

Name \_\_\_\_\_

# UNIT I:

## LIMITS & CONTINUITY

### NOTES, HOMEWORK

### and REVIEW



# Lesson I: The Concept of Instantaneous Rate of Change

## Topic 1.1: Introducing Calculus: Can Change Occur at an Instant?

As students of life, and algebra, you have studied many rates of change for different situations: how fast we drive, how our savings grow over time, the rate that the world population changes, our growth rate every year, unemployment rates, and so much more. In each of these studies, we can express one variable in terms of another, that is, as a function,  $y = f(x)$ .

### Velocity

Consider an object that travels in a straight line. The **average velocity** over some given time interval can be defined as the ratio of the change in position (**displacement or net change**) to the elapsed time period:

Average Velocity =

However, when we think of velocity, we usually mean **instantaneous velocity**, which indicates the speed and direction of some object at a particular moment, which could be faster or slower than the average velocity.

Can you explain why we can't define instantaneous velocity as a ratio?

How can you estimate an instantaneous velocity?

Let's generalize that big idea with this principle:

*Average velocity over a very small time interval is very close to instantaneous velocity.*

**EX #1:** If you climb to the top of the Tower of Pisa and drop a coin to the ground, estimate the instantaneous velocity at  $t = 0.7$  seconds. Use Galileo's formula  $s(t) = 16t^2$  to compute the average velocity, in feet/second, over the time intervals listed in the table below.

Time Interval	Average Velocity
[0.7, 0.71]	22.56
[0.7, 0.705]	
[0.7, 0.7001]	
[0.7, 0.70005]	
[0.7, 0.700001]	

What is your estimate for the **Instantaneous Rate of Change (IROC)** at  $t = 0.7$  second?

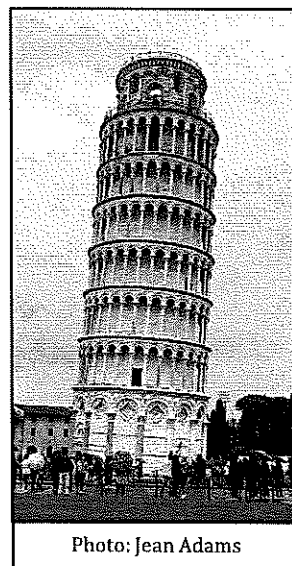


Photo: Jean Adams

## THE STUDY OF RATE OF CHANGE IS FUNDAMENTAL TO THE STUDY OF CALCULUS.

In our first example, we allowed the time interval to shrink to zero, so can say that *the average velocity converges to the instantaneous velocity* or that the *instantaneous velocity is the limit of the average velocity*.

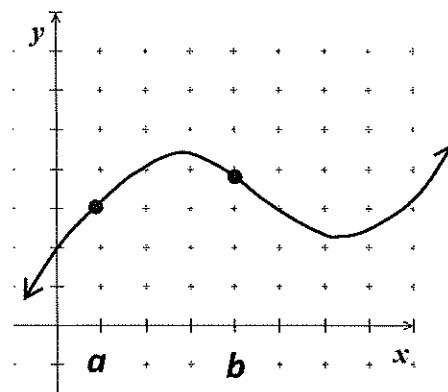
### Graphical Representation of Velocity

**Average Speed (AROC)** is defined by the total distance traveled,  $d$ , divided by the elapsed time period,  $t$ .

**Graphically**, the average speed is the slope of the secant line.

**Analytically**, average speed  $AROC = \frac{\Delta d}{\Delta t}$

Sketch the secant line through  $a$  and  $b$ . Estimate the average velocity.

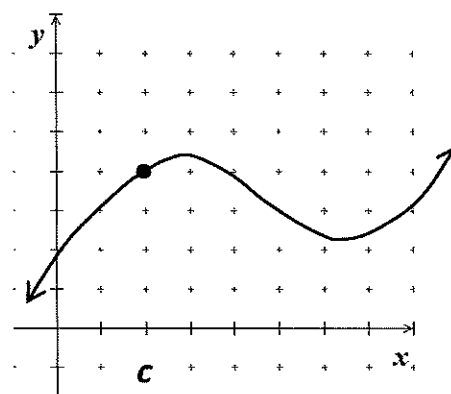


**Instantaneous Speed (IROC)** is the speed at a specific instant in time, where  $t = t_0$  is the limit of the average rates of change.

**Graphically**, the instantaneous speed is the slope of the tangent line at a specific point.

**Analytically**, we need *calculus* to determine this instantaneous speed.

Sketch the tangent line through point  $c$ . Can you estimate the instantaneous velocity at  $x = c$ ?



### EX #2: Spin Class Heart Rate and Exercise

Have you ever taken a Spin Class at the gym? Then you know that after the class is over, your rapid heart rate decreases as time passes. Let  $H(t)$  represent your heart rate. Write an expression for the rate of change of your heart rate over the period from  $t = 0$  seconds to  $t = 90$  seconds, after you have stopped exercising.



### EX #3: Roller Coaster

A **hypercoaster** is a special circuit roller coaster with a height measuring greater than 200 feet. If a new hypercoaster is under design to have its largest drop on the circuit modeled by the equation below, where  $d(t)$  is measured in feet, and time  $t$ , is measured in seconds.

$$d(t) = 1.87t^3 - 20.64t^2 + 112.79t + 132.56$$

- A. Find the average speed (velocity) of the hypercoaster from 2 seconds to 8 seconds.
- B. What is the average speed (velocity) of the hypercoaster from 4 seconds to 6 seconds?
- C. Estimate the instantaneous speed (velocity) of the hypercoaster at exactly 5 seconds.

Time Interval	Average Velocity
[5, 5.1]	
[5, 5.01]	
[5, 5.001]	

- D. Explain the meaning of the instantaneous speed (velocity) in the context of the question at  $t = 5$  seconds.

#### EX #4: Airport Departures



The traffic pattern of departing flights at Orlando International Airport on a fall afternoon can be modeled by the function  $V$  defined by  $V(t) = 92 - 15 \sin\left(\frac{t}{3}\right)$  where  $V(t)$  is measured in vehicles and  $t$  is measured in minutes  $0 \leq t \leq 30$ .

A. Find  $V(t)$  for  $t = 18$ .



B. Make a sketch of the graph of time vs. number of vehicles. Be sure your graph has a scale.

C. What is the average rate of change of the vehicles at the departing flights over the time interval  $15 \leq t \leq 20$ ? Indicate units of measure.

D. Estimate the instantaneous rate of change of vehicles at  $t = 18$  by finding the average rates from  $t = 18$  to  $t = 18.1$ ,  $t = 18$  to  $t = 18.01$ , and  $t = 18$  to  $t = 18.001$ .

Time Interval	Average Velocity
[18, 18.1]	
[18, 18.01]	
[18, 18.001]	

E. Why can't the instantaneous rate of change of traffic in the departure lane with respect to time be calculated using the method in part D?



# The Concept of Instantaneous Rate of Change

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

1. An object dropped from a state of rest at time  $t = 0$  travels a distance  $s(t) = 4.9t^2$  meters in  $t$  seconds.

A. How far does the object travel during the time interval  $[2.5, 3]$ .

B. Find the average velocity of the object over  $[2.5, 3]$ .

C. Estimate the object's instantaneous velocity at  $t = 2.5$  seconds.

Interval	$[2.5, 2.51]$	$[2.5, 2.505]$	$[2.5, 2.5001]$
Average Velocity			

2. Suppose Neil Armstrong decided to throw a golf ball into the air while he was standing on the moon and that the height of the golf ball was modeled by the equation below, where  $s$  is measured in feet and  $t$  is measured in seconds  $s(t) = -2.72t^2 + 26.9t + 6$ . Find the best approximation for the instantaneous rate of change (velocity) of the golf ball at 7 seconds.

3. A pendulum swings from the ceiling. Its distance,  $d$ , in feet, from one wall of the room depends on the number of seconds,  $t$ , since it was set in motion. Assume that the equation for  $d$  as a function of  $t$  is  $d(t) = 20 + 16 \cos\left(\frac{\pi t}{3}\right)$ . You want to find out how fast the pendulum is moving at a given instant,  $t$ , and whether it is approaching or going away from the wall.

A. Find  $d$  when  $t = 4$ . What mode should your calculator be set?

B. Estimate the instantaneous rate of change of  $d$  with respect to  $t$  when  $t = 2.5$ . At that time, is the pendulum approaching the wall or moving away from it? Explain how you know.

4. Let  $C(t)$  be the concentration of a drug in the bloodstream. As the body eliminates the drug,  $C(t)$  decreases at a rate that is proportional to the amount of the drug that is present at the time. Write an algebraic expression to represent the average rate of change of the drug's concentration for the period  $t = 0$  hour to  $t = 4$  hours after the drug has been administered.

5. Compute  $\Delta y/\Delta x$  for the interval  $[2,6]$ , where  $y = 4x - 7$ . What is the instantaneous rate of change of  $y$  with respect to  $x$  at  $x = 2$ ?

6. An initial deposit of \$500 in your bank will have a balance after  $t$  years given by the equation  $P(t) = 500(1.06)^t$  dollars.

A. What are the units of the rate of change of  $P(t)$ ?

B. Find the average rate of change over  $[0, 1]$ .

C. Estimate the instantaneous rate of change at  $t = 1$  by computing the average rate of change over intervals to the left and right of  $t = 1$ .

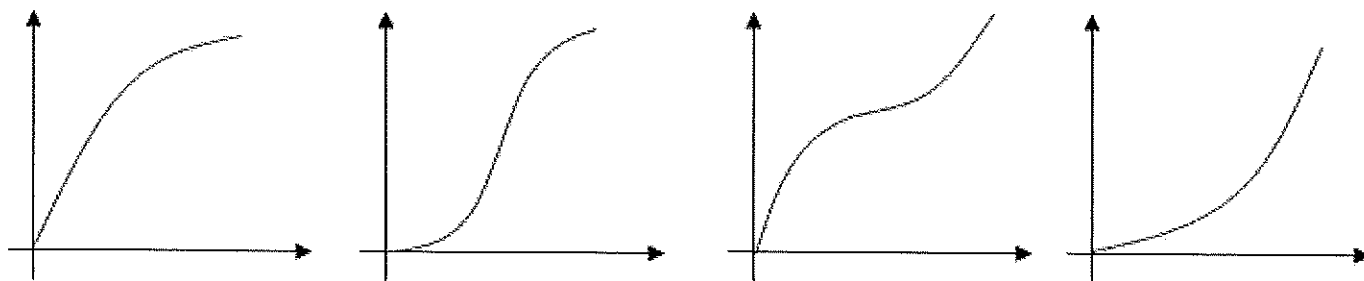
7. Each position graph shown below represents particle motion as a function of time. Match each graph with the proper description:

A) Speeding up

B) Speeding up and then slowing down

C) Slowing down

D) Slowing down and then speeding up



# Lesson 2: Understanding Limits Graphically and Numerically

## Topic 1.2: Defining Limits and Using Limit Notation

Limits are the “backbone” of understanding that connect algebra and geometry to the mathematics of calculus. In basic terms, a limit is just a statement that tells you what height a function *INTENDS TO REACH* as you get close to a specific  $x$ -value. Recall from Pre-Calculus that you evaluated three types of limits. Complete the table below:

### PROPER LIMIT NOTATIONS

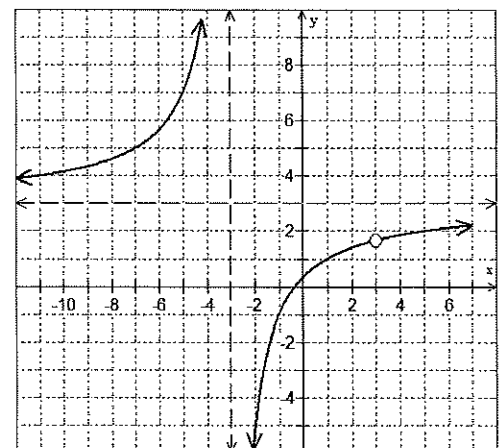
TYPE OF LIMIT	PROPER NOTATION	VERBALLY:
Right-hand limit		
Left-hand limit		
General limit		

Let's begin our discussion of limits by analyzing a rational function and examining the graph.

**EX #1:** Use the equation for  $f(x)$  and the graph of the function to analyze completely.

$$f(x) = \frac{3x^2 - 8x - 3}{x^2 - 9}$$

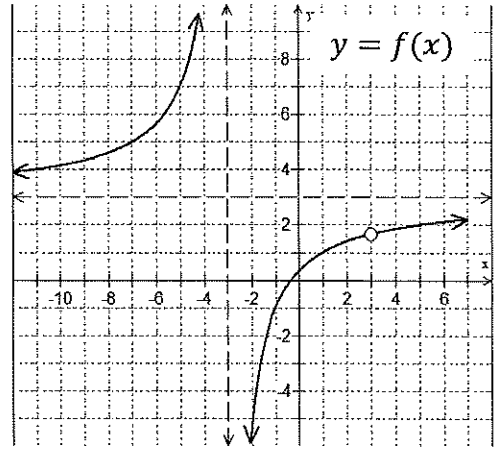
Factor and Simplify	
Coordinates of Hole	
Domain	
Range	
Vertical Asymptote	
Horizontal Asymptote	



## Understanding Limit Notation

Let's revisit the notation we learned in Pre-Calculus. We need to convert these old methods of explaining extreme behaviors into limit notation for use in calculus.

**EX #2:** Use the graph to complete the table below.



### Pre-Calculus Notation vs. Calculus Notation

As $x \rightarrow -\infty$ , the graph of $f(x) \rightarrow$ _____	As $x \rightarrow \infty$ , the graph of $f(x) \rightarrow$ _____
As $x \rightarrow -3$ from the right, the graph of $f(x) \rightarrow$ _____	As $x \rightarrow 3$ from the left, the graph of $f(x) \rightarrow$ _____
As $x \rightarrow -3$ from the left, the graph of $f(x) \rightarrow$ _____	As $x \rightarrow 3$ from the right, the graph of $f(x) \rightarrow$ _____

*Can you explain what the value of a limit represents in terms of the graph?*

Informally, a **limit is a y-value** which a function approaches as  $x$  approaches some value.

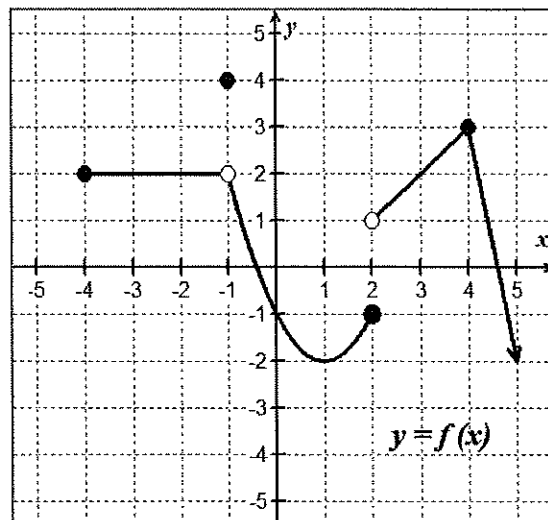
$\lim_{x \rightarrow c} f(x) = L$  means as  $x$  approaches  $c$ ,  $f(x)$  approaches the  $y$ -value of  $L$ .

### Topic 1.3: Estimating Limit Values from Graphs

Consider the function shown at right.

Say you want to find  $\lim_{x \rightarrow 4^+} f(x)$ , the positive sign in the limit notation indicates a right-hand limit. If you think of the function as a highway and imagine you are traveling along the graph of  $f(x)$  toward  $x = 4$  **FROM THE RIGHT, NOT TO THE RIGHT**, and you stop at the vertical line  $x = 4$ , the  $y$ -value where you stop is 3.

Therefore,  $\lim_{x \rightarrow 4^+} f(x) = 3$ .



**EX #3: Use the graph above to evaluate each of the following limits:**

A. $f(2)$	B. $f(-1)$
C. $\lim_{x \rightarrow 4^-} f(x)$	D. $\lim_{x \rightarrow 2^+} f(x)$
E. $\lim_{x \rightarrow 2^-} f(x)$	F. $\lim_{x \rightarrow -1^+} f(x)$
G. $\lim_{x \rightarrow -1^-} f(x)$	H. $\lim_{x \rightarrow -4^+} f(x)$
I. $\lim_{x \rightarrow -4^-} f(x)$	J. $\lim_{x \rightarrow -1} f(x)$
K. $\lim_{x \rightarrow 2} f(x)$	L. $\lim_{x \rightarrow 5} f(x)$
M. $\lim_{x \rightarrow 0} f(x)$	N. $\lim_{x \rightarrow 1} f(x)$

#### Think about this!

If we think of the function as a highway, then the point at  $(2, -1)$  could be considered the end of the road, while the point at  $(-1, 2)$  is more like a "pothole." How would you describe the points located at

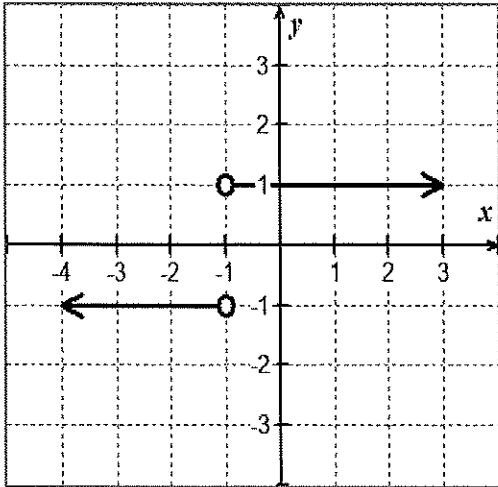
$(2, 1)$  : \_\_\_\_\_

$(4, 3)$  : \_\_\_\_\_

Hopefully, this analogy gives you a visual reference for understanding limits from a graphical approach. Let's get a little more formal with our definition now.

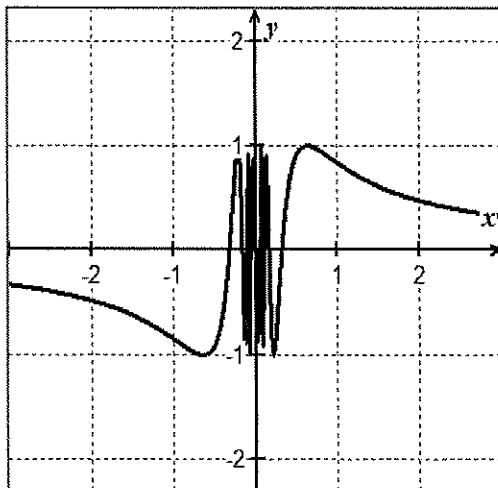
**EX #4: Limits can fail to exist in three situations:**

**CASE 1:** \_\_\_\_\_



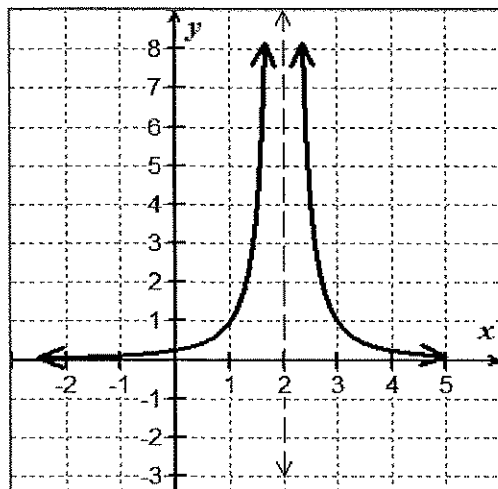
Justify why the limit does not exist at  $x = -1$  for  $f(x) = \frac{|x+1|}{x+1}$

**CASE 2:** \_\_\_\_\_



Justify why the limit does not exist at  $x = 0$  for  $f(x) = \sin\left(\frac{1}{x}\right)$

**CASE 3:** \_\_\_\_\_



Justify why the limit does not exist at  $x = 2$  for  $f(x) = \frac{1}{(x-2)^2}$

## Topic 1.4: Estimating Limit Values from Tables



EX #5: Now, consider the function  $f(x) = \frac{x-3}{x^2+2x-15}$ . Complete the table below to find the limit as  $x \rightarrow 3$ .

