NATIONAL SENIOR CERTIFICATE EXAMINATION
NOVEMBER 2023

MATHEMATICS: PAPER II

MARKING GUIDELINES

Time: 3 hours | 150 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

NOTE:

- If a candidate answers a question more than once, only mark the FIRST attempt.
- Consistent Accuracy applies in all aspects of the marking memorandum.

SECTION A

QUESTION 1

(a)(1)	$95 - a = 80$ $a = 15$	$a = 15$
(a)(2)	$b - 40 = 30$ $b = 70$	$b = 70$
(b)	$Q_1 - 1{,}5 \times \text{IQR}$ $= 40 - 1{,}5 \times 30$ $= -5$ Since the minimum value is greater than -5, it is not an outlier.	–5 not an outlier
(c)(1)	$y = a + bx$ $y = 19{,}259 + 0{,}552x$ $y = 19{,}259 + 0{,}552(180)$ $y = 118{,}603$ The learner would obtain 100%.	$19{,}259$ $0{,}552$ Learner would obtain 100%.
(c)(2)	No, 180 min is outside of the data set, therefore extrapolation. Implies that anyone who studies for over 180 min will obtain 100%.	No Explanation

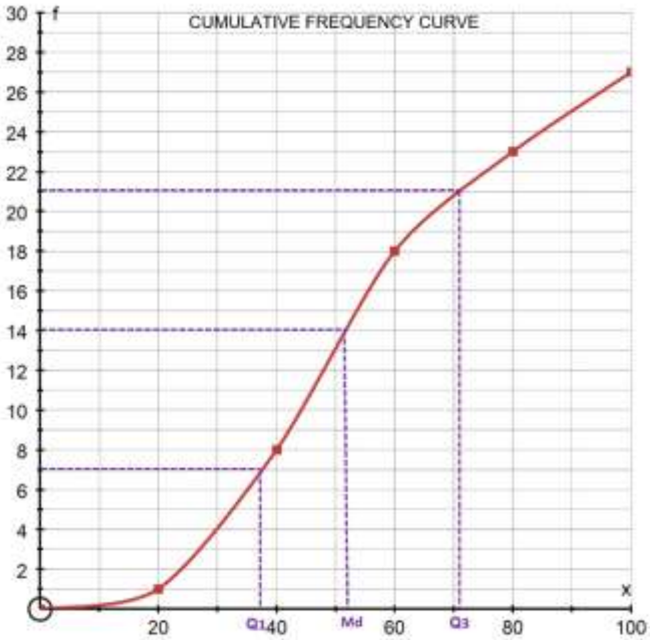
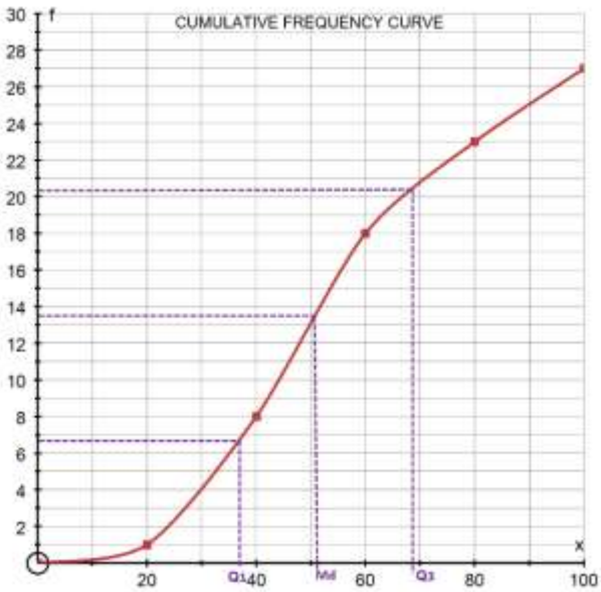
QUESTION 2

