"Just Missing Fermat"

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1 Introduction

While the physical sciences strive to describe the real world, mathematics is concerned only with the ideal world. Sometimes mathematicians are inspired to propose conjectures which, after being proven, become theorems. The development of computer technology has given mathematicians a new method of finding inspiration: experimental mathematics. Experimental mathematics is essentially the use of computers to perform extensive numerical computations to investigate patterns and gain insight. An example of an accomplishment of experimental mathematics is the discovery of the *BBP formula for* π,

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right) \qquad (1)$$

The *BBP formula for* π is the first method able to calculate the *n*th digit of π without calculating the first $n - 1$ digits, and it was published by David H. Bailey, Peter Borwein and Simon Plouffe in 1995[1]. According to David H. Bailey, Peter Borwein and Simon Plouffe [2], equation (1) was discovered while searching for integer relations using the PSLQ algorithm. This is one example of the way in which experimental mathematics is useful. Computers can produce data and numbers that, otherwise, would be very difficult to produce in large quantity, or at all. It then falls upon the mathematician to take this information and find patterns and ideas within the numbers. It is the purpose of this paper to illustrate how the use of experimental mathematics can be used to search for patterns and gain insight into mathematical problems at the undergraduate level.

In *The Simpsons* episode *The Treehouse of Horrors VI*, during $Homer^3$, a mathematical formula passes along the screen which reads: $1782^{12} + 1841^{12} = 1922^{12}$ [3]. This equation is in direct violation of Fermat's Last Theorem[4] which states:

The equation $a^n + b^n = c^n$*, where* $a, b, c, n \in \mathbb{Z}$*, cannot be satisfied for any* $n > 2$

The question then arises: exactly how close is this equation? When expanded out it is found that the LHS (left hand side) of the equation differs from the RHS (right hand side) by approximately 7×10^{29}; this difference is an error of 10^{-7} percent, which is close enough to fool simple scientific calculators. The next natural question is then: given that such an equality can never happen, exactly how close can one actually get to such an equality? Thus the problem can be stated as such:

$$\text{For} \qquad a^n + b^n = c^n + D \qquad (2)$$
$$\text{where } a, b, c, n \in \mathbb{Z}, \text{what is the smallest possible value of } |D|?$$

We will begin by describing how this problem was approached and illustrate some of the interesting results produced. Next the results of the analysis will be discussed. Then the problem will be linked to work done by Euler and several other mathematicians. Finally, we will conclude with a summary of what has been accomplished.

2 Method of Investigation and Initial Results

The extensive computations of this research were carried out by a C++ computer program designed to search through a given range of integers for a particular n value, input them into equation (2), and then determine D. This program was run for $n = 3, \ldots, 10$ with an integer range up to 11000, with the exception of the $n = 3$ case. The results of these computations are given in Table 1. As can be seen in Table 1, a wide range of D values were found, the smallest being $|D| = 1$ for the $n = 3$ case. It makes sense that the $n = 3$ case would have the smallest $|D|$ value because it has the smallest exponent. After all, higher exponents produce larger numbers which would seem to decrease the chances of getting smaller $|D|$ values. In addition to this however, was the observation that for most values of n, the smallest $|D|$ value was only observed once and was always in the lower range of the search. Contrary to this trend, the $n = 3$ case presented its lowest $|D|$ value of one repeatedly and consistently throughout the entire range of search. Upon closer inspection, it was found that some of these "off-by-one" solutions came in pairs that appeared to display a recognizable pattern. We will refer to these pairs of solutions as "typical" solutions, and all other off-by-one solutions as "atypical".

Table 1: Smallest $|D|$ Values for Each n Value

n	$\|D\|$	Range
3	1	33000
4	16	11000
5	12	11000
6	666	11000
7	2070	11000
8	6561	11000
9	242973	11000
10	1000000	11000

Of further interest is the fact that these typical solutions presented themselves persistently as the bounds of the search range were extended. Table 2 gives several examples of off-by-one solutions, both typical and atypical. The ubiquity of these off-by-one solutions, along with the possible pattern of the typical solutions, made it difficult to conclude that these observations were merely a random occurrence.