$x$	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$							

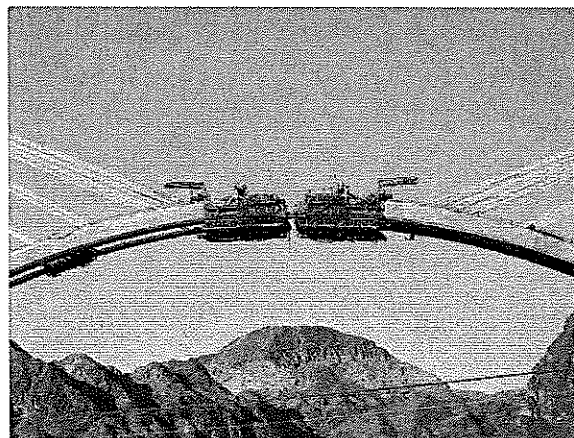
Based on your analysis, what are the values of each of the limits below?

$\lim_{x \rightarrow 3^-} f(x) =$	$\lim_{x \rightarrow 3^+} f(x) =$	$\lim_{x \rightarrow 3} f(x) =$
-----------------------------------	-----------------------------------	---------------------------------

By evaluating certain types of functions at a particular value, we may not necessarily have a sufficient understanding of the function's behavior at a specific point. This is especially true for rational functions that contain discontinuities.

This is why the idea of a **limit is so important!** Rather than just going directly to some  $x$ -value using direct substitution, we can **approach a point from either side** to get some sense of behavior in the **neighborhood**.

Look at the photo of the arch support construction for the Hoover Dam Bypass Bridge (2009). We can tell where the missing section is going to be. The actual height at that point would be the limit. With limits, we can discuss behavior at a point whether the point exists or not. With limits, we are looking at the  $y$ -value the graph approaches, not the actual  $y$ -value at that point.



This Photo by Unknown Author is licensed under [CC BY-SA](https://creativecommons.org/licenses/by-sa/4.0/)

When finding limits, ask yourself, "What is happening to  $y$  as  $x$  gets close to a certain number?" You are finding the  **$y$ -value** for which the function is approaching as  $x$  approaches  $c$ .

### LIMIT EXISTENCE THEOREM:

$$\lim_{x \rightarrow c} f(x) \text{ exists if and only if } \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

where  $L$  is a real number.

**Verbally:** The limit as  $x$  approaches  $c$  on  $f(x)$  will exist if and only if the limit as  $x$  approaches  $c$  from the left is equal to the limit as  $x$  approaches  $c$  from the right.

**EX #6: YOU OWN IT!** In the box below, complete the sentence in your own words.

In order for the GENERAL LIMIT to exist, the function:

**EX #7: Sketch a graph to satisfy each set of conditions.**

1.  $f(a)$  is undefined

2.  $x = a$  is a point discontinuity

3.  $\lim_{x \rightarrow a} f(x)$  exists

1.  $\lim_{x \rightarrow a} f(x)$  DNE

2.  $x = a$  is a jump discontinuity

3.  $f(a)$  is defined

# I.2

## Understanding Limits Graphically & Numerically

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Problems 1 - 4, complete the table and use the result to estimate the limit. Graph the function and verify your result.



1.  $\lim_{x \rightarrow 2} \frac{x-2}{2x^2-9x+10}$

x	1.99	1.999	2	2.001	2.01
f(x)					

2.  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

x	-0.01	-0.001	0	0.001	0.01
f(x)					

3.  $\lim_{x \rightarrow 4} \frac{x^2-16}{x-4}$

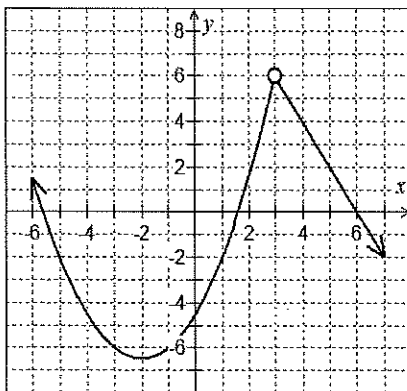
x	3.99	3.999	4	4.001	4.01
f(x)					

4.  $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3}$

x	2.99	2.999	3	3.001	3.01
f(x)					

Problems 5 - 8, use the graph to find the limit, if it exists. If the limit does not exist, explain why.

5.



A.  $\lim_{x \rightarrow -3} f(x)$

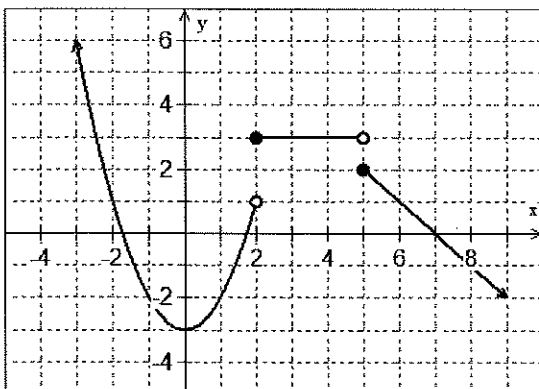
B.  $\lim_{x \rightarrow -\infty} f(x)$

C.  $\lim_{x \rightarrow 6} f(x)$

D.  $\lim_{x \rightarrow 1} f(x)$

E. Does  $\lim_{x \rightarrow 3} f(x)$  exist? Why or why not?

6.



A.  $\lim_{x \rightarrow 0} f(x)$

B.  $\lim_{x \rightarrow 2^-} f(x)$

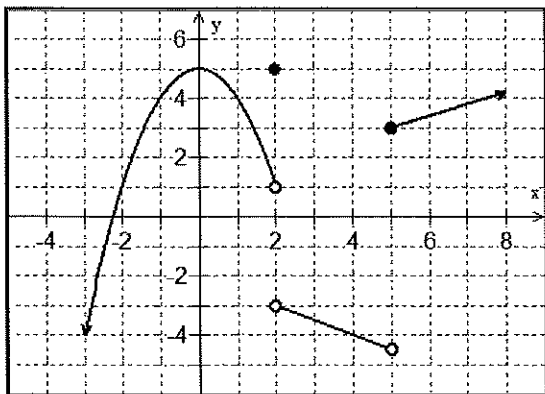
C.  $\lim_{x \rightarrow 4} f(x)$

D.  $\lim_{x \rightarrow 5} f(x)$

E.  $\lim_{x \rightarrow -\infty} f(x)$

F.  $\lim_{x \rightarrow 7} f(x)$

7.



A.  $\lim_{x \rightarrow 5^-} f(x)$

B.  $\lim_{x \rightarrow 2} f(x)$

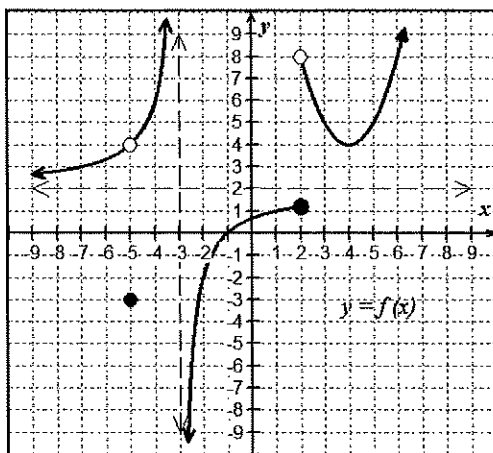
C.  $\lim_{x \rightarrow 0} f(x)$

D.  $\lim_{x \rightarrow \infty} f(x)$

E.  $\lim_{x \rightarrow 4} f(x)$

F.  $\lim_{x \rightarrow 2^+} f(x)$

8.



A.  $\lim_{x \rightarrow -3^-} f(x)$

B.  $\lim_{x \rightarrow -3^+} f(x)$

C.  $\lim_{x \rightarrow 2^-} f(x)$

D.  $\lim_{x \rightarrow 2^+} f(x)$

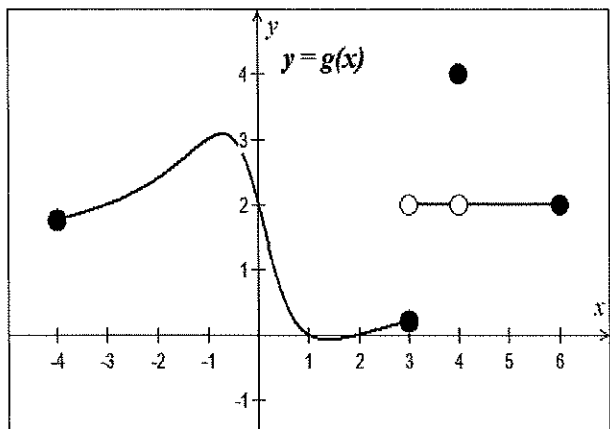
E.  $\lim_{x \rightarrow \infty} f(x)$

F.  $\lim_{x \rightarrow -\infty} f(x)$

G.  $\lim_{x \rightarrow -3} f(x)$

H.  $\lim_{x \rightarrow 2} f(x)$

Problems 9 - 11, Use the graph of  $g(x)$  below to determine if the statements are true or false. If false, explain why.



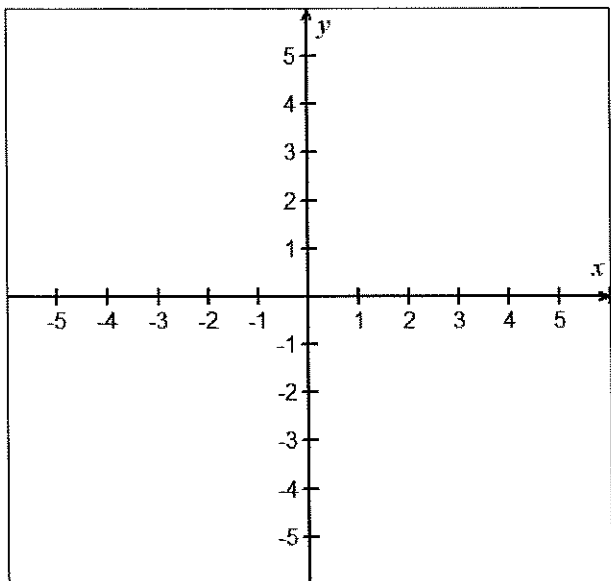
9.  $\lim_{x \rightarrow 4} g(x) = 4$

10.  $\lim_{x \rightarrow 5} g(x) = 2$

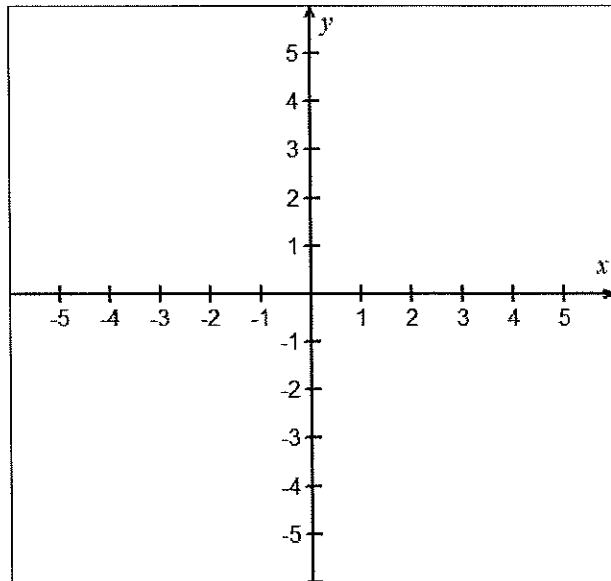
11.  $\lim_{x \rightarrow c} g(x)$  exists for every value of  $c$  in the interval  $(-4, 6)$

Problems 12 - 13, Sketch a graph of a function that satisfies each of the following conditions.

12.  $\lim_{x \rightarrow 1^-} f(x) = 2$      $\lim_{x \rightarrow 1^+} f(x) = -3$   
 $f(1) = 4$



13.  $\lim_{x \rightarrow -3^-} f(x) = 1$      $\lim_{x \rightarrow -3^+} f(x) = -4$   
 $\lim_{x \rightarrow 2^-} f(x) = \infty$      $\lim_{x \rightarrow 2^+} f(x) = -\infty$



14. Use the table below of the rational function  $y = H(x)$  to find the indicated limits. For limits that do not exist, write D.N.E.

$x$	-1000	-4.001	-4	-3.999	1.999	2	2.001	1000
$H(x)$	-0.998	0.666	undefined	0.666	-4497	undefined	4504	3.0002

A.  $\lim_{x \rightarrow -4^-} H(x) =$

B.  $\lim_{x \rightarrow -4^+} H(x) =$

C.  $\lim_{x \rightarrow -\infty} H(x) =$

D.  $\lim_{x \rightarrow -4} H(x) =$

E.  $\lim_{x \rightarrow 2^-} H(x) =$

F.  $\lim_{x \rightarrow 2^+} H(x) =$

G.  $\lim_{x \rightarrow 2} H(x) =$

H.  $\lim_{x \rightarrow \infty} H(x) =$

# Lesson 3: Properties of Limits

## Topic 1.5: Determining Limits Using Algebraic Properties of Limits

There will be times when you need to consider limits of functions symbolically, such as,  $f(x)$ ,  $g(x)$ , and  $h(x)$ . When you are given combinations, compositions, and other modifications of these nonspecific functions, there are some important properties of limits that will prove essential to these general function types.

### LIMIT PROPERTIES

Let  $b$  and  $c$  be real numbers, let  $n$  be a positive integer, and let  $f$  and  $g$  be functions where  $\lim_{x \rightarrow c} f(x)$  and  $\lim_{x \rightarrow c} g(x)$  both exist, then

#### 1. Constant Rule:

$$\lim_{x \rightarrow c} k = k$$

#### 2. Identity Rule:

$$\lim_{x \rightarrow c} x = c$$

#### 3. Coefficient Rule:

$$\lim_{x \rightarrow c} [b \cdot f(x)] = b \cdot \lim_{x \rightarrow c} f(x)$$

#### 4. Sum or Difference Rule:

$$\lim_{x \rightarrow c} [f(x) \pm g(x)] = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x)$$

#### 5. Product Rule:

$$\lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$$

#### 6. Quotient Rule:

$$\lim_{x \rightarrow c} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} ; \lim_{x \rightarrow c} g(x) \neq 0$$

#### 7. Power Rule:

$$\lim_{x \rightarrow c} [f(x)]^n = \left( \lim_{x \rightarrow c} f(x) \right)^n, n \text{ is rational}$$

#### 8. Composite Function Rule:

$$\lim_{x \rightarrow c} f(g(x)) = f \left( \lim_{x \rightarrow c} g(x) \right)$$

#### 9. Root Rule:

$$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)} \text{ if the root on the right side exists.}$$

Being skilled at reciting these rules in English will help you truly understand them. For example, The Sum or Difference Rule can be stated as "The limit of a sum (difference) is the sum (difference) of the limits. Can you give a description for each property in your own words?"

**EX#1: Use the properties of limits to find each of the following limits. If a limit does not exist, state why.**

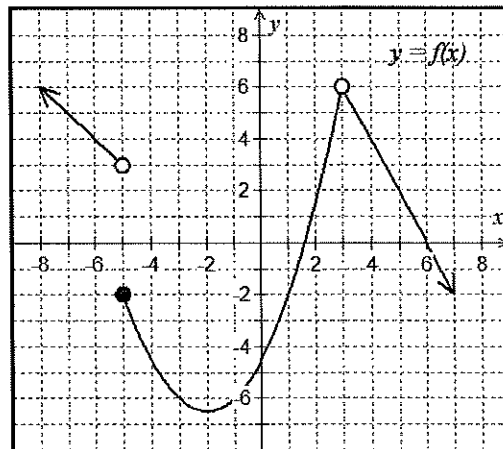
**A.**  $\lim_{x \rightarrow 3^+} [f(x) + g(x)]$

**B.**  $\lim_{x \rightarrow 5} [3f(x) - 2g(x)]$

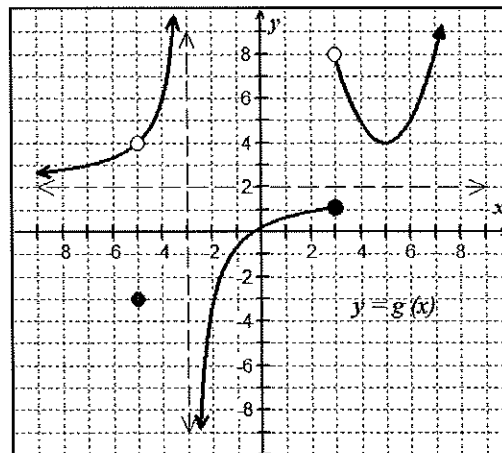
**C.**  $\lim_{x \rightarrow -5^-} [f(x) - g(x)]$

**D.**  $\lim_{x \rightarrow -7} \frac{3f(x)}{g(x)}$

Graph of  $f(x)$



Graph of  $g(x)$



**E.**  $\lim_{x \rightarrow 3} 4[f(x) \cdot g(x)]$

**F.**  $\lim_{x \rightarrow -3} [f(x)]^2$

**G.**  $\lim_{x \rightarrow 4} \sqrt{8g(x)}$

**H.**  $\lim_{x \rightarrow -3} [f(x) + g(x)]$

**J.**  $\lim_{x \rightarrow 6} \frac{f(x)}{g(x)}$

**K.**  $\lim_{x \rightarrow -2} f(g(x))$

**EX #2:** Find the limits if  $\lim_{x \rightarrow c} f(x) = 3$ ,  $\lim_{x \rightarrow c} g(x) = -2$ , and  $\lim_{x \rightarrow c} h(x) = 4$

A.  $\lim_{x \rightarrow c} \sqrt{3h(x) - 2g(x)}$

B.  $\lim_{x \rightarrow c} [f(x) \cdot 5g(x)]$

C.  $\lim_{x \rightarrow c} [7 - g(x)]^2$

D.  $\lim_{x \rightarrow c} \frac{2f(x) + 3h(x)}{h(x) - g(x)}$

E.  $\lim_{x \rightarrow c} [h(x) \cdot (f(x) + 6)]$

F.  $\lim_{x \rightarrow c} \frac{f(x)^2}{4 - g(x)}$

**EX #3:** Given  $\lim_{x \rightarrow c} f(x) = 8$  and  $\lim_{x \rightarrow c} g(x) = -4$ , then find the following:

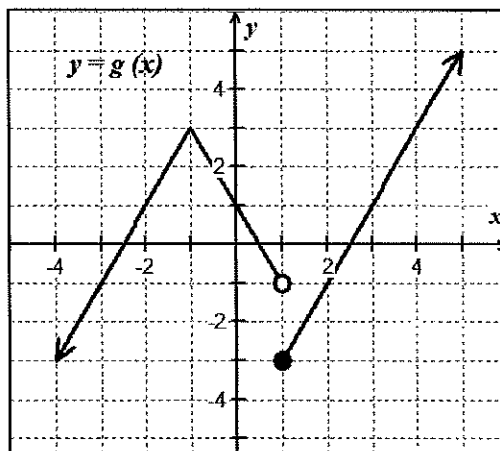
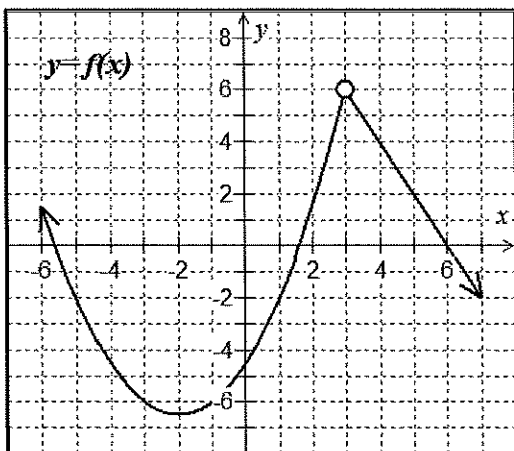
$$\lim_{x \rightarrow c} \left\{ 2f(x) \cdot g(x) - \sqrt[3]{f(x)} + 2[g(x)]^2 \right\}$$

