National College of Business Administration & Economics Lahore

RECENT DEVELOPMENTS FOR ESTIMATION OF A FINITE POPULATION MEAN USING MULTI-STAGE SAMPLING

BY

RIFFAT JABEEN

DOCTOR OF PHILOSOPHY
IN
STATISTICS

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A dissertation submitted to School of Social Sciences

In Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY IN STATISTICS

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بسم الله الرحمن الرحيم

IN THE NAME OF ALLAH

THE MOST BENEFICENT

AND THE MERCIFUL

NATIONAL COLLEGE OF BUSINESS ADMINISTRATION & ECONOMICS LAHORE

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Dissertation Committee:

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Administration & Economics

DECLARATION

It is to declare that this research work has not been submitted for obtaining similar degree from any other university / college.

RIFFAT JABEEN
August, 2014

DEDICATED

TO

My Parents

ACKNOWLEDGEMENT

I cannot find words to express my gratitude for the bounty blessings of Almighty Allah whose help enabled me to present this research and a due respect for the Muhammad (Peace be upon him), who is forever a torch of guidance for mankind.

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Riffat Jabeen

RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis entitled **“Recent Develpoments for Estimation of A Finite Population Mean using Multi-Stage Sampling”** has been carried out and completed by **Riffat Jabeen** under my supervision during her **Ph.D. Statistics** Programme.

(Prof. Dr. Muhammad Hanif)
Supervisor

SUMMARY

In this dissertation, some generalized estimators have been proposed to estimate finite population mean using the information of single and two auxiliary variables. The estimators have been proposed for three different sampling designs such as, two-stage single-phase sampling, two-stage two-phase sampling and two-stage adaptive cluster sampling designs.

In chapter 1, the discussion about the survey sampling and the use of auxiliary information in survey sampling has been given. The sampling designs and their uses such as two-stage sampling, two-phase sampling, adaptive cluster sampling and two-stage adaptive cluster sampling have also been discussed. A brief history of the main developments in the estimation of population mean has been discussed in Chapter 2. Some well known existing estimators in two-stage single phase sampling, two phase sampling have been discussed in chapter 3.

The foremost part of the dissertation starts from chapter 4 by making an effort to introduce some generalized estimators for two-stage single-phase sampling using information of single as well as two auxiliary variables. The estimators have been developed for equal and unequal size of the clusters considering weighted and un-weighted mean for first stage units. Some special cases for each suggested generalized estimator have been discussed. The expressions for mean square error (MSE) and bias have also been derived for the generalized estimators, up to first degree of approximation. The optimal conditions, for which a generalized estimator attains minimum MSE have been discussed. Asymptotical optimal estimators along with their MSE have also been presented.

In Chapter 5, some generalized exponential-type estimators have been proposed by utilizing the information of two auxiliary variables for estimating the population mean in two-stage two-phase sampling design. The estimators have been developed for three different situations in two-stage sampling and furthermore each situation is discussed for three different cases regarding the availability of information on two auxiliary variables. The expressions of MSE and bias for the generalized estimators, up to first order of approximation have also been derived. The optimal conditions have been discussed for which generalized estimator are optimal. The asymptotical optimal estimators along with the MSE have also been presented.

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ABBREVIATIONS

FSU	First Stage Unit
SSU	Second Stage Unit
MSE	Mean Square Error
ACS	Adaptive Cluster Sampling
TSACS	Two Stage Adaptive Cluster Sampling
PRE	Percentage Relative Efficiency
FI	Full Information Case
PI	Partial Information Case
NI	No Information Case

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CHAPTER 1

INTRODUCTION

1.1 SURVEY SAMPLING

Sample surveys are conducted to obtain data on a variety of matters in many fields of life. If properly planned and executed, sample surveys are the main source of information about a desired population characteristic. The information about education, household income and expenditure, nutrition and health conditions etc. may also be obtained by planning sampling surveys. A sample survey will be less expensive than a census survey and the desired information will be obtained in less time.

Survey sampling may be divided broadly in two types such as probability sampling and non probability sampling. Probability sampling design implements a sampling plan with specified probabilities. In academic and government surveys, probability sampling is a standard procedure.

1.2 AUXILIARY INFORMATION

The foremost interest of survey Statistician is to acquire an estimate of the desired population characteristics at high precision by using appropriate sampling design. One way of increasing the precision of estimation is to make use of the auxiliary information either at the stage of planning or inference.

The auxiliary information may be used under the following situations regarding the relationships among the study variable & auxiliary information:

- If the study variable and the auxiliary information have a positive relationship, the ratio estimator may be taken into consideration [Cochran (1940 and 1942)].

- If the study variable and the auxiliary information have a negative relationship, the product estimator may be taken into consideration. [Robson (1957) and Murthy (1964)].

- If the relationship between the study variable and auxiliary information is exponential, either positive or negative, the exponential estimator may be used [Bahl and Tuteja (1991)].

Tripathi (1970) and Das (1988) suggested the following four ways in which the auxiliary information may be available:

(i) "*The values of one or more auxiliary variables may be available a priority only for some units of a finite population.*"

(ii) "*Values of one or more parameters of auxiliary variables i.e. population mean(s), population proportion(s), variance, coefficient(s) of variation, coefficient of skewness may be known. In other words, one or more parameters are known.*"

(iii) "*The exact values of the parameters are not known but their estimated values are known.*"

(iv) "*The values of one or more auxiliary variables may be known for all units of a finite population.*"

Tripathi (1970, 1973 and 1976) also discussed the following four ways regarding the use of auxiliary information in the estimation of population parameters:

(i) "*At the planning and designing stage of a survey, the auxiliary information may be used, for example, in stratifying the population; the strata can be made according to auxiliary information.*"

(ii) "*The auxiliary information may be used for the purpose of estimation i.e. ratio, regression, difference and product estimators etc.*"

(iii) "*The auxiliary information may be used while selecting the sample i.e. probabilities proportional to size sampling.*"

(iv) "*The auxiliary information may be used in mixed ways i.e. combining at least two of the above.*"

1.3 TWO-STAGE SAMPLING

In large scale surveys, it is a usual practice to prefer Multi-stage sampling to estimate the population characteristics over single-stage sampling. The main purpose to use multi-stage sampling is the clear reduction in the cost of survey operations even if estimates derived from multi-stage sampling are likely to be less efficient than those of the unrestricted single-stage sampling.

When a sample is selected from a population and information is recorded from the selected units then the sampling scheme is referred to as the single-stage sampling. Stratified random sampling and simple random sampling can be the form of single-stage sampling scheme.

In two-stage sampling, a population is divided into clusters with equal or unequal sizes as first stage units (fsu's) and then each cluster is further divided into sub-units as second stage units (ssu's). At first stage, a sample of clusters is selected by simple random sampling with or without replacement as fsu's and at second stage another sample of sub-units from each cluster (fsu) is selected as ssu's. The information about the study and auxiliary variables is collected from ssu's. In multi-stage sampling, this process is repeated beyond more than two-stages to select ultimate sampling units. [Thompson (1992) and Goldstein (1995)].

1.4 TWO-PHASE SAMPLING

Two-phase sampling can be used when cost of drawing large sample is too high or when information of auxiliary variable is not readily available for population. The work of Neyman (1938) may be referred to as the initial work wherein auxiliary information has been used and the concept of two-phase sampling is introduced in ratio estimator.

Two-phase sampling is powerful cost effective procedure, equally a larger sample is selected from population and information of auxiliary variable is recorded at the first phase while at second phase usually variable of interest is recorded for estimation. Numerous other works in the field of two-phase sampling has been completed.

Bhal and Tuteja (1991) proposed some exponential estimators for single phase sampling. Motivated by Bhal and Tuteja (1991) and following them, the work of exponential estimators for single-phase sampling have been extended for two-phase sampling.

1.5 ADAPTIVE CLUSTER SAMPLING (ACS)

Adaptive sampling refers to designs in which the procedure for selecting units to be included in the sample may depend on values of the variable of interest observed during the survey. One of the mostly used adaptive procedure is Adaptive Cluster Sampling

Adaptive cluster sampling was motivated by Thompson (1990) for sampling rare and clustered population. The design typically starts with a random sample, although it can also be applied to systematic sampling (Thompson 1991a; Acharya et al. 2000), stratified sampling (Thompson 1991b; Brown 1999) and two-stage sampling (Salehi and Seber 1997). For such populations conventional sampling procedures are not applicable as the results obtained from such sampling may not be consistent due to lack of information on sizes or characteristic of the population and often it is not possible to know the shape of the cluster in advance, for such population Thompson (1990) introduced an ACS design.

In ACS, earlier to sampling, a threshold value (C) is chosen, and if any of the units in the initial sample meet or exceed this threshold, $y_i \geq C$, then neighboring units are sampled. If any of these neighboring units meet this condition, their neighboring units are selected and so on. As sampling continues for any cluster that is detected in the initial sample, the shape and size of the cluster can be described. The final sample is the collection of clusters that were detected in the initial sample, including any of the sample units that were in the initial sample but below the threshold.

Survey effort is targeted to searching in the neighborhood of the location of where any plant (or animal) that is found in the initial sample. This feature of the design is very appealing. The design uses the intuitive behavior of a field biologist that once a rare plant is found they want to search in the immediate neighborhood, and puts this behavior in the framework of (unequal) probability sampling. The set consisting on those units that met the condition is known as network. The units that do not satisfy the condition are known as edge units. Cluster is a combination of network units with connected edge units.

1.6 TWO-STAGE ADAPTIVE CLUSTER SAMPLING (TSACS)

In two-stage Adaptive Cluster Sampling method, the population is divided into a number of primary units and each of the primary unit is divided into a number of subunits. At the first stage a number of primary units are drawn using SRS method and from each of the selected primary units a predetermined number of subunits is drawn using a two-stage sampling method and then the ACS is used to select the neighboring subunits of the selected subunits**.** Salehi and Seber (1997) in their paper described the method for two-stage adaptive cluster sampling reproduced as under:

"In first stage of sampling, we choose a simple random sample of m of them primary units without replacement, though this requirement of simple random sampling can be relaxed in some situations, At the second stage, we take an initial simple random sample of ni units without replacement from primary unit $I\,(i = 1, 2, \ldots, m)$ *so that* $n_0 = \sum_{i=1}^{m} n_i$ *is the total initial sample size. According to the above scheme we the add neighbourhoods adaptively and build up clusters."*

Salehi et al. (2013) described the procedure for regression estimation for sequential two-stage sampling.

1.7 OBJECTIVE OF THE STUDY

To reduce the cost of survey, it is a common practice of survey Statisticians to search for the efficient estimators. The efficient estimators are available in single-stage sampling as well as two-phase sampling design. In the present study our task is to propose new generalized simple and exponential estimators for two-stage sampling in consort with the work which has been done by utilizing double sampling in two-stage sampling. It will be useful when information on auxiliary variable is not readily available. The aim of the study is also to propose exponential estimators in two-stage adaptive cluster sampling using single auxiliary variable. The new generalized simple and exponential estimators will be proposed in the following cases:

- When a first-stage sample say 'n' is selected from equal size fsu's say 'N' and then a second-stage sample say 'm' is selected from second-stage units say 'M'.
- When a first-stage sample say 'n' is selected from unequal size fsu's say 'N' and then a second-stage sample say 'mi' is selected from second-stage units say 'Mi' and weighted mean is used.
- When a first-stage sample say 'n' is selected from unequal size fsu's say 'N' and then a second-stage sample say 'mi' is selected from second-stage units say 'Mi' and un-weighted mean is used.

CHAPTER 2

LITERATURE REVIEW

2.1 USE OF AUXILIARY INFORMATION

Graunt (1662) was the first one who made use of auxiliary information in estimating the population of England. Later on, to estimate the population of France, the use of auxiliary information in survey sampling was recommended by Laplace (1980). The modern sampling theory was established by Bowley (1926) and Neyman (1934). Neyman (1938) gave the concept of two-phase sampling and further stretched it to the stratified sampling. Hansen and Hurwitz (1943) made use of auxiliary information in the selection of a sample at the outset in history of survey sampling.

The use of the auxiliary information in a ratio estimator was first time completed by Cochran (1940) after a positive correlation between the study and auxiliary variable was observed. Robson (1957) proposed a product estimator for negative correlation among the study variable and auxiliary information. Under simple random sampling, Olkin (1956) proposed a multivariate ratio estimator.

By making use of auxiliary information, Sukhatme (1962) proposed a generalized ratio estimator for two-phase sampling. The use of auxiliary information was protracted by William (1963), Mohanty (1967), Tripathi (1970), Srivenkataramana (1980), and Breidt and Fuller (1993) etc.

Murthy (1964) derived the conditions, under which, the ratio, product & an unbiased estimator are relatively more efficient. The use of two-phase sampling was extended by Snedecor and King (1942), Spurr (1952), Freese (1962), Armstrong and St-Jean (1993), Unnikrishan and Kunte (1995), Samiudin and Hanif (2007), Singh and Vishwakarma (2007), Noor-ulamin and Hanif (2012), and Sanaullah et al. (2012) etc.

Bahl and Tuteja (1991) introduced exponential-type ratio and product estimators for the estimation of population mean under single-phase sampling design. Upadhyaya et al. (2011) generalized Bahl and Tuteja (1991) exponential-type ratio and product estimators by utilizing information of single auxiliary variable.

Khoshnevisan et al. (2007) suggested a general family of estimators by taking the information of some known characteristics which were already available by utilizing the information of single auxiliary variable. Koyuncu and Kadilar (2009) introduced the family of efficient estimators in simple random sampling getting motivation from Khoshnevisan et al. (2007).

Singh and Espejo (2007) advised a ratio-cum-product estimator under two-phase sampling design. To illustrate the efficiency of the estimators, a numerical study was conducted.

Singh and Vishwakarma (2007) developed exponential ratio and product estimators using single auxiliary variable under two-phase sampling once population mean of the auxiliary variable is not readily available. An empirical study was also completed to demonstrate the performance of the estimators. Singh and Choudhury (2012) derived exponential-type chain ratio and product estimators under two-phase sampling for the estimation of population mean. Theoretical and empirical comparison has been made to demonstrate the performance of the estimators.

Utilizing the information of single auxiliary variable; Koyuncu and Kadilar (2009) has also advised family of estimators under stratified random sampling. However, this work was extended to two auxiliary variables by Koyuncu and Kadilar (2009).

Solanki et al. (2012) gave a class of estimators for the estimation of population mean using single auxiliary variable. Performance of the estimator has been illustrated in theory and numerical study with Bahl and Tuteja (1991) and, Kadilar and Cingi (2003). Khan et al. (2012) planned a class of improved estimators for estimating population mean in two-phase sampling regarding partial information case.

Sanaullah et al. (2012) suggested some improved exponential type estimators for the purpose of the estimation of the population mean. The properties of the estimators have been analyzed for independent and dependent sample cases independently. Khan et al. (2013) developed new class of improved estimators to observe the exactness of the population mean in single and two-phase sampling using full information case.

Singh et al. (2014) introduced an efficient class of estimators for population mean using auxiliary information. They also presented that the proposed class of estimators is more efficient than Khoshnevisan et al. (2007) and usual regression estimator.

Sanaullah (2014) advised exponential estimators in stratified two-phase sampling using auxiliary information and made its mathematical and empirical comparison with the unbiased sample mean estimator, modified stratified two-phase ratio and product estimators.

2.2 TWO-STAGE SAMPLING DESIGN

Seelbinder (1951) determined the method on the selection of first-stage and second-stage sampling units using without replacement considering the probability of inclusion proportional to size. Design of multi-stage surveys for the estimation of sampling errors was discussed by Durbin (1967). Brewer and Hanif (1970) extended the work of Durbin (1967) to a general case. When the units are sampled in more than one-stage; the work of Cochran (1977), Kalton (1983), and Sarndal et al. (1992) may be taken into consideration.

For rare and clustered populations; Seber (1982) used two-stage sampling design in various situations. The similar idea was developed by Jensen (1994) individualistically. Whittemore (1997) provided the use of multi-stage sampling design & inference using maximum likelihood estimator (MLE) and Horvitz-Thompson (1952) estimator. Sukhatme et al. (1984) discussed two-stage sampling design and introduced regression & ratio estimators under two-stage sampling using equal and unequal first stage units.

To estimate the population total using information on two auxiliary variables in two-stage sampling design; Sahoo and Pandey (1999) proposed general family of estimators. Sahoo and Pandey (1999) also proposed a predictive regression-type estimator in two-stage sampling by using the predictive approach of Basu (1971) and made his comparison with usual regression estimator proposed by Sukhatme et al. (1984).

Srivastava and Garg (2009) proposed a general class of estimator for estimating population mean using multi-auxiliary variables under two-stage sampling for unequal & equal first stage units. Sahoo et al. (2011) proposed a general class of estimators in two-stage sampling using information of two auxiliary variables with varying probabilities. Sahoo and Sahoo (2009) introduced a class of predictive estimators by following Basu (1971) prediction approach under two-stage sampling using the information of two auxiliary variables.

Saini and Bahl (2012) advised some difference and ratio estimators using multi-auxiliary information and double sampling for stratifications in two-stage sampling for estimating population mean for heterogeneous population. A comparison among three designs i.e., single-stage, two-stage and three-stage sampling design was made by Nafiu (2012) and concluded that three stage sampling worked more efficiently than the other designs. Mishra (2012) proposed some of ratio estimators and compared their efficiencies in two-stage sampling design without use of auxiliary information.

Singh et al. (2013) contributed ratio, product and regression estimators for two-stage sampling design for unequal first stage units when the population mean is not known in advance. They obtained the bias and mean square error for the suggested estimators. Efficiency conditions were derived and it was concluded that the proposed estimators are more efficient than Sukhatme et al. (1984).

2.3 ADAPTIVE CLUSTER SAMPLING (ACS)

The concept of Adaptive Cluster Sampling (ACS) for population (to be studied) having rare and clustered nature was introduced by Thompson (1990). In ACS, a first sample (primary sample) is selected by simple random sampling with or without replacement. Then, following initial sample, the neighborhood of each initially selected unit is considered to search additional units (secondary units) that meet a defined condition say *C*. Thompson (1990) made an attempt to modify Hansen-Hurwitz (1946) estimator and Horvitz-Thompson (1952) estimator under ACS. Rao-Blackwell method may also be applied to attain the improved unbiased estimators (Rao 1945; Blackwell 1947). Thompson (1991a) extended the work given in Thompson (1990) by selecting initial sample through systematic sampling strip sampling or some other classical sampling design and then added units to the secondary sample adaptively. Under stratified sampling design; Thompson (1991b) adapted Thompson (1990) and Thompson (1991a).

Thompson et al. (1992) presented stratified adaptive cluster sampling under two different sampling strategies i.e., model unbiased and design unbiased strategy.Smith et al. (1995) applied the ACS developed by Thompson (1990) to population of three different species of waterfowl wintering in U.S.A. Under ACS associated with simple random sampling without replacement as initial design; Salehi and Seber (1997a) proposed Thompson (1990) estimator and improvements of estimators has also been discussed using Rao-Blackwell theorem.

Salehi and Seber (1997b) suggested Hansen-Hurwitz (1943) and Horvitz-Thompson (1952) estimators under two-stage adaptive cluster sampling design. The design proposed by the authors is a mixture of two-stage sampling and ACS designs. They developed the estimators under two schemes i.e., clusters are allowed or not to overlap primary unit boundaries. The cost function and efficiency conditions are also discussed.

The restricted adaptive cluster sampling design to reduce the variability in the final sample size was proposed initially by Brown (1994) and then Brown and Manly (1998). Brown (1999) discussed the two-phase adaptive cluster sampling and stratified adaptive cluster sampling strategies. He has made a comparison of two strategies and presented that stratified adaptive cluster sampling is more efficient for that specific study.

Adaptive cluster sampling performs better in univariate settings is proved from the previous literature. Dryver (2002) made a study to see the efficiency of the ACS design under multivariate setting and advised that the efficiency depends on the relationship between the variables.

Muttlak and Khan (2002) presented the adjusted two-stage adaptive cluster sampling once the networks may be of unequal sizes. They used two-stage sampling for the networks of large sizes while single stage sampling was performed for all the other networks. The estimators for population mean and its properties were discussed for the design.

Dryver and Thompson (2003) developed the estimators for adaptive cluster sampling when clusters were selected using without-replacement sampling. Felix-Medina (2003) studied the asymptotic properties of Horvitz-Thompson (1952) and Hansen-Hurwitz (1943) type estimators under ACS also an asymptotic framework which based on the assumptions that number of initial sample size and number of networks in the population are infinitely large while sizes of networks are bounded was proposed. Finally, he proved that under the suggested asymptotic framework both kinds of estimators are asymptotic normal and design-consistent along with the ordinary estimators for the variances are also design consistent.

Chao (2004) made use of auxiliary information under ACS and proposed a ratio estimator based on single auxiliary variable. A simulation study was carried out to show the relative performance of the proposed ratio estimator in ACS with the ratio estimator in simple random sampling.

Naddeo and Pisani (2005) addressed the problem of imperfect delectability in adaptive cluster sampling by using a pure design based approach and proposed a two-stage adaptive procedure where the abundance in the selected units is estimated by replicated counts.Chaudhuri et al. (2005) proposed a sample-size restrictive adaptive sampling design for such situations where some values of the study variable are sufficiently large while some are negligibly or zero.

Dryver and Thompson (2005) proposed unbiased estimator under ACS with and without imposing the condition of minimal sufficient statistic on the estimators. They compared both type of estimators by making a simulation study. Dryver and Chao (2007) presented the ratio estimators under ACS. To see the relative performance of the proposed estimators, a comparison with classical ratio and mean per unit estimator was conducted. The work of Chao et al. (2008) was the extension of Dryver and Chao (2007) where they modified the ratio estimators using Rao-blackwellization. The proposed estimators were simple and more efficient than Dryver and Chao (2007).

Conroy et al. (2008) introduced two-phase adaptive sampling using Bayesian approach. This type of adaptive sampling with Bayesian approach is more suitable when the population is dispersed in plots of land and it become impossible to collect samples from all plots.

Rocco (2008) combined Brown and Manly (1998) restricted sampling design and Salehi and Seber (1997b) by introducing the new concept of two-stage restricted sampling design. They proposed an estimator under this new design and made a simulation study to see the relative efficiency of the proposed estimator.

Chutiman (2010) modified Kadilar and Cingi (2005) ratio estimators under stratified ACS. He carried out a simulation study to evident the efficiency if the ratio estimator over Hansen-Hurwitz estimator under stratified ACS.

Chao et al. (2010) improved some ratio estimators based on the Hansen-Hurwitz and Horvitz-Thompson under ACS associated with simplified systematic expressions in Rao-Blackwell version using simple random sampling as initial design. Chao et al. (2011) used Rao-blackwellization to improve Hansen and Horvitz and Horvitz and Thompson unbiased estimator. The simulation study showed the relative importance of the improved estimators is over original ACS.

Lei et al. (2012) made a study to see the performance of ACS, simple random sampling, and systematic sampling, for estimation of the population of two species in Hainan of China. They concluded that estimators under ACS provide less variance than the estimators from other two sampling designs.Thompson (2013) provided a motivation for ACS and adaptive web sampling. He also discussed a new class of adaptive sampling designs and their applications in ecological and environmental studies.

Salehi et al. (2013) discussed the new concept of adaptive two-stage sequential sampling design and proposed regression-type estimators under ATSSS. The idea of the design was to study very rare individuals and to lower the variance of estimation so that higher efficiency can be achieved as compared to the conventional design. Under the proposed design, some regression-type estimators were developed.

Noor-Ul-Amin (2014) propsed some exponential estiamtors under Adaptive sampling design by taking initial sample from SRSWOR. Sanaullah (2014) also advised exponential estimators under Stratified ACS and presented an empirical studies about the performance of the proposed estimators.

CHAPTER 3

SOME AVAILABLE ESTIMATORS

3.1 INTRODUCTION

A brief discussion on some existing ratio, product and exponential estimators proposed by different survey Statisticians for estimating population mean along with their mean square error is discussed in this chapter. The estimators presented here are of two-stage sampling, simple random sampling, stratified random sampling, and adaptive cluster sampling.

3.2 SOME ESTIMATORS FOR TWO-STAGE SAMPLING DESIGN

In this section, we will discuss the estimators for two-stage sampling design for the following three cases:

Case-I: when first stage units are unequal and weighted mean is used.
Case-II: when first stage units are unequal and un-weighted mean is used.
Case-III: when first stage units are of equal size.

i) Sukhatme et al. (1984)

Case-I:

Sukhatme et al. (1984) estimator in two-stage sampling when first stage units are unequal and weighted mean is used, may be reproduce as

$$t_1' = \frac{1}{n}\sum_{i=1}^{n} \eta_i \bar{y}_{i.} \tag{3.2.1}$$

The mean square error may be given as

$$MSE(\mathrm{t}_1') = \left(\frac{1}{n} - \frac{1}{N}\right) S_{yb}'^{2} + \frac{1}{nN}\sum_{i=1}^{N} \eta_i^{2}\left(\frac{1}{m_i} - \frac{1}{M_i}\right) S_{ywi}'^{2} \tag{3.2.2}$$

Case-II:

Sukhatme et al. (1984) estimator when first stage units are unequal and un-weighted mean is used, may be defined as,

$$t_1'' = \frac{1}{n}\sum_{i=1}^{n} \bar{y}_{i.} \tag{3.2.3}$$

The mean square error may be given as

$$MSE(t_1'') = \left(\frac{1}{n} - \frac{1}{N}\right) S_{yb}''^{2} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_i} - \frac{1}{M_i}\right) S_{wi}''^{2} \tag{3.2.4}$$

Case-III:

Sukhatme et al. (1984) estimator for equal first stage units may be given as

$$t_1 = \frac{1}{n}\sum_{i=1}^{n} \eta_i \bar{y}_{i.} \tag{3.2.5}$$

The mean square error may be given as

$$MSE(t_1) = \left(\frac{1}{n} - \frac{1}{N}\right) S_{yb}^{2} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_i} - \frac{1}{M_i}\right) S_{ywi}^{2} \tag{3.2.6}$$

Case-I:

Sukhatme et al. (1984) estimator in two-stage sampling when first stage units are unequal and weighted mean is used, may be reproduce as

$$t_2' = \frac{\bar{y}_s'}{\bar{x}_s'} \bar{X}_s \tag{3.2.7}$$

The mean square error may be given as

$$MSE(t_2') \approx \left(\frac{1}{n} - \frac{1}{N}\right)\left(S_{by}'^{2} - 2RS_{bxy}' + R^2 S_{bx}'^{2}\right)$$
$$+ \frac{1}{nN}\sum_{i=1}^{N} \eta_i^2 \left(\frac{1}{m_i} - \frac{1}{M_i}\right)\left(S_{wiy}'^{2} - 2RS_{wixy}' + R^2 S_{wix}'^{2}\right) \tag{3.2.8}$$

Case-II:

Sukhatme et al. (1984) estimator when first stage units are unequal and un-weighted mean is used, may be defined as,

$$t_2'' = \frac{\overline{y}_s''}{\overline{x}_s''}\overline{X}_s \tag{3.2.9}$$

The mean square error may be given as

$$MSE(\mathrm{t}_2'') \approx \left(\frac{1}{n} - \frac{1}{N}\right)\left(S_{by}''^{\,2} - 2RS_{bxy}'' + R^2 S_{bx}''^{\,2}\right)$$
$$+ \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i} - \frac{1}{M_i}\right)\left(S_{wiy}''^{\,2} - 2RS_{wixy}'' + R^2 S_{wix}''^{\,2}\right) \tag{3.2.10}$$

Case-III:

Sukhatme et al. (1984) estimator for equal first stage units may be given as

$$t_2 = \frac{\overline{y}_s}{\overline{x}_s}\overline{X}_s \tag{3.2.11}$$

The mean square error may be given as

$$MSE(\mathrm{t}_2) \approx \left(\frac{1}{n} - \frac{1}{N}\right)\left(S_{by}^{\,2} - 2RS_{bxy} + R^2 S_{bx}^{\,2}\right)$$
$$+ \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i} - \frac{1}{M_i}\right)\left(S_{wiy}^{\,2} - 2RS_{wixy} + R^2 S_{wix}^{\,2}\right) \tag{3.2.12}$$

ii) Sahoo and Panda (1999)

Sahoo and Panda(1999) proposed a general class of predictive estimator for two-stage sampling that may be reproduce hereunder as:

$$t_3 = \hat{Y}^{(a)} + \gamma\left(\tilde{Z} - Z\right), \tag{3.2.12}$$

where $\tilde{Y}_i^{(a)} = \tilde{Y}_i + \gamma_i\left(\tilde{X}_i - X_i\right)$

The variance of the predictive class of the estimator is given as under:

$$Var(t_3) = \sigma_y^2 + 2\gamma\sigma_{yz} + \gamma^2\sigma_z^2 + 2\gamma\sum_{i=1}^{N}\left(\sigma_{iyz} + \gamma_i\sigma_{ixz}\right)\Big/\pi_i$$

$$+\gamma^2\sum_{i=1}^{N}\frac{\sigma_{iz}^2}{\pi_i} + \sum_{i=1}^{N}\left(\sigma_{iy}{}^2 + 2\gamma_i\sigma_{ixy} + \gamma_i{}^2\sigma_{ix}{}^2\right)\Big/\pi_i \tag{3.2.13}$$

The variance in (3.2.13) will be minimized for the following:

$$\gamma_i = -\left(\beta_{iyx} + \gamma^*\beta_{izx}\right) = \gamma_i^* \text{ and } \gamma = \frac{\sigma_{yz} + \sum_{i=1}^{N}\sigma_{iz}{}^2\left(\beta_{iyz} - \beta_{iyx}\beta_{izx}\right)\Big/\pi_i}{\sigma_z{}^2 + \sum_{i=1}^{N}\sigma_{iz}{}^2\left(1-\rho_{ixz}\right)\Big/\pi_i} = \gamma^*$$

iii) Srivastava and Garg (2009)

Case-I:

Srivastava and Garg (2009) proposed an estimator for the estimation of population mean in two-stage sampling design. The estimator may be defined as:

$$t_4' = \frac{1}{n}\sum_{i=1}^{n}\eta_i t_i^1, \tag{3.2.13}$$

where $t_i^1 = \frac{\bar{y}_i}{\bar{x}_i}\bar{X}_i$ is a classical ratio estimator for i^th fsu's.

The MSE (t_4') for two-stage sampling design is,

$$MSE(t_4') = \frac{f}{N-1}\sum_{i=1}^{N}\left[\eta_i z_i - E(\eta_i z_i)\right]^2 + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2 v_i, \tag{3.2.14}$$

where $z_i = \bar{Y}_i\left[1 + f_{m_i}\left(C_{y_i}^2 - \rho_i C_{x_i} C_{y_i}\right)\right]$ and $v_i = \bar{Y}_i^2 f_{m_i}\left(C_{y_i}^2 + C_x^2 - 2\rho_i C_{x_i} C_{y_i}\right)$

Srivastava and Garg (2009) proposed a regression estimator for the estimation of population mean under two-stage sampling design as,

$$t_5' = \frac{1}{n}\sum_{i=1}^{n}\eta_i t_i^2, \tag{3.2.15}$$

where $t_i^2 = \bar{y}_i + b_i(\bar{X}_i - \bar{x})$ is a classical regression estimator for i^th fsu's.

The MSE (t_5') for two-stage sampling design is given as under:

$$MSE(t_5') = \frac{f}{N-1}\sum_{i=1}^{N}\left[\eta_i z_i - E(\eta_i z_i)\right]^2 + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2 v_i\ , \tag{3.2.16}$$

where $z_i = \bar{Y}_i$ and $v_i = f_{m_i}\left(S_{y_i}^2 + b_i^2 S_{x_i}^2 - 2b_i\rho_i S_{x_i} S_{y_i}\right)$.

iv) Sahoo et al. (2011)

Sahoo et al. (2011) proposed a general class of estimators in two-stage sampling using two auxiliary variables given as under:

$$t_6 = \frac{1}{M}\left[\sum_{i\in s}\left\{m_i \bar{y}_i + \left(M_i - m_i\right)h_i\left(e_i\right)\right\}\right] + \frac{N-n}{N}h(e)\ , \tag{3.2.17}$$

where

$$h_i\left(e_i\right) = h_i\left(E_i\right) + h_{i0}\left(\bar{y}_i - \bar{Y}_i\right) + h_{i1}\left(\bar{x}_i - \bar{X}_i\right)$$
$$+h_{i2}\left(\bar{z}_i - \bar{Z}_i\right) + h_{i3}\left(\bar{X}_{ir} - \bar{X}_i\right) + h_{i4}\left(\bar{Z}_{ir} - \bar{Z}_i\right)$$

and

$$h\left(e\right) = h\left(E\right) + h_0\left(\bar{y} - \bar{Y}\right) + h_1\left(\bar{x} - \bar{X}\right) + h_3\left(\bar{X}_r - \bar{X}\right)$$

where $h_{i0}, h_{i1}, h_{i2}, h_{i3}$ are partial derivatives with respect to $\bar{y}_i, \bar{x}_i, \bar{z}_i, \bar{X}_{ir}, \bar{Z}_{ir}$.

The variance of the t_6 may be obtained by means of:

$$V(t_6) = \frac{1-f}{n}\left(S_{by}^2 + h_1^2 S_{bx}^2 + 2h_1 S_{byx}\right) + \frac{1}{nN}\sum_{i=1}^{N}u_i^2\frac{1-f_i}{m_i}\left(S_{yi}^2 + h_{1i}^2 S_{xi}^2 + 2h_{1i}S_{iyx}\right). \tag{3.2.18}$$

The variance of t_6 will be minimum for

$$h_{i1} = -\frac{1}{f}\frac{\beta_{iyx} - \beta_{iyz}\beta_{izx}}{1-\beta_{izx}\beta_{ixz}} = h_{i1}^{\ *},$$

$$h_{i2} = -\frac{1}{f}\frac{\beta_{iyz} - \beta_{iyx}\beta_{ixz}}{1-\beta_{izx}\beta_{ixz}} = h_{i2}^{\ *} \text{ and } h_1 = -\beta_{byx}$$

v) Mishra (2012)

Case-I:

Mishra (2012) advised some ratio estimators as followings,

$$t'_7 = \frac{\frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{y}_i}{\frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{X}_i}\bar{X} \tag{3.2.19}$$

$$t'_8 = \frac{\frac{1}{n}\sum_{i=1}^{n}\eta_i \frac{\bar{y}_i}{\bar{x}_i}\bar{X}_i}{\frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{X}_i}\bar{X} \tag{3.2.20}$$

$$t'_9 = \frac{\frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{y}_i}{\frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{x}_i}\frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{X}_i \tag{3.2.21}$$

The mean square error for (3.2.20) –(3.2.22) may be given as,

$$MSE(t'_7) = \left(\frac{1}{n}-\frac{1}{N}\right)\left(S'^2_{by} + R^2 S'^2_{bx} - 2RS'_{bxy}\right) + \frac{1}{nN}\sum_{i=1}^{N}u_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)S'^2_{yi} \tag{3.2.22}$$

$$MSE(t'_8) = \left(\frac{1}{n}-\frac{1}{N}\right)\left(S'^2_{by} + R^2 S'^2_{bx} - 2RS'_{bxy}\right)$$
$$+\frac{1}{nN}\sum_{i=1}^{N}u_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\left(S'^2_{yi} + R_i^2 S'^2_{xi} - 2R_i S'_{ixy}\right) \tag{3.2.23}$$

$$MSE(t'_9) = \left(\frac{1}{n}-\frac{1}{N}\right)\left(S'^2_{by} + R^2 S'^2_{bx} - 2RS'_{bxy}\right)$$
$$+\frac{1}{nN}\sum_{i=1}^{N}u_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\left(S'^2_{yi} + R^2 S'^2_{xi} - 2RS'_{ixy}\right) \tag{3.2.24}$$

3.3 SOME ESTIMATORS FOR SIMPLE RANDOM SAMPLING

In this section, some estimators are given under simple random sampling:

i) Koyuncu and Kadilar (2009)

Koyuncu and kadilar (2009) introduced a general family of estimator as:

$$t_{10} = \lambda \bar{y} \left[\frac{(a\bar{X} + b)}{\beta(a\bar{x} + b) + (1-\beta)\left(a\bar{X} + b\right)} \right]^{g} \tag{3.3.1}$$

The mean square error of t_{10} is,

$$MSE(t_{10}) = \bar{Y}^2 \left(\frac{N-n}{nN} \lambda^2 C_y^2 + \lambda^2 \left(2g^2 + g\right) - \lambda\left(g^2 + g\right) \frac{N-n}{nN} \beta^2 \upsilon^2 C_x^2 \right.$$
$$\left. -2\beta g \upsilon \frac{N-n}{nN} \left(2\lambda^2 - \lambda\right) \rho C_x C_y + (\lambda - 1)^2 \right). \tag{3.3.2}$$

where $\upsilon = \dfrac{a\bar{X}}{a\bar{X} + b}$.

ii) Upadhyaya et al. (2011)

For simple random sampling, Upadhyaya et al. (2011) has proposed exponential estimator as:

$$t_{11} = \bar{y} \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + (a-1)\bar{x}} \right), \tag{3.3.3}$$

where a is an unknown constant whose value may be determined from large scale surveys.

The mean square error may also be given as

$$MSE\left(t_{11}\right) = \left(\frac{1}{n} - \frac{1}{N} \right) \left(C_y^2 + \frac{C_x^2}{a^2}(1 - 2ak) \right). \tag{3.3.4}$$

where $k = \dfrac{\rho C_y}{C_x}$.