(a)(1)	$m_{BC} = \frac{7-4}{6-0}$ $\therefore m_{BC} = \frac{1}{2}$	$m_{BC} = \frac{7-4}{6-0}$ $m_{BC} = \frac{1}{2}$
(a)(2)	$m_{CD} = -2$ Eq. of CD: $y = -2x + c$ ... sub. $(6;7)$ $c = 19$ $\therefore y = -2x + 19$ Alternate: $m_{CD} = -2$ $y - 7 = -2(x - 6)$ $y = -2x + 19$	$m_{CD} = -2$ sub (6;7) $y = -2x + 19$
(b)	Equation AD: $y = \frac{1}{2}x + c$ ... sub $(4;1)$ $c = -1$ $\therefore y = \frac{1}{2}x - 1$ Point of intersection of CD and AD: $-2x + 19 = \frac{1}{2}x - 1$ $-5x = -40$ $x = 8$ $\therefore y = -2(8) + 19$ $y = 3$ $\therefore D(8;3)$	$y = \frac{1}{2}x + c$ $c = -1$ $-2x + 19 = \frac{1}{2}x - 1$ $x = 8$ $D(8;3)$

	Alternate: Equation AD: sub (4;1) $y-1=\frac{1}{2}(x-4)$ $y=\frac{x}{2}-1$ Point of intersection of CD and AD: $-2x+19=\frac{x}{2}-1$ $-5x=-40$ $x=8$ $\therefore y=-2(8)+19$ $y=3$ $\therefore D(8;3)$	
(c)(1)	$E\left(\frac{6+4}{2};\frac{7+1}{2}\right)$ $\therefore E(5;4)$ Midpoint AE $\left(\frac{9}{2};\frac{5}{2}\right)$ $F\left(\frac{x+8}{2};\frac{y+3}{2}\right)$ $\therefore F(1;2)$	$E(5;4)$ Midpoint AE $\left(\frac{9}{2};\frac{5}{2}\right)$ $F(1;2)$
(c)(2)	From Q2b: $m_{AD}=\frac{1}{2}$ $\tan\theta_1=\frac{1}{2}$ $\theta_1=26,6^\circ$ Gradient AC: $m_{AC}=\frac{7-1}{6-4}$ $\therefore m_{AC}=3$ $\tan\theta_2=3$ $\therefore \theta_2=71,6^\circ$ $E\hat{A}D=71,6^\circ-26,6^\circ$ $\therefore E\hat{A}D=45^\circ$	$\theta_1=26,6^\circ$ $m_{AC}=3$ $\theta_2=71,6^\circ$ $E\hat{A}D=45^\circ$

(d)(2)	Area EAD $=\frac{1}{2}\text{EA}\times\text{AD}\times\sin\text{E}\hat{\text{A}}\text{D}$ Dist EA $=\sqrt{10}$ Dist AD $=2\sqrt{5}$ Area EAD $=\frac{1}{2}\times\sqrt{10}\times2\sqrt{5}\times\sin45^\circ$ $\therefore$ Area EAD $=5$ units^2 $\therefore$ Area EFA $=5$ units^2 ... diag of //^{m} bisect area $\therefore$ Area ADEF $=10$ units^2	Dist EA $=\sqrt{10}$ Dist AD $=2\sqrt{5}$ Area EAD $=\frac{1}{2}\times\sqrt{10}\times2\sqrt{5}\times\sin45^\circ$ Area EAD $=5$ units^2 Area ADEF $=10$ units^2

QUESTION 3

(a)	$\bar{x} = \frac{(10\times1)+(30\times7)+(50\times10)+(70\times5)+(90\times4)}{27}$ $\bar{x} = 53$	midpt x frequency divide by 27 $\bar{x} = 53$
(b)	$\frac{5}{27}\times100$ $=18{,}5\%$	$=18{,}5\%$
(c)	 Alt: 	(0;0) Plotting Endpoints Cumulative Frequency

