An Exponential Polynomial and its Connections

Derek Bryant
University of Virginia's College at Wise
Advisor: Dr. Matthew Harvey

April 12, 2013

Abstract

There exists a closed-form expression equivalent to an integral that is commonly encountered in the study of Calculus. After proof of this formula, special cases are investigated and connections to other famous formulas are found. Finally, it is shown that the sums of permutations of roots allow for a solution to be found for any given constant values present in the integral.

Contents

1 Introduction

There is a very simple recursive formula for $\int e^{rx}x^n \mathrm{d}x$ that is derived by the method of Integration by Parts:

$$\int e^{rx}x^n \mathrm{d}x = \frac{1}{r}e^{rx}x^n - \frac{n}{r}\int x^{n-1}e^{rx}\mathrm{d}x\ . \tag{1.1}$$

This gives the recursive formula. Most math texts I have seen stop there and delve further into integral equations and the like, but I was curious to see if a pattern developed. To my surprise, there was one.

1.1 Derivation of a Summation Formula

Evaluating the integral on the right-hand side of the formula above expands it into the following:

$$\int e^{rx}x^n \mathrm{d}x = \frac{1}{r}e^{rx}x^n - \frac{n}{r}\left(\frac{1}{r}x^{n-1}e^{rx} - \frac{(n-1)}{r}\int x^{n-2}e^{rx}\mathrm{d}x\right).$$

Distributing the n term through the large parenthesis and evaluating the integral, we see a falling factorial term $n(n-1)(n-2)\cdots(n-k+1)$ after k derivations develop. So we have that $\int e^{rx}x^n \mathrm{d}x$ can be expressed as

$$e^{rx}x^n - \frac{n}{r^2}x^{n-1}e^{rx} - \frac{n(n-1)}{r^2}\left(\frac{1}{r}x^{n-2}e^{rx} - \frac{(n-2)}{r}\int x^{n-3}e^{rx}\mathrm{d}x\right).$$

Continuing the pattern, it becomes obvious that each term, including the last, will have an e^{rx} multiplied by it. Noting that the falling factorial notation $n^{\underline{i}}$, read "n to the i falling," can be expressed as $n(n-1)\cdots(n-i+1) = \frac{n!}{(n-i)!}$, and restricting $n \in \mathbb{N}$ and $r \in \mathbb{R}$, we can define the summation formula as such:

$$\int e^{rx}x^n \mathrm{d}x = \frac{e^{rx}}{r^{n+1}}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right) + C.$$

To simplify the work, I defined a new function $\sigma_r^n(x)$:

$$\sigma_r^n(x) := \left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right).$$

This makes the summation formula for $\int e^{rx}x^n \mathrm{d}x$ into something a bit less cumbersome to read:

$$\int e^{rx}x^n \mathrm{d}x = \frac{e^{rx}}{r^{n+1}}\sigma_r^n(x) + C.$$

From formula 1.1 above, consider the trivial case when $n = 0$. It should be easy to see that if $n = 0$, the integrand reduces to e^{rx}, whose derivative is simple to find:

$$\int e^{rx}\mathrm{d}x = \frac{1}{r}e^{rx} + C.$$

Now let us see what happens when we apply our conjectured formula using the special value of $n = 0$. As with most applications of $n!$, we will let $0! = 1$. So for $n = 0$

$$\int e^{rx}x^0 \mathrm{d}x = \frac{e^{rx}}{r^{0+1}}\sigma_r^0(x) + C = \frac{e^{rx}}{r}\left(\sum_{i=0}^{0}(-1)^i 0^{\underline{i}}\, r^{0-i}x^{0-i}\right) + C.$$

Evaluating this leaves

$$\frac{e^{rx}}{r}((-1)^0 0^{\underline{0}}\, r^0 x^{0-0}) + C = \frac{e^{rx}}{r}((1)(1)(1)(1)) + C = \frac{1}{r}e^{rx} + C.$$