**EX #4:** Find the limits of the following composition functions.

**A.** Given  $f(x) = x^3 - x^2 + 4x - 1$  and  $g(x) = -2x$ , find  $\lim_{x \rightarrow 2} f(g(x))$ .

**B.** Given  $f(x) = x^3 - x^2 + 4x - 1$  and  $g(x) = -2x$ , find  $\lim_{x \rightarrow 2} g(f(x))$

**EX #5:** Use the graphs of  $f(x)$  and  $g(x)$  below to find the limits.



**A.**  $\lim_{x \rightarrow 3} f(g(x))$

**B.**  $\lim_{x \rightarrow -1} g(f(x))$

**C.**  $\lim_{x \rightarrow 1} f(g(x))$

**D.**  $\lim_{x \rightarrow -3} g(f(x))$

## 1.3

## Properties of Limits

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Problems 1 - 6, find the limits if  $\lim_{x \rightarrow c} f(x) = 4$  and  $\lim_{x \rightarrow c} g(x) = -3$

1.  $\lim_{x \rightarrow c} [2f(x) - 3]^3$

2.  $\lim_{x \rightarrow c} [3f(x) \cdot 2g(x)]$

3.  $\lim_{x \rightarrow c} \sqrt{2f(x) - 4g(x)}$

4.  $\lim_{x \rightarrow c} \frac{5f(x) + 2g(x)}{g(x) - f(x)}$

5.  $\lim_{x \rightarrow c} [f(x) \cdot (g(x) + 5)]$

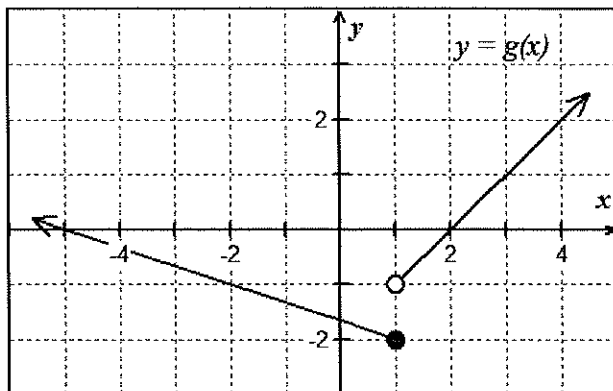
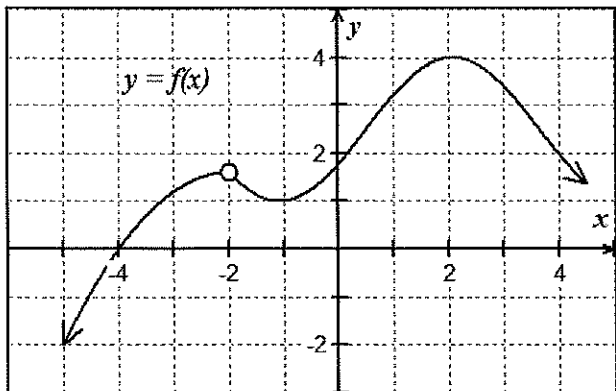
6.  $\lim_{x \rightarrow c} \frac{[7f(x)]^2}{1 - g(x)}$

Problems 7 - 8, use the properties of limits to evaluate. Justify each step with a named rule.

7.  $\lim_{x \rightarrow -3} (5x + 9)$

8.  $\lim_{x \rightarrow 4} \frac{3x}{x-2}$

Problems 9 – 12, use the graphs of  $f$  and  $g$  below to evaluate the limits, if they exist.



9.  $\lim_{x \rightarrow -2} [f(x) + 4g(x)]$

10.  $\lim_{x \rightarrow 1} [f(x) \cdot g(x)]$

11.  $\lim_{x \rightarrow 2} \frac{g(x)}{f(x)}$

12.  $\lim_{x \rightarrow 4} [f(g(x))]^2$

Problems 13 – 16, True or False.

13. If  $\lim_{x \rightarrow 4} f(x) = 2$  and  $\lim_{x \rightarrow 4} g(x) = 0$ , then  $\lim_{x \rightarrow 4} \left[ \frac{f(x)}{g(x)} \right]$  does not exist.

14. If  $\lim_{x \rightarrow 4} f(x) = 0$  and  $\lim_{x \rightarrow 4} g(x) = 0$ , then  $\lim_{x \rightarrow 4} \left[ \frac{f(x)}{g(x)} \right]$  does not exist.

15.  $\lim_{x \rightarrow 1} \frac{x^2 + 4x - 5}{x^2 + 3x - 4} = \frac{\lim_{x \rightarrow 1} x^2 + 4x - 5}{\lim_{x \rightarrow 1} x^2 + 3x - 4}$

16.  $\lim_{x \rightarrow 1} \frac{x^2 - 3}{x^2 + 5x - 4} = \frac{\lim_{x \rightarrow 1} x^2 - 3}{\lim_{x \rightarrow 1} x^2 + 5x - 4}$

Problems 17 – 22, Use the table of values below to find limits for the following:

$x$	-2	3	6	$c$
$f(x)$	5	0	1	$\lim_{x \rightarrow c} f(x) = 6$
$g(x)$	-3	-2	-4	$\lim_{x \rightarrow c} g(x) = -2$
$h(x)$	4	2	3	$\lim_{x \rightarrow c} h(x) = 3$

17.  $\lim_{x \rightarrow c} f(g(x))$

18.  $\lim_{x \rightarrow c} g(h(x))$

19.  $\lim_{x \rightarrow c} \frac{g(f(x))}{h(g(x))}$

20.  $\lim_{x \rightarrow c} \frac{f(x)}{f(h(x))}$

21.  $\lim_{x \rightarrow c} f(g(h(x)))$

22.  $\lim_{x \rightarrow c} g(h(f(x)))$

# Lesson 4: Limits of Composite Functions

## Topic 1.7: Selecting Procedures for Determining Limits

## Topic 1.9: Connecting Multiple Representations of Limits

The limit of a composition is the composition of the limits, provided the outside function is continuous at the limit of the inside function. We often see this IF-THEN STATEMENT presented as follows:

*f and g are functions such that  $\lim_{x \rightarrow c} g(x) = L$  and  $\lim_{x \rightarrow L} f(x) = f(L)$  then*

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L)$$

or stated as,

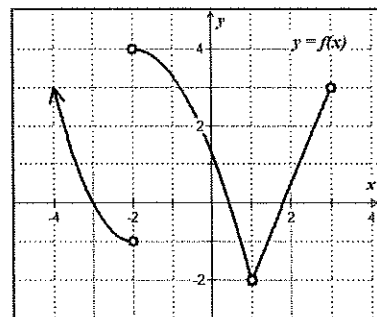
$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) \text{ provided } \lim_{x \rightarrow c} g(x) = L \text{ and } f(x) \text{ is continuous at } x = L$$

We should notice this IF-THEN statement requires certain conditions to be met. Problems occur when  $f(x)$  is not continuous at  $x = L$  or the existence of the limit of  $g(x)$  at  $x = a$ . At issues of this sort, we can't apply the property. You might assume that the limit doesn't exist, but this may or may not be true. So, we will explore some alternative plan to find another approach.

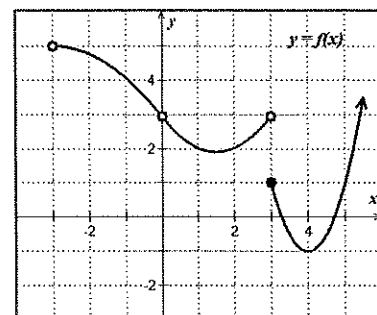
In this case, let's apply the "if and only if" definition of a limit:

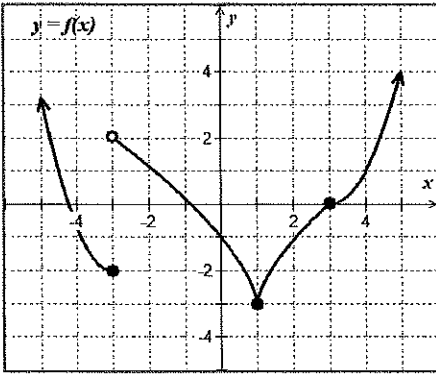
$$\lim_{x \rightarrow c^-} f(x) = L = \lim_{x \rightarrow c^+} f(x) \text{ then } \lim_{x \rightarrow c} f(x) = L$$

**EX #1:**  $\lim_{x \rightarrow 1} f(f(x))$

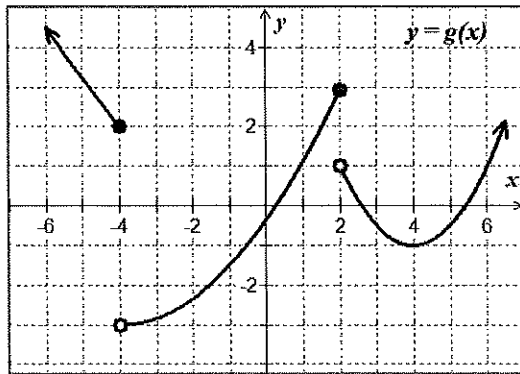


**EX #2:**  $\lim_{x \rightarrow 0} f(f(x))$

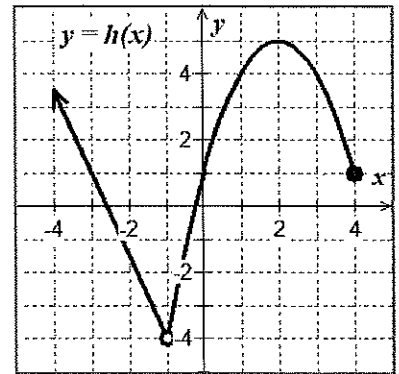




$$y = f(x)$$



$$y = g(x)$$

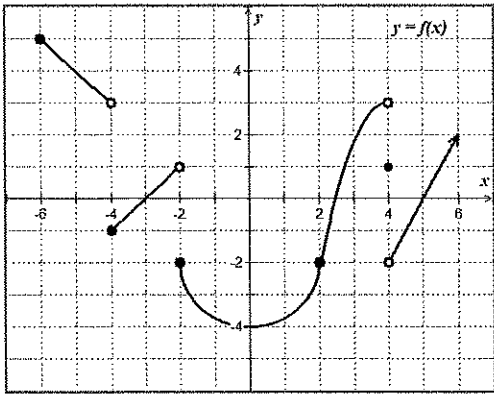


$$y = h(x)$$

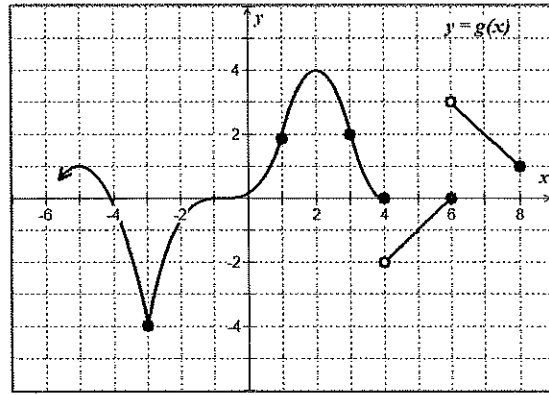
**EX #3:**  $\lim_{x \rightarrow 1} f(f(x))$

**EX #4:**  $\lim_{x \rightarrow 2} h(g(x))$

**EX #5:**  $\lim_{x \rightarrow -1} g(h(x))$



$$y = f(x)$$



$$y = g(x)$$

**EX #6:**  $\lim_{x \rightarrow -3} f(g(x))$

**EX #7:**  $\lim_{x \rightarrow 2} f(g(x))$

**EX #8:**  $\lim_{x \rightarrow 2^+} f(8 - x^2)$

**EX #9:**  $\lim_{x \rightarrow -4^+} g(2 + f(x))$

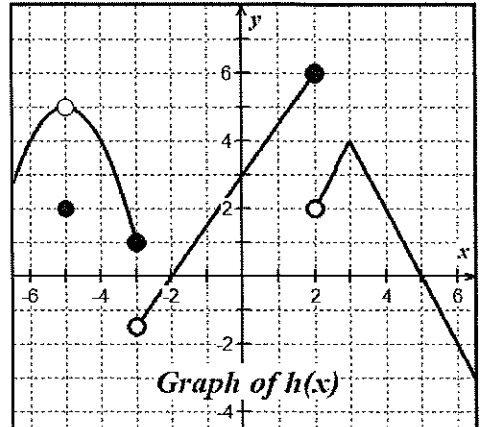
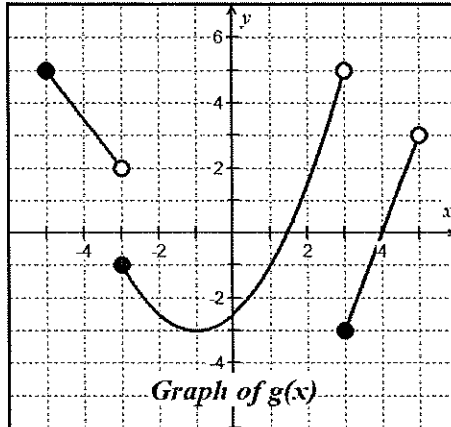
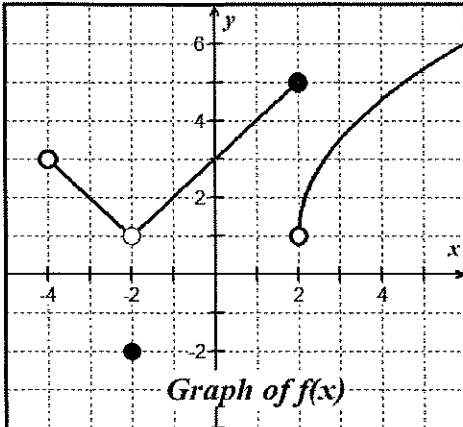
# 1.4

## Limits of Composite Functions

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

**Problems 1 – 8,** Use the graphs of  $f(x)$ ,  $g(x)$  and  $h(x)$  shown below to answer the following limit questions.



1.  $\lim_{x \rightarrow -1} [f(x) + g(x)]$

2.  $\lim_{x \rightarrow 0} [f(x) \cdot h(x)]$

3.  $\lim_{x \rightarrow -2} f(h(x))$

4.  $\lim_{x \rightarrow 1} \sqrt{f(x)}$

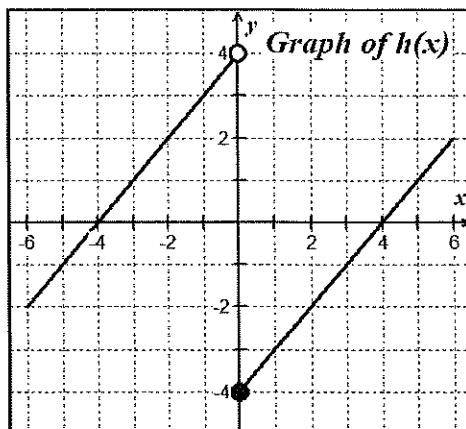
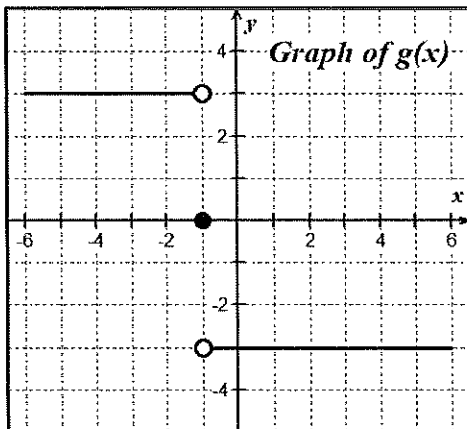
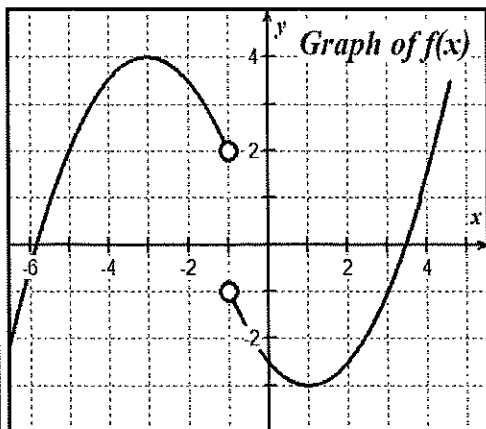
5.  $\lim_{x \rightarrow 0} h(|x| + 3)$

6.  $\lim_{x \rightarrow -5^+} \left[ \frac{g(x)}{h(x)} \right]$

7.  $\lim_{x \rightarrow -5} g(x + 2)$

8.  $\lim_{x \rightarrow -2} f\sqrt{x + 3}$

Problems 9 – 14, use the graphs of  $f$ ,  $g$  and  $h$  below to evaluate the limits, if they exist.



9.  $\lim_{x \rightarrow 5^+} g(25 - x^2)$

10.  $\lim_{x \rightarrow -1} [(g(x))^2 - 5]$

11.  $\lim_{x \rightarrow 0} \frac{h(x)}{h(-x)}$

12.  $\lim_{x \rightarrow 0} h(h(x))$

13.  $\lim_{x \rightarrow -1} f(g(x))$

14.  $\lim_{x \rightarrow -2} f(g(h(x)))$

Problems 15 – 16, Let  $v(x) = \begin{cases} 3, & x = 0 \\ x, & x \neq 0 \end{cases}$  and  $w(x) = \begin{cases} -3, & x < 0 \\ 0, & x = 0 \\ 3, & x > 0 \end{cases}$

15.  $\lim_{x \rightarrow 0} (v(x) \cdot w(x))$

16.  $\lim_{x \rightarrow 0} \frac{w(x)}{v(x)}$

# Lesson 5: Finding Limits by Analytic Methods

## Topic 1.6: Determining Limits Using Algebraic Manipulation

If you only observe the graph of a function, it can be misleading when finding the limit of a function. In this lesson we will explore how to find limits using algebraic techniques and limit theorems.

You will learn to analyze limits by the following methods:

### Methods to Analyze Limits:

1. Direct substitution.
2. Basic Limit Theorems
3. Factor, cancellation technique. Then go back to step 1.
4. The conjugate method, rationalize the numerator. Then, go back to step 2.
5. Special trig limits of  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  or  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$
6. L'Hospital's Rule (presented in Unit 3)

### Substitution Theorem

If  $f$  is a polynomial function or rational function, then  $\lim_{x \rightarrow c} f(x) = f(c)$  provided that if  $f$  is a rational function the value of the denominator does not equal 0.