3.4 SOME ESTIMATORS FOR TWO-PHASE SAMPLING

i) Singh and Vishwakarma's (2007)

Singh and Vishwakarma's (2007) modified exponential ratio-type estimator is given as:

$$t_{12} = \bar{y}_2 \exp\left[\frac{\bar{x}_1 - \bar{x}_2}{\bar{x}_1 + \bar{x}_2}\right], \tag{3.4.1}$$

The mean square error of t_{12} is

$$\mathrm{MSE}(t_{12}) \approx S_y^2\left[\theta_2 + \frac{C_x}{4C_y}(\theta_2 - \theta_1)\left\{\frac{C_x}{C_y} - 4\rho_{xy}\right\}\right]. \tag{3.4.2}$$

The modified form of exponential product-type estimator for two-phase sampling design is

$$t_{13} = \bar{y}_2 \exp\left[\frac{\bar{x}_2 - \bar{x}_1}{\bar{x}_2 + \bar{x}_1}\right], \tag{3.4.3}$$

The mean square error of t_{13} is

$$\mathrm{MSE}(t_{13}) \approx S_y^2\left[\theta_2 + \frac{C_x}{4C_y}(\theta_2 - \theta_1)\left\{\frac{C_x}{C_y} + 4\rho_{xy}\right\}\right]. \tag{3.4.4}$$

ii) Noor-ul-Amin (2014)

The following generalized exponential estimators have been suggested under two-phase sampling design when the information on both of auxiliary variables is unknown.

$$t_{14} = \bar{y}_2 \exp\left[\alpha\frac{\bar{x}_1 - \bar{x}_2}{\bar{x}_1 + \bar{x}_2}\right]\exp\left[\beta\frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_1 + \bar{z}_2}\right], \tag{3.4.5}$$

The mean square error are given as:

$$MSE(t_{14}) \approx \bar{Y}^2 \left[\theta_2 C_y^2 + \frac{\alpha^2}{4}\left(\theta_1 C_x^2 + \theta_2 C_x^2\right) + \frac{\beta^2}{4}\left(\theta_1 C_z^2 + \theta_2 C_z^2\right) \right.$$

$$\left. - \alpha \theta_2 C_x^2 H_{yx} - \beta \theta_2 C_z^2 H_{yz} + (\theta_2 + \theta_1)\frac{\alpha\beta}{2} H_{zx} C_x^2 \right]. \quad (3.4.6)$$

The optimal values of α and β have been obtained as

$$\alpha = \frac{2P}{\left(1+\theta^*\right)} \text{ and } \beta_{NI} = \frac{2}{\left(1+\theta^*\right)}\left[H_{yz} - PH_{xz}\right]. \quad (3.4.7)$$

where $P = \dfrac{H_{yx} - H_{yz}H_{zx}}{\left(1-\rho_{zx}^2\right)}, \theta_1 = \dfrac{1}{n_1} - \dfrac{1}{N}, \theta_2 = \dfrac{1}{n_1} - \dfrac{1}{n_2}.$

3.5 SOME ESTIMATORS FOR ADAPTIVE CLUSTER SAMPLING

i) Thompson (1990)

Thompson (1990) suggested modified Horvitz-Thompson estimator for the population mean of study variable is

$$t_{15} = \frac{1}{N}\sum_{k=1}^{K} \frac{m_k w_{yk}}{\alpha_k}.I\ , \quad (3.5.1)$$

where $\alpha_k = 1 - \dfrac{\binom{N-m_k}{n}}{\binom{N}{n}}$ and $\alpha_{kh} = 1 - \left[\dfrac{\binom{N-m_k}{n} + \binom{N-m_h}{n} - \binom{N-m_k-m_k}{n}}{\binom{N}{n}}\right]$;

are the initial intersection probability and joint probability, respectively.

The variance of t_{15} may be given as:

$$Var(t_{15}) = \frac{1}{N^2}\sum_{k=1}^{K}\sum_{h=1}^{K} \frac{\left(\alpha_{kh} - \alpha_k \alpha_h\right) u_{yk} u_{yh}}{\alpha_k \alpha_h}. \quad (3.5.2)$$

ii) Thompson (2002)

The unbiased estimator based on draw by draw procedure for population mean μ_y may be given as under:

$$t_{16} = \frac{1}{n} \sum_{i \in s_0} w_{yi} \ , \tag{3.5.3}$$

The variance of t_{16} may be given as:

$$Var(t_{16}) = \left(\frac{1}{n} - \frac{1}{N} \right) \frac{1}{N-1} \sum_{i=1}^{N} \left(w_{yi} - \mu_y \right)^2 . \tag{3.5.4}$$

iii) Dryver and Chao (2007)

Dryver and Chao (2007) modified the classical ratio estimator for the transformed population under ACS.

$$t_{17} = \left[\frac{\sum_{i \in s_0} w_{yi}}{\sum_{i \in s_0} w_{xi}} \right] \mu_x = \hat{R} \mu_x \ , \tag{3.5.5}$$

The mean square error of t_{21} is given as:

$$MSE(t_{17}) = \left(\frac{1}{n} - \frac{1}{N} \right) \frac{1}{N-1} \sum_{i=1}^{N} \left(w_{yi} - \mu_y \right)^2 . \tag{3.5.6}$$

iv) **Salehi and Seber (1997)**

Salehi and Seber (1997) introduced modified Horwitz and Thompson (2002) under two-stage sampling design as

$$t_{18} = \frac{1}{N_T} \sum_{i=1}^{m} \frac{\left(N_i \overline{w}_i \right)}{p_i} \tag{3.5.7}$$

where $\bar{w}_i = \sum_{j=1}^{n_i} \frac{w_{ij}}{n_i}$ and $w_{ij} = \frac{p_i n_i Y_{ij}}{N_i E(f_{ij})}$ when p_i and $\frac{n_i}{N_i}$ are the same for all primary units, then $w_{ij} = \frac{Y_{ij}}{\sum_l a_{ij}}$ is the mean of the network associated with unit (i,j).

However the variance of the estimator will be

$$Var(t_{18}) = \frac{1}{N_T^2} M(M-m)\frac{\sigma_M^2}{m} + \frac{1}{N_T^2}\frac{M}{m}\sum_{i=1}^{M} V_i \tag{3.5.8}$$

where $\sigma_M^2 = \frac{\sum_{i=1}^{M}\left(W_i - \bar{W}\right)^2}{M-1}$ and $\bar{W} = \frac{\sum_{i=1}^{M} W_i}{M}$ and $V_i = \frac{N_i(N_i - n_i)s_i^2}{n_i}$.

v) Noor-ul-Amin (2014)

Noor-ul-Amin (2014) suggested exponential estimators under Adaptive Cluster Sampling (ACS) sampling design as.

$$t_{19} = \bar{u}_y \exp\left[\frac{\bar{X} - \bar{u}_x}{\bar{X} + \bar{u}_x}\right]\exp\left[\frac{\bar{Z} - \bar{u}_z}{\bar{Z} + \bar{u}_z}\right] \tag{3.5.9}$$

where $\bar{u}_x$, $\bar{u}_y$, $\bar{u}_z$ are transformed population means of the variables x,y,z.

The mean square error may be given as:

$$MSE(t_{19}) \approx \theta \bar{Y}^2\left[\left(C_{wy}^2 + \frac{C_{wx}^2}{4}(1 - 4H_{wyx})\right) + \frac{C_{wz}^2}{4}\left(1 - 4H_{wyz} + 2H_{wxz}\right)\right] \tag{3.5.10}$$

where $\theta = \left(\frac{1}{n} - \frac{1}{N}\right)$.

CHAPTER 4

GENERALIZED ESTIMATORS FOR POPULATION MEAN USING SINGLE AND TWO AUXILIARY VARIABLES IN TWO-STAGE SAMPLING

4.1 INTRODUCTION

In this chapter, some generalized ratio and product estimators have been proposed for two-stage sampling using information of single and two auxiliary variables. The following three different cases in two stage sampling have been considered

Case-I: when first stage units are unequal and weighted mean is used.
Case-II: when first stage units are unequal and un-weighted mean is used.
Case-III: when first stage units are of equal size.

In order to develop combined-type generalized ratio-type and product-type estimators, the notations of Sukhatme et al. (1984) will be followed in Section 4.1-4.3 and in order to develop separate-type generalized ratio-type and product-type estimators, the notations of Srivastva and Garg (2009) will be followed in Section 4.4 in this chapter.

Case-I: When first stage units are unequal and weighted mean is used

Let a population consists of N first stage units (fsu's) and each fsu consists of M_i second stage units (ssu's). Let a sample of n fsu's is selected and a sample of m_i ssu's from each of n fsu's is selected by assigning weights $\eta_i = \frac{M_i}{\bar{M}}$ to the ssu's. Let $\bar{M}$ be the average number of ssu's belonging to each fsu. Further we assume that the selection of units at each stage has been done using simple random sampling.

In order to obtain the bias and mean square error under two-stage sampling design, we follow the notations given by Sukhatme et al. (1984) as:

i) Notations

y_{ij} = the value of j-th second-stage unit in the i-th fsu $(j = 1, 2, \dots, M_i; i = 1, 2, \dots, N)$

$\overline{Y}_{i.} = \frac{1}{M_i}\sum_{j=1}^{M_i} y_{ij}$ = the mean per second stage unit in the ith fsu in the population. $\overline{y}_{i.} = \frac{1}{m_i}\sum_{j=1}^{m_i} y_{ij}$ = the mean per second stage unit in the ith fsu in the sample.

$\overline{Y}_s' = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\overline{M}}\sum_{j=1}^{M_i} M_i \overline{y}_{i.}$ = the mean per second stage unit in the population.

$\overline{y}_s' = \frac{1}{n}\sum_{i=1}^{n}\frac{M_i}{\overline{M}}\overline{y}_{i.}$ =the mean per second stage units in the sample.

M_i = the number of second stage units in the i-th first-stage unit $(i = 1, 2, \ldots. N)$

$M_0 = \sum_{i=1}^{N} M_i$, the total number of second-stage units in the population.

$\overline{M} = \frac{1}{N}\sum_{i=1}^{N} M_i$, the average of the number of second stage units in the population.

$\eta_i = \frac{M_i}{\overline{M}}$, weight for ith first stage units.

m_i =the number of second-stage units to be selected from ith first stage unit in the sample.

$m_0 = \sum_{i=1}^{n} m_i$, the number of second-stage units in the sample.

$S'^2_{yb} = \frac{1}{N-1}\sum_{i=1}^{N}(y_{ij} - \overline{Y}_s')^2$. $S'^2_{ywi} = \frac{1}{M_i - 1}\sum_{i=1}^{n}(y_{ij} - \overline{Y}_{i.})^2$.

$$C'_{yb} = \frac{S'_{yb}}{\overline{Y}'_s}, C'_{ywi} = \frac{S'_{ywi}}{\overline{Y}'_s}, C'_{xb} = \frac{S'_{xb}}{\overline{X}'_s}, C'_{zb} = \frac{S'_{zb}}{\overline{Z}'_s}, C'_{xwi} = \frac{S'_{xwi}}{\overline{X}'_s}, C'_{zwi} = \frac{S'_{zwi}}{\overline{Z}'_s}$$

$$\rho'_{xyb} = \frac{S'_{xyb}}{S'_{xb}S'_{yb}}, \rho'_{xzb} = \frac{S'_{xzb}}{S'_{xb}S'_{zb}}. \rho'_{yzb} = \frac{S'_{yzb}}{S'_{yb}S'_{zb}},$$

$$\rho'_{xywi} = \frac{S'_{xywi}}{S'_{xwi}S'_{ywi}}, \rho'_{xzwi} = \frac{S'_{xzwi}}{S'_{xwi}S'_{zwi}}. \rho'_{yzwi} = \frac{S'_{yzwi}}{S'_{ywi}S'_{zwi}}$$

ii) Expectations

Let us define

$$\overline{y}'_s = \overline{Y}'_s(1+e'_0), \quad \overline{x}'_s = \overline{X}'_s(1+e'_1), \quad \overline{z}'_s = \overline{Z}'_s(1+e'_2) \tag{4.1.1}$$

$$\left.\begin{array}{l}
E(e'_0) = E(e'_1) = E(e'_2) = 0 \\
E(e'^2_0) = V'_{020}, E(e'^2_1) = V'_{200}, E(e'^2_2) = V'_{002}, \\
\qquad E(e'_0 e'_1) = V'_{110}, E(e'_1 e'_2) = V'_{101}, E(e'_0 e'_2) = V'_{011} \\
\text{where} \\
V'_{020} = \dfrac{1}{\overline{Y}'^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S'^2_{yb} + \dfrac{1}{nN}\sum\limits_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S'^2_{ywi},\right\}, \\
V'_{200} = \dfrac{1}{\overline{X}'^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S'^2_{xb} + \dfrac{1}{nN}\sum\limits_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S'^2_{xwi}\right\} \\
V'_{002} = \dfrac{1}{\overline{Z}'^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S'^2_{zb} + \dfrac{1}{nN}\sum\limits_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S'^2_{zwi}\right\} \\
V'_{110} = \dfrac{1}{\overline{Y}'_s\overline{X}'_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xyb}S'_{yb}S'_{xb} + \dfrac{1}{nN}\sum\limits_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{xywi}S'_{ywi}S'_{xwi}\right\}, \\
V'_{101} = \dfrac{1}{\overline{Z}'_s\overline{X}'_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xzb}S'_{zb}S'_{xb} + \dfrac{1}{nN}\sum\limits_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{xzwi}S'_{zwi}S'_{xwi}\right\}, \\
V'_{011} = \dfrac{1}{\overline{Z}'_s\overline{Y}'_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{yzb}S'_{zb}S'_{yb} + \dfrac{1}{nN}\sum\limits_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{yzwi}S'_{zwi}S'_{ywi}\right\}
\end{array}\right\} \tag{4.1.2}$$

Case-II: When first stage units are unequal and un weighted mean is used

For case-I, in order to obtain a weighted and unbiased estimator for population mean we use a weighting constant $\eta_i = \frac{M_i}{\overline{M}}$ in two-stage sampling design. If we assume equal weights for all unequal first-stage units, it gives us an un-weighted and biased estimator of population mean in two-stage sampling design and this situation will be considered as case-II.

i.e. for $\eta_i = \dfrac{M_i}{\overline{M}} = 1$, and $\overline{y}_s'' = \sum_{i=1}^{n} \overline{y}_i$ (see Sukhatme et al., 1984) an estimator defined in case-I may be turned into the estimator for case-II. So the procedure of two stage sampling design for case II will be same as described in Case I. The notations and expectations may be derived for case II as:

i) Notations

$\overline{Y}_s''^2 = \dfrac{1}{N}\sum_{i=1}^{N} \overline{y}_{i.}$ = the mean per second stage unit in the population.

$\overline{y}_s''^2 = \dfrac{1}{n}\sum_{i=1}^{n} \overline{y}_{i.}$ = the mean per second stage units in the sample.

$$S''^2_{yb} = \frac{1}{N-1}\sum_{i=1}^{N} (y_{ij} - \overline{Y}_s'')^2 \;.\; S''^2_{ywi} = \frac{1}{M_i - 1}\sum_{i=1}^{n} (y_{ij} - \overline{Y}_{i.})^2 \;.$$

$$C''_{yb} = \frac{S''_{yb}}{\overline{Y}_s''}, C''_{ywi} = \frac{S''_{ywi}}{\overline{Y}_s''}, C''_{xb} = \frac{S''_{xb}}{\overline{X}_s''}, C''_{zb} = \frac{S''_{zb}}{\overline{Z}_s''}, C''_{xwi} = \frac{S''_{xwi}}{\overline{X}_s''}, C''_{zwi} = \frac{S''_{zwi}}{\overline{Z}_s''}$$

$$\rho''_{xyb} = \frac{S''_{xyb}}{S''_{xb}S''_{yb}}, \qquad \rho''_{xywi} = \frac{S''_{xywi}}{S''_{xwi}S''_{ywi}},$$

$$\rho''_{xzb} = \frac{S''_{xzb}}{S''_{xb}S''_{zb}}, \qquad \rho''_{xywi} = \frac{S''_{xzwi}}{S''_{xwi}S''_{zwi}},$$

$$\rho''_{yzb} = \frac{S''_{yzb}}{S''_{zb}S''_{yb}}, \qquad \rho''_{yzwi} = \frac{S''_{zywi}}{S''_{zwi}S''_{ywi}},$$

ii) Expectations

Let us define

$$\overline{y}_s'' = \overline{Y}_s''(1 + e_0''), \quad \overline{x}_s'' = \overline{X}_s''(1 + e_1''), \overline{z}_s'' = \overline{Z}_s''(1 + e_2''), \tag{4.1.3}$$

$$\left.\begin{array}{l}
E(e_0'') = E(e_1'') = E(e_2'') = 0 \\
E(e_0''^2) = V_{020}'', E(e_1''^2) = V_{200}'', E(e_2''^2) = V_{002}'', \\
\qquad E(e_0''e_1'') = V_{110}'', E(e_1''e_2'') = V_{101}'', E(e_0''e_2'') = V_{011}'' \\
\text{Where} \\
V_{020}'' = \dfrac{1}{\bar{Y}_s''^2}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S_{yb}''^2 + \dfrac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S_{ywi}''^2,\right\}, \\
V_{200}'' = \dfrac{1}{\bar{X}_s''^2}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S_{xb}''^2 + \dfrac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S_{xwi}''^2\right\} \\
V_{002}'' = \dfrac{1}{\bar{Z}_s''^2}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S_{zb}''^2 + \dfrac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S_{zwi}''^2\right\} \\
V_{110}'' = \dfrac{1}{\bar{Y}_s''\bar{X}_s''}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xyb}S_{yb}''S_{xb}'' + \dfrac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{xywi}S_{ywi}''S_{xwi}''\right\}, \\
V_{101}'' = \dfrac{1}{\bar{Z}_s''\bar{X}_s''}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xyzb}S_{zb}''S_{xb}'' + \dfrac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{xzwi}S_{zwi}''S_{xwi}''\right\}, \\
V_{011}'' = \dfrac{1}{\bar{Z}_s''\bar{Y}_s''}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{yzb}S_{zb}''S_{yb}'' + \dfrac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{yzwi}S_{zwi}''S_{ywi}''\right\}
\end{array}\right\} \quad (4.1.4)$$

Case-III: When first stage units are of equal sizes.

Let a population consists of N first stage units (fsu's) and each fsu consists of M second stage units (ssu's). Let a sample of n fsu's is selected and a sample of m ssu's from each of n fsu's. Further we assume that the selection of units at each stage has been done using simple random sampling.

To obtain the bias and mean square error under two stage sampling design, we follow the notations given by Sukhatme et al. (1984) as follows:

i) Notations

y_{ij} = the value of jth second stage unit in the i-th fsu $(j = 1, 2, \ldots, M\ ; i = 1, 2, \ldots, N)$

$\bar{Y}_{i.} = \dfrac{1}{M}\sum_{j=1}^{M} y_{ij}$ = the mean per second stage unit in the ith fsu in the population.

$\bar{y}_{im} = \frac{1}{m}\sum_{j=1}^{m} y_{ij}$ = the mean per second stage unit in the ith fsu in the sample.

$\bar{Y}_s = \frac{1}{N}\sum_{i=1}^{N} \bar{y}_{i.}$ = the mean per second stage unit in the population.

$\bar{y}_s = \frac{1}{n}\sum_{i=1}^{n} \bar{y}_{i.}$ = the mean per second stage unit in the sample.

M = the number of second stage units in the i-th first-stage unit $(i = 1, 2, \ldots . N)$

m = the number of second-stage units to be selected from ith first stage unit in the sample.

$$S^2_{yb} = \frac{1}{N-1}\sum_{i=1}^{N}(y_{ij} - \bar{Y}_s)^2 \; . \; S^2_{ywi} = \frac{1}{M_i - 1}\sum_{i=1}^{n}(y_{ij} - \bar{Y}_{i.})^2 \; .$$

$$C_{yb} = \frac{S_{yb}}{\bar{Y}_s}, C_{ywi} = \frac{S_{ywi}}{\bar{Y}_s}, C_{xb} = \frac{S_{xb}}{\bar{X}_s}, C_{zb} = \frac{S_{zb}}{\bar{Z}_s}, C_{xwi} = \frac{S_{xwi}}{\bar{X}_s}, C_{zwi} = \frac{S_{zwi}}{\bar{Z}_s}$$

$$\rho_{xyb} = \frac{S_{xyb}}{S_{xb}S_{yb}}, \rho_{xywi} = \frac{S_{xywi}}{S_{xwi}S_{ywi}}, \rho_{xzb} = \frac{S_{xzb}}{S''_{xb}S''_{zb}},$$

$$\rho_{xywi} = \frac{S_{xzwi}}{S_{xwi}S_{zwi}}, \rho_{yzb} = \frac{S_{yzb}}{S_{zb}S_{yb}}, \rho_{yzwi} = \frac{S_{zywi}}{S_{zwi}S_{ywi}},$$

Let us define

$$\bar{y}_s = \bar{Y}_s(1 + e_0), \quad \bar{x}_s = \bar{X}_s(1 + e_1), \bar{z}_s = \bar{Z}_s(1 + e_2) \tag{4.1.5}$$

ii) Expectations

$$\left.\begin{aligned}
&E(e_0) = E(e_1) = E(e_2) = 0\\
&E(e_0^{\,2}) = V_{020},\, E(e_1^{\,2}) = V_{200},\, E(e_2^{\,2}) = V_{002},\\
&\qquad E(e_0e_1) = V_{110},\, E(e_1e_2) = V_{101},\, E(e_0e_2) = V_{011}\\
&\text{Where}\\
&V_{020} = \frac{1}{\overline{Y}_s^{\,2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S^2_{yb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)S^2_{ywi},\right\},\\
&V_{200} = \frac{1}{\overline{X}_s^{\,2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S^2_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)S^2_{xwi}\right\}\\
&V_{002} = \frac{1}{\overline{Z}_s^{\,2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{zb}^{\,2} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)S^2_{zwi}\right\}\\
&V_{110} = \frac{1}{\overline{Y}_s\overline{X}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xyb}S_{yb}S_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\rho_{xywi}S_{ywi}S_{xwi}\right\},\\
&V_{101} = \frac{1}{\overline{Z}_s\overline{X}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xzb}S_{zb}S_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\rho_{xzwi}S_{zwi}S_{xwi}\right\},\\
&V_{011} = \frac{1}{\overline{Z}_s\overline{Y}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yzb}S_{zb}S_{yb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\rho_{yzwi}S_{zwi}S_{ywi}\right\}
\end{aligned}\right\}\qquad (4.1.6)$$

4.2 PROPOSED COMBINED-TYPE GENERALIZED ESTIMATORS IN TWO STAGE SAMPLING USING SINGLE AUXILIARY VARIABLE

In this section two combined-type generalized estimators have been proposed separately using single auxiliary variable. Taking motivation from Bahl and Tuteja (1991) and Sanaullah (2014) estimator-I is proposed for three different cases in two-stage sampling as mentioned in earlier. We have proposed generalized estimator II by following Khoshnevesan et al. (2007) under three different cases in two-stage sampling .

4.2.1 Proposed Generalized Estimator-I

Case I:

i) The exponential ratio-type estimator may be defined as,

$$t'^{R}_{20} = \overline{y}'_s exp\left\{\frac{\overline{X}'_s - \overline{x}'_s}{\overline{X}'_s + \overline{x}'_s}\right\}, \qquad (4.2.1)$$

or

$$t_{20}'^{R} = \bar{y}_s' \exp\left\{1 - \frac{2\bar{x}_s'}{\bar{X}_s' + \bar{x}_s'}\right\}, \tag{4.2.2}$$

ii) the exponential product -type estimator may be defined as,

$$t_{20}'^{P} = \bar{y}_s' exp\left\{-\left(\frac{\bar{X}_s' - \bar{x}_s'}{\bar{X}_s' + \bar{x}_s'}\right)\right\}, \tag{4.2.3}$$

or

$$t_{20}'^{P} = \bar{y}_s' \exp\left\{-\left(1 - \frac{2\bar{x}_s'}{\bar{X}_s' + \bar{x}_s'}\right)\right\}, \tag{4.2.4}$$

The estimators in (4.2.2) and (4.2.4) lead to the generalized form as by introducing constants λ_1' and a_1' as:

$$t_{20}'^{G} = \lambda_{20}' \bar{y}_s' \exp\left\{\alpha_{20}'\left(1 - \frac{a_{20}'\bar{x}_s'}{\bar{X}_s' + (a_{20}' - 1)\bar{x}_s'}\right)\right\}, \quad 0 < \lambda_{20} \leq 1 \tag{4.2.5}$$

where $a_{20}'(\neq 0)$ and $\lambda_{20}'(\neq 0)$ in (4.2.5) are suitably chosen constants to be determined such as mean square error (MSE) of $t_{20}'^{G}$ is minimum and α_{20}' being constant takes the values (0,-1,1) for designing different ratio-type and product- type estimators. Also it is to be mentioned that for a different choice of a_{20}', λ_{20}' and α_{20}' we get different estimators under two-stage sampling design.

Table 4.1
Some Special Cases of the Generalized Estimator $t_{20}'^{G}$

Ratio-type estimator $\alpha_{20}' = 1$	**Product-type estimators** $\alpha_{20}' = -1$	a_{20}'	λ_{20}'
$t_{20}'^{1} = \bar{y}_s' \exp\left(\dfrac{\bar{X}_s' - \bar{x}_s'}{\bar{X}_s' + \bar{x}_s'}\right)$	$t_{20}'^{2} = \bar{y}_s' \exp\left(\dfrac{\bar{x}_s' - \bar{X}_s}{\bar{x}_s + \bar{X}_s}\right)$	2	1
$t_{20}'^{3} = \bar{y}_s' \exp\left(\dfrac{\bar{X}_s' - \bar{x}_s'}{\bar{X}_s'}\right)$	$t_{20}'^{4} = \bar{y}_s' \exp\left(\dfrac{\bar{x}_s' - \bar{X}_s'}{\bar{X}_s'}\right)$	1	1
$t_{20}'^{5} = \bar{y}_s' \exp\left(\dfrac{\bar{X}_s' - \bar{x}_s'}{\bar{X}_s' + (a_{20}' - 1)\bar{x}_s'}\right)$	$t_{20}'^{6} = \bar{y}_s' \exp\left(\dfrac{\bar{x}_s' - \bar{X}_s'}{\bar{x}_s' + (a_{20}' - 1)\bar{X}_s'}\right)$	a_{20}'	1
$t_{20}'^{7} = \lambda_{20}'\bar{y}_s' \exp\left(\dfrac{\bar{X}_s' - \bar{x}_s'}{\bar{X}_s' + \bar{x}_s'}\right)$	$t_{20}'^{8} = \lambda_{20}'\bar{y}_s' \exp\left(\dfrac{\bar{x}_s' - \bar{X}_s'}{\bar{x}_s' + \bar{X}_s}\right)$	2	λ_{20}'
$t_{20}'^{9} = \lambda_{20}'\bar{y}_s' \exp\left(\dfrac{\bar{X}_s' - \bar{x}_s'}{\bar{X}_s'}\right)$	$t_{20}'^{10} = \lambda_{20}'\bar{y}_s' \exp\left(\dfrac{\bar{x}_s' - \bar{X}_s'}{\bar{X}_s'}\right)$	1	λ_{20}'
$t_{20}'^{11} = \lambda_{20}'\bar{y}_s' \exp\left(\dfrac{\bar{X}_s' - \bar{x}_s'}{\bar{X}_s' + (a_{20}' - 1)\bar{x}_s'}\right)$	$t_{20}'^{12} = \lambda_{20}'\bar{y}_s' \exp\left(\dfrac{\bar{x}_s' - \bar{X}_s'}{\bar{x}_s' + (a_{20}' - 1)\bar{X}_s'}\right)$	a_{20}'	λ_{20}'

The Bias and Mean Square Error of Generalized Estimator I

Using notations from (4.1.1) the estimator given in (4.2.5) may be expressed in form of e_1' as:

$$t_{20}'^{G} = \lambda_{20}'\bar{Y}_s'(1 + e_0')\exp\left[-\frac{\alpha_{20}'}{a_{20}'}e_1'\left(1 + \frac{(a_{20}' - 1)}{a_{20}'}e_1'\right)^{-1}\right], \tag{4.2.6}$$

We assume that $|e_1'| < 1$, we expand the series , $\left(1 + \dfrac{(a_{20}' - 1)}{a_{20}'}e_1'\right)^{-1}$, we get

$$t_{20}'^{G} = \lambda_{20}'\bar{Y}_s'(1 + e_0')\exp\left[-\frac{\alpha_{20}'}{a_{20}'}e_1'\left(1 - \frac{(a_{20}' - 1)}{a_{20}'}e_1' + \frac{(a_{20}' - 1)^2}{a_{20}'^{2}}e_1'^{2} + \ldots\right)\right], \tag{4.2.7}$$

It is assumed that the contribution of terms involving powers in e_0', and e_1' higher than two is negligible. It is therefore expanding the exponentials and ignoring terms in e_0', and e_1' of order higher than one, we have

$$t'^{G}_{20} - \bar{Y}'_s = \lambda'_{20}\bar{Y}'_s\left[e'_0 - \frac{\alpha'_{20}}{a'_{20}}e'_1 + \frac{\alpha'_{20}(a'_{20}-1)}{a'^{2}_{20}}e'^{2}_1 - \frac{\alpha'_{20}}{a'_{20}}e'_1e'_0\right] + \left(\lambda'_{20}-1\right)\bar{Y}'_s \qquad (4.2.8)$$

In order to obtain the Bias (t'^{G}_{20}), we take expectation of (4.2.8) and using (4.1.2), the bias of t'^{G}_{20} will be as

$$Bias(t'^{G}_{20}) = \lambda'_{20}\bar{Y}'_s\left[\frac{\alpha'_{20}(a'_1-1)}{a'^{2}_{20}}V_{200} - \frac{\alpha'_{20}}{a'_{20}}V_{110}\right] + \left(\lambda'_{20}-1\right)\bar{Y}'_s\ , \qquad (4.2.9)$$

In order to derive the MSE of t'^{G}_1, we take square of (4.2.8) and retain terms in e'_0, and e'_1 up to the order one.

$$\left(t'^{G}_{20} - \bar{Y}'_s\right)^2 = \lambda'^{2}_{20}\bar{Y}'^{2}_s\left[e'^{2}_0 + \frac{\alpha'^{2}_{20}}{a'^{2}_{20}}e'^{2}_1 - \frac{2\alpha'_{20}}{a'_{20}}e'_1e'_0\right] + \left(\lambda'_{20}-1\right)^2\bar{Y}'^{2}_s \qquad (4.2.10)$$

On taking expectation and using (4.1.2), we have MSE (t'^{G}_{20}) as,

$$MSE\left(t'^{G}_{20}\right) \approx \bar{Y}'^{2}_s\left(\lambda'^{2}_{20}\left(V'_{02} + \left(\frac{\alpha'_{20}}{a'_{20}}\right)^2 V'_{200} - 2\left(\frac{\alpha'_{20}}{a'_{20}}\right)V'_{110}\right) + \left(\lambda'_{20}-1\right)^2\right) \qquad (4.2.11)$$

In order to find the optimal value of λ'_{20} and a'_{20}, we differentiate (4.2.11) with respect to λ'_{20} and a'_{20}, then equate to zero, we wil get

$$\lambda'_{20} = \frac{1}{1 + V_{020} + \left(\frac{\alpha'_{20}}{a'_{20}}\right)^2 V_{200} - 2\left(\frac{\alpha'_{20}}{a'_{20}}\right)V_{110}}, \qquad (4.2.12)$$

and

$$a'^{opt}_{20} = \frac{\alpha'_{20}V'_{110}}{V'_{200}} \qquad (4.2.13)$$

By substituting (4.2.13) in (4.2.12), we obtain

$$\lambda_{20}'^{opt} = \frac{1}{1+\left(V_{020}' - \frac{V_{110}'^{2}}{V_{200}'}\right)}, \tag{4.2.14}$$

Now by substituting (4.2.11) and (4.2.12) in (4.2.10), we get minimum MSE as,

$$MSE_{\min}\left(t_{20}'^{G}\right) = AsymtoticVar\left(t_{20}'^{G}\right) = \frac{\bar{Y}_s^{2} V_{020}'\left(1-\rho'^{2}\right)}{\bar{Y}_s^{2} + V_{020}'\left(1-\rho'^{2}\right)} = \frac{MSE(\text{lr})}{1+\bar{Y}_s^{-2} MSE(\text{lr})} \tag{4.2.15}$$

where $MSE(\text{lr})$ is the MSE of usual linear regression estimator in two-stage sampling.

From (4.2.15) it is clearly observed that minimum MSE of $t_{20}'^{G}$ is less than the MSE of **usual regression estimator in two-stage sampling**. We may observe from (4.2.15) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value $\lambda_{20}'^{opt}$ and $a_{20}'^{opt}$ in (4.2.5), we get optimal estimator as:

$$t_{20}'^{G} = \lambda_{20}'^{opt}\bar{y}_s' \exp\left\{\alpha_{20}'\left(1-\frac{a_{20}'^{opt}\bar{x}_s'}{\bar{X}_s' + (a_{20}'^{opt}-1)\bar{x}_s'}\right)\right\},\ 0<\lambda_{20}'\le 1 \tag{4.2.16}$$

In real life situations, it is not possible for the researcher to presume the value of beta and lambda by employ all the resources e.g. see Horvitz and Thompson (1952), Murthy (1967), Singh and Vishwakarma (2008), Singh and Kumar (2008), Singh and Karpe (2010), Upadhyaya et al. (2011), Yadav and Kadilar (2013) and Sanaullah et al. (2014), so it is better to replace these by0 their consistent estimates as,

$$\lambda_{20}'^{opt} = \frac{1}{1+\hat{V}_{020} + \left(\frac{\alpha_{20}'}{\hat{a}_{20}'^{opt}}\right)^{2}\hat{V}_{200} - 2\left(\frac{\alpha_{20}'}{\hat{a}_{20}'^{opt}}\right)\hat{V}_{110}} \text{ and } a_{20}'^{opt} = \frac{\alpha_{20}'\hat{V}_{110}}{\hat{V}_{200}} \tag{4.2.17}$$

So (4.2.17) may be wriiten as

$$t'^{G}_{20} = \lambda'^{opt}_{20}\,\overline{y}'_s \exp\left\{\alpha'_{20}\left(1-\frac{a'^{\;opt}_{20}\,\overline{x}'_s}{\overline{X}'_s+(a'^{\;opt}_{20}-1)\overline{x}'_s}\right)\right\},\quad 0<\lambda'^{opt}_{20}\le 1 \qquad (4.2.18)$$

Also the minimum MSE may be written as:

$$MSE_{\min}\left(t'^{G}_{20}\right) \approx MSE_{\min}\left(t'^{G}_{20}\right) = \frac{\overline{Y}_s^{\,2}\hat{V}_{020}\left(1-\hat{\rho}'^{2}\right)}{1+\hat{V}_{020}\left(1-\hat{\rho}'^{2}\right)}. \qquad (4.2.19)$$

Remark 4.1

i) For $\alpha'_{20}=1$, we get exponential-ratio type estimators given in Table 4.1. The MSE of t'^{G}_{20} is expressed as

$$MSE(t'^{j}_{20}) = \begin{cases} \overline{Y}'^{2}_s\left(\lambda'^{2}_{20}\left(V'_{020}+V'_{200}-2V'_{110}\right)+(\lambda'_{20}-1)^2\right) & j(\in G)=1 \\ \overline{Y}'^{2}_s\left(\lambda'^{2}_{20}\left(V'_{020}+\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^2}_{20}}V'_{200}-2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{20}}V'_{110}\right)+(\lambda'_{20}-1)^2\right) & j(\in G)=3,5,\ldots,11 \end{cases}$$

(4.2.20)

The optimal values which lead to minimum MSE as

$$\lambda'^{opt}_{20} = \frac{1}{1+V'_{020}+\left(\dfrac{\alpha'_{20}}{a'^{\left(\frac{j-1}{2}\right)^{opt}}_{20}}\right)^2 V'_{200}-2\left(\dfrac{\alpha'_{20}}{a'^{\left(\frac{j-1}{2}\right)^{opt}}_{20}}\right)V'_{110}},$$

$$\text{and } a'^{\left(\frac{j-1}{2}\right)^{opt}}_{20} = \frac{V'_{110}}{V'_{200}}.$$

ii) For $\alpha'_{20}=-1$, we get exponential-product type estimators given in Table 4.1. The MSE of t^{G}_{20} is expressed as

$$MSE(t_{20}'^{k}) = \begin{cases} \bar{Y}_s'^2 \left(\lambda_{20}'^2 \left(V_{020}' + a_{20}'^{opt2} V_{200}' + 2a_{20}'^{opt} V_{110}'\right) + (\lambda_{20}' - 1)^2\right) & k(\in G) = 2 \\ \bar{Y}_s'^2 \left(\lambda_{20}'^2 \left(V_{020}' + \dfrac{1}{a_{20}'^{\left(\frac{k}{2}\right)^2}} V_{200}' - 2\dfrac{1}{a_{20}'^{\left(\frac{k}{2}\right)}} V_{110}' \right) + (\lambda_{20}' - 1)^2 \right) & k(\in G) = 2,4,\ldots,12 \end{cases}$$

(4.2.21)

$$\lambda_{20}'^{opt} = \frac{1}{1 + V_{020}' + \left(\dfrac{\alpha_{20}'}{a_{20}'^{\left(\frac{k}{2}\right)^{opt}}} \right)^2 V_{200}' - 2\left(\dfrac{\alpha_{20}'}{a_{20}'^{\left(\frac{k}{2}\right)^{opt}}} \right) V_{110}'}, \text{ and } a_{20}'^{\left(\frac{k}{2}\right)^{opt}} = \frac{V_{110}'}{V_{200}'}$$

Case II:

The estimator-I in (4.2.5) may be adopted for case II when $\eta_i = \frac{M_i}{\bar{M}} = 1$ is assumed as

$$t_{20}''^{G} = \lambda_{20}'' \bar{y}_s'' \exp\left\{ \alpha_{20}'' \left(1 - \frac{a_{20}'' \bar{x}_s''}{\bar{X}_s'' + (a_{20}'' - 1)\bar{x}_s''} \right) \right\}, \quad 0 < \lambda_{20}'' \le 1 \qquad (4.2.22)$$

where $a_{20}''(\neq 0)$ and $\lambda_{20}''(\neq 0)$ in (4.2.22) are suitably chosen constants to be determined such as mean square error (MSE) of $t_{20}''^{G}$ is minimum and α_{20}'' being constant takes the values (0,-1,1) for designing different ratio-type and product- type estimators. Also it is to be mentioned that for a different choice of a_{20}'', λ_{20}'' and α_{20}'' we get different estimators under two-stage sampling design.

The proposed estimator in (4.2.22) follows naturally in exactly the same fashion along with the class of estimator in Table 1, as that for case-I in Section 4.2.1. In addition, the relation between a_{20}'', α_{20}'' and λ_{20}'' in case-II is the same as that for case-I in Section 4.2.1.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 4.2.1.1, we may write $\text{Bias}(t_{20}''^{G})$ and $\text{MSE}(t_{20}''^{G})$ following the same, and we may also produce a class of estimators for similar choices of a_{20}'' α_{20}'' and λ_{20}'' in case-II.