For $n = 1$ through $n = 5$, we have

Integral	Evaluation
$\int e^{rx}x^1\mathrm{d}x$	$\frac{e^{rx}}{r^2}(rx - 1) + C$
$\int e^{rx}x^2\mathrm{d}x$	$\frac{e^{rx}}{r^3}(r^2x^2 - 2rx + 2) + C$
$\int e^{rx}x^3\mathrm{d}x$	$\frac{e^{rx}}{r^4}(r^3x^3 - 3r^2x^2 + 6rx - 6) + C$
$\int e^{rx}x^4\mathrm{d}x$	$\frac{e^{rx}}{r^5}(r^4x^4 - 4r^3x^3 + 12r^2x^2 - 24rx + 24) + C$
$\int e^{rx}x^5\mathrm{d}x$	$\frac{e^{rx}}{r^6}(r^5x^5 - 5r^4x^4 + 20r^3x^3 - 60r^2x^2 + 120rx - 120) + C$

Figure 1: Evaluations from $n = 1$ to $n = 5$.

Compare the constants with the following values of $n^{\underline{i}} = \frac{n!}{(n-i)!}$, remembering that $0 \le i \le n$:

i	0	1	2	3	4	5
$n = 1$	1	1				
$n = 2$	1	2	2			
$n = 3$	1	3	6	6		
$n = 4$	1	4	12	24	24	
$n = 5$	1	5	20	60	120	120

Figure 2: Values of falling factorials.

Note that these values correspond exactly to the constants in front of each of x^{n-i}. Together with the knowledge that the signs alternate, we can see how each term could be represented by $(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}$ if we let our value of i run from 0 to n.

1.2 A Proof of the Formula

First, we reiterate a famous theorem.

Theorem (Second Fundamental Theorem of Calculus)**.** *If f a continuous function on an open interval I and a any point in I, if F is defined by*

$$F(x) = \int_a^x f(t)\mathrm{d}t,$$

then $F'(x) = f(x)$ at each point in I.

Let $F(x) = \frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)$ and $f(x) = e^{rx}x^n$. If we can show $F'(x) = f(x)$, then by the above theorem we will have proven the conjecture.

Note: It is important that the reader realize that we are not proving the Second Fundamental Theorem of Calculus. We are simply using the FTC to prove our conjecture.

Proof. To begin, we will define the function $\sigma_r^n(x)$ as such

$$\sigma_r^n(x) := \left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right).$$

Now we will derive $[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)]'$ by the Product Rule as follows:

$$\left[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)\right]' = \left[\frac{re^{rx}}{r^{n+1}}\sigma_r^n(x) + \frac{e^{rx}}{r^{n+1}}(\sigma_r^n(x))'\right] = e^{rx}\left[\frac{1}{r^n}\sigma_r^n(x) + \frac{1}{r^{n+1}}(\sigma_r^n(x))'\right].$$

Note that our strategy is to prove that $[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)]' = e^{rx}x^n$. Since we just showed that there exists an e^{rx} term on each side, we can cancel each by division and proceed, noting that we are trying to show that $\sigma_r^n(x) + (\sigma_r^n(x))' = x^n$. It therefore it follows that we should find $(\sigma_r^n(x))'$:

$$(\sigma_r^n(x))' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right).$$

Remembering that the derivative of a finite sum is the sum of the derivatives we have

$$(\sigma_r^n(x))' = \left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}\frac{\mathrm{d}}{\mathrm{d}x}x^{n-i}\right).$$

So by the Power Rule this evaluates to

$$(\sigma_r^n(x))' = \left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\, r^{n-i}(n-i)x^{n-i-1}\right).$$

The careful reader may have noticed that the nth term was effectively canceled when the derivative was taken. This is because it was the topmost term and for $i = n$ was a constant. The derivative of a constant is zero. With that said, putting this all back together, we have that $[e^{rx}\sigma_r^n(x)]'$ can be expressed as

$$\left[\frac{1}{r^n}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right) + \frac{1}{r^{n+1}}\left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\, r^{n-i}(n-1)x^{n-i-1}\right)\right].$$

We may now recall that what we are dealing with is assumed to be x^n. Setting our expression equal to x^n sees us arrive at what is the key to our proof, which is that

$$\frac{1}{r^n}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right) + \frac{1}{r^{n+1}}\left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\, r^{n-i}(n-1)x^{n-i-1}\right) = x^n. \tag{1.2}$$