**NOTE:** Always try **DIRECT SUBSTITUTION** first. If you get  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  the goal will be to simplify the expression using algebraic techniques and then try substitution again.

**EX #1:** Find each of the following limits analytically using direct substitution.

A.  $\lim_{x \rightarrow 2} (3x^2 - 5x + 4)$

B.  $\lim_{x \rightarrow 2} \frac{x^3 + 1}{x + 1}$

C.  $\lim_{x \rightarrow e} \frac{\ln x}{3x}$

D.  $\lim_{x \rightarrow 4} \sqrt[3]{x + 4}$

E.  $\lim_{\theta \rightarrow \frac{\pi}{6}} \sin 2\theta$

F.  $\lim_{x \rightarrow 5} \log_3(x + 4)$

## Topic 1.10: Exploring Types of Discontinuities

What is the process for finding discontinuities of a rational function from Pre-calculus?

You can perform the same algebraic analysis to find the limit of the removable, or point discontinuities and the non-removable, or infinite discontinuities using what we will call the **Factoring Method or Cancellation Technique**.

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 6x + 8}$$

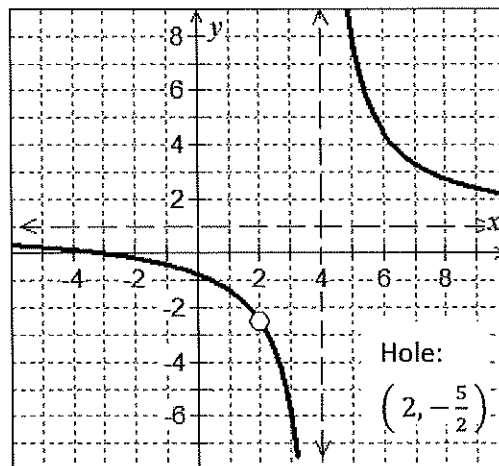
## Topic 1.9: Connecting Multiple Representations of Limits

Graphically looking at the function, we see that just because it is undefined at a specific  $x$ -value doesn't mean that we can't find the limit. *Remember the Hoover Dam construction example.* Use the graph of the function to determine the value of each limit below.

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 6x + 8} = \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow 4^+} \frac{x^2 + x - 6}{x^2 - 6x + 8} = \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow 4^-} \frac{x^2 + x - 6}{x^2 - 6x + 8} = \underline{\hspace{2cm}}$$



### Algebraically Finding Limits of Functions at Undefined Values

Consider what happens when you try to evaluate this limit using direct substitution.

**REMEMBER:** When you get indeterminate form of  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  your goal is to algebraically manipulate the expression in an effort to remove the point(s) of discontinuity. Then, try direct substitution again. Let's explore a few techniques.

### Finding One-Sided Limits

Recall, in our example there is a vertical asymptote at  $x = 4$ .

How can we determine the behavior of a function at this value of  $x$ ?

$$\lim_{x \rightarrow 4^+} \frac{x^2 + x - 6}{x^2 - 6x + 8} =$$

As  $x \rightarrow 4^+$ , pick a value to the right of 4, then analyze the simplified function:

$$\lim_{x \rightarrow 4^+} f(x) =$$

$$\lim_{x \rightarrow 4^-} \frac{x^2 + x - 6}{x^2 - 6x + 8} =$$

As  $x \rightarrow 4^-$ , pick a value to the left of 4, then analyze the simplified function:

$$\lim_{x \rightarrow 4^-} f(x) =$$

**EX #2: Finding limits analytically of piecewise functions. Show your algebraic steps.**

A.  $\lim_{x \rightarrow 1} g(x)$  given,  $g(x) = \begin{cases} x^3 + 1, & x > 1 \\ x + 1, & x \leq 1 \end{cases}$

B.  $\lim_{x \rightarrow 2} h(x)$  given,  $h(x) = \begin{cases} x^2 - 4x + 7, & x \leq 2 \\ -x^2 + 4x - 1, & x > 2 \end{cases}$

## Topic 1.7: Selecting Procedures for Determining Limits

### EX #3: The Factoring or Cancellation Technique

$$\text{A. } \lim_{x \rightarrow -4} \frac{2x^2 + 7x - 4}{x^2 - x - 20}$$

$$\text{B. } \lim_{x \rightarrow 3} \frac{x^3 - 27}{x^2 + x - 12}$$

### EX #4: The LCM/LCM Method for Complex Fractions

$$\text{A. } \lim_{x \rightarrow 0} \frac{x}{\frac{1}{4} + \frac{1}{x-4}}$$

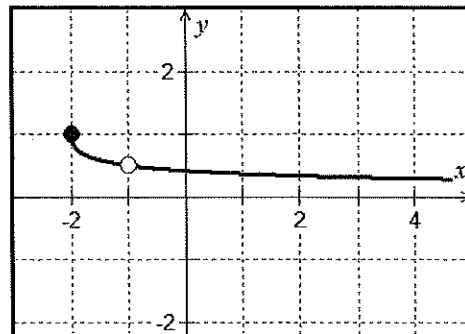
$$\text{B. } \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$$

**EX #5: The Rationalization Technique or Conjugate Method**

The technique of rationalization can be used to find the limit when there is a radical in the numerator or denominator.

A. The graph of  $g(x) = \frac{\sqrt{x+2}-1}{x+1}$  is shown at right.

$$\lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1}$$



B.  $\lim_{x \rightarrow 5} \frac{x-5}{3-\sqrt{x+4}}$

**EX #6: Find each of the following limits analytically. Show your algebraic steps.**

A.  $\lim_{x \rightarrow 3} (5x+1)^{\frac{2}{3}}$

B.  $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3}$

C. 
$$\lim_{x \rightarrow -3^-} \frac{x^2 |2x + 6|}{4x + 12}$$

D. 
$$\lim_{x \rightarrow 3^+} \frac{4x - 12}{\sqrt{x^2 - 6x + 9}}$$

E. 
$$\lim_{x \rightarrow 0} \frac{\frac{1}{x} + \frac{1}{x+3}}{x}$$

F. 
$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$

### Topic 1.8: Determining Limits Using the Squeeze Theorem

The Squeeze Theorem is a technique used to confirm the limit of a function by comparison with two other functions whose limits are known or easily computed. Consider some function  $f(x)$  is “trapped between two functions” on an interval containing point  $c$ . Let  $f, g,$  and  $h$  be functions defined on the interval except possibly at  $c$  itself. Then for  $x \neq c$ , in the interval  $f(x) \leq g(x) \leq h(x)$  and, also that  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$ . Then  $\lim_{x \rightarrow c} g(x) = L$

#### **EX #7: The Squeeze Theorem**

A. If  $1 \leq f(x) \leq x^2 + 2x + 2$ , find  $\lim_{x \rightarrow -1} f(x)$

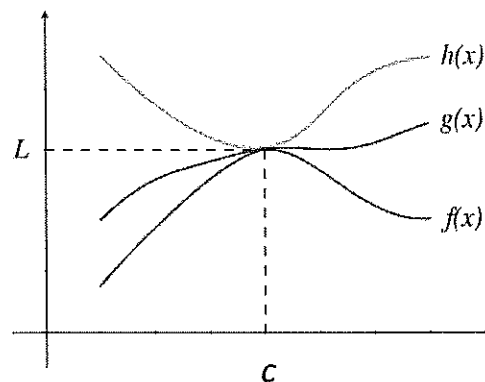
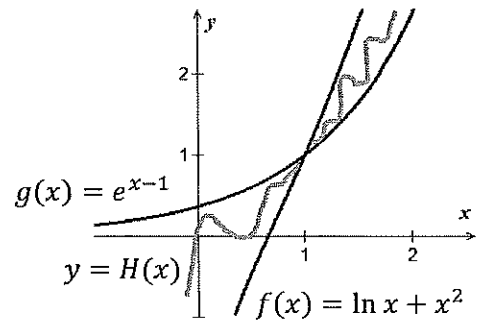


Photo: Public Domain

- B. Suppose  $f(x)$  and  $g(x)$  are boundaries of  $H(x)$ , as shown on the graph below. Given  $f(x) = \ln x + x^2$  and  $g(x) = e^{x-1}$ , for all  $x$  in the interval containing  $x = 1$ , except possibly at  $x = 1$  itself, find  $\lim_{x \rightarrow 1} H(x)$ . Justify.



Write your questions  
and thoughts here!

**Squeeze Theorem:** a.k.a. "Sandwich Theorem" or "Pinching Theorem"

$$\text{If } g(x) \leq f(x) \leq h(x)$$

$$\text{and if } \lim_{x \rightarrow a} g(x) = L \text{ and } \lim_{x \rightarrow a} h(x) = L$$

$$\text{then } \lim_{x \rightarrow a} f(x) = L$$

1. Find  $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^2}\right)$

2. Let  $g$  and  $h$  be the functions defined by  $g(x) = -x^2 + 2x - 3$  and  $h(x) = 2x + 1$ . If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for all  $x$ , what is  $\lim_{x \rightarrow 2} f(x)$ ?

3. Let  $g$  and  $h$  be the functions defined by  $g(x) = \cos\left(\frac{\pi}{2}x\right) + 2$  and  $h(x) = x^2 + 3$ . If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for  $-1 \leq x \leq 5$ , what is  $\lim_{x \rightarrow 0} f(x)$ ?

Let  $f$ ,  $g$ , and  $h$  be the functions defined by  $f(x) = \frac{1 - \cos x}{2x}$ ,  $g(x) = x^3 \sin\left(\frac{1}{x}\right)$ , and  $h(x) = \frac{x}{\sin x}$  for  $x \neq 0$ . All of the following inequalities are true for  $x \neq 0$ . State whether each inequality can be used with the squeeze theorem to find the limit of the function as  $x$  approaches 0?

4.  $-\frac{1}{2} \leq f(x) \leq x^2 + \frac{1}{2}$

5.  $-x^3 \leq g(x) \leq x^3$

6.  $1 - x^2 \leq h(x) \leq x^2 + 1$

## 1.5

Finding Limits by  
Analytic Methods

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Problems 1 – 16, Find each of the following limits analytically. Show your algebraic analysis.

1.  $\lim_{x \rightarrow 3} \left( \frac{2}{3}x^2 + 3x \right)$

2.  $\lim_{t \rightarrow 4} \frac{t-4}{t^2-16}$

3.  $\lim_{x \rightarrow -3} \frac{x^2-5x+6}{2x+6}$

4.  $\lim_{\theta \rightarrow \pi} [\sin^2 \theta - 3 \cos \theta]$

5.  $\lim_{x \rightarrow 2} \frac{x^3-8}{x-2}$

6.  $\lim_{\theta \rightarrow \frac{\pi}{3}} \frac{\tan \theta}{\theta^2}$

7.  $\lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x}$

8.  $\lim_{x \rightarrow 0} \frac{\frac{1}{3+x} - \frac{1}{3}}{x}$

$$9. \lim_{x \rightarrow e} \frac{3x}{\ln x}$$

$$10. \lim_{x \rightarrow 2^+} \frac{3x^2 + 7x + 2}{x^2 - 4}$$

$$11. \lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$

$$12. \lim_{x \rightarrow \frac{3}{2}} \frac{8x^3 - 27}{2x - 3}$$

$$13. \lim_{x \rightarrow \pi^+} \cot x$$

$$14. \lim_{x \rightarrow 0} \cos(x + \sin x)$$

$$15. f(x) = \begin{cases} 3x - 1, & x \leq 1 \\ 3x^2, & x > 1 \end{cases}, \text{ find } \lim_{x \rightarrow 1} f(x)$$

$$16. \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

Problems 17 – 32, find the limits, if they exist, for the given piecewise-defined functions.

$$f(x) = \begin{cases} x + 3, & x < 2 \\ x^2 - 1, & 2 \leq x < 4 \\ \sqrt{x + 5}, & x \geq 4 \end{cases}$$

$$g(x) = \begin{cases} 4 - x^2, & x \leq 1 \\ 7, & 1 < x < 3 \\ 2 - x, & x \geq 3 \end{cases}$$

17.  $\lim_{x \rightarrow 2^+} f(x)$

18.  $\lim_{x \rightarrow 2^-} f(x)$

19.  $\lim_{x \rightarrow 2} f(x)$

20.  $f(2)$

21.  $\lim_{x \rightarrow 4^+} f(x)$

22.  $\lim_{x \rightarrow 4^-} f(x)$

23.  $\lim_{x \rightarrow 4} f(x)$

24.  $f(4)$

25.  $\lim_{x \rightarrow 1^-} g(x)$

26.  $\lim_{x \rightarrow 1^+} g(x)$

27.  $\lim_{x \rightarrow 1} g(x)$

28.  $g(1)$

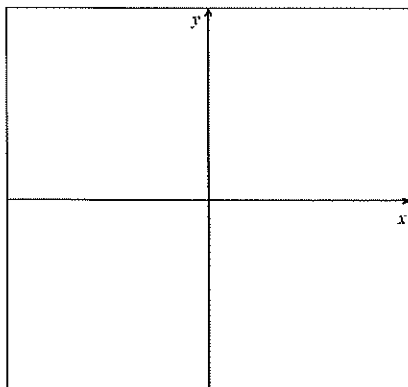
29.  $\lim_{x \rightarrow 3^-} g(x)$

30.  $\lim_{x \rightarrow 3^+} g(x)$

31.  $\lim_{x \rightarrow 3} g(x)$

32.  $g(3)$

33. If  $2 \leq f(x) \leq x^2 + 2$  for all  $x$ ,  
find  $\lim_{x \rightarrow 0} f(x)$ . Sketch a graph to illustrate.



34. If  $\lim_{x \rightarrow c} f(x) = -5$  and  $\lim_{x \rightarrow c} g(x) = 8$ ,  
find  $\lim_{x \rightarrow c} \frac{2f(x)}{g(x) - f(x)}$

## 1.5 The Squeeze Theorem

## Practice

Calculus

Evaluate each limit.

1.  $\lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right)$

2.  $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x^2}\right)$

3.  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x^2}\right)$

4. Let  $g$  and  $h$  be the functions defined by  $g(x) = x^2 - 3x$  and  $h(x) = 2 - 2x$ .

If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for all  $x$ , what is  $\lim_{x \rightarrow 2} f(x)$ ?

5. Let  $g$  and  $h$  be the functions defined by  $g(x) = \cos(\pi(x + 2)) - 3$  and  $h(x) = \frac{x^2}{2} + x - \frac{7}{2}$ .

If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for  $-2 \leq x \leq 0$ , what is  $\lim_{x \rightarrow -1} f(x)$ ?

6. Let  $g$  and  $h$  be the functions defined by  $g(x) = x^2 + x - 1$  and  $h(x) = -x^2 - 4x - 2$ .

If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for all  $x$ , what is  $\lim_{x \rightarrow -1} f(x)$ ?

7. Let  $g$  and  $h$  be the functions defined by  $g(x) = -x^2 - 2x + 5$  and  $h(x) = 2x^2 - x - 4$ .

If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for all  $x$ , what is  $\lim_{x \rightarrow -2} f(x)$ ?

8. Let  $g$  and  $h$  be the functions defined by  $g(x) = \sin\left(\frac{\pi}{2}(x + 1)\right) - 1$  and  $h(x) = \cos(\pi x) - 3$ .

If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for all  $x$ , what is  $\lim_{x \rightarrow 2} f(x)$ ?

9. Let  $g$  and  $h$  be the functions defined by  $g(x) = x^2$  and  $h(x) = \cos(x)$ .

If  $f$  is a function that satisfies  $g(x) \leq f(x) \leq h(x)$  for  $-0.4 \leq x \leq 0.4$ , what is  $\lim_{x \rightarrow 0} f(x)$ ?

10. Let  $f$  and  $g$  be the functions defined by  $f(x) = \frac{\sin x}{5x}$  and  $g(x) = x^2 \cos\left(\frac{1}{x^3}\right)$  for  $x \neq 0$ . The following inequalities are true for  $x \neq 0$ . State whether each inequality can be used with the squeeze theorem to find the limit of the function as  $x$  approaches 0?

a.  $5(\sin(\pi(x+1))) \leq f(x) \leq \frac{1}{5}$

b.  $-x^2 \leq g(x) \leq x^2$

11. Let  $f$  and  $g$  be the functions defined by  $f(x) = \frac{\cos x - 1}{x^2}$  and  $g(x) = x^2 \sin\left(\frac{1}{x}\right)$  for  $x \neq 0$ . The following inequalities are true for  $x \neq 0$ . State whether each inequality can be used with the squeeze theorem to find the limit of the function as  $x$  approaches 0?

a.  $\frac{1}{5} \leq f(x) \leq \frac{1}{2}$

b.  $-x^2 - \frac{1}{2} \leq f(x) \leq x^2 - \frac{1}{2}$

c.  $-x^2 \leq g(x) \leq x^2$

d.  $-\frac{1}{x} \leq g(x) \leq \frac{1}{x}$

12. Let  $f$  and  $g$  be the functions defined by  $f(x) = \frac{x^2 \sin x}{x}$  and  $g(x) = x \cos\left(\frac{1}{|x|}\right)$  for  $x \neq 0$ . The following inequalities are true for  $x \neq 0$ . State whether each inequality can be used with the squeeze theorem to find the limit of the function as  $x$  approaches 0?

a.  $-x \leq f(x) \leq x$

b.  $-\frac{1}{x} \leq g(x) \leq \frac{1}{x}$

c.  $2^{-x} - 1 \leq g(x) \leq 2^x - 1$

# Lesson 6: Limits of Transcendental Functions

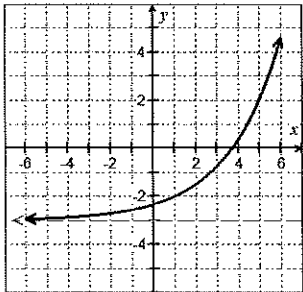
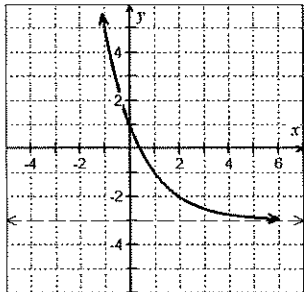
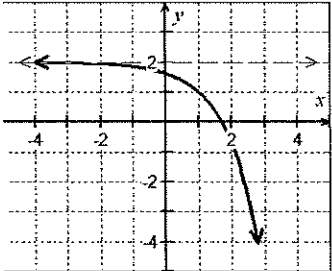
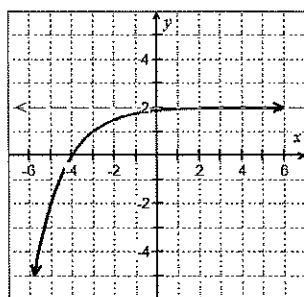
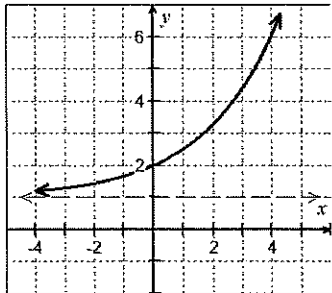
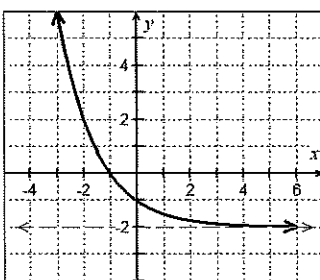
If you research the definition of a **transcendental function** you will find this explanation on Wolfram's Math World:

*A function which is not an algebraic function. In other words, a function which "transcends," i.e., cannot be expressed in terms of, algebra. Examples of transcendental functions include the exponential function, the trigonometric functions, and the inverse functions of both.*

## Topic 1.9: Connecting Multiple Representations of Limits

### Analyzing Limits of Exponential Functions

**EX #1:** Recall that exponential equations are written in the form  $y = ab^{(x-h)} + k$ . You will need to find limits of exponential functions without the aid of a graph or calculator in this course. Do you remember the rules for transformations of exponential functions? Evaluate the limits using the graphs and look for patterns.