Following the notations and expectations for case II presented in Section 4.1, the bias of(4.2.20) may be written directly from (4.2.9),

$$Bias(t''^{G}_{20}) = \lambda''_{20}\bar{Y}''_{s}\left[\frac{\alpha''_{20}(a''_{20}-1)}{a''^{2}_{20}}V''_{200} - \frac{\alpha''_{20}}{a''_{20}}V''_{110}\right] + (\lambda''_{20}-1)\bar{Y}''_{s} \quad . \qquad (4.2.21)$$

Also the expression for MSE (t''^{G}_{20}) may be directly written from (4.2.11) as,

$$MSE\left(t''^{G}_{20}\right) \approx \bar{Y}''^{2}_{s}\left(\lambda_{20}''^{2}\left(V''_{020} + \left(\frac{\alpha''_{20}}{a''_{20}}\right)^{2} V''_{200} - 2\left(\frac{\alpha''_{20}}{a''_{20}}\right)V''_{110}\right) + \left(\lambda''_{20}-1\right)^{2}\right)$$

(4.2.22)

In order to find the minimum MSE of (4.2.22) we will have optimal values of λ''_{20} and a''_{20} as**,**

$$\lambda''_{20} = \frac{1}{1 + V''_{020} + \left(\frac{\alpha_{20}''}{a'^{\,opt}_{20}}\right)^{2} V''_{200} - 2\left(\frac{\alpha''_{20}}{a'^{\,opt}_{20}}\right)V''_{110}}, \text{where } a''_{20} = \frac{\alpha'''_{20}V''_{110}}{V''_{200}}$$

(4.2.23)

The minimum MSE may also be given as

$$MSE_{\min}\left(t''^{G}_{20}\right) = AsymtoticVar\left(t''^{G}_{20}\right) = \frac{\bar{Y}_{s}^{2}V''_{020}\left(1-\rho^{2}\right)}{1 + V''_{020}\left(1-\rho^{2}\right)} = \frac{MSE(\mathrm{lr})}{1 + \bar{Y}_{s}^{-2}MSE(\mathrm{lr})}$$

(4.2.24)

Remark 4.2

i) For $\alpha''_{20} = 1$, we get exponential-ratio type estimators given in Table 4.1. The MSE of t''^{G}_{20} is expressed as

$$MSE(t''^{j}_{20}) = \begin{cases} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{20}\left(V''_{020} + V''_{200} - 2V''_{110}\right) + (\lambda''_{20}-1)^{2}\right) & j(\in G) = 1 \\ \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{20}\left(V''_{02} + \frac{1}{a''^{\left(\frac{j-1}{2}\right)^{2}}_{20}}V'_{200} - 2\frac{1}{a''^{\left(\frac{j-1}{2}\right)}_{20}}V''_{110}\right) + (\lambda''_{20}-1)^{2}\right) & j(\in G) = 3,5,\ldots,11 \end{cases}$$

(4.2.25)

The optimal values which lead to minimum MSE as

$$\lambda_{20}''^{opt} = \frac{1}{1 + V''_{020} + \left(\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)^{opt}}_{20}} \right)^2 V''_{200} - 2\left(\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)^{opt}}_{20}} \right) V''_{110}},$$

and $$a''^{\left(\frac{j-1}{2}\right)^{opt}}_{20} = \frac{V''_{110}}{V''_{200}}$$

ii) For $\alpha''_{20} = -1$, we get exponential-ratio type estimators given in Table 4.1. The MSE of t''^{G}_{20} is expressed as

$$MSE(t''^{k}_{20}) = \begin{cases} \bar{Y}''^{2}_{s} \left(\lambda''^{2}_{20} \left(V''_{020} + a''^{opt2}_{20} V'_{200} + 2a''^{opt}_{20} V''_{110} \right) + (\lambda''_{20} - 1)^2 \right) k(\in G) = 2 \\ \bar{Y}''^{2}_{s} \left(\lambda''^{2}_{20} \left(V''_{020} + \dfrac{1}{a''^{\left(\frac{k}{2}\right)^2}_{20}} V''_{20} + 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{20}} V''_{110} \right) + (\lambda''_{20} - 1)^2 \right) k(\in G) = 2, 4, \ldots, 12 \end{cases}$$

(4.2.26)

The optimal values which lead to minimum MSE as

$$\lambda_{20}''^{opt} = \frac{1}{1 + V''_{020} + \left(\dfrac{1}{a'^{\left(\frac{k}{2}\right)^{opt}}_{20}} \right)^2 V''_{200} + 2\left(\dfrac{1}{a'^{\left(\frac{k}{2}\right)^{opt}}_{20}} \right) V''_{110}}, \quad \text{and} \quad a''^{\left(\frac{k}{2}\right)^{opt}}_{20} = \frac{V''_{110}}{V''_{200}}.$$

Case III:

The estimator I in (4.2.5) may be adopted for Case III may be written as by following (4.2.5),

$$t^{G}_{20} = \lambda_{20} \bar{y}_s \exp\left\{ \alpha_{20} \left(1 - \frac{a_{20} \bar{x}_s}{\bar{X}_s + (a_{20} - 1)\bar{x}_s} \right) \right\}, \quad 0 < \lambda_{20} \le 1 \qquad (4.2.27)$$

The proposed estimator in (4.2.27) follows naturally in exactly the same fashion along with the class of estimator in Table 4.1, as that for case-I in

Section 4.2.1. In addition, the relation between a, α_{20} and λ_{20} in case-II is the same as that for case-I in Section 4.2.1.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 4.2.1.1, we may write $\mathrm{Bias}\,(t_{20}^{G})$ and $\mathrm{MSE}(t_{20}^{G})$ following the same, and we may also produce a class of estimators for similar choices of a, α_{20} and λ_{20} in case-III.

Following the notations and expectations for case II presented in Section 4.1, the bias of(4.2.20) may be obtained,

$$Bias(t_{20}^{G}) = \lambda_{20}\overline{Y}_s\left[\frac{\alpha_{20}(a_{20}-1)}{a_{20}^{2}}V_{200} - \frac{\alpha_{20}}{a_{20}}V_{110}\right] + (\lambda_{20}-1)\overline{Y}_s\ . \qquad (4.2.28)$$

The expression for MSE of t_{20}^{G} may be written as,

$$MSE\left(t_{20}^{G}\right) \approx \overline{Y}_s^{\,2}\left(\lambda_{20}^{2}\left(V_{020} + \left(\frac{\alpha_{20}}{a_{20}}\right)^{2}V_{200} - 2\left(\frac{\alpha_{20}}{a_{20}}\right)V_{110}\right) + (\lambda_{20}-1)^{2}\right)$$

(4.2.29)

The optimal value which leads to the minimum MSE may also be derived exactly in the same manner as given in section 4.2.1.1.as

$$\lambda_{20}^{opt} = \frac{1}{1+V_{020} + \left(\dfrac{\alpha_{20}}{a_{20}^{opt}}\right)^{2}V_{200} - 2\left(\dfrac{\alpha_{20}}{a_{20}^{opt}}\right)V_{110}}, \text{where } a_{20}^{opt} = \frac{\alpha_{20}V_{110}}{V_{200}}\,.$$

(4.2.30)

Remark 4.3

i) For $\alpha_{20} = 1$, we get exponential-ratio type estimators given in Table 4.1. The MSE of t_{20}^{G} is expressed as

$$MSE(t_{20}^{j}) = \begin{cases} \overline{Y}_s^{\,2}\left(\lambda_{20}^{2}\left(V_{020}+V_{200}-2V_{110}\right)+(\lambda_{20}-1)^{2}\right) & j(\in G)=1 \\ \overline{Y}_s^{\,2}\left(\lambda_{20}^{2}\left(V_{020}+\dfrac{1}{a_1^{\left(\frac{j-1}{2}\right)^{2}}}V_{200}-2\dfrac{1}{a_1^{\left(\frac{j-1}{2}\right)}}V_{110}\right)+(\lambda_{20}-1)^{2}\right) & j(\in G)=3,5,\ldots,11 \end{cases}$$

(4.2.31)

The optimal values which lead to minimum MSE as

$$\lambda_{20}^{opt} = \frac{1}{1 + V_{020} + \left(\frac{1}{a_{20}^{\left(\frac{j-1}{2}\right)^{opt}}} \right)^2 V_{200} - 2 \left(\frac{1}{a_{20}^{\left(\frac{j-1}{2}\right)^{opt}}} \right) V_{110}},$$

$$\text{and} \quad a_{20}^{\left(\frac{j-1}{2}\right)^{opt}} = \frac{V_{110}}{V_{200}}$$

ii) For $\alpha_{20} = -1$, we get exponential-ratio type estimators given in Table 4.1. The MSE of t_{20}^{G} is expressed as

$$MSE(t_{20}^{k}) = \begin{cases} \overline{Y}_s^2 \left(\lambda_{20}^2 \left(V_{020} + a_{20}^{opt2} V_{200} + 2 a_{20}^{opt} V_{110} \right) + (\lambda_{20} - 1)^2 \right) k(\in G) = 2 \\ \overline{Y}_s^2 \left(\lambda_{20}^2 \left(V_{020} + \frac{1}{a_1^{\left(\frac{k}{2}\right)^2}} V_{200} - 2 \frac{1}{a_1^{\left(\frac{k}{2}\right)}} V_{110} \right) + (\lambda_{20} - 1)^2 \right) k(\in G) = 2, 4, \ldots, 12 \end{cases}$$

(4.2.32)

$$\lambda_{20}^{opt} = \frac{1}{1 + V_{020} + \left(\frac{1}{a_{20}^{\left(\frac{k}{2}\right)^{opt}}} \right)^2 V_{200} - 2 \left(\frac{1}{a_{20}^{\left(\frac{k}{2}\right)^{opt}}} \right) V_{110}}, \text{and } a_{20}^{\left(\frac{k}{2}\right)^{opt}} = \frac{V_{110}}{V_{200}}.$$

4.2.2 Proposed Generalized Estimator-II

Case I:

Khoshnevesan et al. (2007) estimator may be modified for two stage sampling as

$$t_{21}'^{G} = \lambda_{21}' \bar{y}_s' \left(\frac{c_{21}' \bar{X}_s' + d_{21}'}{a_{21}'(c_{21}' \bar{x}_s' + d_{21}') + (1 - a_{21}')(c_{21}' \bar{X}_s' + d_{21}')} \right)^{\alpha_{21}'} \quad , \quad 0 < \lambda_{21}' \le 1 \tag{4.2.33}$$

where the constant $c_{21}'(\neq 0)$, d_{21}' are either real numbers or the functions of the known parameters of the auxiliary variable, such as the coefficient of variation (C_x) standard deviation (σ_x) , correlation coefficient (ρ), skewness or kurtosis from the population. a_{21}', λ_{21}' is a suitable constants to be determined such that the MSE of $t_{21}'^{G}$ is minimum. a_{21}' being constant takes the values (0,1,-1) for defining different ratio-type and product-type estimators.

Table 4.2

Some Special Cases of the Generalized Estimator t'^{G}_{21}

Ratio Estimator $\alpha'_{21} = 1$	Product Estimator $\alpha'_{21} = -1$	c'_{21}	d'_{21}	λ'_{21}
$t'^{1}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{X}'_s}{\bar{x}'_s}\right)$	$t'^{2}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{x}'_s}{\bar{X}'_s}\right)$			
$t'^{3}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{X}'_s + C_x}{\bar{x}'_s + C_x}\right)$	$t'^{4}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{x}'_s + C_x}{\bar{X}'_s + C_x}\right)$	1	C_{x_i}	1
$t'^{5}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s + C_x}{\beta_2(x)\bar{x}'_s + C_x}\right)$	$t'^{6}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s + C_x}{\beta_2(x)\bar{X}'_s + C_x}\right)$	$\beta_2(\mathrm{x})$	C_{x_i}	1
$t'^{7}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{C_x\bar{X}'_s + \beta_2(x)}{C_x\bar{x}'_s + \beta_2(x)}\right)$	$t'^{8}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{C_x\bar{x}'_s + \beta_2(x)}{C_x\bar{X}'_s + \beta_2(x)}\right)$	C_{x_i}	$\beta_2(\mathrm{x}_i)$	1
$t'^{10}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s + \sigma_x}{\beta_2(x)\bar{x}'_s + \sigma_x}\right)$	$t'^{10}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s + \sigma_x}{\beta_2(x)\bar{X}'_s + \sigma_x}\right)$	$\beta_2(\mathrm{x})$	σ_x	1
$t'^{12}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{X}'_s + \rho}{\bar{x}'_s + \rho}\right)$	$t'^{12}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{x}'_s + \rho}{\bar{X}'_s + \rho}\right)$	1	ρ	1
$t'^{14}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{X}'_s + \beta_2(x)}{\bar{x}'_s + \beta_2(x)}\right)$	$t'^{13}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\bar{x}'_s + \beta_2(x)}{\bar{X}'_s + \beta_2(x)}\right)$	1	$\beta_2(\mathrm{x})$	1
$t'^{15}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + 1}{\sigma_x\bar{x}'_s + 1}\right)$	$t'^{15}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + 1}{\sigma_x\bar{X}'_s + 1}\right)$	σ_{x_i}	1	1
$t'^{18}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + \beta_1(\mathrm{x})}{\sigma_x\bar{x}'_s + \beta_1(\mathrm{x})}\right)$	$t'^{18}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + \beta_1(\mathrm{x})}{\sigma_x\bar{X}'_s + \beta_1(\mathrm{x})}\right)$	σ_{x_i}	$\beta_1(\mathrm{x})$	1
$t'^{19}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + \beta_2(\mathrm{x})}{\sigma_x\bar{x}'_s + \beta_2(\mathrm{x})}\right)$	$t'^{20}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + \beta_2(\mathrm{x})}{\sigma_x\bar{X}'_s + \beta_2(\mathrm{x})}\right)$	σ_x	$\beta_2(\mathrm{x})$	1
$t'^{21}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_1(x)\bar{X}'_s + \beta_2(\mathrm{x})}{\beta_1(x)\bar{x}'_s + \beta_2(\mathrm{x})}\right)$	$t'^{21}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_1(x)\bar{x}'_s + \beta_2(\mathrm{x})}{\beta_1(x)\bar{X}'_s + \beta_2(\mathrm{x})}\right)$	$\beta_1(\mathrm{x})$	$\beta_2(\mathrm{x})$	1
$t'^{23}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s + \beta_1(\mathrm{x})}{\beta_2(x)\bar{x}'_s + \beta_1(\mathrm{x})}\right)$	$t'^{24}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s + \beta_1(\mathrm{x})}{\beta_2(x)\bar{X}'_s + \beta_1(\mathrm{x})}\right)$	$\beta_2(\mathrm{x})$	$\beta_1(x)$	1
$t'^{25}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\rho\bar{X}'_s + 1}{\rho\bar{x}'_s + 1}\right)$	$t'^{26}_{21} = \lambda'_{2}\bar{y}'_s\left(\frac{\rho\bar{x}'_s + 1}{\rho\bar{X}'_s + 1}\right)$	ρ	1	1
$t'^{27}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + C_x}{\sigma_x\bar{x}'_s + C_x}\right)$	$t'^{28}_{21} = \lambda'_{21}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + C_x}{\sigma_x\bar{X}'_s + C_x}\right)$	σ_x	C_x	1

The Bias and Mean Square Error of the Generalized Estimator-II

In order to find bias and mean square error of (4.2.33), we use notations from (4.1.1) and express the estimator given in (4.2.33) in form of $e's$ as:

$$t_{21}'^{G} = \lambda_{21}'\bar{Y}_s'(1+e_0')\left(1+a_{21}'\left(\frac{c_{21}'\bar{X}_s'e_1'}{c_{21}'\bar{X}_s'+d_{21}'}\right)\right)^{-\alpha_{21}} \tag{4.2.34}$$

$$t_{21}'^{G} = \lambda_{21}'\bar{Y}_s'(1+e_0)(1+a_{21}'\upsilon_{21}'e_1')^{-\alpha_2} \quad \text{where} \quad \upsilon_{21}' = \frac{c_{21}'\bar{X}_s'}{c_{21}'\bar{X}_s'+d_{21}'} \tag{4.2.35}$$

It is assumed in (4.2.35) that $\bar{Y}_s' \neq 0$, and $\bar{X}_s' \neq 0$. We assume that $|e_1'| < 1$ so that we may expand the series of $(1+a_{21}'\upsilon_{21}'e_1')^{-\alpha_{21}}$ and we ignore the terms in e's above the order one as:

$$t_{21}'^{G} - \bar{Y}_s' = \lambda_{21}'\bar{Y}_s'\left(e_0' - \alpha_{21}'a_{21}'\upsilon_{21}'e_1' + \frac{\alpha_{21}'(\alpha_{21}'+1)}{2}a_{21}'^{2}\upsilon_{21}'^{2}e_1'^{2} - \alpha_{21}'a_{21}'e_1'e_0'\right) + (\lambda_{21}'-1)\bar{Y}_s', \tag{4.2.36}$$

In order to get the bias, we take expectation on (4.2.36) and get

$$Bias\left(t_{21}'^{G}\right) = \lambda_{21}'\bar{Y}_s'\left(\frac{\alpha_{21}'(\alpha_{21}'+1)}{2}a_{21}'^{2}\upsilon_{21}'^{2}V_{200}' - \alpha_{21}'a_{21}'\upsilon_{21}'V_{110}'\right). \tag{4.2.37}$$

We take Square on both sides on (4.2.36) and retaining terms upto second order in e's and take expectation, we will get MSE($t_{21}'^{G}$) as,

$$MSE\left(t_{21}'^{G}\right) = \bar{Y}_s'^{2}\left(\lambda_{21}'^{2}\left(V_{020}' + \alpha_{21}'^{2}a_{21}'^{2}\upsilon_{21}'^{2}V_{200}' - 2\alpha_{21}'a_{21}'\upsilon_{21}'V_{110}'\right) + (\lambda_{21}'-1)^{2}\right) \tag{4.2.38}$$

The optimal values for which the MSE is minimum will be obtained as,

$$\lambda_{21}' = \frac{1}{1+\left(V_{020}' + \alpha_{21}'^{2}a_{21}'^{2}\upsilon_{21}'^{2}V_{200}' - 2\alpha_{21}'a_{21}'\upsilon_{21}'V_{110}'\right)}, \quad a_2'^{opt} = \frac{V_{11}'}{\alpha_2'\upsilon_2'V_{20}'}. \tag{4.2.39}$$

Now by substituting a'^{opt}_{21} in λ'_{21} we get λ'^{opt}_{21} as,

$$\lambda'^{opt}_{21} = \frac{1}{1+\left(V'_{020} - \frac{V'^{2}_{110}}{V'_{200}}\right)}, \tag{4.2.40}$$

By substituting (4.2.39) and (4.2.40) in (4.2.38), we get asymptotic variance as:

$$MSE_{\min}\left(t'^{G}_{21}\right) = AsymtoticVar\left(t'^{G}_{21}\right) = \frac{\bar{Y}_s^{2} V'_{020}\left(1-\rho'^{2}\right)}{\bar{Y}_s^{2} + V'_{020}\left(1-\rho'^{2}\right)} = \frac{MSE(\mathrm{lr})}{1+\bar{Y}_s^{-2} MSE(\mathrm{lr})}. \tag{4.2.41}$$

where $MSE(\mathrm{lr})$ is the MSE of usual linear regression estimator in two-stage sampling.

From (4.2.41) it is clear that $MSE_{\min}\left(t'^{G}_{21}\right)$ is less than usual linear regression estimator. We may observe from (4.2.34) that proposed generalized estimator is more efficient than its class of estimators under the optimal conditions.

On substituting the optimal value λ'^{opt}_{21} and $a_2'^{opt}$ in (4.2.33), we get optimal estimator as:

$$t'^{G}_{21} = \lambda'^{opt}_{21}\bar{y}'_s\left(\frac{c'_{21}\bar{X}'_s + d'_{21}}{a'^{opt}_{21}\left(c'_{21}\bar{x}'_s + d'_{21}\right) + \left(1-a'^{opt}_{21}\right)\left(c'_{21}\bar{X}'_s + d'_{21}\right)}\right)^{\alpha_{21}} \tag{4.2.42}$$

$$a'^{opt}_{21} = \frac{\hat{V}'_{11}}{\hat{\alpha}'_{21}\hat{\upsilon}'_{21}\hat{V}'_{200}} \; and \; \hat{\lambda}'^{opt}_{21} = \frac{1}{1+\left(\hat{V}'_{020} - \frac{\hat{V}'^{2}_{110}}{\hat{V}'_{200}}\right)}, \tag{4.2.43}$$

This leads to asymptotical optimal estimator as,

$$\hat{t}'^{G}_{21} = \hat{\lambda}'^{opt}_{21}\bar{y}'_s\left(\frac{c'_{21}\bar{X}'_s + d'_{21}}{a'^{opt}_{21}\left(c'_{21}\bar{x}'_s + d'_{21}\right) + \left(1-a'^{opt}_{21}\right)\left(c'_{21}\bar{X}'_s + d'_{21}\right)}\right)^{\alpha'_{21}}, \tag{4.2.44}$$

Also the minimum mean square error may be written as,

$$MSE_{\min}\left(t_{21}'^{G}\right)=\frac{\bar{Y}_s^{\,2}V_{020}'\left(1-\rho'^2\right)}{\bar{Y}_s^{\,2}+V_{020}'\left(1-\rho'^2\right)}. \tag{4.2.45}$$

Remark 4.4

i) For $\alpha'_{21}=1$, some ratio-type estimators are expressed in Table 4.2 The MSE($t_{21}'^{G}$) of these ratio-type estimators may be expressed as,

$$MSE(t_{21}'^{j})=\begin{cases}\bar{Y}_s'^2\left(\lambda_{21}'^2\left(V_{020}'+a_{21}'^{opt2}V_{200}'-2a_{21}'^{opt}V_{110}'\right)+(\lambda_{21}'-1)^2\right) & j(\in G)=1\\ \bar{Y}_s'^2\left(\lambda_{21}'^2\left(V_{020}'+a_{21}'^{opt2}\upsilon_2^{\left(\frac{j-1}{2}\right)^2}V_{200}'-2a_{21}'^{opt}\upsilon_{21}'^{\left(\frac{j-1}{2}\right)}V_{110}'\right)+(\lambda_{21}'-1)^2\right) & j(\in G)=3,5,\ldots,27\end{cases}, \tag{4.2.46}$$

The $MSE(t_{21}'^{j})$ in (4.2.46) is minimum for

$$a_{21}'^{opt}=\frac{V_{110}'}{\upsilon_{21}'^{\left(\frac{j-1}{2}\right)}V_{200}'}\ and\ \lambda_{21}'^{opt}=\frac{1}{1+\left(V_{020}'-\dfrac{V_{110}'^{2}}{V_{200}'}\right)},$$

ii) For $\alpha'_{21}=-1$, some product-type estimators are expressed in Table 4.2, The MSE(t_{21}^{G}) for these product-type estimators may be expressed as,

$$MSE(t_{21}'^{k})=\begin{cases}\bar{Y}_s'^2\left(\lambda_{21}'^2\left(V_{020}'+a_{21}'^{opt2}V_{200}'+2a_{21}'^{opt}V_{110}'\right)+(\lambda_{21}'-1)^2\right) & k(\in G)=1\\ \bar{Y}_s'^2\left(\lambda_{21}'^2\left(V_{020}'+a_{21}'^{opt2}\upsilon_{21}'^{\left(\frac{k}{2}\right)^2}V_{200}'+2a_{21}'^{opt}\upsilon_{21}'^{\left(\frac{k}{2}\right)}V_{110}'\right)+(\lambda_{21}'-1)^2\right) & k(\in G)=2,4,\ldots,28\end{cases}, \tag{4.2.47}$$

The $MSE(t_{21}'^{k})$ in (4.3.47) is minimum for

$$a_{21}'^{opt}=\frac{-V_{11}'}{\upsilon_{21}'^{\left(\frac{j-1}{2}\right)}V_{20}'}\ and\ \lambda'^{opt}=\frac{1}{1+\left(V_{02}'-\dfrac{V_{11}'^{2}}{V_{20}'}\right)}.$$

where

$$\upsilon'^{\,1}_{21} = \frac{\bar{X}'_s}{\bar{X}'_s + C_x},\ \upsilon'^{\,2}_{21} = \frac{\beta_2(x)\bar{X}'_s}{\beta_2(x)\bar{X}'_s + C_x},\ \upsilon'^{\,3}_{21} = \frac{C_x\bar{X}'_s}{C_x\bar{X}'_s + \beta_2(x)},$$

$$\upsilon'^{\,4}_{21} = \frac{\beta_2(x)\bar{X}'_s}{\beta_2(x)\bar{X}'_s + \sigma_x},\ \upsilon'^{\,5}_{21} = \frac{\bar{X}'_s}{\bar{X}'_s + \rho_x},\ \upsilon'^{\,6}_{21} = \frac{\bar{X}'_s}{\bar{X}'_s + \beta_2(x)},$$

$$\upsilon'^{\,7}_{21} = \frac{\sigma_x\bar{X}'_s}{\sigma_x\bar{X}'_s + 1},\ \upsilon'^{\,8}_{21} = \frac{\bar{X}'_s}{\sigma_x\bar{X}'_s + \beta_1(x)},\ \upsilon'^{\,9}_{21} = \frac{\sigma_x\bar{X}'_s}{\sigma_x\bar{X}'_s + \beta_2(x)},$$

$$\upsilon'^{\,10}_{21} = \frac{\beta_1(x)\bar{X}'_s}{\beta_1(x)\bar{X}'_s + \beta_2(x)},\ \upsilon'^{\,11}_{21} = \frac{\beta_2(x)\bar{X}'_s}{\beta_2(x)\bar{X}'_s + \beta_1(x)},\ \upsilon'^{\,12}_{21} = \frac{\rho\bar{X}'_s}{\rho\bar{X}'_s + 1},$$

$$\upsilon'^{\,13}_{21} = \frac{\sigma_x\bar{X}'_s}{\sigma_x\bar{X}'_s + C_x}.$$

Case II:

The estimator II proposed in (4.2.33) may be proposed also for case II as below

$$t''^{G}_{21} = \lambda''_{21}\bar{y}''_s\left(\frac{c''_{21}\bar{X}''_s + d''_{21}}{a''_{21}\left(c''_{21}\bar{x}_s + d''_{21}\right) + (1-a''_{21})\left(c''_{21}\bar{X}''_s + d''_{21}\right)}\right)^{\alpha_{21}},\ 0 < \lambda''_{21} \le 1 \tag{4.2.48}$$

where the constant $c''_{21}(\neq 0)$, d''_{21} are either real numbers or the functions of the known parameters of the auxiliary variable, such as the coefficient of variation (C_x) standard deviation (σ_x), correlation coefficient (ρ), skewness or kurtosis from the population. a''_{21}, λ''_{21} is a suitable constants to be determined such that the MSE of t''^{G}_{21} is minimum. α''_{21} being constant takes the values (0,1,-1) for defining different ratio-type and product-type estimators.

The proposed estimator in (4.2.48) follows indeed the same mode along with the class of estimator in Table 2, as that for case-I in Section 4.2.3. In addition, the relation between a''_{21}, α''_{21} and λ''_{21} in case-II is the same as that for case-I in Section 4.2.3.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 4.2.3.1, we may write $\mathrm{Bias}\,(t''^{G}_{21})$ and MSE (t^{G}_{21}) following the same, and we may also produce a class of estimators for similar choices of a''_{21}, α''_{21} and λ''_{21} in case-II. the bias of (4.2.41) may be obtain by following the notations and expectations for case II presented in Section 4.1,

$$Bias\left(t''^{G}_{21}\right) = \lambda''_{21}\bar{Y}''_{s}\left(\frac{\alpha''_{21}(\alpha''_{21}+1)}{2}a''^{2}_{21}\upsilon''_{21}V''_{200} - \alpha''_{21}a''_{21}\upsilon''_{21}V''_{110}\right). \quad (4.2.49)$$

The expression for MSE of t''^{G}_{21} may be obtained as,

$$MSE\left(t''^{G}_{21}\right) = \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{21}\left(V''_{020} + \alpha''^{2}_{21}a''^{2}_{21}\upsilon''^{2}_{21}V''_{200} - 2\alpha''_{21}a''_{21}\upsilon''_{21}V''_{110}\right) + (\lambda''_{21}-1)^{2}\right), \quad (4.2.50)$$

The optimal values to get the minimum MSE will be

$$a''^{opt}_{21} = \frac{V''_{110}}{\alpha''_{21}\upsilon''_{21}V''_{200}} \; and \; \lambda''^{opt}_{21} = \frac{1}{1+\left(V''_{020} - \frac{V''^{2}_{110}}{V''_{200}}\right)}, \quad (4.2.51)$$

The min MSE will be

$$MSE_{\min}\left(t''^{G}_{21}\right) = AymptoticVar\left(t''^{G}_{21}\right) = \frac{\bar{Y}_{s}^{2}V''_{020}\left(1-\rho''^{2}\right)}{1+V''_{020}\left(1-\rho''^{2}\right)} = \frac{MSE(\mathrm{lr})}{1+\frac{MSE(\mathrm{lr})}{\bar{Y}_{s}^{2}}}. \quad (4.2.52)$$

Remark 4.5

i) For $\alpha''_{21} = 1$, some ratio-type estimators are expressed in Table 4.2. The MSE(t''^{G}_{21}) of these ratio-type estimators may be expressed as,

$$MSE(t''^{j}_{21}) = \begin{cases} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{21}\left(V''_{020} + a''^{opt2}_{21}V''_{200} - 2a''^{opt}_{21}V''_{110}\right) + (\lambda''_{21}-1)^{2}\right) & j(\in G)=1 \\ \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{21}\left(V''_{020} + a''^{opt2}_{21}\upsilon_{2}^{\left(\frac{j-1}{2}\right)^{2}}V''_{200} - 2a''^{opt}_{21}\upsilon_{21}^{\left(\frac{j-1}{2}\right)}V''_{110}\right) + (\lambda''_{21}-1)^{2}\right) & j(\in G)=3,5,\ldots,27 \end{cases}, \quad (4.2.53)$$

The $MSE(t''^{j}_{21})$ in (4.2.53) is minimum for

$$a''^{\;opt}_{21} = \frac{V''_{110}}{\upsilon''^{\left(\frac{j-1}{2}\right)}_{21} V''_{200}} \; and \; \lambda''^{\;opt}_{23} = \frac{1}{1 + \left(V''_{020} - \frac{V''^{\;2}_{110}}{V''_{200}} \right)},$$

ii) For $\alpha''_2 = -1$, some product-type estimators are expressed in Table 4.2. The MSE(t''^{G}_{21}) for these product-type estimators may be expressed as,

$$MSE(t''^{k}_{21}) = \begin{cases} \bar{Y}''^{2}_{s} \left(\lambda''^{\;2}_{21} \left(V''_{020} + a''^{\;opt2}_{21} V''_{200} + 2a''^{\;opt}_{21} V''_{110} \right) + (\lambda''_{21} - 1)^2 \right) k(\in G) = 1 \\ \bar{Y}''^{2}_{s} \left(\lambda''^{\;2}_{21} \left(V''_{020} + a''^{\;opt2}_{21} \upsilon''^{\left(\frac{k}{2}\right)^2}_{21} V''_{200} + 2a'^{\;\prime opt}_{21} \upsilon''^{\left(\frac{k}{2}\right)}_{21} V''_{110} \right) + (\lambda''_{21} - 1)^2 \right) k(\in G) = 2,4,\ldots,28 \end{cases},$$

(4.2.54)

The $MSE(t''^{k}_{21})$ in (4.2.54) is minimum for

$$a''^{\;opt}_{21} = \frac{-V''_{110}}{\upsilon''^{\left(\frac{j-1}{2}\right)}_{21} V''_{200}} \; and \; \lambda''^{\;opt}_{21} = \frac{1}{1 + \left(V''_{020} - \frac{V''^{\;2}_{110}}{V''_{200}} \right)}.$$

where

$$\upsilon''^{\;1}_{21} = \frac{\bar{X}''_s}{\bar{X}''_s + C_x},\; \upsilon''^{\;2}_{21} = \frac{\beta_2(x)\bar{X}''_s}{\beta_2(x)\bar{X}''_s + C_x},\; \upsilon''^{\;3}_{21} = \frac{C_x \bar{X}''_s}{C_x \bar{X}''_s + \beta_2(x)},$$

$$\upsilon''^{\;4}_{21} = \frac{\beta_2(x)\bar{X}''_s}{\beta_2(x)\bar{X}''_s + \sigma_x},\; \upsilon''^{\;5}_{21} = \frac{\bar{X}''_s}{\bar{X}''_s + \rho_x},\; \upsilon''^{\;6}_{21} = \frac{\bar{X}''_s}{\bar{X}''_s + \beta_2(x)},$$

$$\upsilon''^{\;7}_{21} = \frac{\sigma_x \bar{X}''_s}{\sigma_x \bar{X}''_s + 1},\; \upsilon''^{\;8}_{21} = \frac{\bar{X}''_s}{\sigma_x \bar{X}''_s + \beta_1(x)},\; \upsilon''^{\;9}_{21} = \frac{\sigma_x \bar{X}''_s}{\sigma_x \bar{X}''_s + \beta_2(x)},$$

$$\upsilon''^{\;10}_{21} = \frac{\beta_1(x)\bar{X}''_s}{\beta_1(x)\bar{X}''_s + \beta_2(x)},\; \upsilon''^{\;11}_{21} = \frac{\beta_2(x)\bar{X}''_s}{\beta_2(x)\bar{X}''_s + \beta_1(x)},\; \upsilon''^{\;12}_{21} = \frac{\rho \bar{X}''_s}{\rho \bar{X}''_s + 1},$$

$$\upsilon''^{\;13}_{21} = \frac{\sigma_x \bar{X}''_s}{\sigma_x \bar{X}''_s + C_x}.$$

Case III:

We may propose estimator (4.2.33) by considering case III as follows

$$t^{G}_{21} = \lambda_{21}\bar{y}_s \left(\frac{c_{21}\bar{X}_s + d_{21}}{a_{21}(c_{21}\bar{x}_s + d_{21}) + (1 - a_{21})(c_{21}\bar{X}_s + d_{21})} \right)^{\alpha_{21}},\; 0 < \lambda_{21} \le 1,$$

(4.2.55)

where the constant $c_{21}(\neq 0)$, d_{21} are either real numbers or the functions of the known parameters of the auxiliary variable, such as the coefficient of variation (C_x) standard deviation (σ_x), correlation coefficient (ρ), skewness or kurtosis from the population. a_{21}, λ_{21} is a suitable constants to be determined such that the MSE of t_{21}^G is minimum. α_{21} being constant takes the values (0,1,-1) for defining different ratio-type and product-type estimators.

The proposed estimator in (4.2.55) follows the same manner along with the class of estimator in Table 2, as that for case-I in Section 4.2.3. In addition, the relation between a_{21}, α_{21} and λ_{21} in case-III is the same as that for case-I in Section 4.2.3.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 4.2.3.1, we may write $\mathrm{Bias}\,(t_{21}^G)$ and MSE(t_{21}^G) following the same, and we may also produce a class of estimators for similar choices of a_{21}, α_{21} and λ_{21} in case-III.

The bias of (4.2.41) may be obtain by following the notations and expectations for case II presented in Section 4.1 as,

$$Bias\left(t_{21}^G\right) = \lambda_{21}\bar{Y}_s\left(\frac{\alpha_{21}(\alpha_{21}+1)}{2}\beta_{21}{}^2\upsilon_{21}{}^2V_{200} - \alpha_{21}\beta_{21}\upsilon_{21}V_{110}\right). \quad (4.2.56)$$

Similarly the expression for MSE (t_{21}^G) may be as,

$$MSE\left(t_{21}^G\right) = \bar{Y}_s^2\left(\lambda_{21}{}^2\left(V_{02} + \alpha_{21}{}^2a_{21}{}^2\upsilon_{21}{}^2V_{200} - 2\alpha_{21}a_{21}\upsilon_{21}V_{110}\right) + (\lambda_{21}-1)^2\right) \quad (4.2.57)$$

Remark 4.5

i) For $\alpha_{21} = 1$, some ratio-type estimators are expressed in Table 4.2. The MSE(t_2^G) of these ratio-type estimators may be expressed as,

$$MSE(t_{21}^j) = \begin{cases} \bar{Y}_s^2\left(\lambda_{21}{}^2\left(V_{020} + a_{21}{}^{opt2}V_{200} - 2a_{21}{}^{opt}V_{110}\right) + (\lambda_{21}-1)^2\right) & j(\in G) = 1 \\ \bar{Y}_s^2\left(\lambda_{21}{}^2\left(V_{020} + a_{21}{}^{opt2}\upsilon_{21}^{\left(\frac{j-1}{2}\right)^2}V_{200} - 2a_{21}{}^{opt}\upsilon_{21}^{\left(\frac{j-1}{2}\right)}V_{110}\right) + (\lambda_{21}-1)^2\right) & j(\in G) = 3,5,\ldots,27 \end{cases}, \quad (4.2.58)$$

The $MSE(t_{21}^j)$ in (4.2.58) is minimum for

$$a_{21}^{opt} = \frac{V_{110}}{\alpha_{21}\upsilon_{21}^{\left(\frac{j-1}{2}\right)} V_{200}} \; and \; \lambda_{21}^{opt} = \frac{1}{1+\left(V_{020} - \frac{V_{110}^{\;2}}{V_{200}}\right)}, \; .$$

ii) For $\alpha_{21} = -1$, some product-type estimators are expressed in Table 4.2, The MSE(t_{21}^{G}) for these product-type estimators may be expressed as,

$$MSE(t_{21}^{k}) = \begin{cases} \bar{Y}_s^{\,2}\left(\lambda_{21}^{\;2}\left(V_{020} + a_{21}^{opt2}V_{200} + 2a_{21}^{opt}V_{110}\right) + (\lambda_{21}-1)^2\right) k(\in G) = 1 \\ \bar{Y}_s^{\,2}\left(\lambda_{21}^{\;2}\left(V_{020} + a_{21}^{opt2}\upsilon_{21}^{\left(\frac{k}{2}\right)^2} V_{200} + 2a_{21}^{opt}\upsilon_{21}^{\left(\frac{k}{2}\right)}V_{110}\right) + (\lambda_{21}-1)^2\right) k(\in G) = 2,4,\ldots,28 \end{cases}, \tag{4.2.59}$$

The $MSE(t_{21}^{k})$ in (4.2.59) is minimum for

$$a_{21}^{opt} = \frac{V_{110}}{\alpha_{21}\upsilon_{21}^{\left(\frac{k}{2}\right)} V_{200}} \; and \; \lambda_{21}^{opt} = \frac{1}{1+\left(V_{020} - \frac{V_{110}^{\;2}}{V_{200}}\right)} \; .$$

where

$$\upsilon_{21}^{1} = \frac{\bar{X}_s}{\bar{X}_s + C_x},\; \upsilon_{21}^{2} = \frac{\beta_2(x)\bar{X}_s}{\beta_2(x)\bar{X}_s + C_x},\; \upsilon_{21}^{3} = \frac{C_x\bar{X}_s}{C_x\bar{X}_s + \beta_2(x)},$$

$$\upsilon_{21}^{4} = \frac{\beta_2(x)\bar{X}_s}{\beta_2(x)\bar{X}_s + \sigma_x},\; \upsilon_{21}^{5} = \frac{\bar{X}_s}{\bar{X}_s + \rho_x},\; \upsilon_{21}^{6} = \frac{\bar{X}_s}{\bar{X}_s + \beta_2(x)},$$

$$\upsilon_{21}^{7} = \frac{\sigma_x\bar{X}_s}{\sigma_x\bar{X}_s + 1},\; \upsilon_{21}^{8} = \frac{\bar{X}_s}{\sigma_x\bar{X}_s + \beta_1(x)},\; \upsilon_{21}^{9} = \frac{\sigma_x\bar{X}_s}{\sigma_x\bar{X}_s + \beta_2(x)},$$

$$\upsilon_{21}^{10} = \frac{\beta_1(x)\bar{X}_s}{\beta_1(x)\bar{X}_s + \beta_2(x)},\; \upsilon_{21}^{11} = \frac{\beta_2(x)\bar{X}_s}{\beta_2(x)\bar{X}_s + \beta_1(x)},\; \upsilon_{21}^{12} = \frac{\rho\bar{X}_s}{\rho\bar{X}_s + 1},$$

$$\upsilon_{21}^{13} = \frac{\sigma_x\bar{X}_s}{\sigma_x\bar{X}_s + C_x}.$$

4.3 PROPOSED COMBINED-TYPE GENERALIZED ESTIMATOR I IN TWO-STAGE SAMPLING USING TWO AUXILIARY VARIABLE

In this section, we propose two generalized estimators using two auxiliary variables under two-stage sampling design. We have proposed generalized estimator III assuming an exponential relationship among the study variable and two auxiliary variables. And estimator IV is developed by getting motivation from Khushnevisan et al. (2007) under two-stage sampling utilizing the information of two auxiliary variables.