However, we cannot use this result just yet, since we have yet to prove that our formula is actually true. So compacting this down, we must now prove that

$$\frac{1}{r^n}\sigma_r^n(x) + \frac{1}{r^{n+1}}(\sigma_r^n(x))' = x^n.$$

To simplify our work, we will use these identities to distribute each of the fractions through the sums in equation 1.2 to see that we are now trying to prove that

$$\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, \frac{1}{r^i}x^{n-i}\right) + \left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\, \frac{1}{r^{i+1}}(n-1)x^{n-i-1}\right) = x^n.$$

To do this, we will first note that, for $k = n - i$,

$$\frac{r^{n-i}}{r^n} = r^{-i} = \frac{1}{r^i} = \frac{1}{r^{n-k}}.$$

Let $k = n-i$. Note that as i ranges from 0 to n, k ranges from n to 0. Since we are working with a summation, k also ranges from 0 to n. So rewriting $\sigma_r^n(x)$ with our new variable k, noting again that

$$n^{\underline{i}} = n(n-1)\dots(n-i+1) = \frac{n!}{(n-i)!}.$$

From this, we obtain

$$\frac{1}{r^n}\sigma_r^n(x) = \frac{1}{r^n}\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}r^k x^k = \sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\frac{1}{r^{n-k}}x^k.$$

Recall that we found $(\sigma_r^n(x))'$ earlier. Note that when we change our variables using $k = n-i$, we reverse the order of the polynomial. Because of this, instead of running from $i = 0$ to $n-1$ in order to omit the last term $i = n$, we run from $k = 1$ to n to omit the first term $k = 0$. Knowing this, we now have

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))' = \frac{1}{r^{n+1}}\sum_{k=1}^{n}(-1)^{n-k}\frac{n!}{k!}r^k k x^{k-1}.$$

Canceling the k with the first term in $k!$ yields

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))' = \frac{1}{r^{n+1}}\sum_{k=1}^{n}(-1)^{n-k}\frac{n!}{(k-1)!}r^k x^{k-1}.$$

To make this easier to work with, we will shift the index down one to get

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))' = \frac{1}{r^{n+1}}\sum_{k=0}^{n-1}(-1)^{n-(k+1)}\frac{n!}{k!}r^{k+1}x^k.$$

Factoring out a negative term and reincorporating the n^{th} terms gives

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))' = \frac{1}{r^{n+1}}\left[\left(-\sum_{k=0}^{n-1}(-1)^{n-k}\frac{n!}{k!}r^{k+1}x^k\right) - \frac{n!}{n!}r^{n+1}x^n + \frac{n!}{n!}r^{n+1}x^n\right].$$

We will now note the identity

$$\frac{r^{k+1}}{r^{n+1}} = r^{-(n-k+1)-1} = r^{-(n-k-1+1)} = \frac{1}{r^{n-k}}.$$

Distributing through the outermost fraction allows us to cancel both of the r^{n+1} terms attached to their respective x^n terms and gives that

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))' = \left[\left(-\sum_{k=0}^{n-1}(-1)^{n-k}\frac{n!}{k!}\frac{1}{r^{n-k}}x^k\right) - \frac{n!}{n!}x^n + \frac{n!}{n!}x^n\right].$$

Collecting the terms gives us that

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))' = \left(-\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\frac{1}{r^{n-k}}x^k\right) + x^n.$$

Noting that we have just shown that $\frac{1}{r^{n+1}}(\sigma_r^n(x))' = -\frac{1}{r^n}\sigma_r^n(x) + x^n$, surely

$$\frac{1}{r^n}\sigma_r^n(x) + \frac{1}{r^{n+1}}(\sigma_r^n(x))' = x^n.$$

Therefore $[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)]' = e^{rx}x^n$ and by the Second Fundamental Theorem of Calculus,

$$\int e^{rx}x^n \mathrm{d}x = \frac{e^{rx}}{r^{n+1}}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right) + C. \tag{1.3}$$

□

1.3 The Definite Integral

Now we call on the other part of the FTC:

Theorem (First Fundamental Theorem of Calculus)**.** *If f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$ then*

$$\int_a^b f(x)\mathrm{d}x \ = F(b) - F(a).$$

We have found a closed-form expression which is equal to the indefinite integral of $e^{rx}x^n$. Then it would make sense that

$$\int_a^b e^{rx}x^n\mathrm{d}x \ = \frac{1}{r^{n+1}}[e^{br}\sigma_r^n(b) - e^{ar}\sigma_r^n(a)]. \tag{1.4}$$

