward vertical. You should not attempt to find these angles.

This question had the least number of attempts with a relatively small number of substantial attempts. Overall, this question was not done well, with only a small number of candidates achieving half marks or more. A significant number of candidates got 0,1, or 2 marks in total for the question, with the main issue being getting started by knowing a suitable formula for the centre of mass.

In part (i), many candidates overlooked fact that the curve was rotated about the y-axis, rather than the usual x-axis, and didn't change their formula to the correct variables.

Often candidates skipped straight to part (ii), which was done generally done well.

Part (iii) did not have many attempts. Of those candidates that did answer this part, most were only able to get the first few marks for finding the centre of mass of the full shape. The final part (showing there are multiple ways to balance the solid) was not attempted enough to observe any patterns but it was clear that candidates struggled to demonstrate clear understanding of how the equilibrium condition relates to the position of the centre of mass.

Question 10

10 A plank AB of length L initially lies horizontally at rest along the x-axis on a flat surface, with A at the origin.

Point C on the plank is such that AC has length sL, where $0 < s < 1$.

End A is then raised vertically along the y-axis so that its height above the horizontal surface at time t is $\mathrm{h}(t)$, while end B remains in contact with the flat surface and on the x-axis.

The function $\mathrm{h}(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2\mathrm{h}}{\mathrm{d}t^2} = -\omega^2\mathrm{h}, \quad \text{with } \mathrm{h}(0) = 0 \text{ and } \frac{\mathrm{dh}}{\mathrm{d}t} = \omega L \text{ at } t = 0,$$

where ω is a positive constant.

A particle P of mass m remains in contact with the plank at point C.

(i) Show that the x-coordinate of P is $sL\cos\omega t$, and find a similar expression for its y-coordinate.

(ii) Find expressions for the x- and y-components of the acceleration of the particle.

(iii) N and F are the upward normal and frictional components, respectively, of the force of the plank on the particle. Show that

$$N = mg(1 - k\sin\omega t)\cos\omega t,$$

and that

$$F = mgsk + N\tan\omega t$$

where $k = \dfrac{L\omega^2}{g}$.

(iv) The coefficient of friction between the particle and the plank is $\tan\alpha$, where α is an acute angle.

Show that the particle will not slip initially, provided $sk < \tan\alpha$.

Show further that, in this case, the particle will slip

- while N is still positive,
- when the plank makes an angle less than α to the horizontal.

This question did not receive very many attempts.

Part (i) was generally done well, although a significant portion of candidates did not draw a diagram and didn't correctly calculate y, i.e. not performing the necessary subtraction. This meant errors followed through to subsequent parts.

Part (ii) was generally done well.

A good portion of those that attempted part (iii) realised they had to resolve in two directions (either horizontal and vertical or parallel and perpendicular), and made a good attempt to do this. Choosing to resolve horizontally and vertically proved to be more straightforward. Those that chose to resolve parallel and perpendicular often had some difficulties with calculating the resultant acceleration.

Part (iv) was not answered successfully on the whole. Considering the equivalent problem when the plank is not moving may have led to considering $t = \frac{\alpha}{\omega}$, leading to the key idea that $F < \mu N$ when $t = 0$ and $F > \mu N$ when $t = \frac{\alpha}{w}$.

Section C: Probability and Statistics overview

In general, those candidates that attempted the statistics section appeared to be confident with the content and could apply their knowledge successfully, especially on question 12.

Question 11

11 **(i)** Let $\lambda > 0$. The independent random variables $X_1, X_2, \ldots, X_n$ all have probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

and cumulative distribution function $F(x)$.

The value of random variable Y is the largest of the values $X_1, X_2, \ldots, X_n$. Show that the cumulative distribution function of Y is given, for $y \geqslant 0$, by

$$G(y) = \left(1 - e^{-\lambda y}\right)^n.$$

(ii) The values $L(\alpha)$ and $U(\alpha)$, where $0 < \alpha \leqslant \frac{1}{2}$, are such that

$$P(Y < L(\alpha)) = \alpha \quad \text{and} \quad P(Y > U(\alpha)) = \alpha.$$

Show that

$$L(\alpha) = -\frac{1}{\lambda} \ln\left(1 - \alpha^{\frac{1}{n}}\right)$$

and write down a similar expression for $U(\alpha)$.

(iii) Use the approximation $e^t \approx 1 + t$, for $|t|$ small, to show that, for sufficiently large n,

$$\lambda L(\alpha) \approx \ln(n) - \ln\left(\ln\left(\frac{1}{\alpha}\right)\right).$$

(iv) Hence show that the median of Y tends to infinity as n increases, but that the width of the interval $U(\alpha) - L(\alpha)$ tends to a value which is independent of n.