4.3.1 Proposed Generalized Estimator III

Case I

i. Let exponential-type ratio-cum-ratio estimator follows as,

$$t'^{1}_{22} = \bar{y}'_s \exp\left(1 - \frac{\bar{x}'_s}{(\bar{X}'_s + \bar{x}'_s)}\right) \exp\left(1 - \frac{\bar{z}'_s}{(\bar{Z}'_s + \bar{z}'_s)}\right), \tag{4.3.1}$$

ii. Let exponential-type product-cum-product estimator follows as,

$$t'^{2}_{22} = \bar{y}'_s \exp\left(-\left(1 - \frac{\bar{x}'_s}{(\bar{X}'_s + \bar{x}'_s)}\right)\right) \exp\left(-\left(1 - \frac{\bar{z}'_s}{(\bar{Z}'_s + \bar{z}'_s)}\right)\right) \tag{4.3.2}$$

iii. Let exponential-type ratio-cum-product estimator follows as,

$$t'^{3}_{22} = \bar{y}'_s \exp\left(\left(1 - \frac{\bar{x}'_s}{(\bar{X}'_s + \bar{x}'_s)}\right)\right) \exp\left(-\left(1 - \frac{\bar{z}'_s}{(\bar{Z}'_s + \bar{z}'_s)}\right)\right) \tag{4.3.3}$$

iv. Let exponential-type product-cum-ratio estimator follows as,

$$t'^{4}_{22} = \bar{y}'_s \exp\left(-\left(1 - \frac{\bar{x}'_s}{(\bar{X}'_s + \bar{x}'_s)}\right)\right) \exp\left(\left(1 - \frac{\bar{z}'_s}{(\bar{Z}'_s + \bar{z}'_s)}\right)\right) \tag{4.3.4}$$

We may generalize (4.3.1)-(4.3.4) by introducing two real constants α' and β' whose values are known in advance as:

$$t'^{G}_{22} = \lambda'_{22}\,\overline{y}'_s \exp\left(\alpha'_{22}\left(1 - \frac{a'_{22}\,\overline{x}'_s}{\left(\overline{X}'_s + (a'_{22} - 1)\overline{x}'_s\right)}\right)\right)$$

$$\exp\left(\beta'_{22}\left(1 - \frac{b'_{22}\,\overline{z}'_s}{\left(\overline{Z}'_s + (b'_{22} - 1)\overline{z}'_s\right)}\right)\right),\ 0 < \lambda'_{22} \le 1 \qquad (4.3.5)$$

where (a'_{22}, b'_{22}, λ'_{22}) are constants to be determined such that the mean square error (MSE) is minimum. (α'_{22}, β'_{22}) are known constants takes the value(0,1,-1) to produce different ratio-type and product-type estimators.

By substituting different values to the constants in (4.3.5), we get a class of estimators as given in Table 4.3. Some members of estimators t^{G}_{22}

Table 4.3

Some Special Cases of the Generalized Estimator t'^{G}_{22}

Rati0o-cum-product estimator $\alpha'_{22} = 1, \beta'_{22} = 1$	**Product –cum-product estimator** $\alpha'_{22} = -1, \beta'_{22} = -1$	a'_{22}	λ'_{22}
$t'^{1}_{22} = \bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{Z}'_s - \bar{z}'_s}{\bar{Z}'_s + \bar{z}'_s}\right)$	$t'^{2}_{22} = \bar{y}'_s \exp\left(\frac{\bar{x}' - \bar{X}'_s}{\bar{x}' + \bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{z}'_s + \bar{Z}'_s}\right)$	2	1
$t'^{3}_{22} = \bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{Z}'_s - \bar{z}'_s}{\bar{Z}'_s}\right)$	$t'^{4}_{22} = \bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'}{\bar{Z}'_s}\right)$	1	1
$t'^{5}_{22} = \bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{Z}'_s - \bar{z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	$t'^{6}_{22} = \bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	a'_{22}	1
$t'^{7}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{Z}'_s - \bar{z}'_s}{\bar{Z}'_s + \bar{z}'_s}\right)$	$t'^{8}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + \bar{z}'_s}\right)$	2	λ'_{22}
$t'^{9}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{Z}'_s - \bar{z}'_s}{\bar{Z}'_s}\right)$	$t'^{10}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s}\right)$	1	λ'_{22}
$t'^{11}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{Z}'_s - \bar{z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	$t'^{12}_{22} = \lambda'_{3}\bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	a'_{22}	λ'_{22}
Product -cum-ratio estimator $\alpha'_{22} = -1, \beta'_{22} = 1$	**Ratio-cum-product estimator** $\alpha'_{22} = 1, \beta'_{22} = -1$	a'_{22}	λ'_{22}
$t'^{13}_{22} = \bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + \bar{z}'_s}\right)$	$t'^{14}_{22} = \bar{y}'_s \exp\left(\frac{\bar{x}' - \bar{X}'_s}{\bar{x}' + \bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{z}'_s + \bar{Z}'_s}\right)$	2	1
$t'^{15}_{22} = \bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s}\right)$	$t'^{16}_{22} = \bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'}{\bar{Z}'_s}\right)$	1	1
$t'^{17}_{22} = \bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	$t'^{18}_{22} = \bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	a'_{22}	1
$t'^{19}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + \bar{z}'_s}\right)$	$t'^{20}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + \bar{z}'_s}\right)$	2	λ'_{22}
$t'^{21}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s}\right)$	$t'^{22}_{22} = \lambda'_{22}\bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s}\right)$	1	λ'_{22}
$t'^{23}_{22} = \lambda'_{3}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{x}'_s}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	$t'^{24}_{22} = \lambda'_{3}\bar{y}'_s \exp\left(\frac{\bar{x}'_s - \bar{X}'}{\bar{X}'_s + (a'_{22} - 1)\bar{x}'_s}\right)$ $\exp\left(\frac{\bar{z}'_s - \bar{Z}'_s}{\bar{Z}'_s + (b'_{22} - 1)\bar{z}'_s}\right)$	a'_{22}	λ'_{22}

The Bias and Mean Square Error of proposed Estimator III

To derive the bias and mean square error we proceed as follows:

Using (4.1.1) we can express (4.3.5) as

$$t'^{G}_{22} = \lambda'_{22}\bar{Y}'_s(1+e'_0)\exp\left[-\frac{\alpha'_{22}}{a'_{22}}e'_1\left(1+\frac{(a'_{22}-1)}{a'_{22}}e'_1\right)^{-1}\right]$$

$$\exp\left[-\frac{\beta'_{22}}{b'_{22}}e'_2\left(1+\frac{(b'_{22}-1)}{b'_{22}}e'_2\right)^{-1}\right] \qquad (4.3.6)$$

We assume that $|e'_1|<1$, we expand the series, $\left(1+\frac{(a'_{22}-1)}{a'_{22}}e'_1\right)^{-1}$ and $\left(1+\frac{(b'_{22}-1)}{b'_{22}}e'_2\right)^{-1}$, we get

$$t'^{G}_{22} = \lambda'_{22}\bar{Y}'_s(1+e'_0)\exp\left[-\frac{\alpha'_{22}}{a'_{22}}e'_1\left(1-\frac{(a'_{22}-1)}{a'_{22}}e'_1+\frac{(a'_{22}-1)^2}{a'^{2}_{22}}e'^{2}_1+\ldots\right)\right]$$

$$\exp\left[-\frac{\beta'_{22}}{b'_{22}}e'_2\left(1-\frac{(b'_{22}-1)}{b'_{22}}e'_2+\frac{(b'_{22}-1)^2}{b'^{2}_{22}}e'^{2}_2+\ldots\right)\right], \qquad (4.3.7)$$

It is assumed that the contribution of terms involving powers in e'_0, and e'_1 and e'_2 higher than two is negligible. It is therefore expanding the exponentials and ignoring terms in e'_0, and e'_1 of order higher than two, we have

$$t'^{G}_{22} = \lambda'_{22}\bar{Y}'_s(1+e'_0)\left[1-\frac{\alpha'_{22}}{a'_{22}}e'_1+\frac{\alpha'^{2}_{22}}{a'^{2}_{22}}e'^{2}_1\right]\left[1-\frac{\beta'_{22}}{b'_{22}}e'_2+\frac{\beta'^{2}_{22}}{b'^{2}_{22}}e'^{2}_2\right], \qquad (4.3.8)$$

$$t'^{G}_{22} - \bar{Y}'_s = \lambda'_{22}\bar{Y}'_s\left[e'_0-\frac{\alpha'_{22}}{a'_{22}}e'_1+\frac{\alpha'^{2}_{22}}{a'^{2}_{22}}e'^{2}_1-\frac{\beta'_{22}}{b'_{22}}e'_2+\frac{\beta'^{2}_{22}}{b'^{2}_{22}}e'^{2}_2\right.$$

$$\left.-\frac{\beta'_{22}}{b'_{22}}e'_2e'_0-\frac{\alpha'_{22}}{a'_{22}}e'_1e'_0+\frac{\alpha'_{22}}{a'_{22}}\frac{\beta'_{22}}{b'_{22}}e'_2e'_1\right]+\bar{Y}'_s(\lambda'_{22}-1) \qquad (4.3.9)$$

In order to get the bias , we take expectation on (4.3.9) and get

$$Bias(t_{22}'^{G}) = \lambda'_{22}\bar{Y}_s'\left[\frac{\alpha_{22}'^{2}}{a_{22}^{2}}V'_{200} - \frac{\beta'_{22}}{b'_{22}}V'_{002} + \frac{\beta_{22}'^{2}}{b_{22}^{2}}V'_{002} - \frac{\beta'_{22}}{b'_{22}}V'_{011}\right.$$

$$\left. - \frac{\alpha'_{22}}{a'_{22}}V'_{110} + \frac{\alpha'_{22}}{a'_{22}}\frac{\beta'_{22}}{b'_{22}}V'_{101}\right] + \bar{Y}_s'(\lambda'_{22} - 1), \qquad (4.3.10)$$

To get the MSE of the estimator, we take square and retain terms upto Ist order of e's then we take expectation of (4.3.9) and we obtain

$$MSE\left(t_{22}'^{G}\right) = \bar{Y}_s'^{2}\left[\lambda_{22}'^{2}\left(V'_{020} + z_{22}'^{2}V'_{200} + u_{22}'^{2}V'_{002}\right.\right.$$

$$\left.\left. -2z'_{22}V'_{110} - 2u'_{22}V'_{011} + 2t'_{22}u'_{22}V'_{101}\right) + (\lambda'_{22} - 1)^{2}\right], \qquad (4.3.11)$$

where $z'_{22} = \dfrac{\alpha'_{22}}{a'_{22}}$ and $u'_{22} = \dfrac{\beta'_{22}}{b'_{22}}$

For the following optimal value of the constants z'_{22} and u'_{22}, we achieve the minimum MSE among the class of proposed generalized estimator

$$z'_{22} = \frac{V'_{110}V'_{002} - V'_{011}V'_{101}}{V'_{200}V'_{002} - V_{101}'^{2}} \; and \; u'_{22} = \frac{V'_{200}V'_{011} - V'_{110}V'_{101}}{V'_{200}V'_{002} - V_{101}'^{2}}, \lambda'_{22} = \frac{1}{1 + A_{22}'^{G}}$$

where

$$A_{22}'^{G} = \left[V'_{020} + z_{22}'^{2}V'_{200} + u_{22}'^{2}V'_{002} - 2z'_{22}V'_{110} - 2u'_{22}V'_{011} + 2z'_{22}u'_{22}V'_{101}\right]$$

(4.3.12)

By substituting the optimum values of z'_{22} and u'_{22}, we get $\lambda_{22}'^{opt}$ as,

$$\lambda_{22}'^{opt} = \frac{1}{1 + A_{22}'^{*}} \; where \; A_{22}'^{*} = \left(V'_{020} - \frac{V_{110}'^{2}V'_{002} + V_{011}'^{2}V'_{200} - 2V'_{110}V'_{101}V'_{011}}{V'_{200}V'_{002} - V_{101}'^{2}}\right)$$

(4.3.13)

We obtain asymptotic variance of the proposed generalized estimator as the expression for $MSE\left(t_{22}'^{G}\right)$ was considered upto first degree of error term, so minimum MSE may be written as

$$MSE_{\min}\left(t_{22}'^{G}\right) = AsymtoticVar\left(t_{22}'^{G}\right) = \bar{Y}_s'^{2}\left(\frac{A_{22}'^{*}}{1 + A_{22}'^{*}}\right) \qquad (4.3.14)$$

From (4.3.14), we observe that asymptotic variance of the proposed estimator is less than **Usual Linear Regression Estimator**. We may observe from (4.3.14) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value $\lambda_{22}'^{opt}$ and $a_{22}'^{opt}, b_{22}'^{opt}$ in (4.3.5), we get optimal estimator as:

$$\hat{t}_{22}'^{G} = \hat{\lambda}_{22}' \bar{y}_s' \exp\left(\alpha_{22}'\left(1 - \frac{\hat{a}_{22}' \bar{x}_s'}{\left(\bar{X}_s' + (\hat{a}_{22}' - 1)\bar{x}_s'\right)}\right)\right) \exp\left(\beta_{22}'\left(1 - \frac{\hat{b}_{22}' \bar{z}_s'}{\left(\bar{Z}_s' + (\hat{b}_{22}' - 1)\bar{z}_s'\right)}\right)\right), \quad 0 < \lambda_{22}' \le 1 \tag{4.3.15}$$

As described earlier in section (4.2.1.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as

$$\hat{\lambda}_{22}'^{opt} = \frac{1}{1 + \hat{A}_{22}^{*}} \; where \; \hat{A}_{22}^{*} = \left(\hat{V}_{020}' - \frac{\hat{V}_{110}'^{2}\hat{V}_{002}' + \hat{V}_{011}'^{2}\hat{V}_{200}' - 2\hat{V}_{110}'\hat{V}_{101}'\hat{V}_{011}'}{\hat{V}_{200}'\hat{V}_{002}' - \hat{V}_{101}'^{2}}\right) \tag{4.3.16}$$

So (4.3.16) may be wriiten as

$$\hat{t}_{22}'^{G} = \hat{\lambda}_{22}'^{opt} \bar{y}_s' \exp\left(\alpha_{22}'\left(1 - \frac{\hat{a}_{22}'^{opt} \bar{x}_s'}{\left(\bar{X}_s' + (\hat{a}_{22}'^{opt} - 1)\bar{x}_s'\right)}\right)\right) \exp\left(\beta_{22}'\left(1 - \frac{\hat{b}_{22}'^{opt} \bar{z}_s'}{\left(\bar{Z}_s' + (\hat{b}_{22}'^{opt} - 1)\bar{z}_s'\right)}\right)\right), \quad 0 < \lambda_{22}' \le 1 \tag{4.3.17}$$

Also the minimum MSE may be written as:

$$MSE_{\min}\left(\hat{t}_{22}'^{G}\right) = AsymtoticVar\left(\hat{t}_{22}'^{G}\right) = \bar{Y}_s'^{2}\left(\frac{\hat{A}_{22}'^{*}}{1 + \hat{A}_{22}'^{*}}\right) \tag{4.3.18}$$

Remark 4.6

i) For $\alpha'_{22} = 1, \beta'_{22} = 1$, we get exponential ratio cum ratio type estimators given in Table 4.3. The MSE of t'^{G}_{22} is expressed as

$$MSE(t'^{j}_{22}) = \begin{cases} \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{22}\left(V'_{02} + V'_{200} + V'_{002} - 2V'_{110} - 2V_{011} + 2V_{101}\right) + (\lambda'_{22} - 1)^2\right) & j(\in G) = 1 \\ \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{22}\begin{pmatrix} V'_{020} + \dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^2}_{22}}V'_{200} + \dfrac{1}{b'^{\left(\frac{j-1}{2}\right)^2}_{22}}V'_{002} - 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{22}}V'_{110} \\ -2\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{22}}V_{011} + 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{22}}\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{22}}V_{101} \end{pmatrix} + (\lambda'_{22} - 1)^2\right) & j(\in G) = 3, 5, \ldots, 11 \end{cases}$$

(4.3.19)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a'^{\left(\frac{j-1}{2}\right)}_{22}} = \frac{V'_{110}V'_{002} - V'_{011}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}$$

and

$$\frac{1}{b'^{\left(\frac{j-1}{2}\right)}_{22}} = \frac{V'_{200}V'_{011} - V'_{110}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}, \lambda'_{22} = \frac{1}{1 + A'^{G}_{22}}$$

where

$$A'^{G}_{22} = \left[V'_{020} + \frac{1}{a'^{\left(\frac{j-1}{2}\right)^2}_{22}}V'_{200} + \frac{1}{b'^{\left(\frac{j-1}{2}\right)^2}_{22}}V'_{002} - 2\frac{1}{a'^{\left(\frac{j-1}{2}\right)}_{22}}V'_{110} \right.$$

$$\left. -2\frac{1}{b'^{\left(\frac{j-1}{2}\right)}_{22}}V'_{011} + 2\frac{1}{a'^{\left(\frac{j-1}{2}\right)}_{22}}\frac{1}{b'^{\left(\frac{j-1}{2}\right)}_{22}}V'_{101}\right]$$

ii) For $\alpha'_{22} = -1, \beta'_{22} = -1$, we get exponential product cum product estimators given in Table 4.3. The MSE of t'^{G}_{22} is expressed as

$$MSE(t'^{k}_{22}) = \begin{cases} \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{22}\left(V'_{02} + V'_{200} + V'_{002} + 2V'_{110} + 2V_{011} - 2V_{101}\right) + (\lambda'_{22} - 1)^{2}\right) & k(\in G) = 2 \\ \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{22}\left(\begin{array}{l} V'_{020} + \dfrac{1}{a'^{\left(\frac{k}{2}\right)^{2}}_{22}}V'_{200} + \dfrac{1}{b'^{\left(\frac{k}{2}\right)^{2}}_{22}}V'_{002} + 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{22}}V'_{110} \\ +2\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{22}}V'_{011} - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{22}}\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{22}}V'_{101} \end{array}\right) + (\lambda'_{22} - 1)^{2}\right) & k(\in G) = 4, 6, \ldots, 12 \end{cases}$$

(4.3.20)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a'^{\left(\frac{k}{2}\right)}_{22}} = \frac{V'_{110}V'_{002} - V'_{011}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}$$

and

$$\frac{1}{b'^{\left(\frac{k}{2}\right)}_{22}} = \frac{V'_{200}V'_{011} - V'_{110}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}, \lambda'_{22} = \frac{1}{1 + A'^{G}_{22}}$$

where

$$A'^{G}_{22} = \left[V'_{020} + \frac{1}{a'^{\left(\frac{k}{2}\right)^{2}}_{22}}V'_{200} + \frac{1}{b'^{\left(\frac{k}{2}\right)^{2}}_{22}}V'_{002} - 2\frac{1}{a'^{\left(\frac{k}{2}\right)}_{22}}V'_{110} \right.$$

$$\left. -2\frac{1}{b'^{\left(\frac{k}{2}\right)}_{22}}V'_{011} + 2\frac{1}{a'^{\left(\frac{k}{2}\right)}_{22}}\frac{1}{b'^{\left(\frac{k}{2}\right)}_{22}}V'_{101} \right]$$

iii) For $\alpha'_{3} = -1, \beta'_{3} = 1$, we get exponential product cum ratio type estimators given in Table 4.3. The MSE of t'^{G}_{22} is expressed as,

$$MSE(t'^{l}_{22}) = \begin{cases} \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{22}\left(V'_{02} + V'_{200} + V'_{002} + 2V'_{110} - 2V'_{011} - 2V'_{101}\right) + (\lambda'_{22} - 1)^{2}\right) & l(\in G) = 13 \\ \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{22}\left(\begin{array}{l} V'_{020} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^{2}}_{22}}V'_{200} + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^{2}}_{22}}V'_{002} + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{3}}V'_{110} \\ -2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{22}}V'_{011} - 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{22}}\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{22}}V'_{101} \end{array}\right) + (\lambda'_{22} - 1)^{2}\right) & l(\in G) = 15, 17, \ldots, 23 \end{cases}$$

(4.3.21)

The optimal values which lead to minimum MSE as

$$\frac{1}{a_{22}'^{\left(\frac{l-1}{2}\right)}} = \frac{V_{110}'V_{002}' - V_{011}'V_{101}'}{V_{200}'V_{002}' - V_{101}'^{2}}$$

and

$$\frac{1}{b_{22}'^{\left(\frac{l-1}{2}\right)}} = \frac{V_{200}'V_{011}' - V_{110}'V_{101}'}{V_{200}'V_{002}' - V_{101}'^{2}}, \lambda_{22}' = \frac{1}{1 + A_{22}'^{G}}$$

where

$$A_{22}'^{G} = \left[V_{020}' + \frac{1}{a_{22}'^{\left(\frac{l-1}{2}\right)^2}} V_{200}' + \frac{1}{b_{22}'^{\left(\frac{l-1}{2}\right)^2}} V_{002}' - 2\frac{1}{a_{22}'^{\left(\frac{l-1}{2}\right)}} V_{110}' \right.$$

$$\left. -2\frac{1}{b_{22}'^{\left(\frac{l-1}{2}\right)}} V_{011}' + 2\frac{1}{a_{22}'^{\left(\frac{l-1}{2}\right)}} \frac{1}{b_{22}'^{\left(\frac{l-1}{2}\right)}} V_{101}' \right].$$

iv) For $\alpha_{22}' = 1, \beta_{22}' = -1$, we get exponential ratio cum product- type estimators given in Table 4.3. The MSE of $t_{22}'^{G}$ is expressed as

$$MSE(t_{22}'^{m}) = \begin{cases} \bar{Y}_s'^2 \left(\lambda_{22}'^{2} \left(V_{020}' + V_{200}' + V_{002}' - 2V_{110}' + 2V_{101}' - 2V_{011}' \right) + (\lambda_{22}' - 1)^2 \right) & m(\in G) = 14 \\ \bar{Y}_s'^2 \left(\lambda_{22}'^{2} \left(\begin{array}{l} V_{020}' + \frac{1}{a_{22}'^{\left(\frac{k}{2}\right)^2}} V_{200}' + \frac{1}{b_{22}'^{\left(\frac{k}{2}\right)^2}} V_{002}' - 2\frac{1}{a_{22}'^{\left(\frac{k}{2}\right)}} V_{110}' \\ +2\frac{1}{b_{22}'^{\left(\frac{k}{2}\right)}} V_{011}' - 2\frac{1}{a_{22}'^{\left(\frac{k}{2}\right)}} \frac{1}{b_{22}'^{\left(\frac{k}{2}\right)}} V_{101}' \end{array} \right) + (\lambda_{22}' - 1)^2 \right) & m(\in G) = 16, 18, \ldots, 24 \end{cases} \tag{4.3.22}$$

The optimal values which lead to minimum MSE as

$$\frac{1}{a_{22}'^{\left(\frac{m}{2}\right)}} = \frac{V_{110}'V_{002}' - V_{011}'V_{101}'}{V_{200}'V_{002}' - V_{101}'^{2}} \text{ and } \frac{1}{b_{22}'^{\left(\frac{m}{2}\right)}} = \frac{V_{200}'V_{011}' - V_{110}'V_{101}'}{V_{200}'V_{002}' - V_{101}'^{2}}, \lambda_{22}' = \frac{1}{1 + A_{22}'^{G}}$$

where

$$A_{22}'^{G} = \left[V_{020}' + \frac{1}{a_{22}'^{\left(\frac{m}{2}\right)^2}} V_{200}' + \frac{1}{b_{22}'^{\left(\frac{m}{2}\right)^2}} V_{002}' - 2\frac{1}{a_{22}'^{\left(\frac{m}{2}\right)}} V_{110}' \right.$$

$$\left. +2\frac{1}{b_{22}'^{\left(\frac{m}{2}\right)}} V_{011}' - 2\frac{1}{a_{22}'^{\left(\frac{m}{2}\right)}} \frac{1}{b_{22}'^{\left(\frac{m}{2}\right)}} V_{101}' \right]$$

Case II:

The generalized estimator under case II may be proposed following (4.3.1) as

$$t_{22}''^{G} = \lambda_{22}'' \bar{y}_s \exp\left(\alpha_{22}'' \left(1 - \frac{a_{22}'' \bar{x}_s''}{(\bar{X}_s'' + (a_{22}'' - 1)\bar{x}_s'')} \right) \right)$$

$$\exp\left(\beta_{22}'' \left(1 - \frac{b_{22}'' \bar{z}_s''}{(\bar{Z}_s'' + (b_{22}'' - 1)\bar{z}_s'')} \right) \right), \quad 0 < \lambda_{22}'' \le 1 \qquad (4.3.23)$$

where (a_{22}'', b_{22}'', λ_{22}'') are constants to be determined such that the mean square error (MSE) is minimum. (α_{22}'', β_{22}'') are known constants takes the value (0,1,-1) to produce different ratio-type and product-type estimators.

We follow the same routine along with the class of estimator in Table 4.3 for proposed estimator in (4.3.23) , as that for case-I in Section 4.3.1. In addition, the relation between a_{22}'', α_{22}'', λ_{22}'' and b_{22}'', β_{22}'' in case-II is the same as that for case-I in Section 4.3.1.1. Finally, the same is true for the MSE and the Bias. It is therefore directly from Section 4.3.1, we may write $\mathrm{Bias}\,(t_{22}''^{G})$ and $\mathrm{MSE}(t_{22}''^{G})$ following the same, and we may also produce a class of estimators for similar choices of a_{22}'', α_{22}'' and λ_{22}'' in case-II. The bias of (4.3.23) may be obtain by following the notations and expectations for case II presented in Section 4.1,

The bias of (4.3.23) may be obtain by following the notations and expectations for case II presented in Section 4.1 as,

$$Bias(t''^{G}_{22}) = \lambda''_{22}\ \bar{Y}''_s \left[\frac{\alpha''^{2}_{22}}{a''^{2}_{22}} V''_{200} - \frac{\beta''_{22}}{b''_{22}} V''_{002} + \frac{\beta''^{2}_{22}}{b''^{2}_{22}} V''_{002} \right.$$

$$\left. - \frac{\beta''_{22}}{b''_{22}} V''_{011} - \frac{\alpha''_{22}}{a''_{22}} V''_{110} + \frac{\alpha''_{22}}{a''_{22}} \frac{\beta''_{22}}{b''_{22}} V''_{101} \right] + \bar{Y}''_s (\lambda''_{22} - 1)\ , \qquad (4.3.24)$$

Similarly the expression of MSE may also be given as

$$MSE\left(t''^{G}_{22}\right) = \lambda''^{2}_{22} \bar{Y}''^{2}_s \left[V_{020} + z''^{2}_{22} V''_{200} + u''^{2}_{22} V''_{002} - 2 z''_{22} V''_{110} \right.$$

$$\left. - 2 u''_{22} V''_{011} + 2 z''_{22} u''_{22} V''_{101} \right] + \bar{Y}''^{2}_s (\lambda''_{22} - 1)^2 , \qquad (4.3.25)$$

By substituting the optimum values of z''_{22} and u''_{22}, we get λ''^{opt}_{22} as

$$\lambda''^{opt}_{22} = \frac{1}{1 + A''^{*}_{22}} \text{ where } A''^{*}_{22} = \left(V''_{020} - \frac{V''^{2}_{110} V''_{002} + V''^{2}_{011} V''_{200} - 2 V''_{110} V''_{101} V''_{011}}{V''_{200} V''_{002} - V''^{2}_{101}} \right)$$

(4.3.26)

The minimum MSE may be obtained as,

$$MSE_{\min}\left(t''^{G}_{22}\right) = \bar{Y}''^{2}_s \left(\frac{A''^{*}_{22}}{1 + A''^{*}_{22}} \right) . \qquad (4.3.27)$$

Remark 4.7

i) For $\alpha''_{22} = 1, \beta''_{22} = 1$, we get exponential-ratio type estimators given in Table 4.3. The MSE of t''^{G}_{22} is expressed as,

$$MSE(t''^{j}_{22}) = \begin{cases} \bar{Y}''^{2}_s \left(\lambda'^{\prime 2}_{22} \left(V''_{02} + V''_{200} + V''_{002} - 2V''_{110} - 2V''_{011} + 2V''_{101} \right) + (\lambda''_{22} - 1)^2 \right) & j(\in G) = 1 \\ \bar{Y}''^{2}_s \left(\lambda'^{2}_{22} \left(\begin{array}{l} V''_{020} + \dfrac{1}{a''^{\left(\frac{j-1}{2}\right)^2}_{22}} V''_{200} + \dfrac{1}{b''^{\left(\frac{j-1}{2}\right)^2}_{22}} V''_{002} - 2 \dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{22}} V''_{110} \\ -2 \dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{22}} V''_{011} + 2 \dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{22}} \dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{22}} V''_{101} \end{array} \right) + (\lambda''_{22} - 1)^2 \right) & j(\in G) = 3, 5, \ldots, 11 \end{cases}$$

(4.3.28)

The optimal values which lead to minimum MSE as

$$\frac{1}{a_{22}''^{\left(\frac{j-1}{2}\right)}} = \frac{V_{110}''V_{002}'' - V_{011}''V_{101}''}{V_{200}''V_{002}'' - V_{101}''^{\,2}}$$

and

$$\frac{1}{b_{22}''^{\left(\frac{j-1}{2}\right)}} = \frac{V_{200}''V_{011}'' - V_{110}''V_{101}''}{V_{200}''V_{002}'' - V_{101}''^{\,2}}, \lambda_{22}'' = \frac{1}{1 + A_{22}''^{\,G}}$$

where

$$A_{22}''^{\,G} = \left[V_{020}'' + \frac{1}{a_{22}''^{\left(\frac{j-1}{2}\right)^2}} V_{200}'' + \frac{1}{b_{22}''^{\left(\frac{j-1}{2}\right)^2}} V_{002}'' - 2\frac{1}{a_{22}''^{\left(\frac{j-1}{2}\right)}} V_{110}'' \right.$$

$$\left. -2\frac{1}{b_{22}''^{\left(\frac{j-1}{2}\right)}} V_{011}'' + 2\frac{1}{a_{22}''^{\left(\frac{j-1}{2}\right)}} \frac{1}{b_{22}''^{\left(\frac{j-1}{2}\right)}} V_{101}'' \right].$$

ii) For $\alpha_{22}'' = -1, \beta_{22}'' = -1$, we get exponential-ratio product estimators given in Table 4.3. The MSE of $t_{22}''^{G}$ is expressed as,

$$MSE(t_{22}''^{k}) = \begin{cases} \bar{Y}_s''^2 \left(\lambda_{22}''^{\,2} \left(V_{02}'' + V_{200}'' + V_{002}'' + 2V_{110}'' + 2V_{011}'' - 2V_{101}'' \right) + (\lambda_{22}'' - 1)^2 \right) & k(\in G) = 2 \\ \bar{Y}_s''^2 \left(\lambda_{22}''^{\,2} \left(\begin{array}{l} V_{020}'' + \frac{1}{a_{22}''^{\left(\frac{k}{2}\right)^2}} V_{200}'' + \frac{1}{b_{22}''^{\left(\frac{k}{2}\right)^2}} V_{002}'' + 2\frac{1}{a_{22}''^{\left(\frac{k}{2}\right)}} V_{110}'' \\ +2\frac{1}{b_{22}''^{\left(\frac{k}{2}\right)}} V_{011}'' - 2\frac{1}{a_{22}''^{\left(\frac{k}{2}\right)}} \frac{1}{b_{22}''^{\left(\frac{k}{2}\right)}} V_{101}'' \end{array} \right) + (\lambda_{22}'' - 1)^2 \right) & k(\in G) = 4, 6, \ldots, 12 \end{cases}$$

(4.3.29)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{22}''^{\left(\frac{k}{2}\right)}} = \frac{V_{110}''V_{002}'' - V_{011}''V_{101}''}{V_{200}''V_{002}'' - V_{101}''^{\,2}}$$

and

$$\frac{1}{b_{22}''^{\left(\frac{k}{2}\right)}} = \frac{V_{200}''V_{011}'' - V_{110}''V_{101}''}{V_{200}''V_{002}'' - V_{101}''^{\,2}}, \lambda_{22}'' = \frac{1}{1 + A_{22}''^{\,G}}$$

where

$$A''^{\,G}_{22} = \left[V''_{020} + \frac{1}{a''^{\left(\frac{k}{2}\right)^2}_{22}} V''_{200} + \frac{1}{b''^{\left(\frac{k}{2}\right)^2}_{22}} V''_{002} - 2\frac{1}{a''^{\left(\frac{k}{2}\right)}_{22}} V''_{110} \right.$$

$$\left. -2\frac{1}{b''^{\left(\frac{k}{2}\right)}_{22}} V''_{011} + 2\frac{1}{a''^{\left(\frac{k}{2}\right)}_{22}} \frac{1}{b''^{\left(\frac{k}{2}\right)}_{22}} V''_{101} \right].$$

iii) For $\alpha''_{22} = -1, \beta''_{22} = 1$, we get exponential-product cum ratio type estimators given in Table 3. The MSE of t''^{G}_{22} is expressed as

$$MSE(t''^{l}_{22}) = \begin{cases} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{22}\left(V''_{02} + V''_{200} + V''_{002} + 2V''_{110} - 2V''_{011} - 2V''_{101}\right) + (\lambda'{}'_{22} - 1)^2\right) & l(\in G) = 13 \\ \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{22}\left(\begin{array}{l} V''_{020} + \frac{1}{a'^{\left(\frac{l-1}{2}\right)^2}_{22}} V''_{200} + \frac{1}{b'^{\left(\frac{l-1}{2}\right)^2}_{22}} V''_{002} + 2\frac{1}{a'^{\left(\frac{l-1}{2}\right)}_{22}} V''_{110} \\ -2\frac{1}{b''^{\left(\frac{l-1}{2}\right)}_{22}} V''_{011} - 2\frac{1}{a''^{\left(\frac{l-1}{2}\right)}_{22}} \frac{1}{b''^{\left(\frac{l-1}{2}\right)}_{22}} V''_{101} \end{array}\right) + (\lambda''_{22} - 1)^2\right) & l(\in G) = 15, 17, \ldots, 23 \end{cases}$$

(4.3.30)

The optimal values which lead to minimum MSE as

$$\frac{1}{a''^{\left(\frac{l-1}{2}\right)}_{22}} = \frac{V''_{110}V''_{002} - V''_{011}V''_{101}}{V''_{200}V''_{002} - V''^{\,2}_{101}}$$

and

$$\frac{1}{b''^{\left(\frac{l-1}{2}\right)}_{22}} = \frac{V''_{200}V''_{011} - V''_{110}V''_{101}}{V''_{200}V''_{002} - V''^{\,2}_{101}}, \lambda''_{22} = \frac{1}{1 + A''^{G}_{22}}$$

where

$$A''^{G}_{22} = \left[V''_{020} + \frac{1}{a''^{\left(\frac{l-1}{2}\right)^2}_{22}} V''_{200} + \frac{1}{b''^{\left(\frac{l-1}{2}\right)^2}_{22}} V''_{002} - 2\frac{1}{a''^{\left(\frac{l-1}{2}\right)}_{22}} V''_{110} \right.$$

$$\left. -2\frac{1}{b''^{\left(\frac{l-1}{2}\right)}_{22}} V''_{011} + 2\frac{1}{a''^{\left(\frac{l-1}{2}\right)}_{22}} \frac{1}{b''^{\left(\frac{l-1}{2}\right)}_{22}} V''_{101} \right].$$

iv) For $\alpha''_{22} = 1, \beta''_{22} = -1$, we get exponential-ratio cum product type estimators given in Table 3. The MSE of t_{22}^{G} is expressed as,

$$MSE(t''^{m}_{22}) = \begin{cases} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{22}\left(V''_{020} + V''_{200} + V''_{002} - 2V''_{110} + 2V''_{011} - 2V''_{101}\right) + (\lambda''_{22} - 1)^{2}\right) m(\in G) = 14 \\ \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{22}\left(\begin{array}{l} V''_{020} + \dfrac{1}{a''^{\left(\frac{k}{2}\right)^{2}}_{22}}V''_{200} + \dfrac{1}{b''^{\left(\frac{k}{2}\right)^{2}}_{22}}V''_{002} - 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{22}}V''_{110} \\ +2\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{22}}V''_{011} - 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{22}}\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{22}}V''_{101} \end{array}\right) + (\lambda''_{22} - 1)^{2}\right) m(\in G) = 16, 18, \ldots, 24 \end{cases}$$

(4.3.31)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a''^{\left(\frac{m}{2}\right)}_{22}} = \frac{V''_{110}V''_{002} - V''_{011}V''_{101}}{V''_{200}V''_{002} - V''^{\,2}_{101}}$$

and

$$\frac{1}{b''^{\left(\frac{m}{2}\right)}_{22}} = \frac{V''_{200}V''_{011} - V''_{110}V''_{101}}{V''_{200}V''_{002} - V''^{\,2}_{101}}, \lambda''_{22} = \frac{1}{1 + A''^{G}_{22}}$$

where

$$A''^{\ G}_{22} = \left[V_{020} + \frac{1}{a''^{\left(\frac{m}{2}\right)^{2}}_{22}}V''_{200} + \frac{1}{b''^{\left(\frac{m}{2}\right)^{2}}_{22}}V''_{002} - 2\frac{1}{a''^{\left(\frac{m}{2}\right)}_{22}}V''_{110} + 2\frac{1}{b''^{\left(\frac{m}{2}\right)}_{22}}V''_{011} - 2\frac{1}{a''^{\left(\frac{m}{2}\right)}_{22}}\frac{1}{b''^{\left(\frac{m}{2}\right)}_{22}}V''_{101}\right].$$

Case III:

The generalized estimator under case II may be proposed following (4.3.1) as

$$t_{22}^{G} = \lambda_{22}\bar{y}_s \exp\left(\alpha_{22}\left(1 - \frac{a_{22}\bar{x}_s}{\left(\bar{X}_s + (a_{22}-1)\bar{x}_s\right)}\right)\right)$$

$$\exp\left(\beta_{22}\left(1 - \frac{b_{22}\bar{z}_s}{\left(\bar{Z}_s + (b_{22}-1)\bar{z}_s\right)}\right)\right), \quad 0 < \lambda_{22} \le 1 \qquad (4.3.32)$$

where (a_{22}, b_{22}, λ_{22}) are constants to be determined such that the mean square error (MSE) is minimum. (α_{22}, β_{22}) are known constants takes the value (0,1,-1) to produce different ratio-type and product-type estimators.

