MF27 Consolidated Notes

ZENITH
EDUCATION STUDIO

1 Algebra Series

Binomial expansion

$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \ldots + b^n$, where n is a positive integer

and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Maclaurin expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad (|x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \quad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \quad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \quad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \quad (-1 < x \leqslant 1)$$

Partial fractions decomposition

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

2 Trigonometry

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Principal values

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \tfrac{1}{2}\pi \quad (|x| \leqslant 1)$$

$$0 \leqslant \cos^{-1} x \leqslant \pi \quad (|x| \leqslant 1)$$

$$-\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

3 Derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$

4 Integrals

(Arbitrary constants are omitted; a denotes a positive constant.)

$f(x)$	$\int f(x)\,dx$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$	$(x > a)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$
$\tan x$	$\ln(\sec x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$

5 Vectors

The point dividing AB in the ratio $\lambda : \mu$ has position vector $\frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$

Vector product:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

6 Applications of definite integrals

Arc length of a curve defined in cartesian form: $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Surface area of revolution about the x-axis for a curve defined in cartesian form:

$$\int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

7 Functions of two variables

Quadratic approximation of f at (a, b):

$$f(x, y) \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{1}{2}f_{xx}(a, b)(x-a)^2 + f_{xy}(a, b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a, b)(y-b)^2$$

MF27 Consolidated Notes

ZENITH
EDUCATION STUDIO

8 Numerical methods

Trapezium rule (for single strip): $\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$

Simpson's rule (for two strips): $\int_a^b f(x)\,dx \approx \frac{1}{6}(b-a)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$

The Newton-Raphson iteration for approximating a root of $f(x)=0$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

where x_1 is a first approximation.

Euler Method with step size h:

$$y_2 = y_1 + hf(x_1, y_1)$$

Improved Euler Method with step size h:

$$u_2 = y_1 + hf(x_1, y_1)$$
$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, u_2)]$$

9 Standard discrete distributions

Distribution of X	$P(X=x)$	Mean	Variance
Binomial $B(n,p)$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Poisson $Po(\lambda)$	$e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ
Geometric $Geo(p)$	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

10 Standard continuous distribution

Distribution of X	p.d.f.	Mean	Variance
Exponential	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

11 Sampling and testing

Unbiased estimate of population variance:

$$s^2 = \frac{n}{n-1}\left(\frac{\Sigma(x-\bar{x})^2}{n}\right) = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

12 Regression and correlation

Estimated product moment correlation coefficient:

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\{\Sigma(x-\bar{x})^2\}\{\Sigma(y-\bar{y})^2\}}} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{n}}{\sqrt{\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)\left(\Sigma y^2 - \frac{(\Sigma y)^2}{n}\right)}}$$

Estimated regression line of y on x:

$$y - \bar{y} = b(x - \bar{x}), \quad \text{where } b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

13 WILCOXON SIGNED RANK TEST

P is the sum of the ranks corresponding to the positive differences, Q is the sum of the ranks corresponding to the negative differences, T is the smaller of P and Q. For each value of n the table gives the largest value of T which will lead to rejection of the null hypothesis at the level of significance indicated.

Critical values of *T*

	Level of significance			
One	0.05	0.025	0.01	0.005
Two	0.1	0.05	0.02	0.01
$n=6$	2	0		
7	3	2	0	
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37

14 Mathematical Results

AM-GM inequality

For any nonnegative real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \ldots x_n},$$

where the equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Cauchy-Schwarz inequality

For any real numbers $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$,

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leqslant \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right),$$

where the equality holds if and only if there exists a nonzero constant k such that $u_i = kv_i$ for all $i = 1, 2, \ldots, n$.

Triangle inequality

For any real numbers $x_1, x_2, \ldots, x_n$,

$$|x_1 + x_2 + \ldots + x_n| \leqslant |x_1| + |x_2| + \ldots + |x_n|,$$

where the equality holds if $x_1, x_2, \ldots, x_n$ are all nonnegative.

Inclusion-Exclusion Principle

For any subsets $A_1, A_2, \ldots, A_n$ of a set,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$
$$-[|A_1 \cap A_2| + |A_1 \cap A_3| + \ldots + |A_{n-1} \cap A_n|]$$
$$+[|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n|]$$
$$\vdots$$
$$+(-1)^{n-1}|A_1 \cap A_2 \ldots \cap A_{n-1} \cap A_n|$$