2 Relation to the Gamma Function

First, we need an aside. Consider $(-1)^{n+1}(-1)^i(-1)^{n-i}$. Using laws of exponents, we can simplify this expression considerably

$$(-1)^{n+1}(-1)^i(-1)^{n-i} = (-1)^{(n+1)+i+(n-i)} = (-1)^{n+n+(i-i)+1} = (-1)^{2n+1}.$$

Recall that any integer of the form $2n + 1$ is odd by definition, and that any number raised to any odd power is simply retains its sign. Therefore we have that $(-1)^{2n+1} = -1$. This will be useful when considering the case when we have that $r = -1$:

$$\int e^{-x}x^n\mathrm{d}x \ = \frac{e^{-x}}{(-1)^{n+1}}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\,(-1)^{n-i}x^{n-i}\right) + C.$$

Note that $\frac{e^{-x}}{(-1)^{n+1}} = (-1)^{n+1}e^{-x}$. Further, we are allowed to distribute the $(-1)^{n+1}$ term through the sum and gather like bases to obtain

$$\int e^{-x}x^n\mathrm{d}x \ = e^{-x}\left(\sum_{i=0}^{n}(-1)^{n+1}(-1)^i(-1)^{n-i}n^{\underline{i}}\,x^{n-i}\right) + C.$$

Using our aside and recalling that we may factor out the -1 term, we now have that

$$\int e^{-x}x^n\mathrm{d}x \ = -e^{-x}\left(\sum_{i=0}^{n}n^{\underline{i}}\,x^{n-i}\right) + C. \tag{2.1}$$

This is interesting enough on its own just because of its beautiful simplicity, yet it looks quite close to another very famous function.

Definition. *The* gamma function $\Gamma(n)$ *is an extension of the factorial function* $n!$ *from the integers to the real and complex numbers. Since we are not concerned with complex numbers, the gamma function is defined for all positive real numbers. Under these restrictions, it is defined by an improper integral that converges, namely*

$$\Gamma(j) = \int_0^\infty e^{-x}x^{j-1}\mathrm{d}t = \lim_{t\to\infty}\int_0^t e^{-x}x^{j-1}\mathrm{d}x\ . \tag{2.2}$$

We will prove that $\Gamma(n+1) = n!$. Once proven, for $j = n+1$, equation 2.2 becomes

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^{n}\mathrm{d}t = \lim_{t\to\infty}\int_0^t e^{-x}x^{n}\mathrm{d}x\ = n!.$$

2.1 A Proof of $\Gamma(n+1) = n!$

Usually the proof for this identity involves a proof using Integration by Parts and the Principle of Mathematical Induction, but we have developed a summation formula based on Integration by Parts that we may use to do away with an induction proof entirely.

Proof. Recall equation 1.4:

$$\int_a^b e^{rx}x^n\mathrm{d}x\ = \frac{1}{r^{n+1}}[e^{br}\sigma_r^n(b) - e^{ar}\sigma_r^n(a)]. \tag{1.4}$$

Now we will evaluate it for $r = -1$. First, recall that

$$\sigma_{-1}^n(x) = -\sum_{i=0}^{n} n^{\underline{i}}\, x^{n-i}.$$

So we are evaluating

$$\int_a^b e^{-x}x^n\mathrm{d}x\ = e^{-b}\sigma_{-1}^n(b) - e^{-a}\sigma_{-1}^n(a).$$

Expanding each of the $\sigma_{-1}^n(x)$ functions gives that

$$\int_a^b e^{-x}x^n\mathrm{d}x\ = -e^{-b}\left(\sum_{i=0}^{n} n^{\underline{i}}\, b^{n-i}\right) + e^{-a}\left(\sum_{i=0}^{n} n^{\underline{i}}\, a^{n-i}\right).$$

Let us now consider $\sigma_{-1}^n(0)$.

$$\sigma_{-1}^n(0) = -\sum_{i=0}^{n} n^{\underline{i}}\, 0^{n-i} = \left(\sum_{i=0}^{n-1} n^{\underline{i}}\, 0^{n-i}\right) + n^{\underline{n}}\,(1) = n^{\underline{n}}\ = n!.$$