The proposed estimator in (4.3.32) follows the same routine along with the class of estimator in Table 3, as that for case-I in Section 4.3.1. In addition, the relation between the constants a_{22}, α_{22}, b_{22}, β_{22} and λ_{22} in case-III is the same as that for case-I in Section 4.3.1.1. Also, the same is true for the MSE and the bias. It is therefore directly from Section 4.3.3.1, we may write $\mathrm{Bias}\,(t_{22}^{G})$ and $\mathrm{MSE}(t_{22}^{G})$ following the same, and we may also produce a class of estimators for similar choices of a_{22}, α_{22}, b_{22}, β_{22} and λ_{22} in case-III. The bias of (4.3.18) may be obtain by following the notations and expectations for case III presented in Section 4.1,

$$Bias(t_{22}^{G}) = \lambda_{22}\bar{Y}_s\left[\frac{\alpha_{22}}{a_{22}^{2}}V_{200} - \frac{\beta_{22}}{b_{22}}V_{002} + \frac{\beta_{22}^{2}}{b_{22}^{2}}V_{002} - \frac{\beta_{22}}{b_{22}}V_{011}\right.$$

$$\left. - \frac{\alpha_{22}}{a_{22}}V_{110} + \frac{\alpha_{22}}{a_{22}}\frac{\beta_{22}}{b_{22}}V_{101}\right] + \bar{Y}_s(\lambda_{22}-1), \qquad (4.3.33)$$

Similarly the expression of MSE is also given as

$$MSE\left(t_{22}^{G}\right) = \lambda_{22}\bar{Y}_s^{2}\left[V_{020} + t_{22}^{2}V_{200} + u_{22}^{2}V_{002} - 2t_{22}V_{110}\right.$$

$$\left. -2u_{22}V_{011} + 2t_{22}u_{22}V_{101}\right] + \bar{Y}_s^{2}(\lambda_{22}-1)^{2}\ , \qquad (4.3.34)$$

By substituting the optimum values of t_{22} and u_{22}, we get λ_{22}^{opt} as

$$\lambda_{22}^{opt} = \frac{1}{1+A_{22}^{*}} \text{ where } A_{22}^{*} = \left(V_{020} - \frac{V_{110}^{2}V_{002} + V_{011}^{2}V_{200} - 2V_{110}V_{101}V_{011}}{V_{200}V_{002} - V_{101}^{2}}\right)$$

(4.3.35)

The minimum MSE may be obtained as,

$$MSE_{\min}\left(t_{22}^{G}\right) = \overline{Y}_s^{2}\left(\frac{A_{22}^{\;opt}}{1 + A_{22}^{\;opt}}\right) \tag{4.3.36}$$

Remark 4.8

i) For $\alpha_{22} = 1, \beta_{22} = 1$, we get exponential ratio-type estimators given in Table 3. The MSE of $t_{22}'^{G}$ is expressed as,

$$MSE(t_{22}^{j}) = \begin{cases} Y_s^{2}\left(\lambda_{22}^{2}\left(V_{02} + V_{200} + V_{002} - 2V_{110} - 2V_{011} + 2V_{101}\right) + (\lambda_{22} - 1)^{2}\right) & j(\in G) = 1 \\ \overline{Y}_s^{2}\left(\lambda_{22}^{2}\left(\begin{array}{l} V_{020} + \frac{1}{a_{22}^{\left(\frac{j-1}{2}\right)^2}}V_{200} + \frac{1}{b_{22}^{\left(\frac{j-1}{2}\right)^2}}V_{002} - 2\frac{1}{a_{22}^{\left(\frac{j-1}{2}\right)}}V_{110} \\ -2\frac{1}{b_{22}^{\left(\frac{j-1}{2}\right)}}V_{011} + 2\frac{1}{a_{22}^{\left(\frac{j-1}{2}\right)}}\frac{1}{b_{22}^{\left(\frac{j-1}{2}\right)}}V_{101} \end{array}\right) + (\lambda_{22} - 1)^{2}\right) & j(\in G) = 3, 5, \ldots, 11 \end{cases} \tag{4.3.37}$$

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{22}^{\left(\frac{j-1}{2}\right)}} = \frac{V_{110}V_{002} - V_{011}V_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}$$

and

$$\frac{1}{b_{22}^{\left(\frac{j-1}{2}\right)}} = \frac{V_{200}V_{011} - V_{110}V_{101}}{V_{200}V_{002} - V_{101}^{2}}, \lambda_{22} = \frac{1}{1 + A_{22}^{G}}$$

where

$$A_{22}^{G} = \left[V_{020} + \frac{1}{a_{22}^{\left(\frac{j-1}{2}\right)^2}}V_{200} + \frac{1}{b_{22}^{\left(\frac{j-1}{2}\right)^2}}V_{002} - 2\frac{1}{a_{22}^{\left(\frac{j-1}{2}\right)}}V_{110}\right.$$

$$\left. -2\frac{1}{b_{22}^{\left(\frac{j-1}{2}\right)}}V_{011} + 2\frac{1}{a_{22}^{\left(\frac{j-1}{2}\right)}}\frac{1}{b_{22}^{\left(\frac{j-1}{2}\right)}}V_{101}\right].$$

ii) For $\alpha_{22} = -1, \beta_{22} = -1$, we get exponential product cum product type estimators given in Table 4.3. The MSE of t_{22}^{G} is expressed as

$$MSE(t_{22}^{k}) = \begin{cases} \bar{Y}_s^{2}\left(\lambda_{22}^{2}\left(V_{02} + V_{200} + V_{002} + 2V_{110} + 2V_{011} - 2V_{101}\right) + (\lambda_{22} - 1)^{2}\right) & k(\in G) = 2 \\ \bar{Y}_s^{2}\left(\lambda_{22}^{2}\left(\begin{array}{l} V_{020} + \dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)^{2}}}V'_{200} + \dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)^{2}}}V'_{002} + 2\dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)}}V_{110} \\ +2\dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)}}V_{011} - 2\dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)}}V_{101} \end{array}\right) + (\lambda_{22} - 1)^{2}\right) & k(\in G) = 4, 6, \ldots, 12 \end{cases}$$

(4.3.38)

The optimal values which lead to minimum MSE as

$$\frac{1}{a_{22}^{\left(\frac{k}{2}\right)}} = \frac{V_{110}V_{002} - V_{011}V_{101}}{V_{200}V_{002} - V_{101}^{2}}$$

and

$$\frac{1}{b_{22}^{\left(\frac{k}{2}\right)}} = \frac{V_{200}V_{011} - V_{110}V_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}, \lambda_{22} = \frac{1}{1 + A_{22}^{G}}$$

where

$$A_{22}^{G} = \left[\begin{array}{l} V_{020} + \dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)^{2}}}V_{200} + \dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)^{2}}}V_{002} - 2\dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)}}V_{110} \\ \qquad -2\dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)}}V_{011} + 2\dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)}}V_{101} \end{array}\right].$$

iii) For $\alpha_{22} = -1, \beta_{22} = 1$, we get exponential product cum ratio type estimators given in Table 3. The MSE of t'^{G}_{22} is expressed as

$$MSE(t_{22}^{l}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{22}^{2} \left(V_{02} + V_{200} + V_{002} + 2V_{110} - 2V_{011} - 2V_{101} \right) + (\lambda_{22} - 1)^2 \right) & l(\in G) = 13 \\ \bar{Y}_s^2 \left(\lambda_{22}^{2} \left(\begin{array}{l} V_{020} + \dfrac{1}{a_{22}^{\left(\frac{l-1}{2}\right)^2}} V_{200} + \dfrac{1}{b_{22}^{\left(\frac{l-1}{2}\right)^2}} V_{002} + 2 \dfrac{1}{a_{22}^{\left(\frac{l-1}{2}\right)}} V_{110} \\ -2 \dfrac{1}{b_{22}^{\left(\frac{l-1}{2}\right)}} V_{011} - 2 \dfrac{1}{a_{22}^{\left(\frac{l-1}{2}\right)}} \dfrac{1}{b_{22}^{\left(\frac{l-1}{2}\right)}} V_{101} \end{array} \right) + (\lambda_{22} - 1)^2 \right) & l(\in G) = 15,17,\ldots,23 \end{cases}$$

(4.3.38)

The optimal values which lead to minimum MSE as

$$\frac{1}{a_{22}^{\left(\frac{l-1}{2}\right)}} = \frac{V_{110}V_{002} - V_{011}V_{101}}{V_{200}V_{002} - V_{101}^{2}} \text{ and } \frac{1}{b_{22}^{\left(\frac{l-1}{2}\right)}} = \frac{V_{200}V_{011} - V_{110}V_{101}}{V_{200}V_{002} - V_{101}^{2}}, \lambda_{22} = \frac{1}{1 + A_{22}}$$

where

$$A_{22} = \left[V_{020} + \frac{1}{a_{22}^{\left(\frac{l-1}{2}\right)^2}} V_{200} + \frac{1}{b_{22}^{\left(\frac{l-1}{2}\right)^2}} V_{002} - 2 \frac{1}{a_{22}^{\left(\frac{l-1}{2}\right)}} V_{110} \right.$$

$$\left. -2 \frac{1}{b_{22}^{\left(\frac{l-1}{2}\right)}} V_{011} + 2 \frac{1}{a_{22}^{\left(\frac{l-1}{2}\right)}} \frac{1}{b_{22}^{\left(\frac{l-1}{2}\right)}} V_{101} \right].$$

iv) For $\alpha_{22} = 1, \beta_{22} = -1$, we get exponential ratio cum product type estimators given in Table 3. The MSE of t_{22}^{G} is expressed as,

$$MSE(t_{22}^{m}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{22}^{2} \left(V_{020} + V_{200} + V_{002} - 2V_{110} + 2V_{011} - 2V_{101} \right) + (\lambda_{22} - 1)^2 \right) & m(\in G) = 14 \\ \bar{Y}_s^2 \left(\lambda_{22}^{2} \left(\begin{array}{l} V_{020} + \dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)^2}} V_{200} + \dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)^2}} V_{002} - 2 \dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)}} V_{110} \\ +2 \dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)}} V_{011} - 2 \dfrac{1}{a_{22}^{\left(\frac{k}{2}\right)}} \dfrac{1}{b_{22}^{\left(\frac{k}{2}\right)}} V_{101} \end{array} \right) + (\lambda_{22} - 1)^2 \right) & m(\in G) = 16,18,\ldots,24 \end{cases}$$

(4.3.39)

The optimal values which lead to minimum MSE as

$$\frac{1}{a_{22}^{\left(\frac{m}{2}\right)}} = \frac{V_{110}V_{002} - V_{011}V_{101}}{V_{200}V_{002} - V_{101}^{2}} \text{ and } \frac{1}{b_{22}^{\left(\frac{m}{2}\right)}} = \frac{V_{200}V_{011} - V_{110}V_{101}}{V_{200}V_{002} - V_{101}^{2}}, \lambda_{22} = \frac{1}{1 + A_{22}^{G}}$$

where

$$A_{22}^{G} = \left[V_{020} + \frac{1}{a_{22}^{\left(\frac{m}{2}\right)^2}} V_{200} + \frac{1}{b_{22}^{\left(\frac{m}{2}\right)^2}} V_{002} - 2 \frac{1}{a_{22}^{\left(\frac{m}{2}\right)}} V_{110} \right.$$

$$\left. + 2 \frac{1}{b_{22}^{\left(\frac{m}{2}\right)}} V_{011} - 2 \frac{1}{a_{22}^{\left(\frac{m}{2}\right)}} \frac{1}{b_{22}^{\left(\frac{m}{2}\right)}} V_{101} \right].$$

4.3.4 Proposed Generalized Estimator IV

Case I:

We generalize Khoshnevisan et al. (2007) for two stage sampling design using two auxiliary variables as,

$$t'^{G}_{23} = \lambda'_{23} \bar{y}'_s \left(\frac{c'_{23} \bar{X}'_s + d'_{23}}{a'_{23} \left(c'_{23} \bar{x}'_s + d'_{23} \right) + (1 - a'_{23}) \left(c'_{23} \bar{X}'_s + d'_{23} \right)} \right)^{\alpha'_{23}}$$

$$\left(\frac{k'_{23} \bar{Z}'_s + l'_{23}}{b'_{23} \left(k'_{23} \bar{z}'_s + l'_{23} \right) + (1 - b'_{23}) \left(k'_{23} \bar{Z}'_s + l'_{23} \right)} \right)^{\beta'_{23}} \qquad (4.3.40)$$

where the constant $c'_{23}, k'_{23} (\neq 0)$, d'_{23} and l'_{23} are either real numbers or the functions of the auxiliary variable, in form of coefficient of variations, standard deviations, correlation coefficients, skewness or kutosis from the population. (a'_{23}, b'_{23}, λ'_{23}) are constants to be determined such as mean square error (MSE) is minimum.

Table 4.4(a)

Some Special Cases of Generalized Estimator t'^{G}_{23}

Ratio-cum-Ratio Estimator $\alpha'_{23} = 1, \beta'_{23} = 1$	**Product-cum-product Estimator** $\alpha'_{23} = -1, \beta'_{23} = -1$	c'_{23}	d'_{23}	λ'_{23}	k'_{23}	l'_{23}
$t'^{1}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{X}'_s}{\bar{x}'_s}\right)\left(\frac{\bar{Z}'_s}{\bar{z}_s}\right)$	$t'^{2}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s}{\bar{X}'_s}\right)\left(\frac{\bar{z}'_s}{\bar{Z}'_s}\right)$					
$t'^{3}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{X}'_s + C_x}{\bar{x}'_s + C_x}\right)\left(\frac{\bar{Z}'_s + C_z}{\bar{z}'_s + C_z}\right)$	$t'^{4}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s + C_x}{\bar{X}'_s + C_x}\right)\left(\frac{\bar{z}'_s + C_z}{\bar{Z}'_s + C_z}\right)$	1	C_x	1	C_z	1
$t'^{5}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s + C_x}{\beta_2(x)\bar{x}'_s + C_x}\right)\left(\frac{\beta_2(z)\bar{Z}'_s + C_z}{\beta_2(z)\bar{z}'_s + C_z}\right)$	$t'^{6}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s + C_x}{\beta_2(x)\bar{X}'_s + C_x}\right)\left(\frac{\beta_2(z)\bar{z}'_s + C_z}{\beta_2(z)\bar{Z}'_s + C_z}\right)$	$\beta_2(x)$	C_x	1	$\beta_2(z)$	C_z
$t'^{7}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{C_x\bar{X}'_s + \beta_2(x)}{C_x\bar{x}'_s + \beta_2(x)}\right)\left(\frac{C_z\bar{Z}'_s + \beta_2(z)}{C_z\bar{z}'_s + \beta_2(z)}\right)$	$t'^{8}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{C_x\bar{x}'_s + \beta_2(x)}{C_x\bar{X}'_s + \beta_2(x)}\right)\left(\frac{C_z\bar{z}'_s + \beta_2(z)}{C_z\bar{Z}'_s + \beta_2(z)}\right)$	C_x	$\beta_2(x)$	1	C_z	$\beta_2(z)$
$t'^{9}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s + \sigma_x}{\beta_2(x)\bar{x}'_s + \sigma_x}\right)\left(\frac{\beta_2(z)\bar{Z}'_s + \sigma_z}{\beta_2(z)\bar{z}'_s + \sigma_z}\right)$	$t'^{10}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s + \sigma_x}{\beta_2(x)\bar{X}'_s + \sigma_x}\right)\left(\frac{\beta_2(z)\bar{z}'_s + \sigma_z}{\beta_2(z)\bar{Z}'_s + \sigma_z}\right)$	$\beta_2(x)$	σ_x	1		
$t'^{11}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{X}'_s + \rho_{xy}}{\bar{x}'_s + \rho_{xy}}\right)\left(\frac{\beta_2(z)\bar{Z}'_s + \rho_{yz}}{\beta_2(z)\bar{z}'_s + \rho_{yz}}\right)$	$t'^{12}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s + \rho_{xy}}{\bar{X}'_s + \rho_{xy}}\right)\left(\frac{\beta_2(z)\bar{z}'_s + \rho_{yz}}{\beta_2(z)\bar{Z}'_s + \rho_{yz}}\right)$	1	ρ	1		
$t'^{13}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{X}'_s + \beta_2(x)}{\bar{x}'_s + \beta_2(x)}\right)\left(\frac{\bar{Z}'_s + \beta_2(z)}{\bar{z}'_s + \beta_2(z)}\right)$	$t'^{14}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s + \beta_2(x)}{\bar{X}'_s + \beta_2(x)}\right)\left(\frac{\bar{z}'_s + \beta_2(z)}{\bar{Z}'_s + \beta_2(z)}\right)$	1	$\beta_2(x)$	1	1	$\beta_2(z)$
$t'^{15}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + 1}{\sigma_x\bar{x}'_s + 1}\right)\left(\frac{\sigma_z\bar{Z}'_s + 1}{\sigma_x\bar{z}'_s + 1}\right)$	$t'^{16}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + 1}{\sigma_x\bar{X}'_s + 1}\right)\left(\frac{\sigma_z\bar{z}'_s + 1}{\sigma_z\bar{Z}'_s + 1}\right)$	σ_{x_i}	1	1	σ_z	1
$t'^{17}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + \beta_1(x)}{\sigma_x\bar{x}'_s + \beta_1(x)}\right)\left(\frac{\sigma_z\bar{Z}'_s + \beta_1(z)}{\sigma_z\bar{z}'_s + \beta_1(z)}\right)$	$t'^{18}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + \beta_1(x)}{\sigma_x\bar{X}'_s + \beta_1(x)}\right)\left(\frac{\sigma_z\bar{z}'_s + \beta_1(z)}{\sigma_z\bar{Z}'_s + \beta_1(z)}\right)$	σ_{x_i}	$\beta_1(x)$	1		
$t'^{19}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + \beta_2(x)}{\sigma_x\bar{x}'_s + \beta_2(x)}\right)\left(\frac{\sigma_z\bar{Z}'_s + \beta_2(z)}{\sigma_x\bar{z}'_s + \beta_2(z)}\right)$	$t'^{20}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + \beta_2(x)}{\sigma_x\bar{X}'_s + \beta_2(x)}\right)\left(\frac{\sigma_z\bar{z}'_s + \beta_2(z)}{\sigma_x\bar{Z}'_s + \beta_2(z)}\right)$	σ_x	$\beta_2(x)$	1	σ_z	$\beta_2(z)$
$t'^{21}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_1(x)\bar{X}'_s + \beta_2(x)}{\beta_1(x)\bar{x}'_s + \beta_2(x)}\right)\left(\frac{\beta_1(z)\bar{Z}'_s + \beta_2(z)}{\beta_1(z)\bar{z}'_s + \beta_2(z)}\right)$	$t'^{22}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_1(x)\bar{x}'_s + \beta_2(x)}{\beta_1(x)\bar{X}'_s + \beta_2(x)}\right)\left(\frac{\beta_1(z)\bar{z}'_s + \beta_2(z)}{\beta_1(z)\bar{Z}'_s + \beta_2(z)}\right)$	$\beta_1(x)$	$\beta_2(x)$	1	$\beta_1(z)$	$\beta_2(z)$
$t'^{23}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s + \beta_1(x)}{\beta_2(x)\bar{x}'_s + \beta_1(x)}\right)\left(\frac{\beta_2(z)\bar{Z}'_s + \beta_1(z)}{\beta_2(z)\bar{z}'_s + \beta_1(z)}\right)$	$t'^{24}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s + \beta_1(x)}{\beta_2(x)\bar{X}'_s + \beta_1(x)}\right)\left(\frac{\beta_2(z)\bar{z}'_s + \beta_1(z)}{\beta_2(z)\bar{Z}'_s + \beta_1(z)}\right)$	$\beta_2(x)$	$\beta_1(x)$	1		
$t'^{25}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\rho_{xy}\bar{X}'_s + 1}{\rho_{xy}\bar{x}'_s + 1}\right)\left(\frac{\rho_{yz}\bar{Z}'_s + 1}{\rho_{yz}\bar{z}'_s + 1}\right)$	$t'^{26}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\rho_{xy}\bar{x}'_s + 1}{\rho_{xy}\bar{X}'_s + 1}\right)\left(\frac{\rho_{yz}\bar{z}'_s + 1}{\rho_{yz}\bar{Z}'_s + 1}\right)$	ρ_{xy}	1	1	ρ_{yz}	
$t'^{27}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s + C_x}{\sigma_x\bar{x}'_s + C_x}\right)\left(\frac{\sigma_z\bar{Z}'_s + C_z}{\sigma_x\bar{z}'_s + C_z}\right)$	$t'^{28}_{23} = \lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s + C_x}{\sigma_x\bar{X}'_s + C_x}\right)\left(\frac{\sigma_z\bar{z}'_s + C_z}{\sigma_z\bar{Z}'_s + C_z}\right)$	σ_x	C_x	1	σ_z	C_z

Table 4.4 (b).
Some Special Cases of Proposed Generalized Estimator t_{23}^{G}

Product -cum-Ratio Estimator $\alpha'_{23}=-1,\beta'_{23}=1$	**Ratio-cum-product Estimator** $\alpha'_{4}=1,\beta'_{4}=-1$	c'	d'	λ'_{24}	k'	l'
$t'^{29}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s}{\bar{X}'_s}\right)\left(\frac{\bar{Z}'_s}{\bar{z}_s}\right)$	$t'^{30}_{24}=\lambda'_{24}\bar{y}'_s\left(\frac{\bar{X}'_s}{\bar{x}'_s}\right)\left(\frac{\bar{z}'_s}{\bar{Z}'_s}\right)$					
$t'^{31}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s+C_x}{\bar{X}'_s+C_x}\right)\left(\frac{\bar{Z}'_s+C_z}{\bar{z}'_s+C_z}\right)$	$t'^{32}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\bar{X}'_s+C_x}{\bar{x}'_s+C_x}\right)\left(\frac{\bar{z}'_s+C_z}{\bar{Z}'_s+C_z}\right)$	1	C_x	1	C_z	1
$t'^{33}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s+C_x}{\beta_2(x)\bar{X}'_s+C_x}\right)\left(\frac{\beta_2(z)\bar{z}'_s+C_z}{\beta_2(z)\bar{Z}'_s+C_z}\right)$	$t'^{34}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s+C_x}{\beta_2(x)\bar{x}'_s+C_x}\right)\left(\frac{\beta_2(z)\bar{z}'_s+C_z}{\beta_2(z)\bar{Z}'_s+C_z}\right)$	$\beta_2(x)$	C_x	1	$\beta_2(z)$	C_z
$t'^{35}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{C_x\bar{x}'_s+\beta_2(x)}{C_x\bar{X}'_s+\beta_2(x)}\right)\left(\frac{C_z\bar{Z}'_s+\beta_2(z)}{C_z\bar{z}'_s+\beta_2(z)}\right)$	$t'^{36}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{C_x\bar{x}'_s+\beta_2(x)}{C_x\bar{X}'_s+\beta_2(x)}\right)\left(\frac{C_z\bar{Z}'_s+\beta_2(z)}{C_z\bar{z}'_s+\beta_2(z)}\right)$	C_x	$\beta_2(x)$	1	C_z	$\beta_2(z)$
$t'^{37}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s+\sigma_x}{\beta_2(x)\bar{X}'_s+\sigma_x}\right)\left(\frac{\beta_2(z)\bar{Z}'_s+\sigma_z}{\beta_2(z)\bar{z}'_s+\sigma_z}\right)$	$t'^{38}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s+\sigma_x}{\beta_2(x)\bar{x}'_s+\sigma_x}\right)\left(\frac{\beta_2(z)\bar{Z}'_s+\sigma_z}{\beta_2(z)\bar{z}'_s+\sigma_z}\right)$	$\beta_2(x)$	σ_x	1		
$t'^{39}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s+\rho_{xy}}{\bar{X}'_s+\rho_{xy}}\right)\left(\frac{\beta_2(z)\bar{Z}'_s+\rho_{yz}}{\beta_2(z)\bar{z}'_s+\rho_{yz}}\right)$	$t'^{40}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\bar{X}'_s+\rho_{xy}}{\bar{x}'_s+\rho_{xy}}\right)\left(\frac{\beta_2(z)\bar{z}'_s+\rho_{yz}}{\beta_2(z)\bar{Z}'_s+\rho_{yz}}\right)$	1	ρ	1		
$t'^{41}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\bar{x}'_s+\beta_2(x)}{\bar{X}'_s+\beta_2(x)}\right)\left(\frac{\bar{Z}'_s+\beta_2(z)}{\bar{z}'_s+\beta_2(z)}\right)$	$t'^{42}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\bar{X}'_s+\beta_2(x)}{\bar{x}'_s+\beta_2(x)}\right)\left(\frac{\bar{z}'_s+\beta_2(z)}{\bar{Z}'_s+\beta_2(z)}\right)$	1	$\beta_2(\mathrm{x})$	1	1	$\beta_2(z)$
$t'^{43}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s+1}{\sigma_x\bar{X}'_s+1}\right)\left(\frac{\sigma_z\bar{Z}'_s+1}{\sigma_x\bar{z}'_s+1}\right)$	$t'^{44}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s+1}{\sigma_x\bar{x}'_s+1}\right)\left(\frac{\sigma_z\bar{z}'_s+1}{\sigma_z\bar{Z}'_s+1}\right)$	σ_{x_i}	1	1	σ_z	1
$t'^{45}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s+\beta_1(\mathrm{x})}{\sigma_x\bar{X}'_s+\beta_1(\mathrm{x})}\right)\left(\frac{\sigma_z\bar{Z}'_s+\beta_1(z)}{\sigma_z\bar{z}'_s+\beta_1(z)}\right)$	$t'^{46}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s+\beta_1(\mathrm{x})}{\sigma_x\bar{x}'_s+\beta_1(\mathrm{x})}\right)\left(\frac{\sigma_z\bar{z}'_s+\beta_1(z)}{\sigma_z\bar{Z}'_s+\beta_1(z)}\right)$	σ_{x_i}	$\beta_1(x)$	1		
$t'^{47}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s+\beta_2(\mathrm{x})}{\sigma_x\bar{X}'_s+\beta_2(\mathrm{x})}\right)\left(\frac{\sigma_z\bar{Z}'_s+\beta_2(z)}{\sigma_x\bar{z}'_s+\beta_2(z)}\right)$	$t'^{48}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s+\beta_2(\mathrm{x})}{\sigma_x\bar{x}'_s+\beta_2(\mathrm{x})}\right)\left(\frac{\sigma_z\bar{z}'_s+\beta_2(z)}{\sigma_x\bar{Z}'_s+\beta_2(z)}\right)$	σ_x	$\beta_2(x)$	1	σ_z	$\beta_2(z)$
$t'^{49}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_1(x)\bar{x}'_s+\beta_2(x)}{\beta_1(x)\bar{X}'_s+\beta_2(x)}\right)\left(\frac{\beta_1(z)\bar{Z}'_s+\beta_2(z)}{\beta_1(z)\bar{z}'_s+\beta_2(z)}\right)$	$t'^{50}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_1(x)\bar{X}'_s+\beta_2(x)}{\beta_1(x)\bar{x}'_s+\beta_2(x)}\right)\left(\frac{\beta_1(z)\bar{z}'_s+\beta_2(z)}{\beta_1(z)\bar{Z}'_s+\beta_2(z)}\right)$	$\beta_1(\mathrm{x})$	$\beta_2(\mathrm{x})$	1	$\beta_1(z)$	$\beta_2(z)$
$t'^{51}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{x}'_s+\beta_1(\mathrm{x})}{\beta_2(x)\bar{X}'_s+\beta_1(\mathrm{x})}\right)\left(\frac{\beta_2(z)\bar{Z}'_s+\beta_1(z)}{\beta_2(z)\bar{z}'_s+\beta_1(z)}\right)$	$t'^{52}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\beta_2(x)\bar{X}'_s+\beta_1(\mathrm{x})}{\beta_2(x)\bar{x}'_s+\beta_1(\mathrm{x})}\right)\left(\frac{\beta_2(z)\bar{z}'_s+\beta_1(z)}{\beta_2(z)\bar{Z}'_s+\beta_1(z)}\right)$	$\beta_2(x)$	$\beta_1(x)$	1		
$t'^{53}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\rho_{xy}\bar{x}'_s+1}{\rho_{xy}\bar{X}'_s+1}\right)\left(\frac{\rho_{yz}\bar{Z}'_s+1}{\rho_{yz}\bar{z}'_s+1}\right)$	$t'^{54}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\rho_{xy}\bar{X}'_s+1}{\rho_{xy}\bar{x}'_s+1}\right)\left(\frac{\rho_{yz}\bar{z}'_s+1}{\rho_{yz}\bar{Z}'_s+1}\right)$	ρ_{xy}	1	1	ρ_{yz}	
$t'^{55}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{x}'_s+C_x}{\sigma_x\bar{X}'_s+C_x}\right)\left(\frac{\sigma_z\bar{Z}'_s+C_z}{\sigma_x\bar{z}'_s+C_z}\right)$	$t'^{56}_{23}=\lambda'_{23}\bar{y}'_s\left(\frac{\sigma_x\bar{X}'_s+C_x}{\sigma_x\bar{x}'_s+C_x}\right)\left(\frac{\sigma_z\bar{z}'_s+C_z}{\sigma_z\bar{Z}'_s+C_z}\right)$	σ_x	C_x	1	σ_z	C_z

The Bias and Mean Square Error of proposed Generalized Estimator IV

In order to find bias and mean square error of (4.3.40), we use notations from (4.1.1) and express the estimator given in(4.3.40) in form of $e's$ as:

$$t'^{G}_{23} = \lambda'_{23}\bar{Y}'_s(1+e'_0)\left(1+a'_{23}\left(\frac{c'_{23}\bar{X}'_s e'_1}{c'_{23}\bar{X}_s'+d'_{23}}\right)\right)^{-\alpha'_4}\left(1+b'_{23}\left(\frac{k'_{23}\bar{Z}_s e'_2}{k'_{23}\bar{Z}_s'+l'_{23}}\right)\right)^{-\beta'_4} \tag{4.3.41}$$

$$t'^{G}_{23} = \lambda'_{23}\bar{Y}'_s(1+e_0)(1+a'_{23}\upsilon'_{23}e'_1)^{-\alpha'_{23}}(1+b'_{23}\omega'_{23}e'_2)^{-\beta'_{23}}$$

where $\upsilon'_{23} = \dfrac{c'_{23}\bar{X}'_s}{c'_{23}\bar{X}_s'+d'_{23}}, \omega'_{23} = \dfrac{k'_{23}\bar{Z}_s}{k'_{23}\bar{Z}_s'+l'_{23}}$ (4.3.42)

We assume that $|e'_1|<1$ and $|e'_2|<1$ so that we may expand the series of $(1+a'_{23}\upsilon'_{23}e'_1)^{-\alpha_{23}}$ and $(1+b'_{23}\omega'_{23}e'_{23})^{-\beta_{23}}$ we ignore the terms in e's of order more than one as:

$$\begin{aligned} t'^{G}_{23} - \bar{Y}_s' = \lambda'_{23}\bar{Y}_s'\Big[& e'_0 - \alpha'_{23}a'_{23}\upsilon'_{23}e'_1 + \frac{\alpha'_{23}(\alpha'_{23}+1)}{2}a'^{2}_{23}\upsilon'^{2}_{23}e'^{2}_1 \\ & -\beta'_{23}b'_{23}\omega'_{23}e'_2 + \alpha'_{23}\beta'_{23}a'_{23}b'_{23}\upsilon_{23}\omega'_{23}e'_1e'_2 \\ & + \frac{\beta'_{23}(\beta'_{23}+1)}{2}b'^{2}_{23}\omega'^{2}_{23}e'^{2}_2 - \alpha'_{23}a'_{23}\upsilon_{23}e'_1e'_0 \\ & - \beta'_{23}b'_{23}\omega'_{23}e'_2e'_0\Big] + (\lambda'_{23}-1)\bar{Y}_s' \end{aligned} \tag{4.3.43}$$

In order to get the bias, we take expectation on (4.3.43) and get

$$\begin{aligned} Bias\left(t'^{G}_{23}\right) = \lambda'_{23}\bar{Y}_s'\Big[& \frac{\alpha'_{23}(\alpha'_{23}+1)}{2}a'^{2}_{23}\upsilon'^{2}_{23}V'_{200} + \alpha'_{23}\beta'_{23}a'_{23}b'_{23}\upsilon'_{23}\omega'_{23}V'_{011} \\ & + \frac{\beta'_{23}(\beta'_{23}+1)}{2}b'^{2}_{23}\omega'^{2}_{23}V'_{020} \\ & - \alpha'_{23}a'_{23}\upsilon'_{23}V'_{110} - \beta'_{23}b'_{23}\omega'_{23}V'_{101}\Big] + (\lambda'_4-1)\bar{Y}_s' \end{aligned} \tag{4.3.44}$$

We take square on both sides on (4.3.43) and retaining terms in e's upto second order and take expectation, we will get MSE(t'^{G}_{23}) as

$$MSE\left(t_{23}'^{G}\right) = \overline{Y}_s'^2\left[\lambda_{23}'^2\left(V_{020}' + \theta_{23}'^2V_{200}' + \varphi_{23}'V_{002}' - 2\theta_{23}'V_{110}' - 2\varphi_{23}'V_{011}' + 2\theta_{23}'\varphi_{23}'V_{101}'\right) + (\lambda_{23}' - 1)^2\right] \qquad (4.3.45)$$

where $\theta_{23}' = \alpha_{23}'a_{23}'\upsilon_{23}'$, and $\varphi_{23}' = \beta_{23}'b_{23}'\omega_{23}'$

For the following optimal values of the constants θ_4' and φ_3', we achieve the minimum MSE among the class of proposed generalized estimator

$$\theta_{23}' = \frac{V_{110}'V_{002}' - V_{011}'V_{101}'}{V_{200}'V_{002}' - V_{101}'^2} \; and \; \varphi_{23}' = \frac{V_{200}'V_{011}' - V_{110}'V_{101}'}{V_{200}'V_{002}' - V_{101}'^2}, \lambda_{23}' = \frac{1}{1 + A_{23}'^G}$$

where

$$A_{23}'^G = \left[V_{020}' + \theta_{23}'^2V_{200}' + \varphi_{23}'^2V_{002}' - 2\theta_{23}'V_{110}' - 2\varphi_{23}'V_{011}' + 2\theta_{23}'\varphi_{23}'V_{101}'\right] \qquad (4.3.46)$$

By substituting the optimum values of θ_4' and φ_3' in (4.3.45), we get $\lambda_3'^{opt}$ as

$$\lambda_{23}'^{opt} = \frac{1}{1 + A_{23}'^*} \; where \; A_{23}'^* = \left(V_{020}' - \frac{V_{110}'^2V_{002}' + V_{011}'^2V_{200}' - 2V_{110}'V_{101}'V_{011}'}{V_{200}'V_{002}' - V_{101}'^2}\right) \qquad (4.3.47)$$

The min MSE may be obtained as by substituting (4.3.46),(4.3.47) in (4.3.45)

$$MSE_{\min}\left(t_{23}'^G\right) = \overline{Y}_s'^2\left(\frac{A_{23}'^*}{1 + A_{23}^*}\right) \qquad (4.3.48)$$

On substituting the optimal value $\lambda_{23}'^{opt}$ and $a_{23}'^{opt}, b_{23}'^{opt}$ in (4.3.40), we get optimal estimator as:

$$t_{23}'^{G\,opt} = \lambda_{23}'^{opt}\overline{y}_s'\left(\frac{c_{23}'\overline{X}_s' + d_{23}'}{\hat{a}_{23}'^{opt}\left(c_{23}'\overline{x}_s' + d_{23}'\right) + (1 - a_{23}'^{opt})\left(c_{23}'\overline{X}_s' + d_{23}'\right)}\right)^{\alpha_{23}'}$$
$$\left(\frac{k_{23}'\overline{Z}_s' + l_{23}'}{b_{23}'^{opt}\left(k_{23}'\overline{z}_s' + l_{23}'\right) + (1 - b_{23}'^{opt})\left(k_{23}'\overline{Z}_s' + l_{23}'\right)}\right)^{\beta_{23}'} \qquad (4.3.49)$$

In some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as it is discussed in previous section.

$$\hat{\lambda}'^{\,opt}_{23} = \frac{1}{1+\hat{A}'^{*}_{23}} \; where \; \hat{A}'^{*}_{23} = \left(\hat{V}'_{020} - \frac{\hat{V}'^{2}_{110}\hat{V}'_{002} + \hat{V}'^{2}_{011}\hat{V}'_{200} - 2\hat{V}'_{110}\hat{V}'_{101}\hat{V}'_{011}}{\hat{V}'_{200}\hat{V}'_{002} - \hat{V}'^{2}_{101}} \right) \tag{4.3.50}$$

So (4.3.50) may be wriiten as

$$\hat{t}'^{G\,opt}_{23} = \lambda'^{\,opt}_{23}\,\overline{y}'_s \left(\frac{c'_{23}\overline{X}'_s + d'_{23}}{\hat{a}'^{opt}_{23}\left(c'_{23}\overline{x}'_s + d'_{23}\right) + (1-\hat{a}'^{opt}_{23})\left(c'_{23}\overline{X}'_s + d'_{23}\right)} \right)^{\alpha'_{23}} \left(\frac{k'_{23}\overline{Z}'_s + l'_4}{\hat{b}'^{\,opt}_{23}\left(k'_{23}\overline{z}'_s + l'_{23}\right) + (1-\hat{b}'^{opt}_{23})\left(k'_{23}\overline{Z}'_s + l'_{23}\right)} \right)^{\beta'_{23}} \tag{4.3.51}$$

Also the minimum MSE may be written as:

$$MSE_{\min}\left(\hat{t}^{G}_{23}\right) = AsymtoticVar\left(\hat{t}^{G}_{23}\right) = \overline{Y}'^{2}_{s}\left(\frac{\hat{A}'^{*}_{23}}{1+\hat{A}'^{*}_{23}} \right) \tag{4.3.52}$$

Remark 4.7

i) For $\alpha'_{23} = 1, \beta'_{23} = 1$, the mean square error for the estimators given in Table 4.4 can be expressed as,

$$MSE(t'^{j}_{23}) = \begin{cases} \overline{Y}'^{2}_{s}\left(\lambda'^{2}_{23}\left(V'_{020} + a'^{2}_{23}V'_{200} + b'^{2}_{23}V'_{002} - 2a'_{23}V'_{110} - 2b'_{23}V'_{011} + 2a'_{23}b'_{23}V'_{101}\right) + (\lambda'_{23}-1)^2\right) & j(\in G) = 1 \\ \overline{Y}'^{2}_{s}\left(\lambda'^{2}_{23} \left(\begin{array}{c} V'_{020} + a'^{2}_{23}\upsilon'^{\left(\frac{j-1}{2}\right)^2}_{23} V'_{200} + b'^{2}_{23}\omega'^{\left(\frac{j-1}{2}\right)^2}_{23} V'_{002} \\ -2a'_{23}\upsilon'^{\left(\frac{j-1}{2}\right)}_{23} V'_{110} - 2b'_{23}\omega'^{\left(\frac{j-1}{2}\right)}_{23} V'_{011} \\ +2a'_{23}b'_{23}\upsilon'^{\left(\frac{j-1}{2}\right)}_{23}\omega'^{\left(\frac{j-1}{2}\right)^2}_{23} V'_{101} \end{array} \right) + (\lambda'_{23}-1)^2 \right) & j(\in G) = 3,5,\ldots,27 \end{cases} \tag{4.3.53}$$