(d)(1)(i)		see graph
(d)(1)(ii)		see graph
(d)(2)	Mean > Median Positively skewed OR Mean = Median Symmetrical	mean>median positively skewed

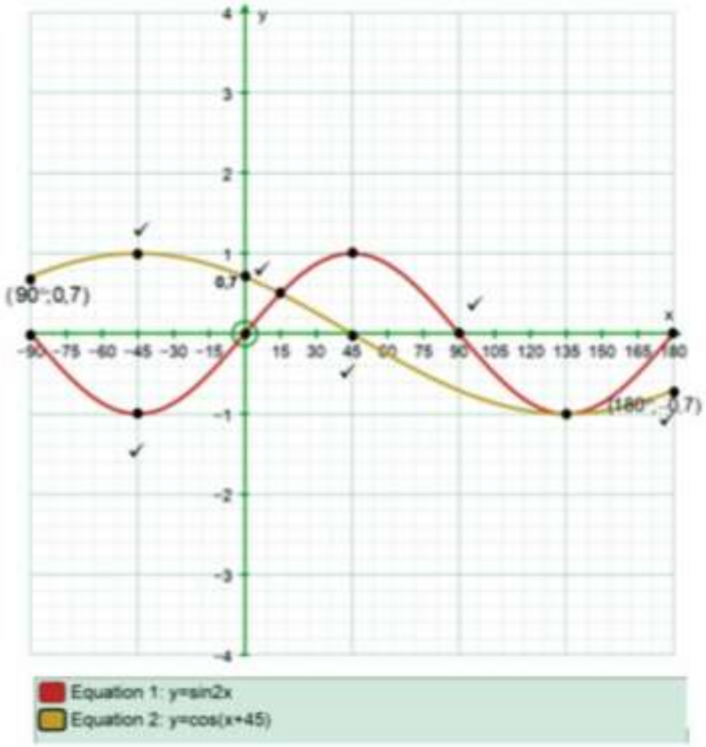
QUESTION 4

(a)	$\hat{C}_1 = 25°$ … angles opp = sides/radii $C\hat{O}A = 180° - 50°$ …int. angles of triangle $= 130°$	$\hat{C}_1 = 25°$ $= 130°$
(b)	$\hat{C}_2 = 25°$ … alt. angles OA//CB $\hat{C}_1 + \hat{C}_2 = \hat{B}$ … angles opp = sides/radii $\therefore \hat{B} = 50°$ $\hat{O}_1 = 180° - 100°$ … int. angles of triangle $\hat{O}_1 = 80°$	$\hat{C}_2 = 25°$ $\hat{C}_1 + \hat{C}_2 = \hat{B}$ $\hat{O}_1 = 80°$

QUESTION 5

(a)	Construct: Join DC and BE $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \frac{\frac{1}{2} \times AD \times EF}{\frac{1}{2} \times DB \times EF}$ $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \frac{AD}{BD}$ ... (1) $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta DEC} = \frac{\frac{1}{2} \times AE \times DG}{\frac{1}{2} \times EC \times DG}$ $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta DEC} = \frac{AE}{CE}$ ... (2) But: $\text{Area } \Delta BDE = \text{Area } \Delta DEC$ … same base, between // lines $\therefore \frac{AD}{BD} = \frac{AE}{CE}$	construction $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \frac{\frac{1}{2} \times AD \times EF}{\frac{1}{2} \times DB \times EF}$ $\frac{\text{Area } \Delta ADE}{\text{Area } \Delta DEC} = \frac{\frac{1}{2} \times AE \times DG}{\frac{1}{2} \times EC \times DG}$ $\text{Area } \Delta BDE = \text{Area } \Delta DEC$ reason $\therefore \frac{AD}{BD} = \frac{AE}{CE}$
(b)	In ΔHIJ: $(HI)^2 = (29)^2 - (21)^2$ … pythag $HI = 20$ units $\frac{IL}{IJ} = \frac{IK}{IH}$ ... line // to one side of Δ $\frac{12}{21} = \frac{IK}{20}$ $\therefore IK = 11\frac{3}{7}$ or IK = 11,4 units	$(HI)^2 = (29)^2 - (21)^2$ $HI = 20$ units $\frac{IL}{IJ} = \frac{IK}{IH}$ reason $IK = 11\frac{3}{7}$