Exponential Growth	Exponential Decay
<p>A. <math>f(x) = \left(\frac{3}{2}\right)^{x-1} - 3</math></p> <p><math>\lim_{x \rightarrow -\infty} f(x) =</math></p> <p><math>\lim_{x \rightarrow \infty} f(x) =</math></p> 	<p>D. <math>f(x) = \left(\frac{1}{2}\right)^{x-2} - 3</math></p> <p><math>\lim_{x \rightarrow -\infty} f(x) =</math></p> <p><math>\lim_{x \rightarrow \infty} f(x) =</math></p> 
<p>B. <math>f(x) = -e^{x-1} + 2</math></p> <p><math>\lim_{x \rightarrow -\infty} f(x) =</math></p> <p><math>\lim_{x \rightarrow \infty} f(x) =</math></p> 	<p>E. <math>f(x) = -\left(\frac{1}{2}\right)^{x+3} + 2</math></p> <p><math>\lim_{x \rightarrow -\infty} f(x) =</math></p> <p><math>\lim_{x \rightarrow \infty} f(x) =</math></p> 
<p>C. <math>f(x) = \left(\frac{2}{3}\right)^{-x} + 1</math></p> <p><math>\lim_{x \rightarrow -\infty} f(x) =</math></p> <p><math>\lim_{x \rightarrow \infty} f(x) =</math></p> 	<p>F. <math>f(x) = 2^{-x} - 2</math></p> <p><math>\lim_{x \rightarrow -\infty} f(x) =</math></p> <p><math>\lim_{x \rightarrow \infty} f(x) =</math></p> 

**EX #2: WHAT JUST HAPPENED?** Did you see that? Basically, the end-behavior of any exponential function tends toward three places.

<b>CASE #1:</b>	<b>CASE #2:</b>	<b>CASE #3:</b>
-----------------	-----------------	-----------------

In PreCalculus, we learned the following rules without considering the behavior for negative  $a$ -values.

$y = ab^{(x-h)} + k$	
$a > 0$ and $b > 1$	$a > 0$ and $0 < b < 1$
$f$ is increasing $b$ is exponential growth factor	$f$ is decreasing $b$ is exponential decay factor

**Growth and decay are not synonymous with increasing and decreasing.**

Consider the behavior of the function in relation to the horizontal asymptote. Notice whether the function is moving away from or toward the horizontal asymptote.

Also, what effect does the negative coefficient on  $x$  do to the function behavior?

**EX #3:** Now, use the equations from example #1 to complete the table below:

Graph of $f(x)$	Base	Increase or Decrease	$\lim_{x \rightarrow -\infty} f(x)$	$\lim_{x \rightarrow \infty} f(x)$
<b>GROWTH</b>	$b > 1$			
$f(x) = \left(\frac{3}{2}\right)^{x-1} - 3$				
$f(x) = -e^{x-1} + 2$				
$f(x) = \left(\frac{2}{3}\right)^{-x} + 1$				
<b>DECAY</b>	$0 < b < 1$			
$f(x) = \left(\frac{1}{2}\right)^{x-2} - 3$				
$f(x) = -\left(\frac{1}{2}\right)^{x+3} + 2$				
$f(x) = 2^{-x} - 2$				

**EX #4:** You got this! Find the limits of each of the following exponential functions.

A.  $\lim_{x \rightarrow -\infty} -(0.3)^x + 2$

B.  $\lim_{x \rightarrow -\infty} e^{-x+1} - 3$

C.  $\lim_{x \rightarrow \infty} -3^{-x+1} - 3$

D.  $\lim_{x \rightarrow \infty} -\left(\frac{1}{3}\right)^{-x} + 1$

E.  $\lim_{x \rightarrow 2} (e^{x-3} - 1)$

F.  $\lim_{x \rightarrow -3} \left[ \left(\frac{1}{2}\right)^{-x-1} + 5 \right]$

## Analyzing Limits of Trigonometric Functions

Most of the work you have done with limits to this point have dealt with polynomial or rational functions. When confronted with trigonometric functions, you will find throughout the course, that there are several methods to use. Let's begin with the basics in this section.

### Topic 1.7: Selecting Procedures for Determining Limits

**EX#5:** Use direct substitution to find each limit.

A.  $\lim_{\theta \rightarrow \frac{\pi}{3}} \frac{\cos 2\theta}{\theta}$

B.  $\lim_{\theta \rightarrow \frac{\pi}{2}} 3 \sin^2 \theta$

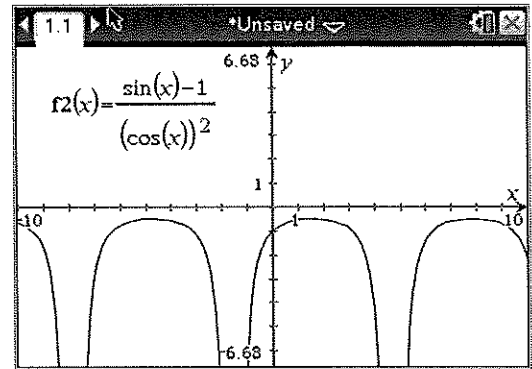
Just like some polynomial functions where a function value is not defined, yet a limit will exist...the same will occur with trigonometric functions.

**EX #6:** Evaluate the limit by direct substitution.  $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin \theta - 1}{\cos^2 \theta}$

## A few thoughts to consider:

If you algebraically determine that a function is undefined at a value, does that mean the limit does not exist? Can you look at the graph? **How do you analyze the limit when no calculator or graph is permitted?**

We need to use some trig identities and rewrite the function. Try that here:



### Topic 1.6: Determining Limits Using Algebraic Manipulation

**EX #7: You Got This!** Use your new skills to evaluate each limit below by first rewriting the function and using identities.

A.  $\lim_{\theta \rightarrow \frac{\pi}{2}} 2 \cos \theta \cot \theta$

B.  $\lim_{\theta \rightarrow \frac{\pi}{4}} \frac{\sin \theta \cos \theta}{\tan \theta}$

C.  $\lim_{\theta \rightarrow \pi} \frac{\sin \theta \csc \theta}{3\theta}$

**HAVE YOU DISCOVERED A PATTERN IN ALL OF THIS YET? If you ALWAYS try direct substitution first, three things will occur!**

Later in the course you will learn a smooth technique known as L'Hospital's Rule. It will be a quicker method for evaluating many functions. For now, you might like to memorize these special rules, as well.

### SPECIAL TRIGONOMETRIC LIMITS

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$$

**EX #8:** Use a graphing calculator and the table feature to evaluate the following trigonometric limits.

A. $\lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta}$	-0.1	-0.01	0	0.01	0.1
B. $\lim_{\theta \rightarrow 0} \frac{\cos 3\theta - 1}{\theta}$	-0.1	-0.01	0	0.01	0.1
C. $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{4\theta}$	-0.1	-0.01	0	0.01	0.1
D. $\lim_{\theta \rightarrow 0} \frac{1 - \cos 3\theta}{2\theta}$	-0.1	-0.01	0	0.01	0.1

**EX#9:** Some algebraic tricks with properties.

A.  $\lim_{\theta \rightarrow 0} \frac{3 \sin 5\theta}{2\theta}$

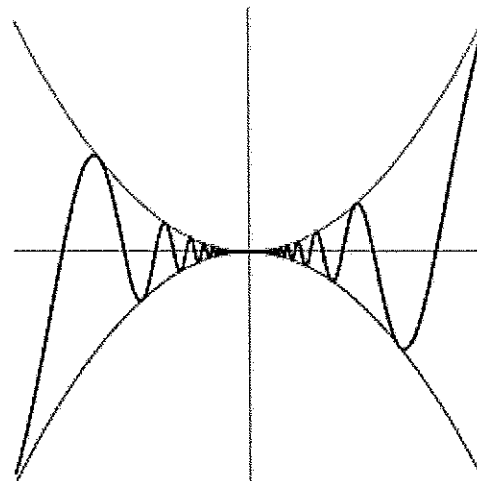
B.  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta + \sin 3\theta}{\theta}$

## Topic 1.8: Determining Limits Using the Squeeze Theorem

**EX #10: Evaluate the limit using the squeeze theorem.**

$$\lim_{x \rightarrow 0} \left[ (x^2) \sin \left( \frac{1}{x} \right) \right]$$

**A.** Explain why you cannot use the Product Limit Law.



**B.** Consider the range of the sine function  $-1 \leq \sin x \leq 1$ . We can conclude that  $-1 \leq \sin \left( \frac{1}{x} \right) \leq 1$ . Multiply through by  $x^2$  to use the Squeeze Theorem.

**EX #11: Find the limit.**  $\lim_{x \rightarrow 0} \left[ (x^2) \cos \left( \frac{1}{x^2} \right) \right]$  .

# I.6

## Limits of Transcendental Functions

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

Problems 1 - 6, find the limits of each exponential function. If necessary, sketch a graph of the function to help you determine the behavior and limit.

1.  $\lim_{x \rightarrow -2} 3^{x+4} - 2$

2.  $\lim_{x \rightarrow \infty} 2^{x-3} + 4$

3.  $\lim_{x \rightarrow -\infty} -(0.25)^{-x+1} + 5$

4.  $\lim_{x \rightarrow 3} \left(\frac{1}{2}\right)^{3-x} - 1$

5.  $\lim_{x \rightarrow -1} -4^{-x-1} + 5$

6.  $\lim_{x \rightarrow \infty} -\left(\frac{1}{3}\right)^{2-x} - 3$

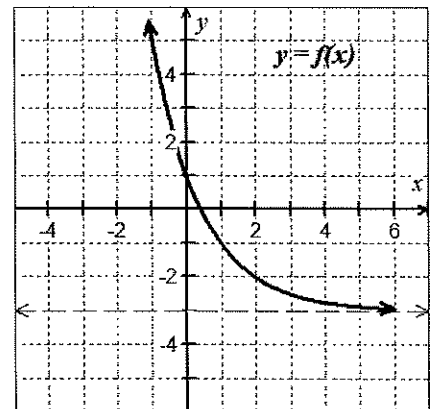
Problems 7 - 10, use the graph of  $y = f(x)$  at right, to find each of the following limits.

7.  $\lim_{x \rightarrow \infty} f(x) =$  \_\_\_\_\_

8.  $\lim_{x \rightarrow -\infty} f(x) =$  \_\_\_\_\_

9.  $\lim_{x \rightarrow 0} f(x) =$  \_\_\_\_\_

10.  $\lim_{x \rightarrow 2} f(x) =$  \_\_\_\_\_



Problems 11 - 28, evaluate the limits, if they exist.

11.  $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$

12.  $\lim_{x \rightarrow 0} \frac{\sin x \sec x}{x}$

13.  $\lim_{x \rightarrow 0} \frac{e^x \cos x}{2}$

14.  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \cos x}{3x}$

15.  $\lim_{x \rightarrow 0} \frac{x + \sin x}{x}$

16.  $\lim_{x \rightarrow 4} e^{3-x} + 4$

17.  $\lim_{\theta \rightarrow \frac{\pi}{3}} \frac{\cos^2 \theta}{1 - \sin \theta}$

18.  $\lim_{x \rightarrow 0} \frac{\sin x}{3x^2 - x}$

$$19. \lim_{x \rightarrow 2} \begin{cases} 3x^2 - 2x, & x < 2 \\ 7 - \cos\left(\frac{\pi x}{3}\right), & x \geq 2 \end{cases}$$

$$20. \lim_{x \rightarrow 3} \begin{cases} \ln x, & 0 < x < 2 \\ x^2 \ln x, & 2 \leq x \leq 4 \end{cases}$$

$$21. \lim_{x \rightarrow \frac{\pi}{3}} \frac{2\sin^2 x + 3\sin x - 2}{2\sin x - 1}$$

$$22. \lim_{x \rightarrow 0} \frac{5x + \sin 2x}{x}$$

$$23. \lim_{x \rightarrow 0} \frac{3^x \cos x}{4}$$

$$24. \lim_{x \rightarrow -3} \left(\frac{1}{4}\right)^{-x-4} + 5$$

$$25. \lim_{\theta \rightarrow 0} \frac{5\cos\theta - 5}{\theta}$$

$$26. \lim_{x \rightarrow -3} \frac{(x+3)\ln(x+5)}{x^2 - 9}$$

27.  $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta + 1 - \cos \theta}{\theta}$

28.  $\lim_{\theta \rightarrow 0} \frac{\cos \theta \sin \theta - \sin \theta}{\theta^2}$

**Problems 29 - 30, Use the Squeeze Theorem to evaluate the following limits.**

29.  $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right)$

30. If  $0 \leq f(x) \leq c$  for some real number  $c$ , prove that  $\lim_{x \rightarrow 0} x^2 f(x) = 0$

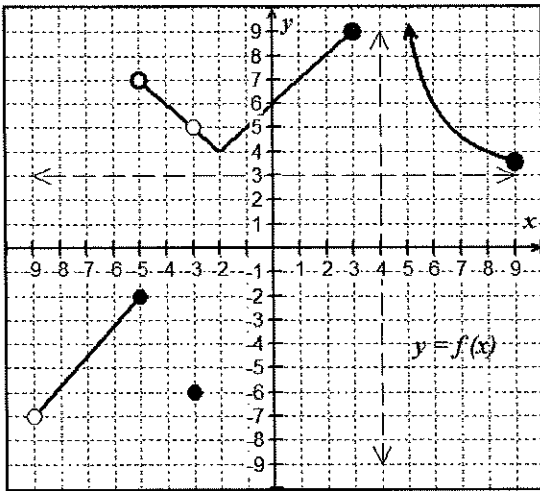
# Lesson 7: Limits and Continuity

Now that we have an understanding of limits and using limit notation, we can use limits to define continuity at a point. *This will be a very important concept throughout our course.*

## Topic 1.11: Defining Continuity at a Point

### EX #1: A Discovery Exploration.

Use the graph below to complete the table. You should *look for three conditions that are necessary to satisfy the definition of continuity. That is, what three conditions must exist in order for  $f(x)$  to be continuous at a point  $x = c$ ?*



Definition of Continuity	

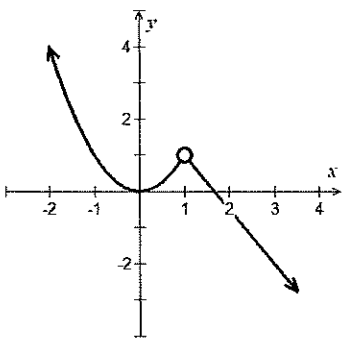
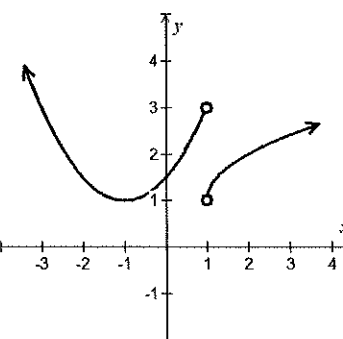
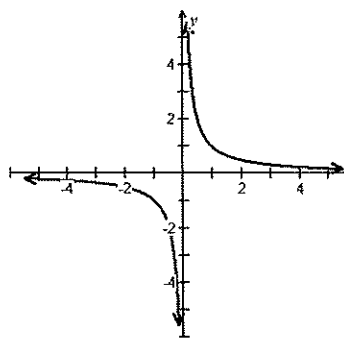
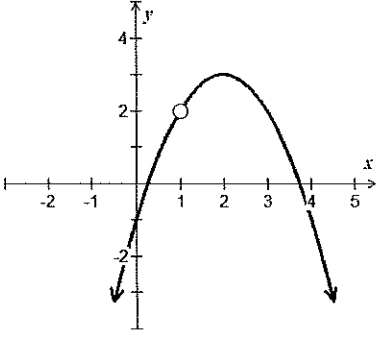
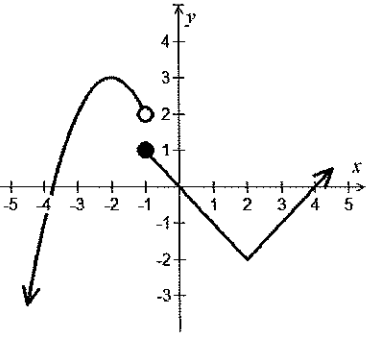
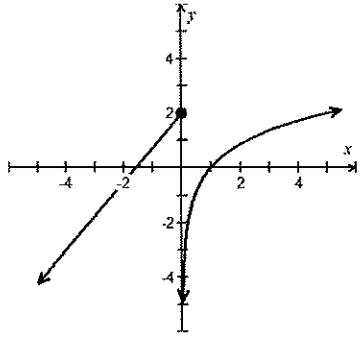
$x = c$	Find the function value, if it exists, for given x-value	Find left-hand limit $\lim_{x \rightarrow c^-} F(x)$	Find right-hand limit $\lim_{x \rightarrow c^+} F(x)$	Find general limit $\lim_{x \rightarrow c} F(x)$	Is $F(x)$ continuous at $x = c$ ?
$x = 9$					
$x = 4$					
$x = 3$					
$x = 0$					
$x = -3$					
$x = -5$					

**Analyze this table and write the three conditions for continuity above. Memorize the rules!**

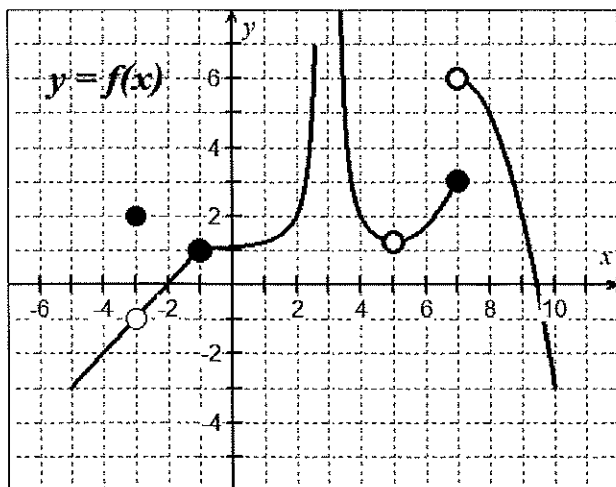
*While **limits** told us where a function intended to go,  
**Continuity** guarantees that the function actually made it there.*

## Classifying Discontinuities

### Topic 1.10: Exploring Types of Discontinuities

Removable or Point (Holes) 2-sided limit exists	Non-Removable	
	Jump 1-sided limits exists	Infinite At least one of the 1-sided limits doesn't exist
		
		

**EX #2: Find the points (intervals) at which the function is continuous, and the points at which the function is discontinuous on the interval  $-5 < x < 10$ .**



## Definition of Continuity - More Facts and Theorems

### One-Sided Continuity

A function  $f(x)$  is called

- Left-continuous at  $x = c$  if  $\lim_{x \rightarrow c^-} f(x) = f(c)$
- Right-continuous at  $x = c$  if  $\lim_{x \rightarrow c^+} f(x) = f(c)$

### Continuity at a Point

Suppose  $f(x)$  is defined on an open interval containing  $x = c$ .