We now consider our integral for $r = -1$, $a = 0$, and $b = t$ as $t \to \infty$:

$$\lim_{t\to\infty}\int_0^t e^{-x}x^n \mathrm{d}x \ = \lim_{t\to\infty}[e^{-t}\sigma_{-1}^{n}(t) - e^{-0}\sigma_{-1}^{n}(0)].$$

Putting the limit with the relevant function and $\sigma_{-1}^{n}(0) = -n!$, we have that

$$\lim_{t\to\infty}\int_0^t e^{-x}x^n \mathrm{d}x \ = \lim_{t\to\infty} e^{-t}\sigma_{-1}^{n}(t) + n!.$$

Expanding the $\sigma_{-1}^{n}(t)$ term, we have

$$\lim_{t\to\infty}\int_0^t e^{-x}x^n \mathrm{d}x \ = -\lim_{t\to\infty} e^{-t}\left(\sum_{i=0}^{n} n^{\underline{i}}\, t^{n-i}\right) + n!.$$

In the numerator of our fraction, there is a polynomial of degree n in the variable t. In the denominator, there is the exponential function e^t. Since for $n > 0$ our limit gives an indeterminate form, we may apply l'Hôpital's rule, which allows us to take the derivative of both the polynomial and the exponential. This reduces the degree of the polynomial by one, but the limit still gives an indeterminate form. We may repeat this process $n-1$ more times to see that

$$\lim_{t\to\infty} e^{-t}\sigma_{-1}^{n}(t) = \lim_{t\to\infty}\frac{\sigma_{-1}^{n}(t)}{e^t} = 0.$$

Then we have that, for $j = n+1$,

$$\lim_{t\to\infty}\int_0^t e^{-x}x^n \mathrm{d}x \ = \int_0^\infty e^{-x}x^{j-1}\mathrm{d}x \ = \Gamma(j) = (j-1)! = \Gamma(n+1) = n!.$$

□

3 Vieta's Formulas

We begin with a definition.

Definition (pth symmetric sum)**.** *The pth symmetric sum, denoted ς_p, of a set to be the sum of the elements multiplied p at a time.*

As an example, consider the set $R_4 = \{r_4, r_3, r_2, r_1\}$, the set of each of the 4 roots of a degree 4 polynomial Then the symmetric sums of R are

$$\begin{aligned} \varsigma_1 &= r_1 + r_2 + r_3 + r_4, \\ \varsigma_2 &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4, \\ \varsigma_3 &= r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4, \\ \varsigma_4 &= r_1r_2r_3r_4. \end{aligned}$$

Knowing this, Vieta's Formulas state that for a polynomial of degree n with roots $r_1, r_2, \ldots, r_n$,

$$\varsigma_p = (-1)^p \cdot \frac{a_{n-p}}{a_n},$$

for $1 \leq p \leq n$ with $a_n \ldots a_0$ being the coefficients to any polynomial of general form

$$P(x) = \sum_{k=0}^{n} a_{n-k} x^{n-k}.$$

Recall our function $\sigma_r^n(x)$. It is a polynomial of degree n, so it is subject to Vieta's Formulas as well. Letting $k = n - i$ as usual, our function is expressed as

$$\sigma_r^n(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} r^k x^k.$$

Then we have our a_n, the first coefficient in our summation, which is when $k = 0$. So it is

$$a_n = (-1)^{n-n} \frac{n!}{n!} r^n = r^n.$$

Note that a_{n-p} is the pth coefficient in our summation, which is when $k = n-p$. So then we have

$$a_{n-p} = (-1)^{n-(n-p)} \frac{n!}{(n-p)!} r^{n-p} = (-1)^p \frac{n!}{(n-p)!} r^{n-p}.$$

Then we have that

$$\varsigma_p = (-1)^p \cdot \frac{a_{n-p}}{a_n} = (-1)^p \cdot \frac{(-1)^p \frac{n!}{(n-p)!} r^{n-p}}{r^n} = (-1)^{2p} \frac{n!}{(n-p)!} r^{n-p-n}.$$