3 Inspired Identities

The most obvious place to start was to attempt to describe the typical solutions as they appeared to have a pattern that was quite evident. This inspired two identities which produce every typical solution when various integer values of x are used.

$$(9x^4 - 3x)^3 + (9x^3 - 1)^3 = (9x^4)^3 - 1 \tag{3}$$
$$(9x^4)^3 + (9x^3 + 1)^3 = (9x^4 + 3x)^3 + 1 \tag{4}$$

By using large values of x, we can then produce off-by-one solutions beyond the range of the computer's search. For example, setting $x = 42$ in equation (3) yields the equation

$$28005138^3 + 666791^3 = 28005264^3 - 1$$

On the other hand these identities are still lacking as they do not return any atypical solutions. After further investigation it was found that another distinct pattern existed within similar solutions that were off-by-two. Table 3 shows some of these numbers. An equation was found that predicts all the off-by-two solutions:

$$(6x^3 - 1)^3 + (6x^2)^3 = (6x^3 + 1)^3 - 2 \tag{5}$$

Given that all off-by-two solutions can be generated by equation (5), it seems reasonable to think that the atypical off-by-one solutions might also follow a quantifiable pattern. However, no definable pattern was found among the atypical solutions. The next course of action considered was to look at related topics in an attempt to find connections that might shed light on the original problem.

Table 2: Off-By-One Solutions for $n = 3$

a	b	c	D
6	8	9	-1
9	10	12	+1
94	64	103	+1
138	71	144	-1
144	73	150	+1
426	372	505	-1
438	334	495	-1
486	426	577	+1
⋮	⋮	⋮	⋮
21588	3086	21609	+1
21609	3088	21630	-1
24965	3453	24987	+1
27630	17328	29737	+1
31180	10876	331615	+1
⋮	⋮	⋮	⋮

4 Taxicab Numbers and Fermat's Last Theorem

The first two connections discovered are of interest primarily because of their mathematical history, rather than having any substantial effect on the research. The second solution given in Table 2 is

$$9^3 + 10^3 = 12^3 + 1^3 \tag{6}$$

This equation has an interesting history, as the number 1729 produced by both sides of the equation is the very first known Taxicab number, $Taxicab(2)$. $Taxicab(n)$ denotes the smallest number that can be written as the sum of two positive cubes in n distinct ways. The term taxicab number was coined because of a famous exchange between the mathematicians G. H. Hardy and Srinivasa Ramanujan. The discussion began with Hardy commenting that the number of the cab he had just ridden in was 1729, and he remarked that it seemed to him a rather dull number. Ramanujan disagreed and noted that the number 1729 was the smallest number that could be written as the sum of two cubes in two distinct ways[5].

Table 3: Off-By-Two Solutions for $n = 3$

a	b	c	D
5	6	7	-2
47	24	49	-2
161	54	163	-2
383	96	385	-2
749	150	751	-2
1295	216	1297	-2
⋮	⋮	⋮	⋮

The original problem investigated by this paper is linked to Taxicab numbers not only through equation (6) but also through FLT. In August 1657 Fermat sent a letter to another mathematician by the name of Kenelm Digby in which he proposed some questions[6]. Two of these questions were:

1. *Find two cube numbers of which the sum is a cube.*
2. *Find two cube numbers of which the sum is equal to two other cube numbers.*

The first question can be recognized as asking for a counterexample to FLT for the $n = 3$ case. The second question is asking for what would later become known as $Taxicab(2)$. As it turns out the original discoverer of $Taxicab(2)$ was a mathematician by the name of Frenicle de Bessy who found it as the solution to Fermat's second question around October in 1657. Thus it can be seen that the problem of this research is connected to Taxicab numbers in two different ways, although perhaps it would be more reasonable to say that FLT is the root of both taxicab numbers and our problem. As stated earlier, the actual significance of these connections is somewhat limited to the interest of mathematical history. While this minor discovery did shed light on the fact that equation (4) can generate infinitely many numbers which fit the second question stated by Fermat, they are not particularly useful or interesting, even more so since Taxicab numbers are only the smallest number to fit each condition. However, this does illuminate something interesting about the process of experimental mathematics: that the search for answers and patterns within large amounts of data forces one to consider many more options than one might do when conventionally investigating a problem. While this might result in something that is not particularly useful to the problem, like we have just seen, it is also possible that it will further the investigation. The next connection is an example of the latter.

5 Euler's Conjectures

In 1769 the mathematician Leonhard Euler conjectured that "at least n nth powers are required, for $n > 2$, to provide a sum that is itself an nth power". This is known as *Euler's Sum of Powers Conjecture* (ESC). This conjecture is essentially a broader version of FLT, since FLT is in fact a special case of ESC.