Then  $f$  is continuous at  $x = c$  if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

### Continuity on an Open Interval

A function is continuous on an open interval  $(a, b)$  if it is continuous at each point in the interval.

### Continuity on a Closed Interval

A function is continuous on a closed interval  $[a, b]$  if it is continuous on the open interval  $(a, b)$  and the function is continuous from the right at  $a$  and continuous from the left at  $b$ .

### Continuity Laws of Some Basic Functions

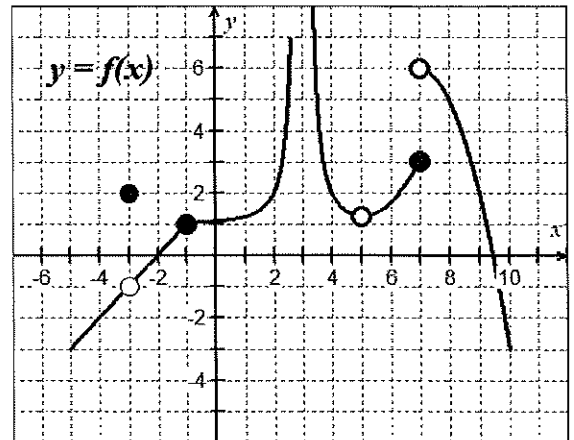
- Polynomial functions  $P(x)$  are continuous over reals
- Rational Functions  $P(x)/Q(x)$  is continuous on its domain such that  $Q(c) \neq 0$ .
- $y = x^{1/n}$  is continuous on all reals if  $n$  is odd and continuous on  $[0, \infty)$  if  $n$  is even.
- $y = \sin x$  and  $y = \cos x$  are continuous over reals
- $y = b^x$  is continuous for  $b > 0, b \neq 1$
- $y = \log_b x$  is continuous for  $x > 0, b > 0, b \neq 1$
- Inverse functions - if  $f(x)$  is continuous on an interval with range  $R$  and  $f^{-1}(x)$  exists, then  $f^{-1}(x)$  is continuous on domain  $R$ .

### Properties of Continuity

Given functions  $f$  and  $g$  continuous at  $x = c$ , then the following functions are also continuous at  $x = c$ .

1. Scalar multiple:  $b \cdot f$
2. Sum or difference:  $f \pm g$
3. Product:  $f \cdot g$
4. Quotient:  $\frac{f}{g}$ ; if  $g(c) \neq 0$
5. Compositions: If  $g$  is continuous at  $c$  and  $f$  is continuous at  $g(c)$ , then the composite function is continuous at  $c$ ,  $(f \circ g)(x) = f(g(x))$

**EX #3:** For  $c = -3$ ,  $c = 3$ , and  $c = 7$ , find  $f(c)$ ,  $\lim_{x \rightarrow c^-} f(x)$ ,  $\lim_{x \rightarrow c^+} f(x)$ , and  $\lim_{x \rightarrow c} f(x)$ . Justify your findings using the three-part definition of continuity.



### Topic 1.12: Confirming Continuity over an Interval

You must be able to confirm continuity without a graph or a calculator by using your knowledge of function behavior for parent functions and their transformations.

**EX #4:** Find the values of  $x$  where the function is discontinuous. Describe the type, infinite or removable. Justify your answer by using the definition of continuity.

A.  $f(x) = \frac{1}{x-3}$

B.  $g(x) = \frac{2x^2 + 7x + 6}{x+2}$

**Topic 1.13: Removing Discontinuities**

**EX #5: Use the definition of continuity to find the value of  $k$  so that the function is continuous for all real numbers.**

A.  $g(x) = \begin{cases} kx^2, & x \leq 2 \\ kx - 6, & x > 2 \end{cases}$

B.  $h(x) = \begin{cases} |x - 4|, & x < 4 \\ \frac{x - 4}{x - 4}, & x = 4 \\ 5k - 4x, & x > 4 \end{cases}$

**EX #6:** Given  $h(x) = \begin{cases} -2x - 5; & x < -2 \\ 3; & x = -2 \\ x^3 - 6x + 3; & x > -2 \end{cases}$  for what values of  $x$  is  $h(x)$  not continuous? Justify.

**EX #6: Use the three-part definition of continuity to create a system of equations. Then, find the values of  $a$  and  $b$  so that  $f(x)$  is continuous for all real numbers.**

$$f(x) = \begin{cases} 2ax - b & ; x < -1 \\ 6 & ; x = -1 \\ ax^2 + b & ; x > -1 \end{cases}$$

You know that the function value and the limit value can exist independently of each other. Let's summarize the big ideas of continuity on intervals, at points, and one-sided limits.

**EX #7: Given  $g(x) = \begin{cases} 2x + 1, & x < 3 \\ x^2, & x \geq 3 \end{cases}$  find each of the following.**

A.  $\lim_{x \rightarrow 3^-} g(x)$

B.  $\lim_{x \rightarrow 3^+} g(x)$

C.  $g(3)$

D.  $\lim_{x \rightarrow 3} g(x)$

F. Is  $g(x)$  continuous at  $x = 3$ ? Justify.

**Summary:**

**If a function is continuous at a point, then the function value and the limit value are the same at that point!**

# 1.7

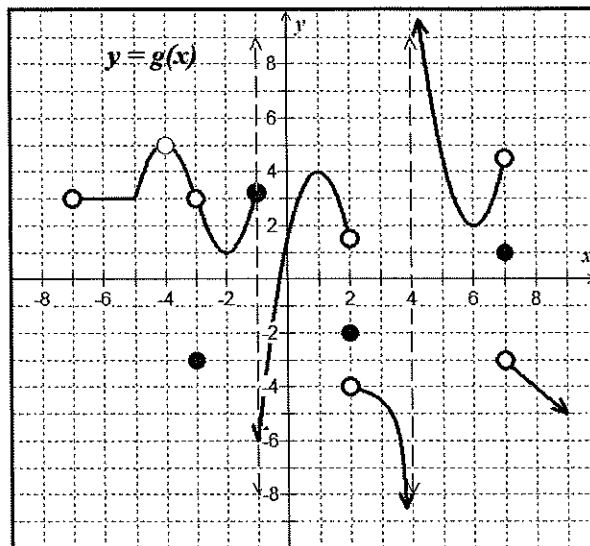
## Limits and Continuity

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

1. Use the graph of the function  $y = g(x)$  shown below, to evaluate each of the following.

A. $\lim_{x \rightarrow -3} g(x) =$	B. $\lim_{x \rightarrow -1^-} g(x) =$
C. $\lim_{x \rightarrow -1^+} g(x) =$	D. $g(-1) =$
E. $\lim_{x \rightarrow -1} g(x) =$	F. $\lim_{x \rightarrow 4^+} g(x) =$
G. $\lim_{x \rightarrow 4^-} g(x) =$	H. $\lim_{x \rightarrow 7} g(x) =$
J. $g(7) =$	K. $\lim_{x \rightarrow 2^+} g(x) =$



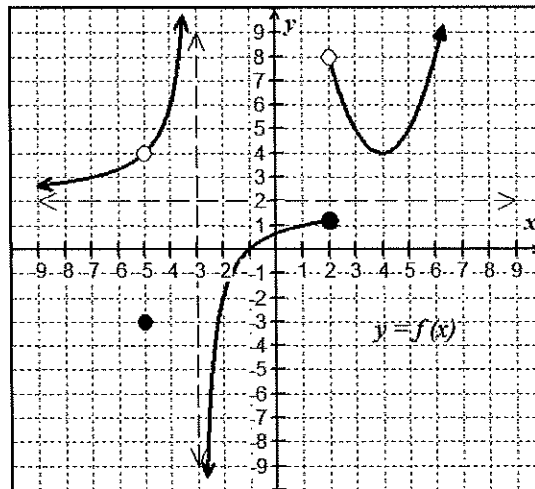
2. Use the graph of the function  $y = f(x)$  shown below, to evaluate each of the following.

A. On the interval  $x \in (-\infty, \infty)$ , list the largest intervals for which  $f(x)$  is continuous.

B. Find the smallest value  $k$ , such that the function is continuous on  $(k, \infty)$

C. Find the smallest value  $k$ , such that the function is continuous on  $[k, \infty)$

D. Find the largest value of  $b$  such that  $y = f(x)$  is continuous on  $(-3, b]$  but not continuous in  $(-3, b + 1]$ . State all values of  $b$  that would work.



**Problem 3 – 6, determine the points, classify the type for each as removable, non-removable, jump, or infinite.**

3.  $f(x) = \frac{1}{(x-3)^2}$

4.  $g(x) = \frac{x-4}{x^2-9x+20}$

5.  $h(x) = \frac{|x+2|}{x+2}$

6.  $f(x) = \begin{cases} x+1 & x < 2 \\ -1 & x = 2 \\ x^2+1 & x > 2 \end{cases}$

**Problems 7 – 8, use the three-part definition of continuity to determine if the given functions are continuous at the indicated values of  $x$ .**

7.  $f(x) = \begin{cases} e^x \cos x, & x \geq \pi \\ e^x \tan\left(\frac{3x}{4}\right), & x < \pi \end{cases}$  at  $x = \pi$

8.  $g(x) = \begin{cases} \frac{x^2-9}{x+3} & x \neq -3 \\ 5 & x = -3 \end{cases}$  at  $x = -3$

Problems 9 – 12, find all value(s) of  $a$ ,  $b$ ,  $c$  or  $k$  that make the function continuous everywhere.

$$9. f(x) = \begin{cases} kx^2 & x \leq 3 \\ 4x - 11 & x > 3 \end{cases}$$

$$10. g(x) = \begin{cases} cx^2 & x < 1 \\ 4 & x = 1 \\ -x^3 + kx & x > 1 \end{cases}$$

$$11. h(x) = \begin{cases} \pi & x < 0 \\ x^2 + ax + b & 0 \leq x \leq 1 \\ 6x + 5 & x > 1 \end{cases}$$

$$12. f(x) = \begin{cases} x^2 & x < 1 \\ \sin(bx) & x \geq 1 \end{cases}$$

13. Consider the function  $y = f(x)$  to answer the following.  $f(x) = \begin{cases} -3 & x \leq -1 \\ mx + k & -1 < x < 4 \\ 3 & x \geq 4 \end{cases}$

A. What two limits must be equal in order for the function to be continuous at  $x = -1$ ?

B. What two limits must be equal in order for the function to be continuous at  $x = 4$ ?

C. Find the values of  $m$  and  $k$  so that the function is continuous everywhere.

14. If  $y = f(x)$  is continuous for all  $x \neq \frac{1}{2}$ , evaluate the following.  $f(x) = \begin{cases} \frac{x^2 - x - 6}{2x^2 + 3x - 2}, & x \neq -2 \\ k, & x = -2 \end{cases}$

A.  $\lim_{x \rightarrow \frac{1}{2}^+} f(x) =$

B.  $\lim_{x \rightarrow 1} f(x) =$

C. What is the value of  $k$ ?

# Lesson 8: Infinite Limits and Limits at Infinity

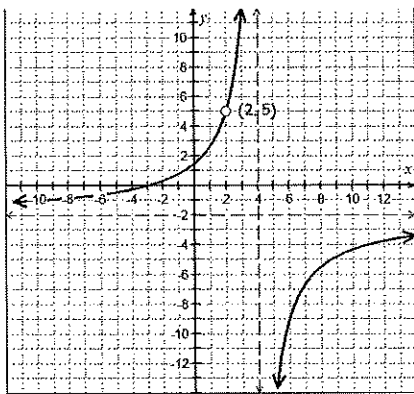
In our first lesson on *Understanding Limits* you were confronted with these two situations. Here we begin to **compare and contrast** the behavior of functions as they **approach infinity**, as well as, functions that **tend toward infinity** in certain circumstances. Let's go...

## Topic 1.14: Connecting Infinite Limits and Vertical Asymptotes

## Topic 1.15: Connecting Limits at Infinity and Horizontal Asymptotes

**EX #1: Use the graphs of  $f(x)$  and  $g(x)$  shown below to compare and contrast behaviors involving infinity. Then, write your discovery and definition.**

$$f(x) = \frac{-2x^2 - 2x + 12}{x^2 - 6x + 8}$$



$$\lim_{x \rightarrow 4^-} f(x) =$$

$$\lim_{x \rightarrow 1^-} g(x) =$$

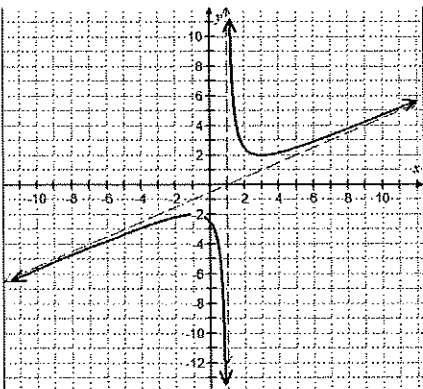
$$\lim_{x \rightarrow 4^+} f(x) =$$

$$\lim_{x \rightarrow 1^+} g(x) =$$

Your Discovery:

**An Infinite Limit is:**

$$g(x) = \frac{x^2 - 2x + 5}{2x - 2}$$



$$\lim_{x \rightarrow -\infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} g(x) =$$

$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow \infty} g(x) =$$

Your Discovery:

**A Limit at Infinity is:**

In Pre-Calculus you learned some basic truths about rational functions.

1. When a factor cancelled from the denominator a \_\_\_\_\_ occurred.
2. When a factor would not cancel from the denominator a \_\_\_\_\_ occurred.

**EX #2: Use the previous equations to find the one-sided limits analytically.**

A.  $\lim_{x \rightarrow 4^-} \frac{-2x^2 - 2x + 12}{x^2 - 6x + 8}$

B.  $\lim_{x \rightarrow 4^+} \frac{-2x^2 - 2x + 12}{x^2 - 6x + 8}$

C.  $\lim_{x \rightarrow 1^-} \frac{x^2 - 2x + 5}{2x - 2}$

D.  $\lim_{x \rightarrow 1^+} \frac{x^2 - 2x + 5}{2x - 2}$

### Definition and Justification of Vertical Asymptotes

Case #1:

$$h(c) = \frac{\text{non-zero}}{\text{zero}}$$

$x = c$  is:

Case #2:

$$h(c) = \frac{\text{zero}}{\text{zero}}$$

$x = c$  is:

**IN CALCULUS, YOU MUST USE NEW LANGUAGE IN ORDER TO JUSTIFY!**

### LIMIT DEFINITION (JUSTIFICATION) OF A VERTICAL ASYMPTOTE

### Topic 1.13: Removing Discontinuities

**EX #3:** Use the function below to find any vertical asymptote(s) that exist. Justify your answer using limits.

$$h(x) = \frac{2x^2 + 9x - 5}{x^2 + 3x - 10}$$

### Limits at Infinity

Next, we will explore **limits at infinity** in order to differentiate between the two conditions. Recall the lessons from Pre-Calculus related to analyzing the **end behavior of functions**. In the exercise below, use this prior knowledge to find each limit at infinity.

**EX #4:** Find each limit at infinity, explain your thinking.

A.  $\lim_{x \rightarrow \infty} (x^2 - 4)(x^2 + 3)$

B.  $\lim_{x \rightarrow -\infty} (5x^3 - 2x + 4)$

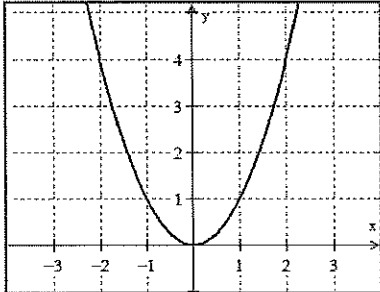
C.  $\lim_{x \rightarrow \infty} \frac{3x^2 - 4}{x^2 + 1}$

D.  $\lim_{x \rightarrow -\infty} \frac{5x - 2}{x^2 + 1}$

**EX #5: There are only four possible outcomes when you explore behavior to the extreme right or left.**

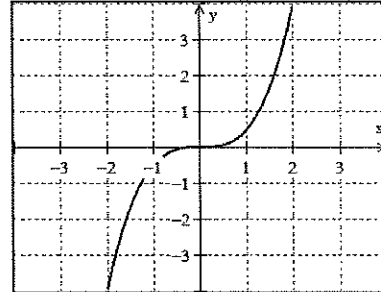
**A. The curve can increase without bound.**

$$\lim_{x \rightarrow \infty} f(x) =$$



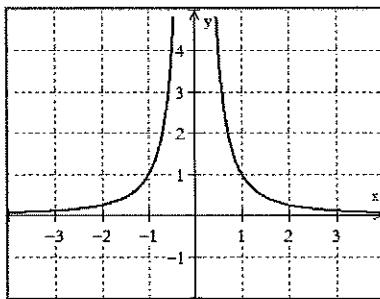
**B. The curve can decrease without bound.**

$$\lim_{x \rightarrow -\infty} f(x) =$$



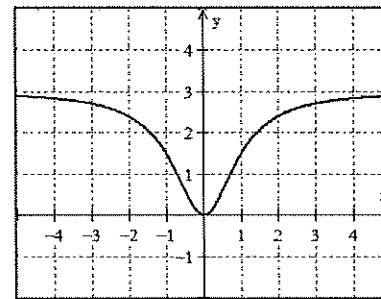
**C. The curve can become asymptotic to the x-axis.**

$$\lim_{x \rightarrow \infty} f(x) =$$



**D. The curve can become asymptotic to a specific y-value.**

$$\lim_{x \rightarrow -\infty} f(x) =$$



## Calculus Knowledge for Asymptotes

Revisiting the rules for finding potential horizontal asymptotes for rational functions from PreCalculus, you can use the idea of a limit and calculus to see why those rules hold true.

1. If degree of numerator is less than degree of denominator (bottom heavy), then limit is zero.

$$\lim_{x \rightarrow \pm\infty} f(x) = 0$$

2. If degree of numerator equals degree of denominator (powers equal), then limit is the ratio of coefficients of the highest degree.

$$\lim_{x \rightarrow \pm\infty} f(x) = \frac{\text{coefficient of numerator's highest power}}{\text{coefficient of denominator's highest power}}$$

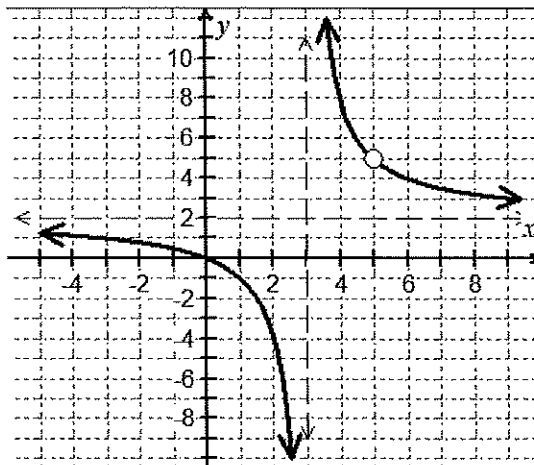
3. If degree of numerator is greater than degree of denominator (top heavy\*), then limit does not exist.

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$$

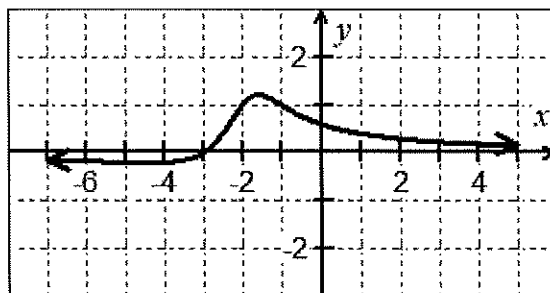
### Topic 1.9: Connecting Multiple Representations of Limits

EX #6: Divide every term in the rational expression by the **highest power of  $x$  that appears in the denominator**. Then, apply the Properties of Limits to evaluate each “piece” to find the limit at infinity, end behavior.