QUESTION 6

(a)		Graph of *f* amplitude frequency turning pts, endpts and intercepts and shape correct Graph of *g* turning pts frequency endpts and intercepts and shape correct
(b)	$\sin 2x = \cos(x+45°)$ $\cos(90° - 2x) = \cos(x+45°)$ $90° - 2x = x + 45° + 360°k \;\; ; \;\; k \in Z$ $x = 15° + 120°k$ $\therefore x = 15°$ OR $90° - 2x = -x - 45° + 360°k$ $x = 135° + 360°k \;\; ; \;\; k \in Z$ $\therefore x = 135°$	$\cos(90° - 2x)$ $\cos(90° - 2x) = \cos(x+45°)$ $90° - 2x = x + 45° + 360°.k$ $\therefore x = 15°$ $\therefore x = 135°$
(c)(1)	$x \in [15°;180°]$ Alt: $15° \le x \le 180°$	$x \in [15°;180°]$
(c)(2)	$x \in (-90°;0°) \cup (45°;90°)$ Alt: $-90° < x < 0$ or $45° < x < 90°$	$x \in (-90°;0°) \cup (45°;90°)$

SECTION B

QUESTION 7

	In $\triangle ABC$: $\hat{B} = 90°$ … tangent perp to radius AO = 7 …radius $\therefore \cos 38° = \frac{14}{AC}$ $AC = \frac{14}{\cos 38°}$ $AC = 17{,}766...$ Construct: DB $A\hat{D}B = 90°$ … angle in semi-circle $\cos 38° = \frac{AD}{14}$ $\therefore AD = 11{,}032...$ $\therefore CD = 17{,}766... - 11{,}032...$ $\therefore CD \approx 6{,}7$ units Alt: $\hat{B} = 90°$ … tangent perp to radius AO = 7 …radius In $\triangle ABC$: $\tan 38° = \frac{CB}{14}$ $\therefore CB = 10{,}9$ units Construct: DB $A\hat{D}B = 90°$ … angle in semi-circle In $\triangle DCB$: $\hat{C} = 52°$ $\therefore \cos 52° = \frac{CD}{CB}$ $\therefore CD = 6{,}7$ units	$\hat{B} = 90°$ and reason $\cos 38° = \frac{14}{AC}$ $AC = 17{,}766...$ $A\hat{D}B = 90°$ reason $\cos 38° = \frac{AD}{14}$ $\therefore AD = 11{,}032...$ $\therefore CD \approx 6{,}7$ units

QUESTION 8

(a)	$a > b > 0$ and $\sin\theta < 0$ $\therefore$ Quadrant 4 $x = a^2 - b^2$ $r = a^2 + b^2$ $\therefore y^2 = \left(a^2+b^2\right)^2 - \left(a^2-b^2\right)^2$ … pythag $y^2 = a^4 + 2a^2b^2 + b^4 - a^4 + 2a^2b^2 - b^4$ $\therefore y^2 = 4a^2b^2$ $y = \pm\sqrt{4a^2b^2}$ $\therefore y = -2ab$ … quad 4 $\therefore \tan\theta = -\dfrac{2ab}{a^2-b^2}$	$\therefore$ Quadrant 4 $\therefore y^2 = \left(a^2+b^2\right)^2 - \left(a^2-b^2\right)^2$ $\therefore y^2 = 4a^2b^2$ $\therefore \tan\theta = -\dfrac{2ab}{a^2-b^2}$
(b)	$\tan\theta = -\dfrac{2ab}{a^2-b^2}$ $\tan\theta = -\dfrac{2\times(3)\times(2)}{(3)^2-(2)^2}$ Ref angle: $-67,4°$ $\theta = -67,4° + 360°(k)$ ... $k \in Z$ $\theta \in \{292,6° ; 652,6°\}$ **Alternate:** $\cos\theta = \dfrac{(3)^2-(2)^2}{(3)^2+(2)^2}$ $\therefore \cos\theta = \dfrac{5}{13}$ Ref angle: $67,4°$ $\theta = 292,6° + 360°(k)$ ... $k \in Z$ $\theta \in \{292,6° ; 652,6°\}$	$\tan\theta = -\dfrac{2\times(3)\times(2)}{(3)^2-(2)^2}$ Ref angle $\theta \in \{292,6°;652,6°\}$