In 1967 ESC was shown to be false by L. J. Lander and T. R. Parkin[7]. In 1998 Ekl defined what he called Euler's Extended Conjecture (EEC)

$The\ Diophantine\ equation\ (k, m, n),$

$$a_1^k + a_2^k + \cdots + a_m^k = b_1^k + b_2^k + \cdots + b_n^k \tag{7}$$

$has\ no\ integral\ solution\ for\ k > m + n\ other\ than\ the\ trivial\ case\ when\ all\ x_i = y_i$ [8]

It just so happens that Euler, even before EEC was even stated, had found the complete rational parametric formula for the special case of $(3,2,2)$[9] which corresponds to the equation:

$$a_1^3 + a_2^3 = b_1^3 + b_2^3 \tag{8}$$

Euler's general formula for equation (8) can be written as

$$(p+q)^3 + (p-q)^3 = (r+s)^3 + (r-s)^3 \tag{9}$$

$where$

$$p = 3(bc - ad)(c^2 + 3d^2),\quad q = (a^2 + 3b^2)^2 - (ac + 3bd)(c^2 + 3d^2),$$
$$r = 3(bc - ad)(a^2 + 3b^2),\quad s = -(c^2 + 3d^2)^2 + (ac + 3bd)(a^2 + 3b^2)$$

It can be seen that equation (8) is equivalent to equation (2) if we set any of the four terms to one. We thought that the particular substitutions required in order to set one term to equal one might yield either equation (3) or (4). If this connection could be made, then it might be possible that there exist other methods of producing an off-by-one equation from equation (9), and that it may account for the atypical solutions. The following substitutions were found to produce equation (3) and equation (4):

Setting $a = 1$, $b = 0$, $c = \frac{3x}{2}$, and $d = \frac{x}{2}$ yields $(9x^4 - 3x)^3 + (9x^3 - 1)^3 = (9x^4)^3 - 1$ which is equivalent to equation (3).

Setting $a = \frac{3x}{2}$, $b = \frac{x}{2}$, $c = 1$, and $d = 0$ yields $(9x^4)^3 + (9x^3 + 1)^3 = (9x^4 + 3x)^3 + 1$ which is equivalent to equation (4).

Next an alternative approach to finding atypical solutions from equation (9) was attempted. A program was used to do a four parameter search through a given range of integers and half-integers,$(0, 0.5, 1, 1.5, 2, \ldots)$. The half-integer values were used in order to account for the $\frac{n}{2}$ in the substitutions used to derive equations (3) and (4) from equation (9). The program then used these values in equation (9) and returned any off-by-one solutions. It was found that all of the typical solutions were returned. Most of the atypical solutions were not found and those that were did not suggest any new patterns. As a result we were not able to create any new equations to describe the atypical solutions.

Due to the fact that very specific decimal numbers, the half-integers, were able to produce off-by-one solutions from equation (9), it was thought that atypical solutions might be produced by other sets of non-integers. The following substitutions for equation (9) produced equation (3) and equation (4):

Setting $a = 0$, $b = \frac{1}{\sqrt{3}}$, $c = \frac{\sqrt{3}}{2}x$ and $d = \frac{\sqrt{3}}{2}x$ yields equation (3).

Setting $= \frac{\sqrt{3}}{2}x$, $b = \frac{\sqrt{3}}{2}x$, $c = 0$ and $d = \frac{1}{\sqrt{3}}$ yields equation (3).

This discovery confirms that it is possible to produce off-by-one equations using irrational parameters in Euler's equation. However, no substitutions were found that lead to any of the atypical solutions. Furthermore, while a search using integer parameters is very simple for a computer to perform, a computer is ill-suited to run a search with irrational parameters since there is no simple way for a computer to increment systematically through irrational numbers.

6 Conclusions

While Fermat's Last Theorem shows that the equation $a^n + b^n = c^n$ has no integer solutions when $n > 2$, we have shown in this paper that it is possible to get remarkably close to such an equality when $n = 3$. Along with the fact that there are a large number of examples that are only off by one, we have shown that a great many of these examples display a very specific pattern. This pattern led to a pair of identities that were able to produce all of the typical solutions found by a computer program as well as predicts typical solutions which had not been calculated. Furthermore, yet another identity was found that was able to describe all solutions that are off-by-two. Although the presence of the atypical solutions was unable to be accounted for, the connection of our new identities to Euler's general equation provides the possibility of finding a way to describe all solutions.

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[1] http://mathworld.wolfram.com/BBPFormula.html

[2] http://www.ams.org/samplings/feature-column/fcarc-pi

[3] http://homepage.smc.edu/nestler_andrew/SimpsonsMath.htm

[4] Kleiner, Israel. "From Fermat to Wiles: Fermat's Last Theorem Becomes a Theorem". Elemente der Mathematik, 55 (1), February 2000

[5] http://euler.free.fr/taxicab.htm

[6] Boyer, Christian. "New Upper Bounds for Taxicab and Cabtaxi Numbers". Article 08.1.6. http://www.cs.uwaterloo.ca/journals/JIS/vol11.html

[7] http://mathworld.wolfram.com/EulersSumofPowersConjecture.html

[8] Ekl, Randy L. "New Results in Equal Sums of Like Powers". Mathematics of Computation, 67, July 1998

[9] Dickson, L. "History of the Theory of Numbers Vol. 2". Carnegie Institute of Washington, 1919. Electronic