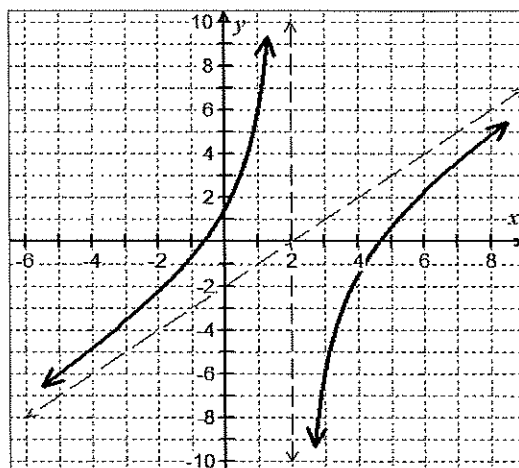
A. 
$$\lim_{x \rightarrow \infty} \frac{2x^2 - 10x}{x^2 - 8x + 15}$$



B. 
$$\lim_{x \rightarrow \infty} \frac{x + 3}{x^2 + 4x + 5}$$



C. 
$$\lim_{x \rightarrow \infty} \frac{x^2 - 4x - 3}{x - 2}$$



**EX #7:** Summarize and discuss characteristics and end behavior at horizontal asymptotes and slant asymptotes based on your observations in EX #6.

**Functions with Horizontal Asymptotes**

$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} f(x) =$$

**Functions with Slant Asymptotes**

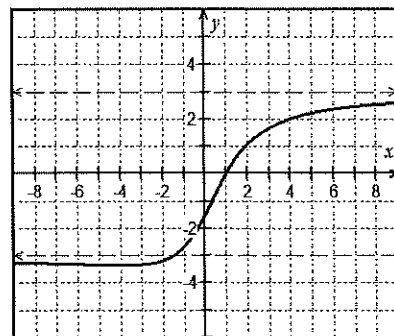
$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} f(x) =$$

**IN CALCULUS, YOU MUST USE NEW LANGUAGE IN ORDER TO JUSTIFY!**

**LIMIT DEFINITION (JUSTIFICATION) OF  
A HORIZONTAL ASYMPTOTE**

**EX #8: CHALLENGE!** Use algebraic techniques to find the limits for  $g(x) = \frac{3x - 3}{\sqrt{x^2 + 4}}$ , whose graph is shown.



$$\lim_{x \rightarrow -\infty} \frac{3x - 3}{\sqrt{x^2 + 4}} =$$

$$\lim_{x \rightarrow \infty} \frac{3x - 3}{\sqrt{x^2 + 4}} =$$

Notes (goes w/ 1.8 Material)

Limits at Infinity

Ex. 1  $g(x) = \frac{3x-3}{\sqrt{x^2+4}}$

$$\lim_{x \rightarrow \infty} g(x)$$

$$\lim_{x \rightarrow -\infty} g(x)$$

Ex. 2

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 10}}{x - 5}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 10}}{x - 5}$$

## 1.8

# Infinite Limits and Limits at Infinity

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

**Problem 1 - 4, Find  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$ , then identify all horizontal or slant asymptotes.**

$$1. f(x) = \frac{7 + 2x - 5x^2}{2x^2 - 7x - 4}$$

$$2. f(x) = \frac{2x - 3}{\sqrt{4x^2 + 3}}$$

$$3. f(x) = \frac{3 + 4x - 2x^3}{x^2 + 1}$$

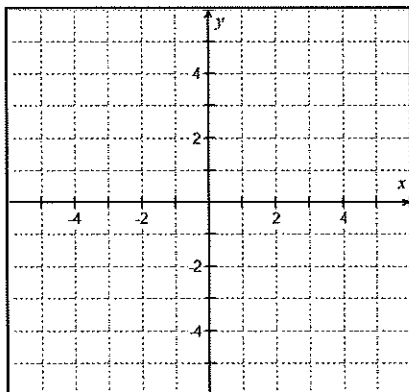
$$4. f(x) = \frac{x^{4/3} + x^{1/3}}{(4x^{2/3} + 1)^2}$$

**Problems 5 - 6, sketch a function that satisfies the stated conditions. Include asymptotes.**

$$5. \lim_{x \rightarrow 3} f(x) = -2 \quad \lim_{x \rightarrow -2^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -2^+} f(x) = +\infty \quad \lim_{x \rightarrow -\infty} f(x) = 3$$

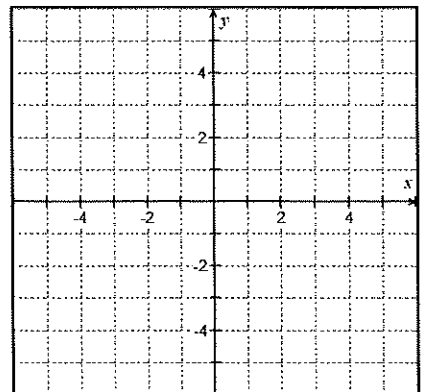
$$\lim_{x \rightarrow +\infty} f(x) = \infty$$



$$6. \lim_{x \rightarrow -3^-} f(x) = +\infty \quad \lim_{x \rightarrow -3^+} f(x) = +\infty$$

$$\lim_{x \rightarrow 2} f(x) = 3 \quad \lim_{x \rightarrow -\infty} f(x) = -3$$

$$\lim_{x \rightarrow +\infty} f(x) = -3$$



**Problems 7 - 10, Identify all vertical asymptotes and find  $\lim_{x \rightarrow a^+} f(x)$ ;  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a} f(x)$ , where  $a$  is the  $x$ -value of the asymptote.**

$$7. f(x) = \frac{x+4}{x^2+9x+20}$$

$$8. f(x) = \frac{2x^2-x-15}{x^2-5x+6}$$

$$9. f(x) = \frac{\ln(x^2+1)}{x+1}$$

$$10. f(x) = \frac{x^2+3x-18}{x^2-6x+9}$$

**Problems 11 - 12, For the piecewise functions, find the limit as  $x \rightarrow \infty$ ,  $x \rightarrow -\infty$ ,  $x \rightarrow 0^-$ , and  $x \rightarrow 0^+$ .**

$$11. f(x) = \begin{cases} \frac{2x+3}{x-1}, & x < 0 \\ \frac{1}{x}, & x \geq 0 \end{cases}$$

$$12. g(x) = \begin{cases} \frac{1}{x^2}, & x < 0 \\ \frac{2x}{x+1}, & x \geq 0 \end{cases}$$

Problems 13 – 20, Evaluate the following limits without the aid of a calculator.

$$13. \lim_{x \rightarrow \infty} \left[ \left( \frac{3x^2 - 1}{x^2} \right) \left( \frac{2}{x} - 1 \right) \right]$$

$$14. \lim_{x \rightarrow -\infty} \left( \frac{5 + 4x - 3x^2}{2x^2 + 1} \right)$$

$$15. \lim_{x \rightarrow \infty} (e^{-x} \sin x)$$

$$16. \lim_{x \rightarrow 3^+} \ln(x^2 - 9)$$

$$17. \lim_{x \rightarrow -1^-} \frac{x+1}{x^4-1}$$

$$18. \lim_{x \rightarrow 0^-} \frac{2^x}{x}$$

$$19. \lim_{x \rightarrow \infty} \frac{x^2 + 3x}{\sqrt{2x^2 + 1}}$$

$$20. \lim_{x \rightarrow \infty} \frac{4x + 9}{2x^2 - x + 6}$$

Problems 21 - 26, Use the graph of  $y = g(x)$  at right to find the limits.

21.  $\lim_{x \rightarrow -3^-} g(x) =$

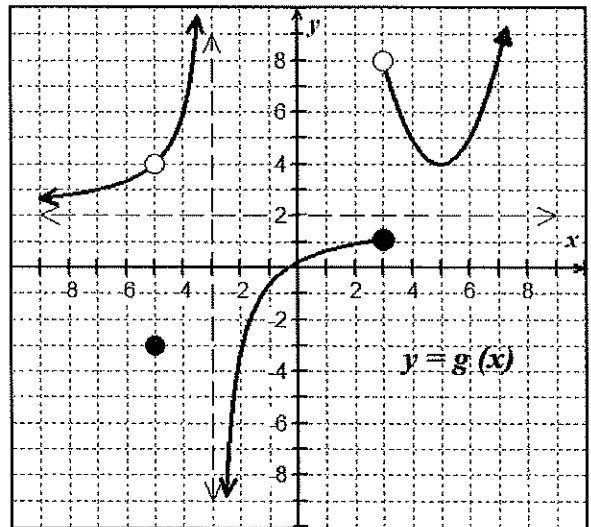
22.  $\lim_{x \rightarrow -3^+} g(x) =$

23.  $\lim_{x \rightarrow -3} g(x) =$

24.  $\lim_{x \rightarrow -\infty} g(x) =$

25.  $\lim_{x \rightarrow \infty} g(x) =$

26.  $\lim_{x \rightarrow 3^+} g(x) =$



Problems 27 - 29, Multiple Choice.

\_\_\_\_\_ 27. Find the limit:  $\lim_{x \rightarrow -\infty} \frac{4+e^{-x}}{1-e^{-x}}$

- (A) 4                      (B) -4                      (C) -1                      (D)  $-\infty$

\_\_\_\_\_ 28. Find the limit:  $\lim_{x \rightarrow 3} \left( 2 - \frac{5}{(x-3)^2} \right)$

- (A)  $\infty$                       (B)  $-\infty$                       (C) 2                      (D) -3

\_\_\_\_\_ 29.  $f(x)$  decreases without bound as  $x$  approaches what value from the right?

$$f(x) = \frac{6}{(x-3)(7-x)}$$

- (A) -7                      (B) 7                      (C) -3                      (D) 3

Goes with 1.8 Material

Find each limit: (Show work for #1-3)

$$1) \lim_{x \rightarrow -\infty} \left( \frac{\sqrt{5x^2 - 3}}{x + 3} \right)$$

$$2) \lim_{x \rightarrow \infty} \left( \frac{5x - 2}{\sqrt{3 + 2x^2}} \right)$$

$$3) \lim_{x \rightarrow -\infty} \left( \frac{7 - 8x}{\sqrt{4x^2 - 1}} \right)$$

Graph each function based on the given information. Answers may vary.

- 4)
- a.  $f(-2) = 5$
  - b.  $\lim_{x \rightarrow -2} f(x) = 1$
  - c.  $\lim_{x \rightarrow 4^+} f(x) = 3$
  - d.  $f$  is increasing on  $x < -2$
  - e.  $\lim_{x \rightarrow 4^-} f(x) < \lim_{x \rightarrow 4^+} f(x)$

- 5)
- a.  $g(1) = 3$
  - b.  $\lim_{x \rightarrow 1} g(x) = -2$
  - c.  $\lim_{x \rightarrow -3^+} g(x) = 5$
  - d.  $g$  is increasing only on  $-5 < x < -3$  and  $x > 1$
  - e.  $\lim_{x \rightarrow -3^-} g(x) > \lim_{x \rightarrow -3^+} g(x)$

- 6)
- a)  $f(2) = 0$
  - b)  $\lim_{x \rightarrow \infty} f(x) = 0$
  - c)  $\lim_{x \rightarrow -\infty} f(x) = 0$
  - d)  $\lim_{x \rightarrow 0} f(x) = \infty$

- e)  $\lim_{x \rightarrow 5^-} f(x) = -\infty$
- f)  $\lim_{x \rightarrow 5^+} f(x) = \infty$

## Some Review

$$7) \lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$$

$$8) \lim_{x \rightarrow 0} \frac{\frac{1}{x-4} + \frac{1}{4}}{x}$$

9) Is  $f(x)$  continuous at  $x=0$ ? Justify using limits.

$$f(x) = \begin{cases} 4-3x^2 & x < 0 \\ 4 & x = 0 \\ \sqrt{16-x^2} & 0 < x \leq 4 \end{cases}$$

## Lesson 9: Intermediate Value Theorem

Continuity of a function will prove to be an important characteristic in many theorems for our course. In order for theorems to be properly applied, we need to meet the certain conditions. Namely, the “if” part, called the ***hypothesis***, must be satisfied before we can apply the “then” part, or ***conclusion*** (consequence) of the theorem.

In this lesson, we will explore our first **existence theorem**. Existence theorems allow us to draw conclusions about a function’s behavior on an interval without precisely locating that behavior.

### Topic 1.16: Working with the Intermediate Value Theorem (IVT)

#### Intermediate Value Theorem

##### The Big Idea:

The **IVT** says that a continuous function on an interval cannot skip values.

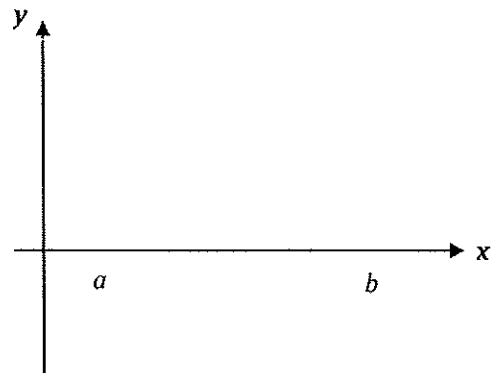
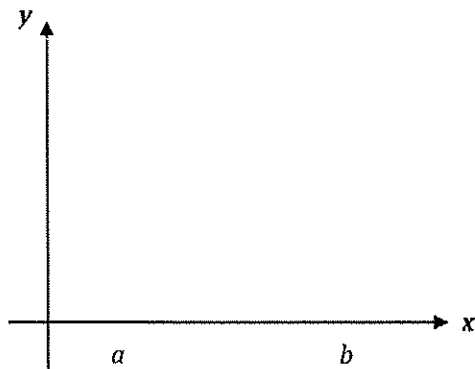
##### Using If/Then statements:

**If** a function  $y = f(x)$  is continuous on a closed interval  $[a, b]$  and  $f(a) \neq f(b)$ , **Then**  $f(x)$  takes on every value between  $f(a)$  and  $f(b)$  on that interval.

##### Another Approach:

Said in a different fashion, if we know a  $y$ -value, say  $y = k$ , that resides between the two endpoints,  $f(a)$  and  $f(b)$ , then we are guaranteed at least one  $x$ -value,  $x = c$ , between the endpoints that generates that  $y$ -value, such that  $f(c) = k$ .

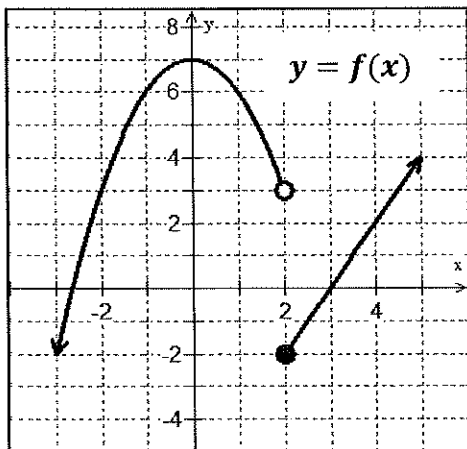
#### EX #1: Graphical representation of the Intermediate Value Theorem and its corollary:



*A continuous function on an interval cannot skip values.*

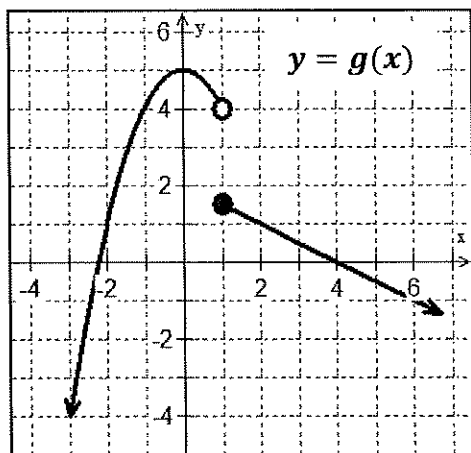
*An important outcome of I.V.T. is that it can be helpful in finding zeros of a continuous function on an interval.*

**EX #2:** Does the Intermediate Value Theorem apply on the specified intervals for the piecewise functions shown below? Be sure to explain completely.



**A.** Is there a value of  $x = c$  on the interval  $[-2, 5]$  such that  $f(c) = 1$ ?

**B.** Does I.V.T. guarantee a value of  $c$  on the interval  $[-2, 5]$  such that  $f(c) = 1$ ? Justify.



**C.** Is there a value of  $x = c$  on the interval  $[1, 6]$  such that  $g(c) = 3$ ?

**D.** Does I.V.T. guarantee a value of  $c$  on the interval  $[1, 6]$  such that  $g(c) = 3$ ? Justify.

**What three conditions are necessary to apply the Intermediate Value Theorem?**

1.

2.

3.

**EX #3:** Apply the IVT, if possible, on  $[0, 5]$  so that  $f(c) = 1$  for the function  $f(x) = x^2 + x - 1$ .

**EX #4:** Use the Intermediate Value Theorem to show that  $f(x) = x^3 + 2x - 1$  has a zero in the interval  $[0, 1]$ .

**EX #5:** A car travels on a straight track. During the time interval  $0 \leq t \leq 60$  seconds, the car's velocity  $v$ , measured in feet per second is a continuous function. The table below shows selected values of the function.

$t$ , in seconds	0	15	25	30	35	50	60
$v(t)$ in ft/sec	-20	-30	-20	-14	-10	0	10

- A. For  $0 < t < 60$ , must there be a time  $t$  when  $v(t) = -5$  ?  
B. Justify your answer.

**EX #6:** Let  $g(x)$  be a continuous function. Selected values of  $y = g(x)$  are given in the table below. For which value of  $k$  will the equation  $g(x) = 3/4$  have **at least two solutions** on the closed interval  $[2, 8]$ ?

$x$	2	3	4	5	8
$g(x)$	3	4	$k$	5	2

(A) 1

(B)  $\frac{3}{4}$

(C)  $\frac{9}{16}$

(D)  $\frac{3}{2}$

**EX #7:** For the function  $f(x) = \begin{cases} (x-3)^2, & x = 5 \\ 6, & 5 < x \leq 10 \end{cases}$ . Find  $f(5)$  and  $f(10)$ . Does IVT guarantee a  $y$ -value  $f(c) = k$ , on  $5 \leq x \leq 10$  such that  $f(5) < f(c) < f(10)$ ? Justify your answer.

**EX #8:** The functions  $f$  and  $g$  are continuous for all real numbers. The table below gives values of the functions at selected values of  $x$ . The function  $h$  is given by  $h(x) = g(f(x)) + 2$ . Explain why there must be a value  $w$  for  $1 < w < 6$  such that  $h(w) = 0$ .

$x$	1	2	6	8
$f(x)$	2	9	8	13
$g(x)$	3	-12	5	28

**I.9****Intermediate Value Theorem**

Name \_\_\_\_\_

Date \_\_\_\_\_ Period \_\_\_\_\_

**Problems 1 - 13, Use the Intermediate Value Theorem to complete.**

1. In the function  $f(x) = x^3 - x - 1$ , it can be shown that  $f(1) = -1$  and  $f(2) = 5$ . Complete the table below to find an approximation for a solution of the interval  $[1, 2]$ .