QUESTION 9

(a)	$\dfrac{\frac{1}{2}\cos(90°+\theta)-\sin\theta.\sin(\theta-90°)}{\cos^2(180°-\theta)-2\cos(-\theta)+\cos^2(\theta+90°)}$ $=\dfrac{-\frac{1}{2}\sin(\theta)-(\sin\theta).(-\cos(\theta))}{\cos^2(\theta)-2\cos(\theta)+\sin^2(\theta)}$ $=\dfrac{-\frac{1}{2}\sin\theta+\sin\theta.\cos\theta}{1-2\cos\theta}$ $=\dfrac{-\frac{1}{2}\sin\theta(1-2\cos\theta)}{1-2\cos\theta}$ $=-\frac{1}{2}\sin\theta$	$-\frac{1}{2}\sin(\theta)$ $(-\cos(\theta))$ $\cos^2(\theta)$ $-2\cos(\theta)$ $\sin^2\theta+\cos^2\theta=1$ $-\frac{1}{2}\sin\theta(1-2\cos\theta)$ $=-\frac{1}{2}\sin\theta$
(b)(1)	$\text{LHS}=\sin\theta\times\dfrac{\sin\theta}{\cos\theta}\div\left[\dfrac{\sin2\theta}{\cos2\theta}\times\left(1-\dfrac{\sin^2\theta}{\cos^2\theta}\right)\right]$ $=\dfrac{\sin^2\theta}{\cos\theta}\div\left[\dfrac{\sin2\theta}{\cos2\theta}\times\left(\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}\right)\right]$ $=\dfrac{\sin^2\theta}{\cos\theta}\div\left[\dfrac{\sin2\theta}{\cos2\theta}\times\left(\dfrac{\cos2\theta}{\cos^2\theta}\right)\right]$ $=\dfrac{\sin^2\theta}{\cos\theta}\div\left[\dfrac{2\sin\theta.\cos\theta}{\cos^2\theta}\right]$ $=\dfrac{\sin^2\theta}{\cos\theta}\times\left[\dfrac{\cos^2\theta}{2\sin\theta.\cos\theta}\right]$ $=\dfrac{\sin\theta}{2}$ =RHS	$\dfrac{\sin\theta}{\cos\theta}$ $\left(\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}\right)$ $\cos2\theta$ $2\sin\theta.\cos\theta$ $=\dfrac{\sin^2\theta}{\cos\theta}\times\left[\dfrac{\cos^2\theta}{2\sin\theta.\cos\theta}\right]$ = RHS
(b)(2)	**Working: Not valid for:** $\sin2\theta=0$ $\therefore\theta=90°k\quad ...\ k\in Z$ $\cos2\theta=0$ $\therefore\theta=45°+90°k\quad ...\ k\in Z$ $\cos\theta=0\quad\therefore\theta=\pm90°+360°k\quad ...\ k\in Z$ $\tan^2\theta=1$ $\therefore\theta=\pm45°+180°k\quad ...\ k\in Z$ **Therefore, not valid for** $\theta=45°k\quad ...\ k\in Z$	✓ $\tan2\theta=0$ ✓General solution ✓ $1-\tan^2\theta=0$ ✓General solution ✓ $\theta=45°k\quad ...\ k\in Z$