The min MSE may be obtain for the following optimal values of the costants

$$a'_{23} = \frac{1}{\upsilon'^{\left(\frac{j-1}{2}\right)}_{23}}\left(\frac{V'_{110}V'_{002} - V'_{011}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}\right)$$

and

$$b'_{23} = \frac{1}{\omega'^{\left(\frac{j-1}{2}\right)}_{23}}\left(\frac{V'_{200}V'_{011} - V'_{110}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}\right), \lambda'_{23} = \frac{1}{1 + A'^{G}_{23}}$$

where

$$A'^{G}_{23} = \left[V'_{020} + a'^{2}_{23}\upsilon'^{\left(\frac{j-1}{2}\right)^2}_{23}V'_{200} + b'_{23}\omega'^{\left(\frac{j-1}{2}\right)^2}_{4}V'_{002} - 2a'_{23}\upsilon'^{\left(\frac{j-1}{2}\right)}_{23}V'_{110}\right.$$

$$\left. -2b'_{23}\omega_{23}^{\left(\frac{j-1}{2}\right)}V'_{011} + 2a'_{23}\upsilon'^{\left(\frac{j-1}{2}\right)}_{23}b'_{23}\omega_{23}^{\left(\frac{j-1}{2}\right)}V'_{101}\right].$$

ii) For $\alpha'_{23} = -1, \beta'_{23} = -1$, the mean square error for the estimators given in Table 4.4 may be expressed as,

$$MSE(t'^{k}_{23}) = \begin{cases} \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{23}\begin{pmatrix} V'_{020} + a'^{2}_{23}V'_{200} + b'^{2}_{23}V'_{002} + 2a'_{23}V'_{110} \\ +2b'_{23}V'_{011} - 2a'_{23}b'_{23}V'_{101}\end{pmatrix} + (\lambda'_{23} - 1)^2\right) & k(\in G) = 2 \\ \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{23}\begin{pmatrix} V'_{020} + a'^{2}_{23}\upsilon'^{\left(\frac{k}{2}\right)^2}_{23}V'_{200} + b'^{2}_{23}\omega'^{\left(\frac{k}{2}\right)^2}_{23}V'_{002} \\ +2a'_{23}\upsilon'^{\left(\frac{k}{2}\right)}_{23}V'_{110} + 2b'_{23}\omega'^{\left(\frac{k}{2}\right)}_{23}V'_{011} \\ -2a'_{23}b'_{23}\upsilon'^{\left(\frac{k}{2}\right)}_{23}\omega'^{\left(\frac{k}{2}\right)}_{23}V'_{101}\end{pmatrix} + (\lambda'_{23} - 1)^2\right) & k(\in G) = 4, 6, \ldots, 28 \end{cases}$$

(4.3.54)

The min MSE may be obtain for the following optimal values of the constants

$$a'_{23} = \frac{1}{\upsilon'^{\left(\frac{j-1}{2}\right)}_{23}}\left(\frac{V'_{110}V'_{002} - V'_{011}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}\right)$$

and

$$b'_{23} = \frac{1}{\omega'^{\left(\frac{j-1}{2}\right)}_{23}}\left(\frac{V'_{200}V'_{011} - V'_{110}V'_{101}}{V'_{200}V'_{002} - V'^{2}_{101}}\right), \lambda'_{23} = \frac{1}{1 + A'^{G}_{23}}$$

where

$$A_{23}^{\prime G} = \left[V'_{020} + a'^{2}_{23} \upsilon'^{\left(\frac{j-1}{2}\right)^2}_{23} V'_{200} + b'_{23} \omega'^{\left(\frac{j-1}{2}\right)^2}_{23} V'_{002} - 2a'_{23} \upsilon'^{\left(\frac{j-1}{2}\right)}_{23} V'_{110} \right.$$

$$\left. -2b'_{23} \omega_{23}^{\left(\frac{j-1}{2}\right)} V'_{011} + 2a'_{23} \upsilon'^{\left(\frac{j-1}{2}\right)}_{23} b'_{23} \omega_{23}^{\left(\frac{j-1}{2}\right)} V'_{101} \right].$$

iii) For $\alpha'_{23} = -1, \beta'_{23} = 1$, the mean square error for the estimators given in Table 4.4 can be expressed as,

$$MSE(t'^{l}_{23}) = \begin{cases} \bar{Y}'^{2}_{s} \left(\lambda'^{2}_{23} \left(\begin{array}{c} V'_{020} + a'^{2}_{23} V'_{200} + b'^{2}_{23} V'_{002} - 2a'_{23} V'_{110} \\ -2b'_{23} V'_{011} + 2a'_{23} b'_{23} V'_{101} \end{array} \right) + (\lambda'_{23} - 1)^2 \right) & l(\in G) = 1 \\ \bar{Y}'^{2}_{s} \left(\lambda'^{2}_{23} \left(\begin{array}{c} V'_{020} + a'^{2}_{23} \upsilon'^{\left(\frac{l-1}{2}\right)^2}_{23} V'_{200} + b'^{2}_{23} \omega'^{\left(\frac{l-1}{2}\right)^2}_{23} V'_{002} \\ -2a'_{23} \upsilon'^{\left(\frac{l-1}{2}\right)}_{23} V'_{110} - 2b'_{23} \omega'^{\left(\frac{l-1}{2}\right)}_{23} V'_{011} \\ +2a'_{23} b'_{23} \upsilon'^{\left(\frac{l-1}{2}\right)}_{23} \omega'^{\left(\frac{l-1}{2}\right)^2}_{23} V'_{101} \end{array} \right) + (\lambda'_{23} - 1)^2 \right) & l(\in G) = 3, 5, \ldots, 27 \end{cases}$$

(4.3.55)

For the following optimal values , we get minimum MSE as,

$$a'_{23} = \frac{1}{\upsilon'^{\left(\frac{l-1}{2}\right)}_{23}} \left(\frac{V'_{110} V'_{002} - V'_{011} V'_{101}}{V'_{200} V'_{002} - V'^{2}_{101}} \right)$$

and

$$b'_{23} = \frac{1}{\omega'^{\left(\frac{l-1}{2}\right)}_{23}} \left(\frac{V'_{200} V'_{011} - V'_{110} V'_{101}}{V'_{200} V'_{002} - V'^{2}_{101}} \right), \qquad \lambda'_{23} = \frac{1}{1 + A'^{G}_{23}}$$

where

$$A_{23}^{\prime G} = \left[\begin{array}{l} V'_{020} + a'^{2}_{23} \upsilon'^{\left(\frac{l-1}{2}\right)^2}_{23} V'_{200} + b'_{23} \omega'^{\left(\frac{l-1}{2}\right)^2}_{23} V'_{002} - 2a'_{23} \upsilon'^{\left(\frac{l-1}{2}\right)}_{23} V'_{110} \\ -2b'_{23} \omega'^{\left(\frac{l-1}{2}\right)}_{23} V'_{011} + 2a'_{23} \upsilon'^{\left(\frac{l-1}{2}\right)}_{23} b'_{23} \omega'^{\left(\frac{l-1}{2}\right)}_{23} V'_{101} \end{array} \right]$$

iv) For $\alpha'_{23} = -1, \beta'_{23} = 1$, the mean square error for the estimators given in Table 4.4 can be expressed as

$$MSE(t'^{k}_{23}) = \begin{cases} \bar{Y}_s'^2 \left(\lambda'^{2}_{23} \left(\begin{array}{l} V'_{020} + a'^{2}_{23} V'_{200} + b'^{2}_{23} V'_{002} + 2a'_{23} V'_{110} \\ -2b'_{23} V'_{011} + 2a'_{23} b'_{23} V'_{101} \end{array} \right) + (\lambda'_{23} - 1)^2 \right) & m(\in G) = 30 \\ \bar{Y}_s'^2 \left(\lambda'^{2}_{23} \left(\begin{array}{l} V'_{020} + a'^{2}_{23} \upsilon'^{\left(\frac{m}{2}\right)^2}_{23} V'_{200} + b'^{2}_{23} \omega'^{\left(\frac{m}{2}\right)^2}_{23} V'_{002} \\ +2a'_{23} \upsilon'^{\left(\frac{m}{2}\right)}_{23} V'_{110} - 2b'_{23} \omega'^{\left(\frac{m}{2}\right)}_{23} V'_{011} \\ +2a'_{23} b'_{23} \upsilon'^{\left(\frac{m}{2}\right)}_{23} \omega'^{\left(\frac{m}{2}\right)}_{23} V'_{101} \end{array} \right) + (\lambda'_{23} - 1)^2 \right) & m(\in G) = 32, 34, \ldots, 56 \end{cases}$$

(4.3.56)

The $MSE(t'^{k}_{23})$ in (4.3.56) is minimum for

$$a'_{23} = \frac{1}{\upsilon'^{\left(\frac{m}{2}\right)}_{23}} \left(\frac{V'_{110} V'_{002} - V'_{011} V'_{101}}{V'_{200} V'_{002} - V'^{2}_{101}} \right)$$

and

$$b'_{23} = \frac{1}{\omega'^{\left(\frac{m}{2}\right)}_{23}} \left(\frac{V'_{200} V'_{011} - V'_{110} V'_{101}}{V'_{200} V'_{002} - V'^{2}_{101}} \right), \lambda'_{23} = \frac{1}{1 + A'^{G}_{23}}$$

where

$$A^{G}{}_{23} = \left[\begin{array}{l} V'_{020} + a'^{2}_{23} \upsilon'^{\left(\frac{m}{2}\right)^2}_{4} V'_{200} + b'_{23} \omega'^{\left(\frac{m}{2}\right)^2}_{23} V'_{002} - 2a'_{23} \upsilon'^{\left(\frac{m}{2}\right)}_{23} V'_{110} \\ -2b'_{23} \omega_{23}^{\left(\frac{m}{2}\right)} V'_{011} + 2a'_{23} \upsilon'^{\left(\frac{m}{2}\right)}_{23} b'_{23} \omega_{23}^{\left(\frac{m}{2}\right)} V'_{101} \end{array} \right]$$

where

$$\upsilon'^{1}_{23} = \frac{\bar{X}'_s}{\bar{X}'_s + C_x}, \; \upsilon'^{2}_{23} = \frac{\beta_2(x) \bar{X}'_s}{\beta_2(x) \bar{X}'_s + C_x}, \; \upsilon'^{3}_{23} = \frac{C_x \bar{X}'_s}{C_x \bar{X}'_s + \beta_2(x)},$$

$$\upsilon'^{4}_{23} = \frac{\beta_2(x) \bar{X}'_s}{\beta_2(x) \bar{X}'_s + \sigma_x}, \; \upsilon'^{5}_{23} = \frac{\bar{X}'_s}{\bar{X}'_s + \rho_x}, \; \upsilon'^{6}_{23} = \frac{\bar{X}'_s}{\bar{X}'_s + \beta_2(x)},$$

$$\upsilon'^{7}_{23} = \frac{\sigma_x \bar{X}'_s}{\sigma_x \bar{X}'_s + 1}, \; \upsilon'^{8}_{23} = \frac{\bar{X}'_s}{\sigma_x \bar{X}'_s + \beta_1(x)}, \; \upsilon'^{9}_{23} = \frac{\sigma_x \bar{X}'_s}{\sigma_x \bar{X}'_s + \beta_2(x)},$$

$$\upsilon'^{\,10}_{23} = \frac{\beta_1(x)\bar{X}'_s}{\beta_1(x)\bar{X}'_s + \beta_2(x)},\ \upsilon'^{\,11}_{23} = \frac{\beta_2(x)\bar{X}'_s}{\beta_2(x)\bar{X}'_s + \beta_1(x)},\ \upsilon'^{\,12}_{23} = \frac{\rho\bar{X}'_s}{\rho\bar{X}'_s + 1},$$

$$\upsilon'^{\,13}_{23} = \frac{\sigma_x\bar{X}'_s}{\sigma_x\bar{X}'_s + C_x},\ \omega'^{\,1}_{23} = \frac{\bar{Z}'_s}{\bar{Z}'_s + C_z},\ \omega'^{\,2}_{23} = \frac{\beta_2(z)\bar{Z}'_s}{\beta_2(z)\bar{Z}'_s + C_z},$$

$$\omega'^{\,4}_{23} = \frac{C_x\bar{Z}'_s}{C_x\bar{Z}'_s + \beta_2(z)},\ \omega'^{\,4}_{23} = \frac{\beta_2(z)\bar{Z}'_s}{\beta_2(z)\bar{Z}'_s + \sigma_x},\ \omega'^{\,5}_{23} = \frac{\bar{Z}'_s}{\bar{Z}'_s + \rho_z},$$

$$\omega'^{\,6}_{23} = \frac{\bar{Z}'_s}{\bar{Z}'_s + \beta_2(z)},\ \omega'^{\,7}_{23} = \frac{\sigma_z\bar{Z}'_s}{\sigma_z\bar{Z}'_s + 1},\ \omega'^{\,8}_{23} = \frac{\bar{Z}'_s}{\sigma_z\bar{Z}'_s + \beta_1(z)},$$

$$\omega'^{\,9}_{23} = \frac{\sigma_x\bar{Z}'_s}{\sigma_x\bar{Z}'_s + \beta_2(z)},\ \omega'^{\,10}_{23} = \frac{\beta_1(z)\bar{Z}'_s}{\beta_1(z)\bar{Z}'_s + \beta_2(z)},$$

$$\omega'^{\,11}_{23} = \frac{\beta_2(z)\bar{Z}'_s}{\beta_2(z)\bar{Z}'_s + \beta_1(z)},\ \omega'^{\,12}_{23} = \frac{\rho\bar{Z}'_s}{\rho\bar{Z}'_s + 1},\ \omega^{\,13}_{23} = \frac{\sigma_z\bar{Z}'_s}{\sigma_z\bar{Z}'_z + C_z}.$$

Case II:

The generalized estimator under case II may be proposed following (4.3.40) as

$$t''^{G}_{23} = \lambda''_{23}\bar{y}''_s\left(\frac{c''_{23}\bar{X}''_s + d''_{23}}{a''_{23}\left(c''_{23}\bar{x}''_s + d''_{23}\right) + (1 - a''_{23})\left(c''_{23}\bar{X}''_s + d''_{23}\right)}\right)^{\alpha''_4}$$
$$\left(\frac{k''_4\bar{Z}''_s + l''_4}{b''_{23}\left(k'\bar{z}''_s + l''\right) + (1 - b''_4)\left(k''_4\bar{Z}''_s + l''_4\right)}\right)^{\beta''_4} \quad (4.3.57)$$

where the constant $c''_{23}, k''_{23}(\neq 0)$, d''_{23} and l''_{23} are either real numbers or the functions of the auxiliary variable, in form of coefficient of variations ,standard deviations, correlation coefficients, skewness or kutosis from the population. (a''_{23}, b''_{23}, λ''_{23}) are constants to be determined such as mean square error (MSE) is minimum.

The proposed estimator in (4.3.57) follows the same way along with the class of estimator in Table 4, as that for case-I in Section 4.3.4. In addition, the relation between a''_{23}, α''_{23}, λ''_{23} and b''_{23}, β''_{23} in case-II is the same as that for case-I in Section 4.3.4.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 4.3.1, we may write Bias (t''^{G}_{23}) and MSE (t''^{G}_{23}) following the same, and we may also produce a class of estimators for similar choices of a''_{23}, α''_{23} and λ''_{23} in case-II. The bias of(4.3.30) may be obtain by following the notations and expectations for case II presented in Section 4.1,

$$Bias\left(t''^{G}_{23}\right) = \lambda''_{23}\bar{Y}''_s\left[\frac{\alpha''_{23}(\alpha''_{23}+1)}{2}a''^{2}_{23}\upsilon''^{2}_{23}V''_{200} + \alpha''_{23}\beta''_{23}a''_{23}b''_{23}\upsilon''_{23}\omega''_{23}V''_{011}\right.$$

$$+\frac{\beta''_{23}(\beta''_{23}+1)}{2}b''^{2}_{23}\omega''^{2}_{23}V''_{020} - \alpha''_{23}a''_{23}\upsilon''_{23}V''_{110}$$

$$\left. - \beta''_{23}b''_{23}\omega''_{23}V''_{101}\right] + (\lambda''_4 - 1)\bar{Y}''_s \qquad (4.3.58)$$

The expression of MSE is also given as

$$MSE\left(t''^{G}_{23}\right) = \bar{Y}''^{2}_s\left[\lambda''^{2}_{23}\left(V''_{020} + \theta''^{2}_{23}V''_{200} + \varphi''_{23}V''_{002} - 2\theta''_{23}V''_{110}\right.\right.$$

$$\left.\left. -2\varphi''_{23}V''_{011} + 2\theta''_{23}\varphi''_{23}V''_{101}\right) + (\lambda''_{23}-1)^2\right]$$

where

$$\theta''_{23} = \alpha''_{23}a''_{23}\upsilon''_{23}, \varphi''_{23} = \beta''_{23}b''_{23}\omega''_{23} \qquad (4.3.59)$$

The $MSE\left(t''^{G}_{23}\right)$ will be minimum for

$$\lambda''^{\,opt}_{23} = \frac{1}{1+A''^{*}_{23}} \; where \; A''^{*}_{23} = \left(V''_{020} - \frac{V''^{2}_{110}V''_{002} + V''^{2}_{011}V''_{200} - 2V''_{110}V''_{101}V''_{011}}{V''_{200}V''_{002} - V''^{2}_{101}}\right)$$

(4.3.60)

The minimum MSE may be obtained as

$$MSE_{\min}\left(t''^{G}_{23}\right) = \bar{Y}''^{2}_s\left(\frac{A''^{*}_{23}}{1+A''^{*}_{23}}\right) \qquad (4.3.61)$$

Remark 4.8

i) For $\alpha''_{23} = 1, \beta''_{23} = 1$, the mean square error for the estimators given in Table 4 can be expressed as

$$MSE(t''^{j}_{23}) = \begin{cases} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{23}\begin{pmatrix} V''_{020} + a''^{2}_{23}V''_{200} + b''^{2}_{23}V''_{002} - 2a''_{23}V''_{110} \\ -2b''_{23}V''_{011} + 2a''_{23}b''_{23}V''_{101} \end{pmatrix} + (\lambda''_{23} - 1)^2\right) & j(\in G) = 1 \\ \bar{Y}'^{2}_{s}\left(\lambda''^{2}_{23}\begin{pmatrix} V''_{020} + a''^{2}_{23}\upsilon''^{\left(\frac{j-1}{2}\right)^2}_{423}V''_{200} + b'^{\prime 2}_{23}\omega''^{\left(\frac{j-1}{2}\right)^2}_{23}V''_{002} \\ -2a''_{23}\upsilon''^{\left(\frac{j-1}{2}\right)}_{23}V''_{110} - 2b''_{23}\omega''^{\left(\frac{j-1}{2}\right)}_{23}V''_{011} \\ +2a''_{23}b''_{23}\upsilon''^{\left(\frac{j-1}{2}\right)}_{23}\omega''^{\left(\frac{j-1}{2}\right)^2}_{23}V''_{101} \end{pmatrix} + (\lambda''_{23} - 1)^2\right) & j(\in G) = 3,5,..,27 \end{cases}$$

(4.3.62)

The min MSE may be obtain for the following optimal values of the costants

$$a''_{23} = \frac{1}{\upsilon''^{\left(\frac{j-1}{2}\right)}_{23}}\left(\frac{V''_{110}V''_{002} - V''_{011}V''_{101}}{V''_{200}V''_{002} - V''^{2}_{101}}\right)$$

and

$$b''_{23} = \frac{1}{\omega''^{\left(\frac{j-1}{2}\right)}_{23}}\left(\frac{V''_{200}V''_{011} - V''_{110}V''_{101}}{V''_{200}V''_{002} - V''^{2}_{101}}\right), \lambda''_{23} = \frac{1}{1 + A''^{G}_{23}}$$

where

$$A''^{G}_{23} = \begin{bmatrix} V''_{020} + a''^{2}_{23}\upsilon''^{\left(\frac{j-1}{2}\right)^2}_{23}V''_{200} + b''_{23}\omega''^{\left(\frac{j-1}{2}\right)^2}_{23}V''_{002} - 2a''_{23}\upsilon''^{\left(\frac{j-1}{2}\right)}_{23}V''_{110} \\ -2b''_{23}\omega''^{\left(\frac{j-1}{2}\right)}_{23}V''_{011} + 2a''_{23}\upsilon''^{\left(\frac{j-1}{2}\right)}_{23}b''_{23}\omega''^{\left(\frac{j-1}{2}\right)}_{23}V''_{101} \end{bmatrix}.$$

ii) For $\alpha''_{23} = -1, \beta''_{23} = -1$, the mean square error for the estimators given in Table 4.4 can be expressed as,

$$MSE(t''^{k}_{23}) = \begin{cases} \bar{Y}''^{2}_{s}\left(\lambda'^{\prime 2}_{23}\begin{pmatrix} V''_{020} + a''^{2}_{23}V''_{200} + b''^{2}_{23}V''_{002} \\ + 2a''_{23}V''_{110} + 2b''_{23}V''_{011} - 2a''_{23}b''_{23}V''_{101} \end{pmatrix} + (\lambda''_{23} - 1)^2\right) & k(\in G) = 2 \\ \bar{Y}'^{2}_{s}\left(\lambda''^{2}_{23}\begin{pmatrix} V''_{020} + a''^{2}_{23}\upsilon''^{\left(\frac{k}{2}\right)^2}_{23}V''_{200} + b''^{2}_{23}\omega'^{\prime\left(\frac{k}{2}\right)^2}_{23}V''_{002} \\ +2a''_{23}\upsilon'^{\prime\left(\frac{k}{2}\right)}_{23}V''_{110} + 2b''_{23}\omega''^{\left(\frac{k}{2}\right)}_{23}V''_{011} \\ -2a''_{23}b''_{23}\upsilon''^{\left(\frac{k}{2}\right)}_{23}\omega''^{\left(\frac{k}{2}\right)}_{23}V''_{101} \end{pmatrix} + (\lambda''_{23} - 1)^2\right) & k(\in G) = 4,6,..,28 \end{cases}$$

(4.3.63)

iii) For $\alpha''_4 = -1, \beta'_4{}' = 1$, the mean square error for the estimators given in Table 4.4 may be expressed as,

$$MSE(t''^{l}_{23}) = \begin{cases} \bar{Y}_s''^{2}\left(\lambda''^{2}_{23}\begin{pmatrix} V''_{020} + a''^{2}_{23}V''_{200} + b''^{2}_{23}V''_{002} - 2a''_{23}V''_{110} \\ -2b''_{23}V''_{011} + 2a''_{23}b''_{23}V''_{101} \end{pmatrix} + (\lambda''_{23}-1)^2\right) & l(\in G) = 1 \\ \bar{Y}_s''^{2}\left(\lambda''^{2}_{23}\begin{pmatrix} V''_{020} + a''^{2}_{23}\upsilon''^{\left(\frac{l-1}{2}\right)^2}_{23}V''_{200} + b''^{2}_{23}\omega''^{\left(\frac{l-1}{2}\right)^2}_{23}V''_{002} \\ -2a''_{23}\upsilon''^{\left(\frac{l-1}{2}\right)}_{23}V''_{110} - 2b''_{23}\omega'^{\,\prime\left(\frac{l-1}{2}\right)}_{23}V''_{011} \\ +2a''_{23}b''_{23}\upsilon''^{\left(\frac{l-1}{2}\right)}_{23}\omega''^{\left(\frac{l-1}{2}\right)^2}_{23}V''_{101} \end{pmatrix} + (\lambda''_{23}-1)^2\right) & l(\in G) = 3,5,\ldots,27 \end{cases}$$

(4.3.64)

For the following optimal values , we get minimum MSE as

$$a''_{23} = \frac{1}{\upsilon''^{\left(\frac{l-1}{2}\right)}_{23}}\left(\frac{V''_{110}V''_{002} - V''_{011}V''_{101}}{V''_{200}V''_{002} - V''^{2}_{101}}\right)$$

and

$$b''_{23} = \frac{1}{\omega''^{\left(\frac{l-1}{2}\right)}_{23}}\left(\frac{V''_{200}V''_{011} - V''_{110}V''_{101}}{V''_{200}V''_{002} - V''^{2}_{101}}\right), \lambda''_{23} = \frac{1}{1 + A''^{G}_{23}}$$

where

$$A''^{G}_{23} = \begin{bmatrix} V''_{020} + a''^{2}_{23}\upsilon''^{\left(\frac{l-1}{2}\right)^2}_{23}V''_{200} + b''_{23}\omega''^{\left(\frac{l-1}{2}\right)^2}_{23}V''_{002} - 2a''_{23}\upsilon''^{\left(\frac{l-1}{2}\right)}_{23}V''_{110} \\ -2b''_{23}\omega''^{\left(\frac{l-1}{2}\right)}_{23}V''_{011} + 2a''_{23}\upsilon''^{\left(\frac{l-1}{2}\right)}_{23}b''_{23}\omega''^{\left(\frac{l-1}{2}\right)}_{23}V''_{101} \end{bmatrix}$$

iv) For $\alpha''_{23} = 1, \beta''_{23} = -1$, the mean square error for the estimators given in Table 4.4 can be expressed as

$$MSE(t_{23}''^{k}) = \begin{cases} \bar{Y}_s'^{\prime 2}\left(\lambda_{23}''^{2}\begin{pmatrix} V_{020}'' + a_{23}''^{2}V_{200}'' + b_{23}''^{2}V_{002}'' + 2a_{23}''V_{110}'' \\ -2b_{23}''V_{011}'' + 2a_{23}''b_{23}''V_{101}'' \end{pmatrix} + (\lambda_{23}'' - 1)^2\right) & m(\in G) = 30 \\ \bar{Y}_s'^{\prime 2}\left(\lambda_{23}''^{2}\begin{pmatrix} V_{020}'' + a_{23}''^{2}\upsilon_{23}''^{\left(\frac{m}{2}\right)^2}V_{200}'' + b_{23}''^{2}\omega_{23}'^{\prime\left(\frac{m}{2}\right)^2}V_{002}'' \\ -2a_{23}''\upsilon_{23}'^{\prime\left(\frac{m}{2}\right)}V_{110}'' + 2b_{23}''\omega_{23}''^{\left(\frac{m}{2}\right)}V_{011}'' \\ +2a_{23}''b_{23}''\upsilon_{23}''^{\left(\frac{m}{2}\right)}\omega_{23}''^{\left(\frac{m}{2}\right)}V_{101}'' \end{pmatrix} + (\lambda_{23}'' - 1)^2\right) & m(\in G) = 32, 34, .., 56 \end{cases}$$

(4.3.65)

where

$$\upsilon_{23}''^{1} = \frac{\bar{X}_s''}{\bar{X}_s'' + C_x},\ \upsilon_{23}''^{2} = \frac{\beta_2(x)\bar{X}_s''}{\beta_2(x)\bar{X}_s'' + C_x},\ \upsilon_{23}''^{3} = \frac{C_x\bar{X}_s''}{C_x\bar{X}_s'' + \beta_2(x)},$$

$$\upsilon_{23}''^{4} = \frac{\beta_2(x)\bar{X}_s''}{\beta_2(x)\bar{X}_s'' + \sigma_x},\ \upsilon_{23}''^{5} = \frac{\bar{X}_s''}{\bar{X}_s'' + \rho_x},\ \upsilon_{23}''^{6} = \frac{\bar{X}_s''}{\bar{X}_s'' + \beta_2(x)},$$

$$\upsilon_{23}''^{7} = \frac{\sigma_x\bar{X}_s''}{\sigma_x\bar{X}_s'' + 1},\ \upsilon_{23}''^{8} = \frac{\bar{X}_s''}{\sigma_x\bar{X}_s'' + \beta_1(x)},\ \upsilon_{23}''^{9} = \frac{\sigma_x\bar{X}_s''}{\sigma_x\bar{X}_s'' + \beta_2(x)}$$

$$\upsilon_{23}''^{10} = \frac{\beta_1(x)\bar{X}_s''}{\beta_1(x)\bar{X}_s'' + \beta_2(x)},\ \upsilon_{23}''^{11} = \frac{\beta_2(x)\bar{X}_s''}{\beta_2(x)\bar{X}_s'' + \beta_1(x)},\ \upsilon_{23}''^{12} = \frac{\rho\bar{X}_s''}{\rho\bar{X}_s'' + 1},$$

$$\upsilon_{23}''^{13} = \frac{\sigma_x\bar{X}_s''}{\sigma_x\bar{X}_s'' + C_x}.$$

$$\omega_{23}''^{1} = \frac{\bar{Z}_s''}{\bar{Z}_s'' + C_z},\ \omega_{23}''^{2} = \frac{\beta_2(z)\bar{Z}_s''}{\beta_2(z)\bar{Z}_s'' + C_z},\ \omega_{23}''^{4} = \frac{C_x\bar{Z}_s''}{C_x\bar{Z}_s' + \beta_2(z)},$$

$$\omega_{23}''^{4} = \frac{\beta_2(z)\bar{Z}_s'}{\beta_2(z)\bar{Z}_s' + \sigma_x},\ \omega_{23}''^{5} = \frac{\bar{Z}_s'}{\bar{Z}_s' + \rho_z},\ \omega_{23}''^{6} = \frac{\bar{Z}_s'}{\bar{Z}_s' + \beta_2(z)},$$

$$\omega_{23}''^{7} = \frac{\sigma_z\bar{Z}_s'}{\sigma_z\bar{Z}_s' + 1},\ \omega_{23}''^{8} = \frac{\bar{Z}_s'}{\sigma_z\bar{Z}_s' + \beta_1(z)},\ \omega_{23}''^{9} = \frac{\sigma_z\bar{Z}_s'}{\sigma_z\bar{Z}_s' + \beta_2(z)},$$

$$\omega_{23}''^{10} = \frac{\beta_1(z)\bar{Z}_s'}{\beta_1(z)\bar{Z}_s' + \beta_2(z)},\ \omega_{23}''^{11} = \frac{\beta_2(z)\bar{Z}_s'}{\beta_2(z)\bar{Z}_s' + \beta_1(z)},\ \omega_{23}''^{12} = \frac{\rho\bar{Z}_s'}{\rho\bar{Z}_s' + 1},$$

$$\omega_{23}''^{13} = \frac{\sigma_z\bar{Z}_s'}{\sigma_z\bar{Z}_z' + C_z}$$

For the following optimal values , we get minimum MSE as

$$a_{23}'' = \frac{1}{\upsilon_{23}''^{\left(\frac{m}{2}\right)}}\left(\frac{V_{110}''V_{002}'' - V_{011}''V_{101}''}{V_{200}''V_{002}'' - V_{101}''^{2}}\right)$$

and

$$b''_{23} = \frac{1}{\omega''^{\left(\frac{m}{2}\right)}_{23}} \left(\frac{V''_{200}V''_{011} - V''_{110}V''_{101}}{V''_{200}V''_{002} - V''^{2}_{101}} \right), \lambda''_{23} = \frac{1}{1 + A''^{G}_{23}}$$

where

$$A''^{G}_{23} = \begin{bmatrix} V''_{020} + a''^{2}_{23}\upsilon''^{\left(\frac{m}{2}\right)^2}_{23} V''_{200} + b''_{23}\omega''^{\left(\frac{m}{2}\right)^2}_{23} V''_{002} - 2a''_{23}\upsilon''^{\left(\frac{m}{2}\right)}_{23} V''_{110} \\ -2b''_{23}\omega''^{\left(\frac{m}{2}\right)}_{23} V''_{101} + 2a''_{23}\upsilon''^{\left(\frac{m}{2}\right)}_{23} b'_{23}\omega''^{\left(\frac{m}{2}\right)}_{23} V''_{011} \end{bmatrix}$$

Case III:

The generalized estimator under case II may be proposed following (4.3.40) as

$$t^{G}_{23} = \lambda_{23}\bar{y}_s \left(\frac{c_{23}\bar{X}_s + d_{23}}{a_{23}\left(c_{23}\bar{x}_s + d_{23}\right) + (1 - a_{23})\left(c_{23}\bar{X}_s + d_{23}\right)} \right)^{\alpha_{23}}$$

$$\left(\frac{k_{23}\bar{Z}_s + l_{23}}{b_{23}\left(k_{23}\bar{z}_s + l_{23}\right) + (1 - b_{23})\left(k_{23}\bar{Z}_s + l_{23}\right)} \right)^{\beta_{23}} \quad (4.3.66)$$

where the constant $c_{23}, k_{23}(\neq 0)$, d_{23} and l_{23} are either real numbers or the functions of the auxiliary variable, in form of coefficient of variations ,standard deviations, correlation coefficients, skewness or kutosis from the population. (a_{23}, b_{23}, λ_{23}) are constants to be determined such as mean square error (MSE) is minimum.

The proposed estimator in (4.3.66) follows the same steps along with the class of estimator as in Table 3, as that for case-I in Section 4.3.4, In addition, the relation between a_4, α_4, λ_4 and b_4, β_4 in case-III is the same as that for case-I in Section 4.3.1.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 4.3.4.1, we may write $\text{Bias}(t^{G}_{4})$ and $\text{MSE}(t^{G}_{4})$ following the same, and we may also produce a class of estimators for similar choices of a_4, α_4, λ_4 and b_4, β_4 in case-III. The bias of (4.3.39) may be obtain by following the notations and expectations for case III presented in Section 4.1.

The expressions of bias and MSE are given as:

$$Bias\left(t_{23}^{G}\right)=\lambda_{23}\overline{Y}_s\left[\frac{\alpha_{23}(\alpha_{23}+1)}{2}a_{23}{}^2\upsilon_{23}{}^2V_{200}+\alpha_{23}\beta_{23}a_{23}b_{23}\upsilon_{23}\omega_{23}V_{011}\right.$$
$$+\frac{\beta_{23}(\beta_{23}+1)}{2}b_{23}{}^2\omega_{23}{}^2V_{020}-\alpha_{23}a_{23}\upsilon_{23}V_{110}$$
$$\left.-\beta_{23}b_{23}\omega_{23}V_{101}\right]+\left(\lambda_{23}-1\right)\overline{Y}_s. \tag{4.3.67}$$

$$MSE\left(t_{23}^{G}\right)=\overline{Y}_s^2\left(\lambda_{23}^2\left(\begin{array}{c}\lambda_{23}{}^2V_{020}+\theta_{23}{}^2V_{200}+\varphi_{23}V_{002}-2\theta_{23}V_{110}\\-2\varphi_{23}V_{011}+2\theta_{23}\varphi_{23}V_{101}\end{array}\right)+(\lambda_{23}-1)^2\right) \tag{4.3.68}$$

where $\theta_{23}=\alpha_{23}a_{23}\upsilon_{23}$, and $\varphi_{23}=\beta_{23}b_{23}\omega_{23}$

The $MSE\left(t_{23}^{G}\right)$ will be minimum for

$$\lambda_{23}{}^{opt}=\frac{1}{1+A_{23}{}^{*}}\,where\;A_{23}{}^{*}=\left(V_{020}-\frac{V_{110}{}^2V_{002}+V_{011}{}^2V_{200}-2V_{110}V_{101}V_{011}}{V_{200}V_{002}-V_{101}{}^2}\right) \tag{4.3.69}$$

The minimum MSE may be obtained as

$$MSE_{\min}\left(t_{23}^{G}\right)=\overline{Y}_s^2\left(\frac{A_{23}{}^{*}}{1+A_{23}{}^{*}}\right) \tag{4.3.70}$$

Remark 4.9

i) For $\alpha_{23}=1,\beta_{23}=1$, the mean square error for the estimators given in Table 4 can be expressed as

$$MSE(t_{23}^{j})=\begin{cases}\bar{Y}_s^2\left(\lambda_{23}^2\begin{pmatrix}V_{020}+a_{23}^2V_{200}+b_{23}^2V_{002}-2a_{23}V_{110}\\-2b_{23}V_{011}+2a_{23}b_{23}V_{101}\end{pmatrix}+(\lambda_{23}-1)^2\right) & j(\in G)=1\\ \bar{Y}_s^2\left(\lambda_{23}^2\begin{pmatrix}V_{020}+a_{23}^2\upsilon_{23}^{\left(\frac{j-1}{2}\right)^2}V_{200}+b_{23}^2\omega_{23}^{\left(\frac{j-1}{2}\right)^2}V_{002}\\-2a_{23}\upsilon_{23}^{\left(\frac{j-1}{2}\right)}V_{110}-2b_{23}\omega_{23}^{\left(\frac{j-1}{2}\right)}V_{011}\\+2a_{23}b_{23}\upsilon_{23}^{\left(\frac{j-1}{2}\right)}\omega_{23}^{\left(\frac{j-1}{2}\right)^2}V_{101}\end{pmatrix}+(\lambda_{23}-1)^2\right) & j(\in G)=3,5,..,27\end{cases}$$

(4.3.71)

The min MSE may be obtain for the following optimal values of the constants

$$a_{23}=\frac{1}{\upsilon_{23}^{\left(\frac{j-1}{2}\right)}}\left(\frac{V_{110}V_{002}-V_{011}V_{101}}{V_{200}V_{002}-V_{101}^2}\right)$$

and

$$b_{23}=\frac{1}{\omega_{23}^{\left(\frac{j-1}{2}\right)}}\left(\frac{V_{200}V_{011}-V_{110}V_{101}}{V_{200}V_{002}-V_{101}^2}\right),\lambda_{23}=\frac{1}{1+A_{23}^G}$$

where

$$A_{23}^G=\begin{bmatrix}V_{020}+a_{23}^2\upsilon_{23}^{\left(\frac{j-1}{2}\right)^2}V_{200}+b_{23}\omega_{23}^{\left(\frac{j-1}{2}\right)^2}V_{002}-2a_{23}\upsilon_{23}^{\left(\frac{j-1}{2}\right)}V_{110}\\-2b_{23}\omega_{23}^{\left(\frac{j-1}{2}\right)}V_{011}+2a_{23}\upsilon_{23}^{\left(\frac{j-1}{2}\right)}b_{23}\omega_{23}^{\left(\frac{j-1}{2}\right)}V_{101}\end{bmatrix}.$$

ii) For $\alpha_{23}=-1,\beta_{23}=-1$, the mean square error for the estimators given in Table 4.4 can be expressed as

$$MSE(t_{23}^{k})=\begin{cases}\bar{Y}_s^2\left(\lambda_{23}^2\begin{pmatrix}V_{020}+a_{23}^2V_{200}+b_{23}^2V_{002}+2a_{23}V_{110}\\+2b_{23}V_{011}-2a_{23}b_{23}V_{101}\end{pmatrix}+(\lambda_{23}-1)^2\right) & k(\in G)=2\\ \bar{Y}_s^2\left(\lambda_{23}''^2\begin{pmatrix}V_{020}+a_{23}^2\upsilon_{23}^{\left(\frac{k}{2}\right)^2}V_{200}+b_{23}^2\omega_{23}^{\left(\frac{k}{2}\right)^2}V_{002}\\+2a_{23}\upsilon_{23}^{\left(\frac{k}{2}\right)}V_{110}+2b_{23}\omega_{23}^{\left(\frac{k}{2}\right)}V_{011}\\-2a_{23}b_{23}\upsilon_{23}^{\left(\frac{k}{2}\right)}\omega_{23}^{\left(\frac{k}{2}\right)}V_{101}\end{pmatrix}+(\lambda_{23}-1)^2\right) & k(\in G)=4,6,...,28\end{cases}$$

(4.3.72)