$x$	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$f(x)$											

2. Find the value of  $c$  guaranteed by the Intermediate Value Theorem.  $f(x) = x^2 + 4x - 13$  on  $[0, 4]$  such that  $f(c) = 8$

3. Show that  $g(x) = 2x^3 - 5x^2 - 10x + 5$  has a root somewhere in the interval  $[-1, 2]$ .

4. Between which of the following two values does the equation  $3x^3 + 5x - 11 = 0$  have a solution?

**(A)**  $[-2, -1]$ **(B)**  $[0, 1]$ **(C)**  $[-1, 0]$ **(D)**  $[1, 2]$

5. Given the function  $f(x) = \frac{2x-3}{2x-5}$ , determine which interval(s) satisfies the conditions for the Intermediate Value Theorem, such that  $f(x) = 0$ .

(A) One solution between  $x = 0$  and  $x = 1$

(B) One solution between  $x = 1$  and  $x = 2$

(C) One solution between  $x = 1$  and  $x = 2$  and one solution between  $x = 2$  and  $x = 3$

(D) One solution between  $x = 2$  and  $x = 3$

6. Apply the Intermediate Value Theorem, if possible, on  $[1, 2]$  so that  $f(c) = 9$  for the function  $f(x) = x^3 + x$ .

7. A delivery van travels along a straight road. During the time interval  $0 \leq t \leq 30$  seconds, the van's velocity in feet per second is a continuous function. Use the table below to find the minimum number of times that the van must have been stopped. Justify your answer.

$t$ (sec)	0	5	7	12	18	22	30
$V(t)$ (ft/sec)	-28	-60	-15	8	24	-4	10

8. Explain why the Intermediate Value Theorem does not apply for guaranteeing that a zero exists for the function  $f(x) = x^2 + 2x + 5$  over  $[0, 6]$ .

9. The functions  $f$  and  $g$  are continuous. The function  $h$  is given by  $h(x) = f(g(x)) - x$ . The table below gives values of the functions. Explain why there must be a value  $c$  for  $1 < c < 5$  such that  $h(c) = -2$ .

$x$	1	2	3	4	5
$f(x)$	0	9	7	-3	8
$g(x)$	4	6	-4	1	3

10. Given  $f(x) = \frac{x}{x-3}$  on the interval  $[-2, 2]$ . Determine if the IVT applies. State why or why not. Then, find the value of  $c$  such that  $f(c) = \frac{1}{3}$ .

11. Show that there is a value  $c$  with  $0 < c < 2$  such that  $x^2 + \cos \pi x = 4$ . Then, use a graphing utility to find the approximate value of  $c$ .

12. Does the IVT apply to the function  $h(x) = \frac{x^2+x}{x-2}$  on the interval  $[2.5, 5]$ ? If so, find the value of  $c$  guaranteed to exist, such that  $h(c) = 12$ .

13. Does the IVT apply to the function  $f(x) = -\left(\frac{1}{2}\right)^{3-x} - 3$  on the interval  $[2,5]$  for  $f(c) = -4$ ?

\_\_\_\_\_ 14. Let  $f$  be a continuous function on the closed interval  $[-2, 7]$ . If  $f(-2) = -3$  and  $f(7) = 4$ , then the Intermediate Value Theorem guarantees that

(A)  $f(0) < 0$

(B)  $-3 \leq f(x) \leq 4$  for all  $x$  between  $-2$  and  $7$ .

(C)  $f(c) = 1$  for at least one  $c$  between  $-2$  and  $7$

(D)  $f(c) = 0$  for at least one  $c$  between  $-3$  and  $4$

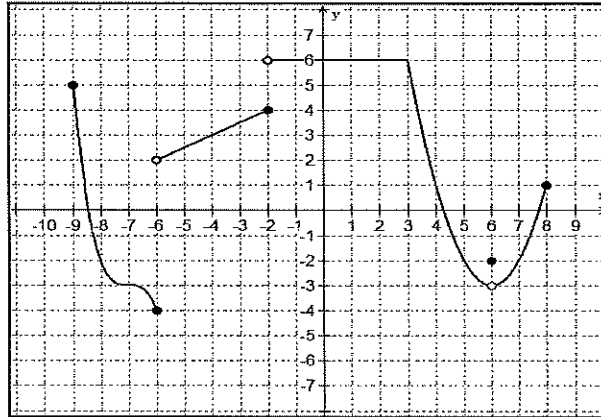
# LIMITS & CONTINUITY

Name \_\_\_\_\_

## QUIZ REVIEW

Show all work! Evaluate all limits analytically, unless a graph or table is provided for you. It is expected that proper mathematical notation throughout.

1. Use the graph of  $h(t)$  shown below to evaluate the following.



A)  $\lim_{t \rightarrow 0} h(t) =$

E)  $\lim_{t \rightarrow -6} h(t) =$

B)  $\lim_{t \rightarrow 6} h(t) =$

F)  $h(6) =$

C)  $\lim_{t \rightarrow -2} h(t) =$

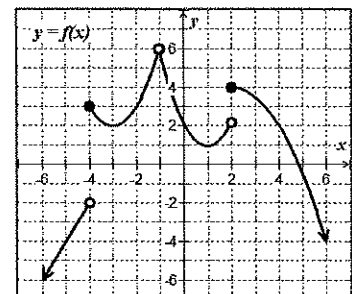
G)  $h(-2) =$

D)  $\lim_{t \rightarrow -6^-} h(t) =$

2. Given  $\lim_{x \rightarrow c} f(x) = 8$  and  $\lim_{x \rightarrow c} g(x) = -3$ , then find  $\lim_{x \rightarrow c} [2f(x)g(x) - \sqrt[3]{f(x)} + 3[g(x)]^2]$

3. Given the graph of a piecewise function, use the **Properties of Limits** to find the limit, if possible. If not possible show work to support your claim.

$\lim_{x \rightarrow 2} f(f(x))$



4. Does  $f(x) = \frac{x-3}{x^2+2x-15}$  have any points of discontinuity? If so, indicate the x-coordinate of the point of discontinuity and indicate whether or not they are removable, by circling the appropriate term.

1<sup>st</sup> x-coordinate of discontinuity: \_\_\_\_\_ Removable or Non-Removable

2<sup>nd</sup> x-coordinate of discontinuity: \_\_\_\_\_ Removable or Non-Removable

3<sup>rd</sup> x-coordinate of discontinuity: \_\_\_\_\_ Removable or Non-Removable

5. Determine the value of  $k$  so that  $f(x)$  is continuous on the entire real line when

$$f(x) = \begin{cases} kx^2; & x \leq 2 \\ 2x + k; & x > 2 \end{cases}. \text{ Be sure to show all of your justifications.}$$

6. Evaluate the following limit by using the output table to complete the following

$$\lim_{x \rightarrow 4} f(x) = \underline{\hspace{2cm}}.$$

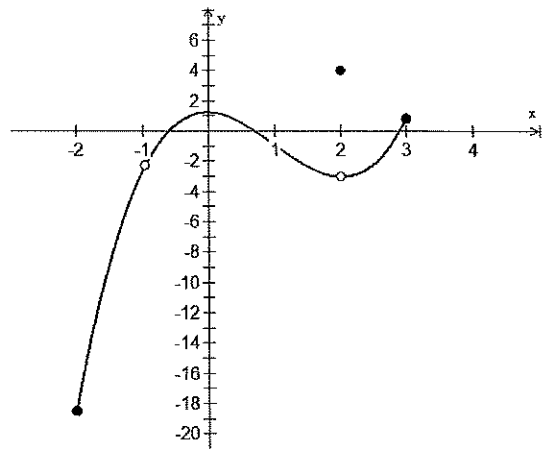
$x$	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				ERROR			

$X$	$Y1$
3.7	.99
3.8	1.64
3.9	2.31
4	ERROR
4.1	3.71
4.2	4.44
4.3	5.19

$X$	$Y1$
3.97	2.7909
3.98	2.8604
3.99	2.9301
4	ERROR
4.01	3.0701
4.02	3.1404
4.03	3.2109

$X$	$Y1$
3.997	2.979
3.998	2.986
3.999	2.993
4	ERROR
4.001	3.007
4.002	3.014
4.003	3.021

7. Is the function continuous everywhere? Explain!  
If not, where is it discontinuous and based upon the definition of continuity, why is it discontinuous?



Evaluate each of the following limits.

8.  $\lim_{x \rightarrow 1} (2x^3 - 4x^2 + x - 2)$

9.  $\lim_{x \rightarrow 5} \left( \frac{\frac{1}{5} - \frac{1}{x}}{5 - x} \right)$

10.  $\lim_{x \rightarrow \frac{\pi}{3}} \left( \frac{1 - \cos x}{x} \right)$

11.  $\lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$

12.  $\lim_{x \rightarrow 3^+} f(x)$ ; where  $f(x) = \begin{cases} 5x + 4 ; & x < 3 \\ 2x^2 - 1 ; & x \geq 3 \end{cases}$

13.  $\lim_{x \rightarrow 0} \frac{\sin x}{8x}$

14.  $\lim_{x \rightarrow 6^-} \frac{|x-6|}{x-6}$

15.  $\lim_{x \rightarrow 4} \left( \csc \frac{\pi x}{3} \right)$

16.  $\lim_{t \rightarrow 0} \frac{\sqrt{t^2+9}-3}{t^2}$

17.  $\lim_{x \rightarrow 2} \frac{x^3-8}{x-2}$

18. What are the three common types of behavior associated with the nonexistence of a limit?

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_

**Important Topics:**

- Limits
- Limit laws, left and right hand limits, limits at infinity
- Continuity
- Intermediate Value Theorem

1.  $\lim_{x \rightarrow 0} \frac{3x^5 + 12x^2}{5x^4 + 6x^2}$       A) 0      B)  $\frac{1}{2}$       C) 2      D) DNE

2.  $\lim_{x \rightarrow 3^-} \frac{4 - 3x}{x - 3}$       A) 0      B)  $\infty$       C)  $-\infty$       D) -3

3. Let  $f$  be the function that is continuous on the closed interval  $[3, 5]$  with  $f(3) = 9$  and  $f(5) = 22$ . Which of the following is guaranteed by the Intermediate Value Theorem?

- A)  $f(x) = 11$  has at least one solution in the open interval  $(3, 5)$ .
- B)  $f(4) = 13$
- C)  $f(x)$  has at least one zero in the open interval  $(3, 5)$ .
- D)  $f$  attains a maximum value on the open interval  $(3, 5)$ .

4. If  $f(x) = 3x^2 + 4x$ , then  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

- A)  $6x$
- B)  $6x + 4$
- C) 0
- D) DNE

$$g(x) = \begin{cases} \sin \frac{x\pi}{3}, & x < 2 \\ x\sqrt{2}, & x = 2 \\ \frac{x\sqrt{3}}{3x-4}, & x > 2 \end{cases}$$

5. For the function above, which of the following would be the reason(s) why the function,  $g(x)$  is not continuous at  $x = 2$ ?

- I.  $g(2)$  is undefined      II.  $\lim_{x \rightarrow 2} g(x)$  does not exist      III.  $\lim_{x \rightarrow 2} g(x) \neq g(2)$

- A) III only  
 B) II only  
 C) I and II only  
 D) II and III only

6.  $\lim_{x \rightarrow \infty} \frac{2x^2 - 4x + 8x^3}{3x^3 - 5x^2 + 7}$       A)  $\infty$       B)  $-\infty$       C)  $\frac{8}{3}$       D)  $\frac{2}{3}$

7.  $\lim_{x \rightarrow -\infty} \frac{5 - 3x}{\sqrt{x^2 + 3}}$       A)  $\infty$       B)  $-\infty$       C)  $-3$       D)  $3$

8. Given the function  $H(x) = \begin{cases} x^2 - 3x, & x < 2 \\ 2x - 5, & x \geq 2 \end{cases}$ . Which of the following statements is/are true?

- I.  $\lim_{x \rightarrow 2^-} H(x) = -2$       II.  $\lim_{x \rightarrow 2} H(x)$  exists      III.  $H(x)$  is continuous at  $x = 2$

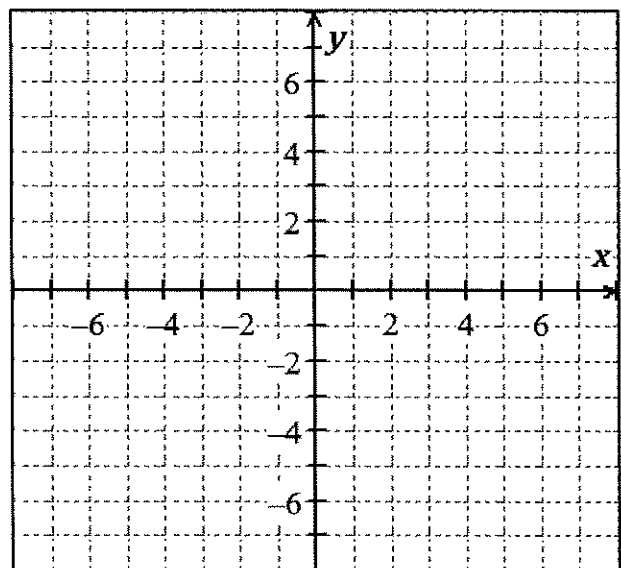
- A) I only      B) II only      C) I and II only      D) None of these

## True or False

9.  $\lim_{x \rightarrow 4} \left( \frac{2x}{x-4} - \frac{8}{x-4} \right) = \lim_{x \rightarrow 4} \frac{2x}{x-4} - \lim_{x \rightarrow 4} \frac{8}{x-4}$
10.  $\lim_{x \rightarrow 1} \left( \frac{x^2+6x-7}{x^2+5x-6} \right) = \frac{\lim_{x \rightarrow 1} (x^2+6x-7)}{\lim_{x \rightarrow 1} (x^2+5x-6)}$
11.  $\lim_{x \rightarrow 1} \frac{x-3}{x^2+2x-4} = \frac{\lim_{x \rightarrow 1} (x-3)}{\lim_{x \rightarrow 1} (x^2+2x-4)}$
12. If  $\lim_{x \rightarrow 5} f(x) = 2$  and  $\lim_{x \rightarrow 5} g(x) = 0$ , then  $\lim_{x \rightarrow 5} [f(x)/g(x)]$  does not exist.
13. If  $\lim_{x \rightarrow 5} f(x) = 0$  and  $\lim_{x \rightarrow 5} g(x) = 0$ , then  $\lim_{x \rightarrow 5} [f(x)/g(x)]$  does not exist.
14. If  $\lim_{x \rightarrow 6} f(x)g(x)$  exists, then the limit must be  $f(6)g(6)$ .
15. If  $p$  is a polynomial, then  $\lim_{x \rightarrow b} p(x) = p(b)$
16. If  $\lim_{x \rightarrow 5} f(x) = \infty$  and  $\lim_{x \rightarrow 5} g(x) = \infty$ , then  $\lim_{x \rightarrow 5} [f(x) - g(x)] = 0$ .
17. If the line  $x = 1$  is a vertical asymptote of  $y = f(x)$ , then  $f$  is not defined at 1.
18. If  $f(1) > 0$  and  $f(3) < 0$ , then there exists a number  $c$  between 1 and 3 such that  $f(c) = 0$ .
19. If  $f$  is continuous at 5 and  $f(5) = 2$  and  $f(4) = 3$ , then  $\lim_{x \rightarrow 2} f(4x^2 - 11) = 2$ .
20.  $\lim_{x \rightarrow 0} \frac{|x|}{x} = 1$
21. If  $\lim_{x \rightarrow c} f(x) = L$ , then  $f(c) = L$ .
22. If  $f$  is undefined at  $x = c$ , then the limit of  $f(x)$  as  $x$  approaches  $c$  does not exist.

23. Sketch the graph of an example of a function  $f$  that satisfies all of the following conditions:

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= -2 & \lim_{x \rightarrow 0^-} f(x) &= 1 \\ f(0) &= -1 & \lim_{x \rightarrow \infty} f(x) &= 3 \\ \lim_{x \rightarrow 2^-} f(x) &= \infty & \lim_{x \rightarrow 2^+} f(x) &= -\infty \\ \lim_{x \rightarrow -\infty} f(x) &= 4 \end{aligned}$$



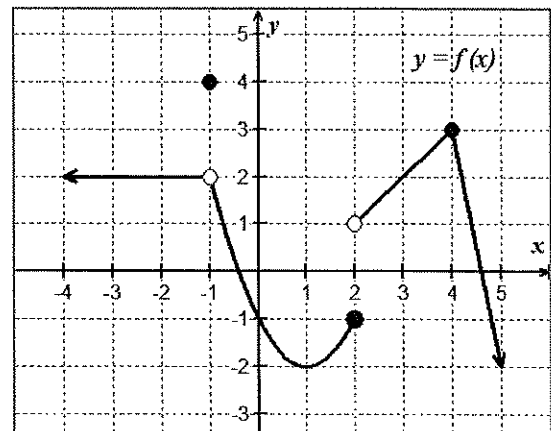
24. Use the graph of  $f(x)$  shown in the figure at the right. Let  $g(x) = \sqrt{4 + x^2}$ . Evaluate the limits, if they exist.

A.)  $\lim_{x \rightarrow 2^+} [f(x) + 5g(x)]$

B.)  $\lim_{x \rightarrow 1} [f(x) \cdot g(x)]$

C.)  $\lim_{x \rightarrow 2^-} \frac{f(x)}{g(x)}$

D.)  $\lim_{x \rightarrow 4} [g^2(x)]$



25. Find the values of  $k$  and  $m$  so that the function below is continuous for all reals.

$$f(x) = \begin{cases} x^2 - kx + 3, & x < -2 \\ 2x - 3, & -2 \leq x \leq 3 \\ 4 - 2m, & x > 3 \end{cases}$$

## 26. Limits at Infinity

Function	Limit as $x \rightarrow a$	Limit as $x \rightarrow \pm\infty$
A.) $f(x) =  x $	$\lim_{x \rightarrow 0} f(x)$	$\lim_{x \rightarrow -\infty} f(x)$
B.) $f(x) = \frac{ x }{x}$	$\lim_{x \rightarrow 0} f(x)$	$\lim_{x \rightarrow \infty} f(x)$
	$\lim_{x \rightarrow -5} f(x)$	
C.) $f(x) = 6$	$\lim_{x \rightarrow 2} 6$	$\lim_{x \rightarrow -\infty} 6$
D.) $f(x) = \cos x$	$\lim_{x \rightarrow 0} f(x)$	$\lim_{x \rightarrow \infty} f(x)$
E.) $f(x) = \frac{1}{x}$	$\lim_{x \rightarrow 0} f(x)$	$\lim_{x \rightarrow \infty} f(x)$
		$\lim_{x \rightarrow -\infty} f(x)$
F.) $f(x) = \frac{x+4}{x-2}$	$\lim_{x \rightarrow 1} f(x)$	$\lim_{x \rightarrow \infty} f(x)$
	$\lim_{x \rightarrow 2} f(x)$	$\lim_{x \rightarrow -\infty} f(x)$
G.) $f(x) = \frac{x-2}{x^2-4}$	$\lim_{x \rightarrow 0} f(x)$	$\lim_{x \rightarrow \infty} f(x)$
	$\lim_{x \rightarrow -2} f(x)$	$\lim_{x \rightarrow -\infty} f(x)$
	$\lim_{x \rightarrow 2} f(x)$	
H.) $f(x) = \frac{x^2-4}{x-2}$	$\lim_{x \rightarrow 2} f(x)$	$\lim_{x \rightarrow \infty} f(x)$
	$\lim_{x \rightarrow 0} f(x)$	$\lim_{x \rightarrow -\infty} f(x)$