	Alt: $\tan 2\theta = 0$ $\therefore 2\theta = 180°k$ $\therefore \theta = 90°k$ and $\tan 2\theta$ is undefined for: $\theta = 45° + 90°k$ Thus: $\theta = 45°k$ ... $k \in Z$	

QUESTION 10

(a)	$\hat{A}_1 = 90°$ … tangent perp to radius $\hat{D}_3 = 90°$ … tangent perp to radius $\therefore$ AODE is a cyclic quad… Opp. angles suppl.	$\hat{A}_1 = 90°$ tangent perp to radius $\hat{D}_3 = 90°$ Opp angles suppl.
(b)	$\hat{O}_2 = 2(68°)$ … angle at centre $\hat{O}_2 = 136°$ $\therefore \hat{E} = 44°$ … opp. angles of cyclic quad	$\hat{O}_2 = 136°$ reason $\therefore \hat{E} = 44°$ reason

QUESTION 11

(a)	$x^2 - 12x + y^2 - 4y = -p$ $(x-6)^2 + (y-2)^2 = -p + 36 + 4$ $(x-6)^2 + (y-2)^2 = -p + 40$ $C(6;2) \quad \therefore \text{radius} = 2$ $-p + 40 = 2^2$ $\therefore p = 36$	$(x-6)^2$ $(y-2)^2$ radius = 2 $-p + 40 = 2^2$ $\therefore p = 36$
(b)	Draw CB $m_{CB} = \frac{0-2}{12-6}$ $m_{CB} = -\frac{1}{3}$ Draw CD: radius perp. to tangent OB at D $\tan C\hat{B}D = \frac{1}{3}$ $\therefore C\hat{B}D = 18,4349...$ $\Delta ECB \equiv \Delta DCB$ … RHS $\therefore D\hat{B}A = 2 \times 18,4349...$ $D\hat{B}A = 36,8698...$ $m_{AB} = \tan(180° - 36,8689...)$ $m_{AB} \approx -0,75 = -\frac{3}{4}$ $y = -\frac{3}{4}x + c$ …sub. (12 ; 0) $c = 9$ $\therefore y = -\frac{3}{4}x + 9$	$m_{CB} = -\frac{1}{3}$ $\therefore C\hat{B}D = 18,4349...$ $\Delta ECB \equiv \Delta DCB$ … RHS $D\hat{B}A = 36,8698...$ $m_{AB} = -\frac{3}{4}$ $c = 9$ $\therefore y = -\frac{3}{4}x + 9$

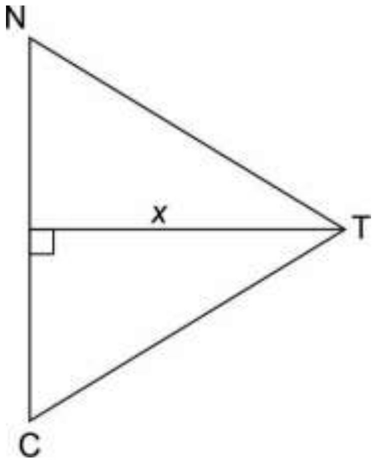
(c)	Second Circle: $x^2+(y-9)^2=r^2$ … sub. (2;3) $r^2=40$ Alt: $r = \sqrt{(2-0)^2+(3-9)^2} = \sqrt{40}$ Distance between centres $=\sqrt{(6-0)^2+(2-9)^2}$ $=\sqrt{85}$ $\approx 9{,}2$ Sum of radii $=\sqrt{40}+2$ $\approx 8{,}3$ They do not intersect since the distance between centres is greater than the sum of the radii.	$r^2=40$ $\approx 9{,}2$ $\approx 8{,}3$ They do not intersect. The distance between centres is greater that the sum of the radii.