Noting that $(-1)^{2p} = 1$ and $r^{n-p-n} = r^{-p}$, we have that

$$\varsigma_p = \frac{1}{r^p} \cdot \frac{n!}{(n-p)!} = \frac{n^{\underline{p}}}{r^p}. \tag{3.1}$$

This is simultaneously both powerful and fascinating. For instance, consider $\varsigma_n = r_1 r_2 \cdots r_n$. By Vieta's Formulas, we know that this is simply

$$\varsigma_n = \frac{1}{r^p} \cdot \frac{n!}{(n-n)!} = \frac{n!}{r^n}.$$

Even more powerful is $\varsigma_1 = r_1 + r_2 + \cdots + r_n$, which can be evaluated as

$$\varsigma_1 = \frac{1}{r^p} \cdot \frac{n!}{(n-1)!} = \frac{n}{r}.$$

Since each of these symmetric sums are the sums of roots taken p at a time with order mattering, they are permutations. We now make an observation that $n^{\underline{p}} = \frac{n!}{(n-p)!} = P_p^n$, where P_p^n is the permutation of n things taken p at a time, with order mattering.

3.1 Roots of Our Integral

For our integral $\int e^{rx}x^n \mathrm{d}x$, let $n = k$. Then there exists a vector

$$\hat{\xi_k} = \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{k-1} \\ \varsigma_k \end{pmatrix}.$$

Define a function $\mathbf{R}(\hat{\xi_k})$ that maps the $k \times 1$ vector $\hat{\xi_k}$ onto a square $k \times k$ matrix via

$$\mathbf{R}(\hat{\xi_k}) = \hat{\xi_k}^{\mathrm{T}} \mathbf{I}_{(k,k)},$$

where $\mathbf{I}_{(k,k)}$ is the $k \times k$ identity matrix. Then

$$\mathbf{R}(\hat{\xi_k}) = \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{k-1} \\ \varsigma_k \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \varsigma_1 & 0 & \cdots & 0 \\ 0 & \varsigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varsigma_k \end{pmatrix}.$$

Recalling equation 3.1 and that $n^{\underline{p}} = \frac{n!}{(n-p)!} = P_p^n$, we know that $\varsigma_k = \frac{P_p^n}{r^p}$. In order to condense this matrix equation into something more manageable, let there exist a $j \times 1$ vector $\hat{\beta}$ so that

$$\hat{\beta}^{\mathrm{T}} = \begin{pmatrix} \frac{P_1^k}{r^1} & \frac{P_2^k}{r^2} & \cdots & \frac{P_{j-1}^k}{r^{j-1}} & \frac{P_j^k}{r^j} \end{pmatrix} \quad (3.2)$$

with $1 \leq j \leq k$. Let $\hat{\rho}$ be a $j \times 1$ column vector containing in order each of

$$(\frac{1}{r^1}, \frac{1}{r^2}, \ldots, \frac{1}{r^{j-1}}, \frac{1}{r^j})$$

and $\hat{\psi}$ be a $1 \times j$ row vector containing in order each of j permuations

$$(P_1^k, P_2^k, \ldots, P_{j-1}^k, P_j^k).$$

Then we may rewrite equation 3.2 as

$$\hat{\beta} = \hat{\rho}\hat{\psi}. \quad (3.3)$$

Then to find solutions to the equation

$$\int e^{rx}x^n \mathrm{d}x = 0,$$

one must find all solutions to the nonlinear system $\mathbf{R}(\hat{\xi_k}) = \hat{\beta}$.

3.1.1 Using the Quadratic Formula to Find Roots

Say we wished to find all roots of the equation $\int e^{rx}x^2\mathrm{d}x\ = 0$. We know by equation 1.3 that we may rewrite the integral as follows:

$$\int e^{rx}x^2\mathrm{d}x\ = \frac{e^{rx}}{r^{2+1}}\sum_{i=0}^{2}(-1)^i 2^{\underline{i}}\, r^{2-i}x^{2-i} = 0.$$