The min MSE may be obtain for the following optimal values of the constants

$$a_{23} = \frac{1}{\upsilon_{23}^{\left(\frac{j-1}{2}\right)}}\left(\frac{V_{110}V_{002} - V_{011}V_{101}}{V_{200}V_{002} - V_{101}^{2}}\right)$$

and

$$b_{23} = \frac{1}{\omega_{23}^{\left(\frac{j-1}{2}\right)}}\left(\frac{V_{200}V_{011} - V_{110}V_{101}}{V_{200}V_{002} - V_{101}^{2}}\right), \lambda_{23} = \frac{1}{1 + A_{23}^{G}}$$

where

$$A_{23}^{G} = \begin{bmatrix} V_{020} + a_{23}^{2}\upsilon^{\left(\frac{j-1}{2}\right)^{2}} V_{200} + b_{23}\omega_{4}^{\left(\frac{j-1}{2}\right)^{2}} V_{002} - 2a_{23}\upsilon_{23}^{\left(\frac{j-1}{2}\right)} V_{110} \\ -2b_{23}\omega_{4}^{\left(\frac{j-1}{2}\right)} V_{011} + 2a_{23}\upsilon_{23}^{\left(\frac{j-1}{2}\right)} b_{23}\omega_{23}^{\left(\frac{j-1}{2}\right)} V_{101} \end{bmatrix}$$

iii) For $\alpha_{23} = -1, \beta_{23} = 1$, the mean square error for the estimators given in Table 4 can be expressed as

$$MSE(t_{23}^{l}) = \begin{cases} \bar{Y}_{s}^{2}\left(\lambda_{23}^{2}\begin{pmatrix} V_{020} + a_{23}^{2}V_{200} + b_{23}^{2}V_{002} - 2a_{23}V_{110} \\ -2b_{23}V_{011} + 2a_{23}b_{23}V_{101} \end{pmatrix} + (\lambda_{23} - 1)^{2}\right) & l(\in G) = 1 \\ \bar{Y}_{s}^{2}\left(\lambda_{23}^{2}\begin{pmatrix} V_{020} + a_{23}^{2}\upsilon_{4}^{\left(\frac{l-1}{2}\right)^{2}} V_{200} + b_{23}^{2}\omega_{23}^{\left(\frac{l-1}{2}\right)^{2}} V_{002} \\ - 2a_{23}\upsilon_{23}^{\left(\frac{l-1}{2}\right)} V_{110} - 2b_{23}\omega_{23}^{\left(\frac{l-1}{2}\right)} V_{011} \\ + 2a_{23}b_{23}\upsilon_{23}^{\left(\frac{l-1}{2}\right)}\omega_{23}^{\left(\frac{l-1}{2}\right)^{2}} V_{101} \end{pmatrix} + (\lambda_{23} - 1)^{2}\right) & l(\in G) = 3, 5, .., 27 \end{cases}$$

(4.3.73)

For the following optimal values, we get minimum MSE as

$$a_{23} = \frac{1}{\upsilon_{23}^{\left(\frac{l-1}{2}\right)}}\left(\frac{V_{110}V_{002} - V_{011}V_{101}}{V_{200}V_{002} - V_{101}^{2}}\right)$$

and

$$b_{23} = \frac{1}{\omega_{23}^{\left(\frac{l-1}{2}\right)}}\left(\frac{V_{200}V_{011} - V_{110}V_{101}}{V_{200}V_{002} - V_{101}^{2}}\right), \lambda_{23} \frac{1}{1 + A_{23}^{G}}$$

where

$$A_{23}^{G} = \begin{bmatrix} V_{020} + a_{23}^{2}\upsilon_{23}^{\left(\frac{l-1}{2}\right)^2} V_{200} + b_{23}\omega_{23}^{\left(\frac{l-1}{2}\right)^2} V_{002} \\ -2a_{23}\upsilon_{23}^{\left(\frac{l-1}{2}\right)} V_{110} - 2b_{23}\omega_{23}^{\left(\frac{l-1}{2}\right)} V_{011} \\ +2a_{23}\upsilon_{23}^{\left(\frac{l-1}{2}\right)} b_{23}\omega_{23}^{\left(\frac{l-1}{2}\right)} V_{101} \end{bmatrix}.$$

iv) For $\alpha'_{23} = 1, \beta'_{23} = -1$, the mean square error for the estimators given in Table 4 can be expressed as,

$$MSE(t_{23}^{k}) = \begin{cases} \bar{Y}_s^{2}\left(\lambda_{23}^{2}\left(\begin{array}{c} V_{020} + a_{23}^{2}V_{200} + b_{23}^{2}V_{002} + 2a_{23}V_{110} \\ -2b_{23}V_{011} + 2a_{23}b_{23}V_{101} \end{array}\right) + (\lambda_{23} - 1)^{2}\right) & m(\in G) = 30 \\ \bar{Y}_s^{2}\left(\lambda_{23}^{2}\left(\begin{array}{c} V_{020} + a_{23}^{2}\upsilon_{23}^{\left(\frac{m}{2}\right)^2} V_{200} + b_{23}^{2}\omega_{23}^{\left(\frac{m}{2}\right)^2} V_{002} \\ -2a_{23}\upsilon_{23}^{\left(\frac{m}{2}\right)} V_{110} + 2b_{23}\omega_{23}^{\left(\frac{m}{2}\right)} V_{011} \\ +2a_{23}b_{23}\upsilon_{23}^{\left(\frac{m}{2}\right)}\omega_{23}^{\left(\frac{m}{2}\right)} V_{101} \end{array}\right) + (\lambda_{23} - 1)^{2}\right) & m(\in G) = 32, 34, \ldots, 56 \end{cases}$$

(4.3.74)

For the following optimal values , we get minimum MSE as

$$a_{23} = \frac{1}{\upsilon_{23}^{\left(\frac{m}{2}\right)}}\left(\frac{V_{110}V_{002} - V_{011}V_{101}}{V_{200}V_{002} - V_{101}^{2}}\right)$$

and

$$b_{23} = \frac{1}{\omega_{23}^{\left(\frac{m}{2}\right)}}\left(\frac{V_{200}V_{011} - V_{110}V_{101}}{V_{200}V_{002} - V_{101}^{2}}\right), \lambda_{23} = \frac{1}{1 + A_{23}^{G}}$$

where

$$A_{23}^{G}=\begin{bmatrix} V_{020}+a_{23}^{2}\upsilon_{23}^{\left(\frac{m}{2}\right)^{2}}V_{200}+b_{23}\omega_{23}^{\left(\frac{m}{2}\right)^{2}}V_{002}-2a_{23}\upsilon_{23}^{\left(\frac{m}{2}\right)}V_{110} \\ -2b_{23}\omega_{23}^{\left(\frac{m}{2}\right)}V_{011}+2a_{23}\upsilon_{23}^{\left(\frac{m}{2}\right)}b_{23}\omega_{23}^{\left(\frac{m}{2}\right)}V_{101} \end{bmatrix}$$

where

$$\upsilon_{23}^{1}=\frac{\overline{X}_s}{\overline{X}_s+C_x},\ \upsilon_{23}^{2}=\frac{\beta_2(x)\overline{X}_s}{\beta_2(x)\overline{X}_s+C_x},\ \upsilon_{23}^{3}=\frac{C_x\overline{X}_s}{C_x\overline{X}_s+\beta_2(x)},$$

$$\upsilon_{23}^{4}=\frac{\beta_2(x)\overline{X}_s}{\beta_2(x)\overline{X}_s+\sigma_x},\ \upsilon_{23}^{5}=\frac{\overline{X}_s}{\overline{X}_s+\rho_x},\ \upsilon_{23}^{6}=\frac{\overline{X}_s}{\overline{X}_s+\beta_2(x)},$$

$$\upsilon_{23}^{7}=\frac{\sigma_x\overline{X}_s}{\sigma_x\overline{X}_s+1},\ \upsilon_{23}^{8}=\frac{\overline{X}_s}{\sigma_x\overline{X}_s+\beta_1(x)},\ \upsilon_{23}^{9}=\frac{\sigma_x\overline{X}_s}{\sigma_x\overline{X}_s+\beta_2(x)},$$

$$\upsilon_{23}^{10}=\frac{\beta_1(x)\overline{X}_s}{\beta_1(x)\overline{X}_s+\beta_2(x)},\upsilon_{23}^{11}=\frac{\beta_2(x)\overline{X}_s}{\beta_2(x)\overline{X}_s+\beta_1(x)},\ \upsilon_{23}^{12}=\frac{\rho\overline{X}_s}{\rho\overline{X}_s+1},$$

$$\upsilon_{23}^{13}=\frac{\sigma_x\overline{X}_s}{\sigma_x\overline{X}_s+C_x}.$$

$$\omega_{23}^{1}=\frac{\overline{Z}'_s}{\overline{Z}'_s+C_z},\ \omega_{23}^{2}=\frac{\beta_2(z)\overline{Z}_s}{\beta_2(z)\overline{Z}_s+C_z},\omega_{23}^{3}=\frac{C_x\overline{Z}_s}{C_x\overline{Z}_s+\beta_2(z)},$$

$$\omega_{23}^{4}=\frac{\beta_2(z)\overline{Z}_s}{\beta_2(z)\overline{Z}_s+\sigma_x},\ \omega_{23}^{5}=\frac{\overline{Z}_s}{\overline{Z}_s+\rho_z},\ \omega_{23}^{6}=\frac{\overline{Z}_s}{\overline{Z}_s+\beta_2(z)},$$

$$\omega_{23}^{7}=\frac{\sigma_z\overline{Z}_s}{\sigma_z\overline{Z}_s+1},\omega_{23}^{8}=\frac{\overline{Z}_s}{\sigma_z\overline{Z}_s+\beta_1(z)},\omega_{23}^{9}=\frac{\sigma_x\overline{Z}_s}{\sigma_x\overline{Z}_s+\beta_2(z)},$$

$$\omega_{23}^{10}=\frac{\beta_1(z)\overline{Z}_s}{\beta_1(z)\overline{Z}_s+\beta_2(z)}\ \ \omega_{23}^{11}=\frac{\beta_2(z)\overline{Z}_s}{\beta_2(z)\overline{Z}_s+\beta_1(z)},\ \omega_{23}^{12}=\frac{\rho\overline{Z}_s}{\rho\overline{Z}_s+1},$$

$$\omega_{23}^{13}=\frac{\sigma_z\overline{Z}_s}{\sigma_z\overline{Z}_z+C_z}.$$

4.4 PROPOSED SEPARATE-TYPE GENERALIZED ESTIMATOR USING SINGLE AUXILIARY VARIABLE

In this section, a new form of the proposed generalized estimator t_{24}^{G} has been defined by following the estimator given by Srivastva and Garg (2009). The estimator has been defined under above mentioned three cases.

i) Notations

The notations which are commonly used in two-stage sampling are given as follows:

N	Total number of first stage units(fsu) in the population
n	Total number of first stage units in the sample
M_i	Total number of second stage units belonging to ith fsu in the population
M_0	Total number of second stage units in population
$\bar{M}$	Average size of fsu's
m_i	Total number of second stage units belonging to ith fsu
m_0	Total number of second stage units in the sample
Y	Variable under study
$\bar{Y}_i$	Population mean of ssu's in the ith fsu
$\bar{Y}$	Population mean $= \frac{1}{M_0}\sum_{i=1}^{N}\sum_{j=1}^{M_i} Y_{ij} = \frac{1}{N}\sum_{j=1}^{M_i}\bar{Y}_i$
y_{ij}	Observation on jth ssu belonging to the ith fsu in the sample; $i = 1, 2, \ldots n$ and $j = 1, 2, \ldots m_i$
$\bar{y}_i$	Sample mean of ssu's in ith fsu $= \frac{1}{m_i}\sum_{j=1}^{m_i} y_{ij},$
$\bar{x}_i$	Sample mean of auxiliary variable for ssu in ith fsu
α_i	Weight for ith fsu
S_y^2	Population mean square error for Y variable $= \frac{1}{M_0 - 1}\sum_{i=1}^{N}\sum_{j=1}^{M_i}\left(Y_{ij} - \bar{Y}\right)^2$

S^2_{yi} Population mean square error for Y variable for i^th^ fsu

$$= \frac{1}{M_i - 1}\sum_{j=1}^{M_i}\left(Y_{ij} - \bar{Y}\right)^2$$

$S^2_{x_i}$ Population mean square error of auxiliary variable for i^th^ fsu

$$= \frac{1}{M_i - 1}\sum_{j=1}^{M_i}\left(X_{ij} - \bar{X}_{ij}\right)^2$$

C^2_{yi} Coefficient of variation of Y for i^th^ fsu $= \dfrac{S^2_{yi}}{\bar{Y}^2_i}$

ρ_i Correlation coefficient between the variables Y and auxiliary variable for i^th^ fsu.

b_{ij} Regression coefficient between Y and auxiliary variable for j^th^ ssu in i^th^ fsu in the sample.

B_i Regression coefficient between Y and auxiliary variable for j^th^ ssu in i^th^ fsu in the population.

$$f = \left(\frac{1}{n} - \frac{1}{N}\right), \quad f_i = \left(\frac{1}{m_i} - \frac{1}{M_i}\right)$$

ii) Expectations

$$\bar{y}_i = \bar{Y}_i(e_{\bar{y}_i} + 1), \qquad \bar{x}_i = \bar{X}_i(e_{\bar{x}_i} + 1), \quad i = 1, 2, \ldots, n, \tag{4.4.1}$$

where $e_{\bar{y}i}$ and $e_{\bar{x}i}$ are the sampling error. Further we assume that $E(e_{\bar{y}_i}) = E(e_{\bar{x}_i}) = 0$, and some expectations under two-stage sampling design are obtained in order to obtain the bias and mean square error as,

$$\left.\begin{array}{l} E(e^2_{\bar{y}_i}) = f_{m_i} C^2_{y_i}, E(e^2_{\bar{x}_i}) = f_{m_i} C^2_{x_i}, E(e_{\bar{y}_i} e_{\bar{x}_i}) = f_{m_i} \rho_i C_{y_i} C_{x_i} \\ \text{where} \quad f = \left(\frac{1}{n} - \frac{1}{N}\right), f_{m_i} = \left(\frac{1}{m_i} - \frac{1}{M_i}\right) \end{array}\right\} \tag{4.4.2}$$

4.4.1 Proposed Generalized Estimator V

We have proposed estimator in two-stage sampling design by taking motivation from Srivastva and Garg (2009) under the three different cases in two-stage sampling as mentioned earlier.

Case-I:

We propose a weighted generalized estimator for unequal fsu's in two-stage sampling design as,

$$t'^{G}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^G \ , \tag{4.4.3}$$

where t^{G}_{24} is proposed weighted generalized estimator in two-stage sampling design, α_i is weighting known constant and t_i^G is proposed ratio type estimator for population mean for ssu's belonging to the i^{th} fsu's as,

$$t_i^G = \bar{y}_i \left[\frac{(a_i \bar{X}_i + b_i)}{\beta(a_i \bar{x}_i + b_i) + (1-\beta)\left(a_i \bar{X}_i + b_i\right)} \right]^g \ , \tag{4.4.4}$$

where β and g are assumed to be the unknown constants whose values are to be estimated. $a_i (\neq 0)$, and b_i are assumed to be known as either real numbers or (Linear or Non-linear) functions of some known parameters of auxiliary variable x such as standard deviation (σ_{xi}), coefficient of variation (C_{xi}), skewness $(\beta_{1i}(x))$, kurtosis $(\beta_{2i}(x))$ and correlation coefficient (ρ_i) for ssu's belonging to i^{th} fsu's from the population. For different values of the constants in (4.4.2), we may get different ratio and product type estimators as shown in Table 4.5.

Table 4.5
Some Special Cases of Generalized Estimators of t_i^G

Ratio Estimator g **=1**		**Product Estimator** g **= -1**		a_i	b_i	β
$t'^{1}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^1$	$t_i^1 = \bar{y}_i\left(\frac{\bar{X}_i}{\bar{x}_i}\right)$	$t'^{2}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^2$	$t_i^2 = \bar{y}_i\left(\frac{\bar{x}_i}{\bar{X}_i}\right)$	1	0	1
$t'^{3}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^3$	$t_i^3 = \bar{y}_i\left(\frac{\bar{X}_i + C_{x_i}}{\bar{x}_i + C_{x_i}}\right)$	$t'^{4}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^4$	$t_i^4 = \bar{y}_i\left(\frac{\bar{x}_i + C_{\bar{x}_i}}{\bar{X}_i + C_{\bar{x}_i}}\right)$	1	C_{x_i}	1
$t'^{5}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^5$	$t_i^5 = \bar{y}_i\left(\frac{\beta_2(x_i)\bar{X} + C_{x_i}}{\beta_2(x_i)\bar{x}_i + C_{x_i}}\right)$	$t'^{6}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^6$	$t_i^6 = \bar{y}_i\left(\frac{\beta_2(x_i)\bar{x}_i + C_{x_i}}{\beta_2(x_i)\bar{X}_i + C_{x_i}}\right)$	$\beta_2(x_i)$	C_{x_i}	1
$t'^{7}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^7$	$t_i^7 = \bar{y}_i\left(\frac{C_{x_i}\bar{X}_i + \beta_2(x_i)}{C_{x_i}\bar{x}_i + \beta_2(x_i)}\right)$	$t'^{8}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^8$	$t_i^8 = \bar{y}_i\left(\frac{C_{x_i}\bar{x}_i + \beta_2(x_i)}{C_{x_i}\bar{X}_i + \beta_2(x_i)}\right)$	C_{x_i}	$\beta_2(x_i)$	1
$t'^{9}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^9$	$t_i^9 = \bar{y}_i\left(\frac{\beta_2(x_i)\bar{X} + \sigma_{x_i}}{\beta_2(x_i)\bar{x}_i + \sigma_{x_i}}\right)$	$t'^{10}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{10}$	$t_i^{10} = \bar{y}_i\left(\frac{\beta_2(x_i)\bar{x}_i + \sigma_{x_i}}{\beta_2(x_i)\bar{X}_i + \sigma_{x_i}}\right)$	$\beta_2(x_i)$	σ_{x_i}	1
$t'^{11}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{11}$	$t_i^{11} = \bar{y}_i\left(\frac{\bar{X}_i + \rho_i}{\bar{x}_i + \rho_i}\right)$	$t'^{12}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{12}$	$t_i^{12} = \bar{y}_i\left(\frac{\bar{x}_i + \rho_i}{\bar{X}_i + \rho_i}\right)$	1	ρ_i	1
$t^{13}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{13}$	$t_i^{13} = \bar{y}_i\left(\frac{\bar{X}_i + \beta_2(x_i)}{\bar{x}_i + \beta_2(x_i)}\right)$	$t^{14}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{14}$	$t_i^{14} = \bar{y}_i\left(\frac{\bar{x}_i + \beta_2(x_i)}{\bar{X}_i + \beta_2(x_i)}\right)$	1	$\beta_2(x_i)$	1
$t^{15}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{15}$	$t_i^{15} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{X}_i + 1}{\sigma_{x_i}\bar{x}_i + 1}\right)$	$t^{16}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{16}$	$t_i^{16} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{x}_i + 1}{\sigma_{x_i}\bar{X}_i + 1}\right)$	σ_{x_i}	1	1
$t^{17}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{17}$	$t_i^{17} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{X}_i + \beta_1(x_i)}{\sigma_{x_i}\bar{x}_i + \beta_1(x_i)}\right)$	$t^{18}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{18}$	$t_i^{18} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{x}_i + \beta_1(x_i)}{\sigma_{x_i}\bar{X}_i + \beta_1(x_i)}\right)$	σ_{x_i}	$\beta_1(x)$	1
$t^{19}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{19}$	$t_i^{19} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{X}_i + \beta_2(x_i)}{\sigma_{x_i}\bar{x}_i + \beta_2(x_i)}\right)$	$t^{20}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{20}$	$t_i^{20} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{x}_i + \beta_2(x_i)}{\sigma_{x_i}\bar{X}_i + \beta_2(x_i)}\right)$	σ_{x_i}	$\beta_2(x_i)$	1
$t^{21}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{21}$	$t_i^{21} = \bar{y}_i\left(\frac{\beta_1(x_i)\bar{x}_i + \beta_2(x_i)}{\beta_1(x_i)\bar{X}_i + \beta_2(x_i)}\right)$	$t^{22}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{22}$	$t_i^{22} = \bar{y}_i\left(\frac{\beta_1(x_i)\bar{x}_i + \beta_2(x_i)}{\beta_1(x_i)\bar{X}_i + \beta_2(x_i)}\right)$	$\beta_1(x)$	$\beta_2(x_i)$	1
$t^{23}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{23}$	$t_i^{23} = \bar{y}_i\left(\frac{\beta_2(x_i)\bar{X}_i + \beta_1(x_i)}{\beta_2(x_i)\bar{x}_i + \beta_1(x_i)}\right)$	$t^{24}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{24}$	$t_i^{24} = \bar{y}_i\left(\frac{\beta_2(x_i)\bar{x}_i + \beta_1(x_i)}{\beta_2(x_i)\bar{X}_i + \beta_1(x_i)}\right)$	$\beta_2(x_i)$	$\beta_1(x_i)$	1
$t^{25}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{25}$	$t_i^{25} = \bar{y}_i\left(\frac{\rho_i\bar{X}_i + 1}{\rho_i\bar{x}_i + 1}\right)$	$t^{26}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{26}$	$t_i^{26} = \bar{y}_i\left(\frac{\rho_i\bar{x}_i + 1}{\rho_i\bar{X}_i + 1}\right)$	ρ_i	1	1
$t^{27}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{27}$	$t_i^{27} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{X}_i + C_{x_i}}{\sigma_{x_i}\bar{x}_i + C_{x_i}}\right)$	$t^{28}_{24} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{28}$	$t_i^{28} = \bar{y}_i\left(\frac{\sigma_{x_i}\bar{x}_i + C_{x_i}}{\sigma_{x_i}\bar{X}_i + C_{x_i}}\right)$	ρ_i	C_{x_i}	1

The Bias and Mean Square Error of t_{1N}^{G} in Two-Stage Sampling Design

In order to obtain the bias expression, let us define the expectation of t_{1N}^{G} in two-stage sampling design as,

$$E\left(t_{24}'^{G}\right) = E_1 E_2\left(t_{24}'^{G}\right), \tag{4.4.5}$$

$$= E_1 E_2\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i t_i^{G}\right), \tag{4.4.6}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\alpha_i z_i \qquad where\ \ z_i = E_2(t_i^{G}). \tag{4.4.7}$$

Now the bias of t_{24}^{G} may be written as,

$$\text{Bias}(t_{24}'^{G}) = E\left(t_{24}^{G}\right) - \bar{Y}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\alpha_i\left(z_i - \bar{Y}_i\right). \tag{4.4.8}$$

In order to find expression for the MSE of t_{1N}^{G} in two-stage sampling, let us define

$$MSE(t_{24}'^{G}) = MSE_1\left(\left(E_2(t_{24}'^{G})\right)\right) + E_1\left(MSE_2(t_{24}'^{G})\right) \tag{4.4.9}$$

where

$$MSE_1\left(E_2(t_{24}^{G})\right) = MSE_1\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i z_i^{*}\right) \tag{4.4.10}$$

$$\left.\begin{aligned} &= \frac{f}{N-1}\sum_{i=1}^{N}\left(\alpha_i z_i^{*} - E\left(\alpha_i z_i^{*}\right)\right)^2 \\ &where \\ &E\left(\alpha_i z_i\right) = \frac{1}{N}\sum_{i=1}^{N}\alpha_i z_i^{*} \end{aligned}\right\} \tag{4.4.11}$$

and

$$\left.\begin{aligned} E_1\left(MSE_2(t_{24}^G)\right) &= E_1\left(\frac{1}{n^2}\sum_{i=1}^{n}\alpha_i^2 v_i\right) \\ &\text{where} \\ v_i &= MSE_2(t_i^G) \end{aligned}\right\} \tag{4.4.12}$$

In order to derive the bias and mean square error(MSE) of the proposed class of estimator, we need to find z_i, z_i^* and v_i from the units of second stage. It is therefore

We rewrite (4.4.3) in the form of e's as

$$t_i^G = \bar{Y}_i(1+e_{\bar{y}_i})\left[1+\beta\frac{a_i\bar{X}_i}{(a_i\bar{X}_i+b_i)}e_{\bar{x}_i}\right]^{-g}, \tag{4.4.13}$$

or

$$t_i^G = \bar{Y}_i(1+e_{\bar{y}_i})\left[1+\beta\lambda_i e_{\bar{x}_i}\right]^{-g} \text{ where, } \lambda_i = \frac{a_i\bar{X}_i}{(a_i\bar{X}_i+b_i)}, \tag{4.4.14}$$

Consider $\left|\beta\lambda_i e_{\bar{x}i}\right| < 1$ so that we can expand the series of $(1+\beta\lambda_i e_i)^{-g}$ in (4.4.13). On ignoring the terms in $e_{\bar{y}i}$ and $e_{\bar{x}i}$ of order higher than one as,

$$t_i^G = \bar{Y}_i\left[1+e_{\bar{y}_i}-\beta g\lambda_i e_{\bar{x}_i}+\frac{g(g+1)}{2}\beta^2\lambda_i^2 e_{\bar{x}_i}^2-\beta g\lambda_i e_{\bar{x}_i}e_{\bar{y}_i}\right], \tag{4.4.15}$$

We take expectation of (4.4.15) for ssu's belonging to every i^{th} fsu as,

$$E(t_i^G) = \bar{Y}_i\left[1+\frac{g(g+1)\beta^2\lambda_i^2}{2}C_{x_i}^2-\beta g\lambda_i\rho_i C_{x_i}C_{y_i}\right] = z_i, \tag{4.4.16}$$

Now from (4.4.16) and (4.4.8), we will have the bias of t_{24}^G as,

$$\text{Bias}(k_{TS}^G) = \frac{1}{N}\sum_{i=1}^{N}\alpha_i\bar{Y}_i\left[\frac{g(g+1)\beta^2\lambda_i^2}{2}C_{x_i}^2-\beta g\lambda_i\rho_i C_{x_i}C_{y_i}\right]. \tag{4.4.17}$$

We can rewrite (4.4.15) as,

$$t_i^G - \bar{Y}_i = \bar{Y}_i\left[e_{\bar{y}_i} - \beta g \lambda_i e_{\bar{x}_i} + \frac{g(g+1)}{2}\beta^2\lambda_i^2 e_{\bar{x}_i}^2 - \beta g \lambda_i e_{\bar{x}_i} e_{\bar{y}_i}\right], \tag{4.4.18}$$

We retain the terms up to the order one and then take expectation of (4.4.18). We get,

$$E(t_i^G) = \bar{Y}_i = z_i^* \tag{4.4.19}$$

In order to obtain the MSE, we take square of (4.4.18), retain terms in e's upto the order one and take expectation,

$$E(t_i^G - \bar{Y}_i)^2 = \bar{Y}_i^2 f_{m_i}\left(C_{y_i}^2 + \beta^2 g^2 \lambda_i^2 C_x^2 - 2\beta g \lambda_i \rho_i C_{x_i} C_{y_i}\right) = v_i \tag{4.4.20}$$

By substituting (4.4.17) and (4.4.20) respectively in (4.4.11) and (4.4.12), we have

$$MSE_1\left(E_2(\mathrm{t}_{1N}^G)\right) = \frac{f}{N-1}\sum_{i=1}^{N}\left(\alpha_i\bar{Y}_i - E\left(\alpha_i\bar{Y}_i\right)\right)^2 \text{ where } E(\alpha_i z_i) = \frac{1}{N}\sum_{i=1}^{N}\alpha_i\bar{Y}_i \tag{4.4.21}$$

and

$$E_1\left(MSE_2(\mathrm{t}_{24}^G)\right) = \frac{1}{nN}\sum_{i=1}^{n}\alpha_i^2\bar{Y}_i^2 f_{m_i}\left(C_{y_i}^2 + \beta^2 g^2\lambda_i^2 C_x^2 - 2\beta g\lambda_i\rho_i C_{x_i}C_{y_i}\right) \tag{4.4.22}$$

The MSE of t_{24}^G is finally obtained in two-stage sampling as,

$$MSE(\mathrm{t}_{24}^G) = \frac{f}{N-1}\sum_{i=1}^{N}\left(\alpha_i\bar{Y}_i - \frac{\sum_{i=1}^{N}\alpha_i\bar{Y}_i}{N}\right)$$
$$+\frac{1}{nN}\sum_{i=1}^{n} f_{m_i}\alpha_i^2\bar{Y}_i^2\left(C_{y_i}^2 + \beta^2 g^2\lambda_i^2 C_x^2 - 2\beta g\lambda_i\rho_i C_{x_i}C_{y_i}\right) \tag{4.4.23}$$

The values of mean square errors of the ratio-type and product-type estimators mentioned in Table 1 may also be obtained directly by substituting different values of g , a_i , b_i, and β to the expression (4.4.23).

The MSE of t_{24}^{G} in (4.4.23) is minimum if $v_i \left(= f_{m_i} \bar{Y}_i^2 \left(C_{y_i}^2 + \beta^2 g^2 \lambda_i^2 C_x^2 - 2\beta g \lambda_i \rho_i C_{x_i} C_{y_i}\right)\right)$ is minimum. It is therefore the minimization of v_i with respect to β yields its optimum value as,

$$\beta^{opt} = \frac{\rho_i C_{y_i}}{\lambda_i g C_{x_i}}. \tag{4.4.24}$$

The minimum MSE for i^{th} fsu on substituting the optimum value in (4.4.20), may be written as,

$$v_i^{\min} = \bar{Y}_i^2 f_{m_i} C_{y_i}^2 (1-\rho_i^2), \tag{4.4.25}$$

Now the minimum MSE of t_{24}^{opt} in two-stage sampling is obtained on substituting (4.4.25) in (4.4.23) as,

$$\min .MSE(t_{24}^{opt}) = \frac{f}{N-1} \sum_{i=1}^{N} \left[\alpha_i \bar{Y}_i - \frac{\sum_{i=1}^{N} \alpha_i \bar{Y}_i}{N} \right]^2 + \frac{1}{nN} \sum_{i=1}^{N} f_{m_i} \alpha_i^2 \bar{Y}_i^2 C_{y_i}^2 (1-\rho_i^2). \tag{4.4.26}$$

which is asymptotical equal to the MSE of regression estimator in two-stage sampling design (see, Srivastava and Garg, 2009).

On using the optimal value β^{opt}, we get an asymptotically optimal estimator (AOE) in two-stage sampling as,

$$t_{24}^{AOE} = \frac{1}{n} \sum_{i=1}^{n} \alpha_i t_i^{opt} \text{ where } t_i^{opt} = \bar{y}_i \left[\frac{(a_i \bar{X}_i + b_i)}{\beta^{opt}(a_i \bar{x}_i + b_i) + (1-\beta^{opt})\left(a_i \bar{X}_i + b_i\right)} \right]^g \tag{4.4.27}$$

The values of β^{opt} can be searched out from the previous surveys or omay be guessed from the knowledge drawn in due course of time, for case in point, see Horvitz and Thompson (1952), Murthy (1967), Singh & Vishwakarma (2008), Singh and Kumar (2008), Singh and Karpe (2010), Upadhyaya et al. (2011), Yadav and Kadilar (2013) and Sanaullah et al. (2014).

In many real life situations, it is not possible for the researcher to presume the value β^{opt} by employ all the resources, so it is better to replace β^{opt} in (4.4.27) by their consistent estimates as,

$$\hat{\beta}^{opt} = \frac{\hat{\rho}_i \hat{C}_{y_i}}{\lambda_i g \hat{C}_{x_i}} \tag{4.4.28}$$

So an estimator in (4.4.27) may be written as

$$t_{24}^{AOE} = \frac{1}{n}\sum_{i=1}^{n} \alpha_i t_i^{opt}$$

where

$$t_i^{opt} = \bar{y}_i \left[\frac{(a_i \bar{X}_i + b_i)}{\hat{\beta}^{opt}(a_i \bar{x}_i + b_i) + (1-\hat{\beta}^{opt})\left(a_i \bar{X}_i + b_i\right)} \right]^g \tag{4.4.29}$$

Similarly an unbiased estimator for the MSE of t_{24}^{opt} is given as,

$$\min.MSE(\mathrm{t}_{24}^{opt}) = \frac{f}{n-1}\sum_{i=1}^{N}\left(\alpha_i \bar{Y}_i - \frac{\sum_{i=1}^{n} \alpha_i \bar{Y}_i}{n} \right)^2 + \frac{1}{nN}\sum_{i=1}^{N} \alpha_i^2 v_i^{opt} \tag{4.4.30}$$

Remark 4.10

i) For $\mathrm{g} = 1$, some ratio-type estimators are expressed in Table 4.5.we may express the MSE(t_i^G)=v_i given in (4.4.18) for these ratio-type estimators for i^{th} fsu as,

$$MSE(t_i^j) = v_i^j = \begin{cases} \bar{Y}_i^2 f_{m_i}\left(C_{y_i}^2 + \beta^{\bullet 2} C_{x_i}^2 - 2\beta^{\bullet}\rho_i C_{y_i} C_{x_i}\right) & j(\in G) = 1 \\ \bar{Y}_i^2 f_{m_i}\left(C_{y_i}^2 + \beta^{\bullet 2}\lambda_i^{\left(\frac{j-1}{2}\right)^2} C_{x_i}^2 - 2\beta^{\bullet}\lambda_i^{\left(\frac{j-1}{2}\right)}\rho_i C_{y_i} C_{x_i} \right) & j(\in G) = 3,5,..,27 \end{cases},$$

(4.4.33)

The $MSE(t_i^j)=v_i^j$ in (4.4.31) is minimum for $\beta^{\bullet}=\dfrac{\rho_i C_{y_i}}{\lambda_i^{\frac{j-1}{2}} C_{x_i}}$.

ii) For $g=-1$, some product-type estimators are expressed in Table 4.5. We may express MSE(t_i^G)=v_i given in (4.4.18) for these product-type estimators for i^{th} fsu may be expressed as,

$$MSE(t_i^k)=v_i^k=\begin{cases}\overline{Y}_i^2 f_{m_i}\left(C_{y_i}^2+\beta^{\circ 2}C_{x_i}^2+2\beta^{\circ}\rho_i C_{y_i}C_{x_i}\right) & k(\in G)=2\\ \overline{Y}_i^2 f_{m_i}\left(C_{y_i}^2+\beta^{\circ 2}\lambda_i^{\left(\frac{k}{2}\right)^2}C_{x_i}^2+2\beta^{\circ}\lambda_i^{\left(\frac{k}{2}\right)}\rho_i C_{y_i}C_{x_i}\right) & k(\in G)=4,6,..,28\end{cases},$$

(4.4.32)

The $MSE(t_i^k)=v_i^k$ in (4.4.32) is minimum for $\beta^{\circ}=-\dfrac{\rho_i C_{y_i}}{\lambda_i^{\frac{k}{2}} C_{x_i}}$.

where

$$\lambda_i^1=\frac{\overline{X}_i}{\overline{X}_i+C_{x_i}},\ \lambda_i^2=\frac{\beta_2(x_i)\overline{X}_i}{\beta_2(x_i)\overline{X}_i+C_{x_i}},\quad \lambda_i^3=\frac{C_{x_i}\overline{X}_i}{C_{x_i}\overline{X}_i+\beta_2(x_i)},$$

$$\lambda_i^4=\frac{\beta_2(x_i)\overline{X}_i}{\beta_2(x_i)\overline{X}_i+\sigma_{x_i}},\quad \lambda_i^5=\frac{\overline{X}_i}{\overline{X}_i+\rho_{xy}},\quad \lambda_i^6=\frac{\overline{X}_i}{\overline{X}_i+\beta_2(x_i)},\quad \lambda_i^7=\frac{\sigma_{x_i}\overline{X}_i}{\sigma_{x_i}\overline{X}_i+1},$$

$$\lambda_i^8=\frac{\sigma_{x_i}\overline{X}_i}{\sigma_{x_i}\overline{X}_i+\beta_1(x_i)},\ \lambda_i^9=\frac{\sigma_{x_i}\overline{X}_i}{\sigma_{x_i}\overline{X}_i+\beta_2(x_i)},\ \lambda_i^{10}=\frac{\beta_1(x_i)\overline{X}_i}{\beta_1(x_i)\overline{X}_i+\beta_2(x_i)},$$

$$\lambda_i^{11}=\frac{\beta_2(x_i)\overline{X}_i}{\beta_2(x_i)\overline{X}_i+\beta_1(x_i)},\ \lambda_i^{12}=\frac{\rho_{xy}\overline{X}_i}{\rho_{xy}\overline{X}_i+1},\ \lambda_i^{13}=\frac{\sigma_{x_i}\overline{X}_i}{\sigma_{x_i}\overline{X}_i+C_{x_i}}.$$

Case-II:

We propose an un weighted generalized estimator for unequal fsu's in two-stage sampling design as,

$$t_{24}''^{G}=\frac{1}{n}\sum_{i=1}^{n}t_i^G, \tag{4.4.33}$$

where t_{1N}^G where is proposed unweighted generalized estimator in two-stage sampling design and t_i^G is proposed ratio type estimator for population mean for ssu's belonging to the i^{th} fsu's as,

$$t_i^G = \bar{y}_i \left[\frac{(a_i \bar{X}_i + b_i)}{\beta(a_i \bar{x}_i + b_i) + (1-\beta)\left(a_i \bar{X}_i + b_i\right)} \right]^g , \tag{4.4.34}$$

where β and g are assumed to be the unknown constants whose values are to be estimated. $a_i (\neq 0)$, and b_i are assumed to be known as either real numbers or (Linear or Non-linear) functions of some known parameters of auxiliary variable x such as standard deviation (σ_{xi}), coefficient of variation (C_{xi}), skewness $(\beta_{1i}(x))$, kurtosis $(\beta_{2i}(x))$ and correlation coefficient (ρ_i) for ssu's belonging to i^{th} fsu's from the population.

The proposed estimator in (4.4.33) follows naturally in exactly the same fashion along with the class of estimator in Table 4.5, as that for case-I in Section 4.4.1. The same is true for the MSE and the bias. It is therefore directly from Section 4.4.1.1, we may write $\text{Bias}(t_{24}''^{G})$ and $\text{MSE}(t_{24}''^{G})$ following the same, and we may also produce a class of estimators for similar choices of the constants in case-II.

Following the notations and expectations for case II presented in Section 4.4.1, the bias of(4.4.33) may be written directly from (4.4.17),

$$\text{Bias}(t_{24}''^{G}) = \frac{1}{N} \sum_{i=1}^{N} \bar{Y}_i \left[\frac{g(g+1)\beta^2 \lambda_i^2}{2} C_{x_i}^2 - \beta g \lambda_i \rho_i C_{x_i} C_{y_i} \right] \tag{4.4.35}$$

$$MSE(t_{24}''^{G}) = \frac{f}{N-1} \sum_{i=1}^{N} \left(\bar{Y}_i - \frac{\sum_{i=1}^{N} \bar{Y}_i}{N} \right)$$

$$+ \frac{1}{nN} \sum_{i=1}^{n} f_{m_i} \bar{Y}_i^2 \left(C_{y_i}^2 + \beta^2 g^2 \lambda_i^2 C_x^2 - 2\beta g \lambda_i \rho_i C_{x_i} C_{y_i} \right) \tag{4.4.36}$$

Case-III:

We propose a generalized estimator for equal fsu's in two-stage sampling design as,

$$t_{24}^G = \frac{1}{n} \sum_{i=1}^{n} t_i^G , \tag{4.4.37}$$

where t_{24}^{G} is proposed generalized estimator in two-stage sampling design and t_i^G is proposed ratio type estimator for population mean for ssu's belonging to the i^{th} fsu's as,

$$t_i^G = \bar{y}_i \left[\frac{(a_i \bar{X}_i + b_i)}{\beta(a_i \bar{x}_i + b_i) + (1-\beta)\left(a_i \bar{X}_i + b_i\right)} \right]^g , \tag{4.4.38}$$

where β and g are assumed to be the unknown constants whose values are to be estimated. $a_i(\neq 0)$, and b_i are assumed to be known as either real numbers or (Linear or Non-linear) functions of some known parameters of auxiliary variable x such as standard deviation (σ_{xi}), coefficient of variation (C_{xi}), skewness $(\beta_{1i}(x))$, kurtosis $(\beta_{2i}(x))$ and correlation coefficient (ρ_i) for ssu's belonging to i^{th} fsu's from the population.