QUESTION 12

(a)	In ΔABC and ΔOFC: $\hat{C}_1 = \hat{C}_2$ … given $\hat{B} = 90°$ … angle in semi-circle $\therefore \hat{B} = \hat{F}_1$ $\hat{A}_2 = \hat{O}_2$ … third angle $\therefore \Delta ABC /// \Delta OFC$ … equiangular	$\hat{C}_1 = \hat{C}_2$ $\hat{B} = 90°$ … angle in semi-circle $\therefore \Delta ABC /// \Delta OFC$ … equiangular
(b)	$\frac{BC}{FC} = \frac{AC}{OC}$ … /// triangles; sides in prop Let: $OC = x$ $\therefore AC = 2x$ …radii $\therefore BC : FC = 2:1$	$\frac{BC}{FC} = \frac{AC}{OC}$ $\therefore BC : FC = 2:1$
(c)	$LHS = \left(\frac{AC}{2}\right)^2 - \left(\frac{AB}{2}\right)^2$ $LHS = \frac{(AC)^2 - (AB)^2}{4}$ $= \frac{(BC)^2}{4}$ pythag $\frac{(BC)^2}{4} = (CF)^2$ from 12(b) RTP: $(FC)^2 = DF \times FA$ In ΔCFA and ΔDFE $\hat{E} = \hat{A}$ … angle in same segment $\hat{C}_1 = \hat{D}$ … angle in same segment $\therefore \Delta CFA /// \Delta DFE$ … equiangular $\therefore \frac{CF}{DF} = \frac{FA}{FE}$ $CF = FE$ … line from centre perp to chord $\therefore \frac{CF}{DF} = \frac{FA}{CF}$ $\therefore (FC)^2 = DF \times FA$	$\frac{(BC)^2}{4}$ $\frac{(BC)^2}{4} = (CF)^2$ $\hat{E} = \hat{A}$ and $\hat{C}_1 = \hat{D}$ reason $\therefore \Delta CFA /// \Delta DFE$ $\therefore \frac{CF}{DF} = \frac{FA}{FE}$ $CF = FE$ $\therefore (CF)^2 = DF \times FA$

	Alternate: In ΔOFC: $(OC)^2 = (CF)^2 + (OF)^2$ ... pythag $(CF)^2 = (OC)^2 - (OF)^2$ Since AC=2.OC $(CF)^2 = \left(\frac{AC}{2}\right)^2 - \left(\frac{AB}{2}\right)^2$ RTP: $(CF)^2 = DF \times FA$ In ΔCFA and ΔDFE $\hat{E} = \hat{A}$ … angle in same segment $\hat{C}_1 = \hat{D}$ … angle in same segment $\therefore \Delta CFA /// \Delta DFE$ … equiangular $\therefore \frac{CF}{DF} = \frac{FA}{FE}$ $CF = FE$ … line from centre perp to chord $\therefore \frac{CF}{DF} = \frac{FA}{CF}$ $\therefore (CF)^2 = DF \times FA$ Then: $\therefore \left(\frac{AC}{2}\right)^2 - \left(\frac{AB}{2}\right)^2 = DF \times FA$	

QUESTION 13

	$N\hat{T}C = 46°$ ... int ∠ of Δ For CT: $\frac{10}{\sin 46°} = \frac{CT}{\sin 69°}$ $CT = 12,978295...$ In ΔATC: $\tan 43,5° = \frac{\text{Height of tree}}{12,978...}$ $\therefore \text{Height} = 12,3 \text{ m}$  $\frac{x}{12,978295} = \sin 65°$ $\therefore x = 11,7623...$ ∴ The tree will hit the house.	$N\hat{T}C = 46°$ ... int ∠ of Δ $\frac{10}{\sin 46°} = \frac{CT}{\sin 69°}$ $CT = 12,978295...$ $\tan 43,5° = \frac{\text{Height of tree}}{12,978...}$ ∴ Height = 12,3 m $\frac{x}{12,978295} = \sin 65°$ $\therefore x = 11,7623...$ conclusion

Total: 150 marks