Evaluating the sum gives $\frac{e^{rx}}{r^3}(r^2x^2 - 2rx + 2r) = 0$. We may multiply through both sides by r^3 and then use the zero factor property to note that either the exponential or the polynomial must be zero in order for their product to be zero. Since we know that exponentials can never equal zero, we must conclude that we are solving $r^2x^2 - 2rx + rx = 0$. This is a simple application of the quadratic formula. We have

$$x = \frac{-(-2r) \pm \sqrt{(-2r)^2 - 4(r^2)(2)}}{2(r^2)} = \frac{2r \pm \sqrt{4r^2 - 8r^2}}{2r^2} = \frac{2r \pm 2ri}{2r^2} = \frac{1 \pm i}{r}.$$

3.1.2 Using a Nonlinear System to Find Roots

The symmetric equations for a second degree polynomial are

$$\begin{aligned} \varsigma_1 &= r_1 + r_2 \\ \varsigma_2 &= r_1 r_2. \end{aligned}$$

Form $\mathbf{R}(\hat{\xi_k}) = \hat{\beta}$ as

$$\mathbf{R}(\hat{\xi_k}) = \begin{pmatrix} \varsigma_1 & 0 \\ 0 & \varsigma_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & 0 \\ 0 & r_1 r_2 \end{pmatrix} = \begin{pmatrix} \frac{P_1^2}{r^1} \\ \frac{P_2^2}{r^2} \end{pmatrix}.$$

So then we are solving the nonlinear matrix equation

$$\begin{pmatrix} r_1 + r_2 & 0 \\ 0 & r_1 r_2 \end{pmatrix} = \begin{pmatrix} \dfrac{2}{r^1} \\ \dfrac{2}{r^2} \end{pmatrix},$$

or written as a system

$$\begin{aligned} (1) \quad r_1 + r_2 &= \frac{2}{r} \\ (2) \quad r_1 r_2 &= \frac{2}{r^2}. \end{aligned}$$

Solving equation (1) for r_1 gives $r_1 = \frac{2}{r} - r_2$. Plugging this into equation (2) gives $(\frac{2}{r} - r_2)r_2 = \frac{2}{r^2} \Rightarrow \frac{2r_2}{r} - r_2^2 = \frac{2}{r^2}$. Placing all terms on one side, we now have the following quadratic:

$$-r_2^2 + \frac{2r_2}{r} - \frac{2}{r^2} = 0.$$

Then $a = -1$, $b = \frac{2}{r}$, and $c = -\frac{2}{r^2}$. Plugging into the quadratic equation then gives that

$$r_2 = \frac{-\left(\frac{2}{r}\right) \pm \sqrt{\left(\frac{2}{r}\right)^2 - 4(-1)\left(-\frac{2}{r^2}\right)}}{-2} = \frac{-\frac{2}{r} \pm \sqrt{\frac{4-8}{r^2}}}{-2} = \frac{-\frac{2}{r} \pm \frac{2i}{r}}{-2} = \frac{1 \pm i}{r}.$$

Since we know that complex roots always occur in conjugate pairs, we have found both roots. Yet we will plug both results back into equation (1) for the sake of checking our work.

$$\begin{array}{llll} (+) & \frac{2}{r} - \left(\frac{1+i}{r}\right) & = \frac{2-1-i}{r} & = \frac{1-i}{r} \\ (-) & \frac{2}{r} - \left(\frac{1-i}{r}\right) & = \frac{2-1+i}{r} & = \frac{1+i}{r}. \end{array}$$

So for either value of r_2, the value of r_1 assumes the conjugate, as required.

3.1.3 Advantages of Using a Nonlinear Approach

As we saw, using the nonlinear method still resulted in us solving a quadratic. Indeed, any application of this method on a degree n polynomial will result in a system with n equations and n unknowns. When solving them directly, one could expect to find a degree n polynomial set equal to zero after some intense basic algebra.

The strength in this approach is that while we cannot easily solve for the roots of most polynomials above degree four, there are several fields of mathematics that can use the language of nonlinear algebra in order to apply all the tools from that discipline. Algebraic geometry is an entire field concerned with zeros of polynomial equations, and differential geometry has plenty of uses for both nonlinear and multilinear algebra, and could provide an interesting way to view this problem.

While it is beyond the scope of this paper, further study could be done in these fields to see if there exists a formula dependent on n and r that could give the roots of

$$\int e^{rx} x^n \mathrm{d}x = 0.$$

With such a deep connection to permutations, the gamma function, and combinatorics in general, such a proof linking any form of advanced geometry to such topics would be extraordinary.