(v) You are given that, for $|t|$ small, $\ln(1 + t) \approx t$ and that $e^3 \approx 20$.

Show that, for sufficiently large n, there is an interval of width approximately $4\lambda^{-1}$ in which Y lies with probability 0.9.

Parts (i) and (ii) were generally well done.

Candidates were often able to make good progress with parts (iv) and (v) even if they had found difficulty with part (iii) (since the answer to part (iii) was given in the question).

In part (iii), many candidates incorrectly assumed that $\alpha^{\frac{1}{n}} \to 0$, leading to the incorrect approximation $\ln\left(1 - \alpha^{\frac{1}{n}}\right) \approx -\alpha^{\frac{1}{n}}$

A significant number of candidates ignored the word 'hence' in part (iv), either:

- not realising that $L\left(\frac{1}{2}\right)$ was the median
- instead solving $G(m) = \frac{1}{2}$ directly to find the median m.Most candidates who attempted part (v) focused entirely on estimating the size of $U(0.05) - L(0.05)$, without ever stating that $P\big(L(0.05) < Y < U(0.05)\big) = 0.9$).

In parts (iii), (iv) and (v), a number of candidates did not give sufficient precision in the use of approximations/limits, for example writing asymptotic results as equalities which held for all n

Approximately half of the candidates implicitly utilised the identity $U(\alpha) = L(1 - \alpha)$. Whilst, formally, the bound $0 < \alpha \leq \frac{1}{2}$ given in the question invalidated this method unless the range of the arguments of L and U were first extended to $0 < \alpha < 1$, the identity allowed candidates to save considerable repetition of work and candidates who employed this method were not penalised on account of this technical ubtlety.

Question 12

12 **(i)** Show that, for any functions f and g, and for any $m \geqslant 0$,

$$\sum_{r=1}^{m+1} \left(\mathrm{f}(r) \sum_{s=r-1}^{m} \mathrm{g}(s) \right) = \sum_{s=0}^{m} \left(\mathrm{g}(s) \sum_{r=1}^{s+1} \mathrm{f}(r) \right).$$

(ii) The random variables $X_0, X_1, X_2, \ldots$ are defined as follows

- X_0 takes the value 0 with probability 1;
- X_{n+1} takes the values $0, 1, \ldots, X_n + 1$ with equal probability, for $n = 0, 1, \ldots$.

(a) Write down $\mathrm{E}(X_1)$.

Find $\mathrm{P}(X_2 = 0)$ and $\mathrm{P}(X_2 = 1)$ and show that $\mathrm{P}(X_2 = 2) = \frac{1}{6}$.

Hence calculate $\mathrm{E}(X_2)$.

(b) For $n \geqslant 1$, show that

$$\mathrm{P}(X_n = 0) = \sum_{s=0}^{n-1} \frac{\mathrm{P}(X_{n-1} = s)}{s + 2}$$

and find a similar expression for $\mathrm{P}(X_n = r)$, for $r = 1, 2, \ldots, n$.

(c) Hence show that $\mathrm{E}(X_n) = \frac{1}{2}\left(1 + \mathrm{E}(X_{n-1})\right)$.

Find an expression for $\mathrm{E}(X_n)$ in terms of n, for $n = 1, 2, \ldots$.

Most candidates answered all parts of this question well, with many candidates earning full or close to full marks.

In part (i), a small number of candidates erroneously believed that

$$\sum_{r=1}^{m+1}\left(f(r)\sum_{s=r-1}^{m} g(s)\right) = \left(\sum_{r=1}^{m+1} f(r)\right) \times \left(\sum_{s=r-1}^{m} g(s)\right)$$

and likewise for the second sum. Such attempts earned no credit.

In part (ii) (a), a significant number of candidates did not understand the concept of X_{n+1} being uniformly distributed on $\{0,1,\dots,X_n+1\}$, usually leading to the incorrect values.

In part (ii) (b), a number of candidates gave no justification of the written result, simply writing

$$P(X_n = 0) = \frac{1}{2}P(X_{n-1} = 0) + \frac{1}{3}P(X_{n-1} = 1) + \cdots + \frac{1}{n+1}P(X_{n-1} = n-1) = \sum_{s=0}^{n-1}\frac{P(X_{n-1} = s)}{s = 2};$$

such attempts earned no credit.

In part (ii) (c), most candidates solved this part either by inductively proving that $E(X_n) = 1 - 2^{-n}$ or by noting that $E(X_n) - 1 = \frac{1}{2}[E(X_{n-1}) - 1]$ and applying recursion. A smaller number of candidates applied recursion directly to the formula $E(X_n) = \frac{1}{2}[E(X_{n-1}) + 1]$ leading to a correct solution via geometric series.