The proposed estimator in (4.4.33) follows naturally in exactly the same fashion along with the class of estimator in Table 4.5, as that for case-I in Section 4.4.1. The same is true for the MSE and the bias. It is therefore directly from Section 4.4.1.1, we may write $\text{Bias}\,(t_{24}^{G})$ and $\text{MSE}(t_{24}^{G})$ following the same, and we may also produce a class of estimators for similar choices of the constants in case-II.

Following the notations and expectations for case II presented in Section 4.4.1, the bias of(4.4.33) may be written directly from (4.4.17),

$$\text{Bias}(t_{24}^{G}) = \frac{1}{N}\sum_{i=1}^{N} \bar{Y}_i \left[\frac{g(g+1)\beta^2\lambda_i^2}{2} C_{x_i}^2 - \beta g \lambda_i \rho_i C_{x_i} C_{y_i} \right] \tag{4.4.39}$$

$$MSE(t_{24}^{G}) = \frac{f}{N-1}\sum_{i=1}^{N} \left(\bar{Y}_i - \frac{\sum_{i=1}^{N} \bar{Y}_i}{N} \right)$$

$$+ \frac{1}{nN}\sum_{i=1}^{n} f_{m_i} \bar{Y}_i^2 \left(C_{y_i}^2 + \beta^2 g^2 \lambda_i^2 C_x^2 - 2\beta g \lambda_i \rho_i C_{x_i} C_{y_i} \right) \tag{4.4.40}$$

CHAPTER 5

GENERALIZED EXPONENTIAL ESTIMATORS FOR POPULATION MEAN FOR TWO-PHASE SAMPLING USING TWO AUXILIARY VARIABLES IN TWO-STAGE SAMPLING

5.1 INTRODUCTION

In this chapter, some generalized exponential-type estimators have been proposed by utilizing the information of two auxiliary variables separately for estimating the population mean in two-stage sampling design. The generalized estimators proposed in two-stage sampling design have been discussed in three different situations for two-phase sampling discussed by Samiuddin and Hanif (2007) regarding the availability of information for auxiliary two variables.

Situation I:

When complete information on desired population characteristics is readily available for the two auxiliary variables. This situation is considered as full-information case (FIC).

Situation II:

When complete information on desired population characteristics is readily available for one auxiliary variable. This situation is considered as partial-information case (PIC).

Situation III:

When information of desired population characteristics for both auxiliary variables is not available. This situation is taken as no-information case (NIC).

The availability of information will be dealt according to three cases (as mentioned in chapter 4) in two-stage sampling for each situation regarding the availability of information. .

In order to obtain the bias and mean square error under two-stage sampling design, we follow the notations given as:

Notations:

Case-I: When first-stage units are unequal and weighted mean is used.

Let a population consists of N first stage units (fsu's) and each fsu consists of M_i second stage units (ssu's). Let n be a sample of fsu's and $m_{i(1)}$ be the sample of first-phase units of which $m_{i(2)}$ $(m_{i(2)} \subset m_{i(1)})$ be another sample of second-phase units selected from second-stage units and each of n fsu's is selected by assigning weights $\eta_i = \frac{M_i}{\overline{M}}$ to it. Further we assume that the selection of units at each stage has been made by simple random sampling.

M_0 Total number of second-stage units (ssu's) in the population,

$\overline{Y}_i$ Mean of the study variable of ssu's belonging to the ith fsu,

$\overline{Y}$ Population mean $= \frac{1}{M_0}\sum_{i=1}^{N}\sum_{j=1}^{M_i} Y_{ij} = \frac{1}{N\overline{M}}\sum_{i=1}^{M_i}\overline{Y}_i$.

$y_{ij(2)}$ observation made in 2nd-phase sample of jth ssu belonging to the ith fsu. $j = 1, 2, \ldots m_i$ and $i = 1, 2, \ldots n$.

$\overline{y}_{i(2).}$ Sample mean for second-phase sample of ssu's belonging to ith fsu

$$\overline{y}_{i(2)} = \frac{1}{m_{i(2)}}\sum_{j=1}^{m_{i(2)}} y_{ij(2)} .$$

$\overline{x}_{i(1)}$ Sample mean of for first-phase sample of ssu's belonging to ith fsu,

$$\overline{x}_{i(1)} = \frac{1}{m_{i(1)}}\sum_{j=1}^{m_{i(1)}} x_{ij(1)} .$$

$\overline{x}'_{s(1)}$ the mean of first-phase sample in two-stage sampling for auxiliary variable x, $\overline{x}'_{s(1)} = \frac{1}{n}\sum_{i=1}^{n}\eta_i\overline{x}_{i(1).}$

$\overline{z}'_{s(2)}$ the mean of second-phase sample in two-stage sampling for auxiliary variable z, $\overline{z}'_{s(2)} = \frac{1}{n}\sum_{i=1}^{n}\eta_i\overline{z}'_{i(2).}$

$\overline{y}'_{s(2)}$ the mean of two-stage sample for study variable

$$\bar{y}'_{s(2)} = \frac{1}{n}\sum_{i=1}^{n} \eta_i \bar{y}_{i(2)}.$$

ii) Expectation
Let us define

$$\begin{gathered}\bar{y}'_{s(2)} = \bar{Y}'_s(1+e'_{0(2)}),\ \bar{x}'_{s(1)} = \bar{X}'_s(1+e'_{1(1)}),\ \ \bar{z}'_{s(2)} = \bar{Z}'_s(1+e'_{2(2)}),\\ \bar{x}'_{s(2)} = \bar{X}'_s(1+e'_{1(2)}),\ \ \bar{z}'_{s(1)} = \bar{Z}'_s(1+e'_{2(1)})\end{gathered} \tag{5.1.1}$$

$$\left.\begin{aligned}
&E(e'_{0(2)}) = E(e'_{1(2)}) = E(e'_{2(2)}) = E(e'_{1(1)}) = E(e'_{2(1)}) = 0\\
&E(e'^{2}_{0(2)}) = V'_{020(2)},\ E(e'^{2}_{1(2)}) = V'_{200(2)},\ E(e'^{2}_{2(2)}) = V'_{002(2)},\ E(e'^{2}_{1(1)})\\
&\qquad = V'_{200(1)},\ E(e'^{2}_{2(1)}) = V'_{002(1)},\ E(e'_{1(2)}e'_{1(1)}) = V'_{200(1)}\\
&E(e'_{0(2)}e'_{1(2)}) = V'_{110(2)},\ E(e'_{1(2)}e'_{2(2)}) = V'_{101(2)},\ E(e'_{0(2)}e'_{2(2)}) = V'_{011(2)},\\
&\qquad E(e'_{0(2)}e'_{1(1)}) = V'_{110(1)},\ E(e'_{1(2)}e'_{2(1)}) = V'_{101(1)},\ E(e'_{0(2)}e'_{2(1)}) = V'_{011(1)}\\
&\text{Where}\\
&V'_{020(2)} = \frac{1}{\bar{Y}'^{2}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{yb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{m_{i(1)}}\right)S'^{2}_{ywi},\right\},\ V'_{200(1)} = \frac{1}{\bar{X}'^{2}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)S'^{2}_{xwi}\right\}\\
&V'_{200(2)} = \frac{1}{\bar{X}'^{2}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{m_{i(1)}}\right)S'^{2}_{xwi}\right\},\ V'_{002(1)} = \frac{1}{\bar{Z}'^{2}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{zb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)S'^{2}_{zwi}\right\},\\
&V'_{002(2)} = \frac{1}{\bar{Z}'^{2}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{zb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{m_{i(1)}}\right)S'^{2}_{zwi}\right\},\\
&V'_{110(1)} = \frac{1}{\bar{Y}'_s\bar{X}'_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xyb}S'_{yb}S'_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)\rho_{xywi}S'_{ywi}S'_{xwi}\right\},\\
&V'_{110(2)} = \frac{1}{\bar{Y}'_s\bar{X}'_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xyb}S'_{yb}S'_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{m_{i(1)}}\right)\rho_{xywi}S'_{ywi}S'_{xwi}\right\},\\
&V'_{101(1)} = \frac{1}{\bar{Z}'_s\bar{X}'_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xzb}S'_{zb}S'_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)\rho_{xzwi}S'_{zwi}S'_{xwi}\right\},\\
&V'_{101(2)} = \frac{1}{\bar{Z}'_s\bar{X}'_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xzb}S'_{zb}S'_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{m_{i(1)}}\right)\rho_{xzwi}S'_{zwi}S'_{xwi}\right\},\\
&V'_{200(1)} = \frac{1}{\bar{Z}'_s\bar{X}'_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'_{xb}S'_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)S'_{xwi}S'_{xwi}\right\}\\
&V'_{011(2)} = \frac{1}{\bar{Z}'_s\bar{Y}'_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yzb}S'_{zb}S'_{yb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{m_{i(1)}}\right)\rho_{yzwi}S'_{zwi}S'_{ywi}\right\}\\
&V'_{011(1)} = \frac{1}{\bar{Z}'_s\bar{Y}'_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yzb}S'_{zb}S'_{yb} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)\rho_{yzwi}S'_{zwi}S'_{ywi}\right\}
\end{aligned}\right\} \tag{5.1.2}$$

Case-II: When first-stage units are unequal and un weighted mean is used

For case-I, in order to obtain a weighted and unbiased estimator for population mean we used a weighting constant $\eta_i = \frac{M_i}{\bar{M}}$ in two-stage sampling design. If we assume equal weights for all unequal first-stage units, it will give us an un-weighted and biased estimator of population mean in two-stage sampling design and this situation will be considered as case-II.

i.e. for $\eta_i = \frac{M_i}{\bar{M}} = 1$, and $\bar{y}_{i(2)} = \frac{1}{m_{i(2)}} \sum_{j=1}^{m_{i(2)}} y_{ij(2)}$ (see Sukhatme et al., 1984) an estimator defined in case-I may be turned into the estimator for case-II. So the procedure of two stage sampling design for case II will be same as described in Case I. The notations and expectations may be derived for case II as:

i) Notations

$\bar{Y}_s''$ the mean of two-stage population for study variable.

$$\bar{Y}_s'' = \frac{1}{N} \sum_{i=1}^{N} \bar{y}_{i(2).}$$

$\bar{y}_{s(2)}''$ the mean of two-stage sample for study variable

$$\bar{y}_{s(2)}'' = \frac{1}{n} \sum_{i=1}^{n} \bar{y}_{i(2).}$$

$\bar{x}_{s(1)}''$ the mean of first-phase sample in two-stage sampling for auxiliary variable x,

$$\bar{x}_{s(1)}'' = \frac{1}{n} \sum_{i=1}^{n} \bar{x}_{i(1).}$$

$\bar{z}_{s(2)}''$ the mean of second-phase sample in two-stage sampling for auxiliary variable z

$$\bar{z}_{s(2)}'' = \frac{1}{n} \sum_{i=1}^{n} \bar{z}_{i(2).}$$

$$S''^2_{yb} = \frac{1}{N-1}\sum_{i=1}^{N}(y_{ij} - \bar{Y}''_s)^2 \; . \; S''^2_{ywi} = \frac{1}{M_i - 1}\sum_{i=1}^{n}(y_{ij(2)} - \bar{Y}_{i.})^2 \; .$$

$$C''_{yb} = \frac{S''_{yb}}{\bar{Y}''_s}, C''_{ywi} = \frac{S''_{ywi}}{\bar{Y}''_s}, C''_{xb} = \frac{S''_{xb}}{\bar{X}''_s}, C''_{xb} = \frac{S''_{xb}}{\bar{X}''_s}, C''_{zb} = \frac{S''_{zb}}{\bar{Z}''_s},$$

$$C''_{xwi} = \frac{S''_{xwi}}{\bar{X}''_s}, C''_{xwi} = \frac{S''_{xwi}}{\bar{X}''_s}, C''_{zwi} = \frac{S''_{zwi}}{\bar{Z}''_s}, C''_{zwi} = \frac{S''_{zwi}}{\bar{Z}''_s}, C''_{zb} = \frac{S''_{zb}}{\bar{Z}''_s}, C''_{zwi} = \frac{S''_{zwi}}{\bar{Z}''_s}.$$

$$\rho''_{xyb} = \frac{S''_{xyb}}{S''_{xb}S''_{yb}}, \qquad \rho''_{xywi} = \frac{S''_{xywi}}{S''_{xwi}S''_{ywi}}, \rho''_{xzb} = \frac{S''_{xzb}}{S''_{xb}S''_{zb}}, \; \rho''_{xywi} = \frac{S''_{xzwi}}{S''_{xwi}S''_{zwi}},$$

$$\rho''_{yzb} = \frac{S''_{yzb}}{S''_{zb}S''_{yb}}, \qquad \rho''_{yzwi} = \frac{S''_{zywi}}{S''_{zwi}S''_{ywi}}, \rho''_{xyb} = \frac{S''_{xyb}}{S''_{xb}S''_{yb}}, \; \rho''_{xywi} = \frac{S''_{xywi}}{S''_{xwi}S''_{ywi}},$$

$$\rho''_{xzb} = \frac{S''_{xzb}}{S''_{xb}S''_{zb}}, \qquad \rho''_{xywi} = \frac{S''_{xzwi}}{S''_{xwi}S''_{zwi}}, \rho''_{yzb} = \frac{S''_{yzb}}{S''_{zb}S''_{yb}}, \; \rho''_{yzwi} = \frac{S''_{zywi}}{S''_{zwi}S''_{ywi}},$$

ii) Expectations

$$y''_{s(2)} = \bar{Y}_s(1 + e''_{0(2)}), \bar{x}''_{s(1)} = \bar{X}_S(1 + e''_{1(1)}), \; \bar{z}''_{s(2)} = \bar{Z}_s(1 + e''_{2(2)}),$$
$$\bar{x}''_{s(2)} = \bar{X}_s(1 + e''_{1(2)}), \; \bar{z}''_{s(1)} = \bar{Z}_s(1 + e''_{2(1)}) \qquad (5.1.3)$$

$$\left.\begin{array}{l}
E(e''_{0(2)}) = E(e''_{1(2)}) = E(e''_{2(2)}) = E(e''_{1(1)}) = E(e''_{2(1)}) = 0 \\
E({e''_{0(2)}}^2) = V''_{020(2)},\ E({e''_{1(2)}}^2) = V''_{200(2)},\ E({e''_{2(2)}}^2) = V''_{002(2)}, \\
\qquad E({e''_{1(1)}}^2) = V''_{200(1)},\ E({e''_{2(1)}}^2) = V''_{002(1)}, \\
E(e''_{0(2)}e''_{1(2)}) = V''_{110(2)},\ E(e''_{1(2)}e''_{2(2)}) = V''_{101(2)},\ E(e''_{0(2)}e''_{2(2)}) = V''_{011(2)}, \\
\qquad E(e''_{0(2)}e''_{1(1)}) = V''_{110(1)},\ E(e''_{1(2)}e''_{2(1)}) = V''_{101(1)},\ E(e''_{0(2)}e''_{2(1)}) = V''_{011(1)} \\
\text{Where} \\
V''_{020(2)} = \dfrac{1}{\bar{Y}''^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^2_{yb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(2)}}-\dfrac{1}{m_{i(1)}}\right)S''^2_{ywi}\right\},\ V''_{200(1)} = \dfrac{1}{\bar{X}''^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^2_{xb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(1)}}-\dfrac{1}{M_i}\right)S''^2_{xwi}\right\} \\
V''_{200(2)} = \dfrac{1}{\bar{X}''^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^2_{xb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(2)}}-\dfrac{1}{m_{i(1)}}\right)S''^2_{xwi}\right\},\ V'_{002(1)} = \dfrac{1}{\bar{Z}'^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^2_{zb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(1)}}-\dfrac{1}{M_i}\right)S''^2_{zwi}\right\}, \\
V''_{002(2)} = \dfrac{1}{\bar{Z}'^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^2_{zb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(2)}}-\dfrac{1}{m_{i(1)}}\right)S''^2_{zwi}\right\}, \\
V''_{110(1)} = \dfrac{1}{\bar{Y}''_s\bar{X}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xyb}S''_{yb}S''_{xb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(1)}}-\dfrac{1}{M_i}\right)\rho_{xywi}S''_{ywi}S''_{xwi}\right\}, \\
V''_{110(2)} = \dfrac{1}{\bar{Y}''_s\bar{X}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xyb}S''_{yb}S''_{xb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(2)}}-\dfrac{1}{m_{i(1)}}\right)\rho_{xywi}S''_{ywi}S''_{xwi}\right\}, \\
V''_{101(1)} = \dfrac{1}{\bar{Z}''_s\bar{X}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xzb}S''_{zb}S''_{xb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(1)}}-\dfrac{1}{M_i}\right)\rho_{xzwi}S''_{zwi}S''_{xwi}\right\}, \\
V''_{101(2)} = \dfrac{1}{\bar{Z}''_s\bar{X}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{xzb}S''_{zb}S''_{xb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(2)}}-\dfrac{1}{m_{i(1)}}\right)\rho_{xzwi}S''_{zwi}S''_{xwi}\right\}, \\
V'_{200(1)} = \dfrac{1}{\bar{X}''^2_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''_{xb}S''_{xb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(1)}}-\dfrac{1}{M_i}\right)S''_{xwi}S''_{xwi}\right\} \\
V''_{011(2)} = \dfrac{1}{\bar{Z}''_s\bar{Y}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{yzb}S''_{zb}S''_{yb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(2)}}-\dfrac{1}{m_{i(1)}}\right)\rho_{yzwi}S''_{zwi}S''_{ywi}\right\} \\
V''_{011(1)} = \dfrac{1}{\bar{Z}''_s\bar{Y}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{yzb}S''_{zb}S''_{yb} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_{i(1)}}-\dfrac{1}{M_i}\right)\rho_{yzwi}S''_{zwi}S''_{ywi}\right\}
\end{array}\right\}$$

(5.1.4)

Case-III: When first-stage units are of equal sizes.

Let a population consists of N first stage units (fsu's) and each fsu consists of *M* second stage units (ssu's). Let n be a sample of fsu's and $m_{(1)}$ be the sample of first-phase units of which $m_{(2)}(m_{(2)} \subset m_{(1)})$ be another sample of second-phase units selected from second-stage units and each of n fsu's is selected by assigning equal weights to it. Further we assume that the selection of units at each stage has been done using simple random sampling.

i) Notations

$y_{ij(2)}$ observation made in 2nd-phase sample of jth ssu belonging to the ith fsu. $j = 1, 2, \ldots m_i$ and $i = 1, 2, \ldots n$.

$(j = 1, 2, \ldots, M \, ; i = 1, 2, \ldots, N)$

$\bar{y}_{i(2).}$ Sample mean for second-phase sample of ssu's belonging to ith fsu

$$\bar{y}_{i(2)} = \frac{1}{m_{(2)}} \sum_{j=1}^{m_{(2)}} y_{ij(2)} \, .$$

$\bar{x}_{i(1)}$ Sample mean of for first-phase sample of ssu's belonging to ith fsu,

$$\bar{x}_{i(1)} = \frac{1}{m_{(1)}} \sum_{j=1}^{m_{(1)}} x_{ij(1)} \, .$$

$\bar{x}_{s(1)}$ the mean of first-phase sample in two-stage sampling for auxiliary variable x, $\bar{x}_{s(1)} = \frac{1}{n} \sum_{i=1}^{n} \bar{x}_{i(1).}$

$\bar{z}_{s(2)}$ the mean of second-phase sample in two-stage sampling for auxiliary variable z, $\bar{z}_{s(2)} = \frac{1}{n} \sum_{i=1}^{n} \bar{z}_{i(2).}$

$\bar{y}_{s(2)}$ the mean of two-stage sample for study variable

$$\bar{y}_{s(2)} = \frac{1}{n} \sum_{i=1}^{n} \bar{y}_{i(2).}$$

$\bar{Y}_{i.}$ the mean of second-phase unit at jth second-stage unit in the ith fsu in the population.

$$\bar{Y}_{i.} = \frac{1}{M} \sum_{j=1}^{M} y_{ij(2)}$$

$\bar{Y}_s$ the mean per second-stage unit in the population.

$$\bar{Y}_s = \frac{1}{N} \sum_{i=1}^{N} \bar{y}_{i(2)}$$

$$S^2_{yb} = \frac{1}{N-1}\sum_{i=1}^{N}(y_{ij} - \overline{Y}_s)^2 . \qquad S^2_{ywi} = \frac{1}{M-1}\sum_{i=1}^{N}(y_{ij} - \overline{Y}_{i.})^2 .$$

$$C_{yb} = \frac{S_{yb}}{\overline{Y}_s}, C_{ywi} = \frac{S_{ywi}}{\overline{Y}_s}, C_{xb} = \frac{S_{xb}}{\overline{X}_s}, C_{zb} = \frac{S_{zb}}{\overline{Z}_s}, C_{xwi} = \frac{S_{xwi}}{\overline{X}_s}, C_{zwi} = \frac{S_{zwi}}{\overline{Z}_s}$$

$$\rho_{xyb} = \frac{S_{xyb}}{S_{xb}S_{yb}}, \qquad \rho_{xywi} = \frac{S_{xywi}}{S_{xwi}S_{ywi}}, \rho_{xzb} = \frac{S_{xzb}}{S_{xb}S_{zb}}, \ \rho_{xywi} = \frac{S_{xzwi}}{S_{xwi}S_{zwi}},$$

$$\rho_{yzb} = \frac{S_{yzb}}{S_{zb}S_{yb}}, \qquad \rho_{yzwi} = \frac{S_{zywi}}{S_{zwi}S_{ywi}},$$

ii) Expectation

Let us define

$$y_{s(2)} = \overline{Y}_s(1 + e_{0(2)}), \overline{x}_{s(1)} = \overline{X}_S(1 + e_{1(1)}), \ \overline{z}_{s(2)} = \overline{Z}_s(1 + e_{2(2)}),$$
$$\overline{x}_{s(2)} = \overline{X}_s(1 + e_{1(2)}), \ \overline{z}_{s(1)} = \overline{Z}_s(1 + e_{2(1)}) \qquad (5.1.5)$$

$$
\left.
\begin{array}{l}
E(e_{0(2)}) = E(e_{1(2)}) = E(e_{2(2)}) = E(e_{1(1)}) = E(e_{2(1)}) = 0 \\
E(e_{0(2)}^{\;2}) = V_{020(2)}, E(e_{1(2)}^{\;2}) = V_{200(2)}, E(e_{2(2)}^{\;2}) = V_{002(2)}, \\
\qquad E(e_{1(1)}^{\;2}) = V_{200(1)}, E(e_{2(1)}^{\;2}) = V_{002(1)}, \\
E(e_{0(2)}e_{1(2)}) = V_{110(2)}, E(e_{1(2)}e_{2(2)}) = V_{101(2)}, E(e_{0(2)}e_{2(2)}) = V_{011(2)}, \\
\qquad E(e_{0(2)}e_{1(1)}) = V_{110(1)}, E(e_{1(2)}e_{2(1)}) = V_{101(1)}, E(e_{0(2)}e_{2(1)}) = V_{011(1)} \\
\text{Where} \\
V_{020(2)} = \frac{1}{\bar{Y}_s'^{2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{yb}^{2} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(2)}}-\frac{1}{m_{(1)}}\right)S_{ywi}^{2},\right\}, V_{200(1)} = \frac{1}{\bar{X}_s^{2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{xb}^{2} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(1)}}-\frac{1}{M}\right)S_{xwi}^{2}\right\} \\
V_{200(2)} = \frac{1}{\bar{X}_s^{2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{xb}^{2} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(2)}}-\frac{1}{m_{(1)}}\right)S_{xwi}^{2}\right\}, V_{002(1)} = \frac{1}{\bar{Z}_s^{2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{zb}^{2} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(1)}}-\frac{1}{M}\right)S_{zwi}^{2}\right\}, \\
V_{002(2)} = \frac{1}{\bar{Z}_s'^{2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S^{2}_{zb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(2)}}-\frac{1}{m_{(1)}}\right)S^{2}_{zwi}\right\}, \\
V_{110(1)} = \frac{1}{\bar{Y}_s'\bar{X}_s'}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xyb}S_{yb}S_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(1)}}-\frac{1}{M}\right)\rho_{xywi}S_{ywi}S_{xwi}\right\}, \\
V_{110(2)} = \frac{1}{\bar{Y}_s\bar{X}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xyb}S_{yb}S_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(2)}}-\frac{1}{m_{(1)}}\right)\rho_{xywi}S_{ywi}S_{xwi}\right\}, \\
V_{101(1)} = \frac{1}{\bar{Z}_s\bar{X}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xzb}S_{zb}S_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(1)}}-\frac{1}{M}\right)\rho_{xzwi}S_{zwi}S_{xwi}\right\}, \\
V_{101(2)} = \frac{1}{\bar{Z}_s\bar{X}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xzb}S_{zb}S_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(2)}}-\frac{1}{m_{(1)}}\right)\rho_{xzwi(1)}S_{zwi}S_{xwi}\right\}, \\
V_{200(1)} = \frac{1}{\bar{X}_s^{2}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{xb}S_{xb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(1)}}-\frac{1}{M}\right)S_{xwi}S_{xwi}\right\} \\
V_{011(2)} = \frac{1}{\bar{Z}_s\bar{Y}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yzb}S_{zb}S_{yb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(2)}}-\frac{1}{m_{(1)}}\right)\rho_{yzwi}S_{zwi}S_{ywi}\right\} \\
V_{011(1)} = \frac{1}{\bar{Z}_s\bar{Y}_s}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yzb}S_{zb}S_{yb} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m_{(1)}}-\frac{1}{M}\right)\rho_{yzwi}S_{zwi}S_{ywi}\right\}
\end{array}
\right\}
\qquad (5.1.6)
$$

5.2 PROPOSED GENERALIZED ESTIMATORS UNDER SITUATION II USING TWO AUXILIARY VARIABLES

In this section generalized estimator VI has been developed by assuming an exponential relationship between study variable and the two auxiliary variable for three different cases (defined in chapter 4) under situation I.

5.2.1 Proposed Generalized Exponential Estimator VI

Case I

i) The exponential-type ratio-cum-ratio estimator may be defined as,

$$t'^{RR}_{25} = \bar{y}'_{s(2)} exp\left(1 - \frac{2\bar{x}'_{s(1)}}{\bar{X}'_s + \bar{x}'_{s(1)}}\right) \exp\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}'_s + \bar{z}'_{s(2)}}\right), \tag{5.2.1}$$

ii) The exponential-type product-cum-ratio estimator may be defined as,

$$t'^{PR}_{25} = \bar{y}'_{s(2)} exp\left(-\left(1 - \frac{2\bar{x}'_{s(1)}}{\bar{X}'_s + \bar{x}'_{s(1)}}\right)\right) \exp\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}'_s + \bar{z}'_{s(2)}}\right), \tag{5.2.2}$$

iii) The exponential-type ratio-cum –product estimator may be defined as,

$$t'^{RP}_{25} = \bar{y}'_{s(2)} \exp\left(1 - \frac{2\bar{x}'_{s(1)}}{\bar{X}'_s + \bar{x}'_{s(1)}}\right) \exp\left(-\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}'_s + \bar{z}'_{s(2)}}\right)\right), \tag{5.2.3}$$

iv) The exponential-type product-cum -product estimator may be defined as,

$$t'^{PP}_{25} = \bar{y}'_{s(2)} exp\left(-\left(1 - \frac{2\bar{x}'_{s(1)}}{\bar{X}'_s + \bar{x}'_{s(1)}}\right)\right) \exp\left(-\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}'_s + \bar{z}'_{s(2)}}\right)\right), \tag{5.2.4}$$

From (5.2.1)-(5.2.4), we may write a generalized form by introducing some real constants α'_{25}, β'_{25}, a'_{25} λ'_{25} and b'_{25}.

$$t'^{G}_{25} = \lambda'_{25}\bar{y}'_s \exp\left(\alpha'_{25}\left(1 - \frac{a'_{25}\bar{x}'_{s(1)}}{\left(\bar{X}'_s + (a'_{25} - 1)\bar{x}'_{s(1)}\right)}\right)\right)$$

$$\exp\left(\beta'_{25}\left(1 - \frac{b'_{25}\bar{z}'_{s(2)}}{\left(\bar{Z}'_s + (b'_{25} - 1)\bar{z}'_{s(2)}\right)}\right)\right), \quad 0 < \lambda'_{25} \le 1 \tag{5.2.5}$$

where (a'_{25}, b'_{25}, λ'_{25}) are constants to be determined such that the mean square error (MSE) is minimum. (α'_{25}, β'_{25}) are known constants takes the value(0,1,-1) to produce different ratio-type and product-type estimators.

Table 5.1
Some Special Casse of Generalized Estimator t'^{G}_{25}

Ratio-cum-product estimator $\alpha'_{25}=1,\ \beta'_{25}=1$	**Product-cum-product estimator** $\alpha'_{25}=-1,\ \beta'_{25}=-1$	a'_{25}	λ'_{25}	b'_{25}
$t'^{1}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{Z}'_s-\bar{z}'_{s(2)}}{\bar{Z}'_s+\bar{z}'_{s(2)}}\right)$	$t'^{2}_{25}=\bar{y}'_s\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{x}'_{s(1)}+\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{z}'_{s(2)}+\bar{Z}'_s}\right)$	2	1	2
$t'^{3}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{Z}'_s-\bar{z}'_{s(2)}}{\bar{Z}'_s}\right)$	$t'^{4}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s}\right)$	1	1	1
$t'^{5}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+(a'_5-1)\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{Z}'_s-\bar{z}'_{s(2)}}{\bar{Z}'_s+(b'_5-1)\bar{z}'_{s(2)}}\right)$	$t'^{6}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s+(a'_5-1)\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+(b'_5-1)\bar{z}'_{s(2)}}\right)$	a'_{25}	1	b'_{25}
$t'^{7}_{25}=\lambda'_{25}\bar{y}'_s\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{Z}'_s-\bar{z}'_{s(2)}}{\bar{Z}'_s+\bar{z}'_{s(2)}}\right)$	$t'^{8}_{25}=\lambda'_5\bar{y}'_s\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+\bar{z}'_{s(2)}}\right)$	2	λ'_{25}	2
$t'^{9}_{25}=\lambda'_5\bar{y}'_s\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{Z}'_s-\bar{z}'_{s(2)}}{\bar{Z}'_s}\right)$	$t'^{10}_{25}=\lambda'_5\bar{y}'_s\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s}\right)$	1	λ'_{25}	1
$t'^{11}_{25}=\lambda'_{25}\bar{y}'_s\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+(a'_5-1)\bar{x}'_s}\right)\exp\left(\frac{\bar{Z}'_s-\bar{z}'_{s(2)}}{\bar{Z}'_s+(b'_5-1)\bar{z}'_{s(2)}}\right)$	$t'^{12}_{25}=\lambda'_{25}\bar{y}'_s\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'}{\bar{X}'_s+(a'_{25}-1)\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+(b'_{25}-1)\bar{z}'_{s(2)}}\right)$	a'_{25}	λ'_{25}	b'_{25}
Product-cum-ratio estimator $\alpha'_{25}=-1,\ \beta'_{25}=1$	**Ratio-cum-product estimator** $\alpha'_{25}=1,\ \beta'_{25}=-1$	a'_{25}	λ'_{25}	b'_{25}
$t'^{13}_{25}=\bar{y}'_s\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+\bar{z}'_{s(2)}}\right)$	$t'^{14}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{x}'_{s(1)}+\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{z}'_{s(2)}+\bar{Z}'_s}\right)$	2	1	2
$t'^{15}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(1)}-\bar{Z}'_s}{\bar{Z}'_s}\right)$	$t'^{16}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'}{\bar{Z}'_s}\right)$	1	1	1
$t'^{17}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+(a'_5-1)\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+(b'_5-1)\bar{z}'_{s(2)}}\right)$	$t'^{18}_{25}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s+(a'_5-1)\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+(b'_5-1)\bar{z}'_{s(2)}}\right)$	a'_{25}	1	b'_{25}
$t'^{19}_{25}=\lambda'_5\bar{y}'_s\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+\bar{z}'_{s(2)}}\right)$	$t'^{20}_{25}=\lambda'_{25}\bar{y}'_s\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+\bar{z}'_{s(2)}}\right)$	2	λ'_{25}	2
$t'^{21}_{25}=\lambda'_5\bar{y}'_s\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s}\right)$	$t'^{22}_{25}=\lambda'_5\bar{y}'_s\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s}\right)$	1	λ'_{25}	1
$t'^{23}_{25}=\lambda'_{25}\bar{y}'_s\exp\left(\frac{\bar{X}'_s-\bar{x}'_{s(1)}}{\bar{X}'_s+(a'_{25}-1)\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+(b'_{25}-1)\bar{z}'_{s(2)}}\right)$	$t'^{24}_{25}=\lambda'_{25}\bar{y}'_s\exp\left(\frac{\bar{x}'_{s(1)}-\bar{X}'_s}{\bar{X}'_s+(a'_{25}-1)\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_s}{\bar{Z}'_s+(b'_{25}-1)\bar{z}'_{s(2)}}\right)$	a'_{25}	λ'_{25}	b'_{25}

The Bias and Mean Square Error of Generalized Estimator VI

To derive the bias and mean square error we proceed as follows:

Using (5.1.1) we can express (5.2.5) as,

$$t_{25}'^{G} = \lambda_{25}' \bar{Y}_s' \left(1 + e_{0(2)}'\right) \exp\left[-\frac{\alpha_{25}'}{a_{25}'} e_{1(1)}' \left(1 + \frac{(a_{25}' - 1)}{a_{25}'} e_{1(1)}'\right)^{-1} \right]$$

$$\exp\left[-\frac{\beta_{25}'}{b_{25}'} e_{2(2)}' \left(1 + \frac{(b_{25}' - 1)}{b_{25}'} e_{2(2)}'\right)^{-1} \right], \qquad (5.2.6)$$

We assume that $\left|e_{1(1)}'\right| < 1$, we expand the series , $\left(1 + \frac{(a_{25}' - 1)}{a_{25}'} e_{1(1)}'\right)^{-1}$ and $\left(1 + \frac{(b_{25}' - 1)}{b_{25}'} e_{2(2)}'\right)^{-1}$, we get,

$$t_{25}'^{G} = \lambda_{25}' \bar{Y}_s' \left(1 + e_{0(2)}'\right) \exp\left[-\frac{\alpha_{25}'}{a_{25}'} e_{1(1)}' \left(1 - \frac{(a_{25}' - 1)}{a_{25}'} e_{1(1)}' + \frac{(a_{25}' - 1)^2}{a_{25}'^2} e_{1(1)}'^2 + \ldots\right) \right]$$

$$\exp\left[-\frac{\beta_{25}'}{b_{25}'} e_{2(2)}' \left(1 - \frac{(b_{25}' - 1)}{b_{25}'} e_{2(2)}' + \frac{(b_{25}' - 1)^2}{b_{25}'^2} e_{2(2)}'^2 + \ldots\right) \right], \qquad (5.2.7)$$

It is assumed that the contribution of terms involving powers in $e_{0(2)}'$, $e_{1(1)}'$ and $e_{2(2)}'$ higher than one is negligible. It is therefore expanding the exponentials and ignoring terms in $e_{0(2)}'$, and $e_{1(1)}'$ of order higher than two, we have

$$t_{25}'^{G} = \lambda_{25}' \bar{Y}_s' \left(1 + e_{0(2)}'\right) \left[1 - \frac{\alpha_{25}'}{a_{25}'} e_{1(1)}' + \frac{\alpha_{25}'^2}{a_{25}'^2} e_{1(1)}'^2 \right] \left[1 - \frac{\beta_{25}'}{b_{25}'} e_{2(2)}' + \frac{\beta_{25}'^2}{b_{25}'^2} e_{2(2)}'^2 \right], \qquad (5.2.8)$$

$$t'^{G}_{25} - \overline{Y}'_s = \lambda'_{25}\overline{Y}'_s\left[e'_{0(2)} - \frac{\alpha'_{25}}{a'_{25}} e'_{1(1)} + \frac{\alpha'^{2}_{25}}{a'^{2}_{25}} e'^{2}_{1(1)} - \frac{\beta'_{25}}{b'_{25}} e'_{2(2)} + \frac{\beta'^{2}_{25}}{b'^{2}_{25}} e'^{2}_{2(2)} \right.$$

$$\left. - \frac{\beta'_{25}}{b'_{25}} e'_{2(2)} e'_{0(2)} - \frac{\alpha'_{25}}{a'_{25}} e'_{1(1)} e'_{0(2)} \right] + \overline{Y}'_s(\lambda'_{25} - 1), \qquad (5.2.9)$$

In order to get the bias, we take expectation on (5.2.9) and get,

$$Bias(t'^{G}_{25}) = \lambda'_{25}\overline{Y}'_s\left[\frac{\alpha'^{2}_{25}}{a'^{2}_{25}} V'_{200(1)} + \frac{\beta'^{2}_{25}}{b'^{2}_{25}} V'_{002(2)} - \frac{\beta'_{25}}{b'_{25}} V'_{011(2)} \right.$$

$$\left. - \frac{\alpha'_{25}}{a'_{25}} V'_{110(1)} + \frac{\alpha'_{25}}{a'_{25}} \frac{\beta'_{25}}{b'_{25}} V'_{101(1)} \right] + \overline{Y}'_s(\lambda'_{25} - 1), \qquad (5.2.10)$$

To get the MSE of the estimator, we take square and retain terms upto second order of e’s then we take expectation of (5.2.9) and we obtain,

$$MSE\left(t'^{G}_{25}\right) = \overline{Y}'^{2}_s \left[\lambda'^{2}_{25} \left(V'_{020(2)} + z'^{2}_{25} V'_{200(1)} + u'^{2}_{25} V'_{002(2)} - 2 z'_{25} V'_{110(1)} \right.\right.$$

$$\left.\left. - 2u'_{25} V'_{011(2)} + 2 z'_{25} u'_{25} V'_{101(1)} \right) + (\lambda'_{25} - 1)^2 \right], \qquad (5.2.11)$$

where $z'_{25} = \dfrac{\alpha'_{25}}{a'_{25}}$ and $u'_{25} = \dfrac{\beta'_{25}}{b'_{25}}$

For the following optimal value of the constants z'_{25} and u'_{25}, we achieve the minimum Variance among the class of proposed generalized estimator,

$$z'_{25} = \frac{V'_{110(1)} V'_{002(2)} - V'_{011(1)} V'_{101(2)}}{V'_{200(1)} V'_{002(2)} - V'^{2}_{101(2)}}$$

and

$$u'_{25} = \frac{V'_{200(1)} V'_{011(1)} - V'_{110(1)} V'_{101(2)}}{V'_{200(1)} V'_{002(2)} - V'^{2}_{101(2)}}, \lambda'_{25} = \frac{1}{1 + A'^{G}_{25}}$$

where

$$A'^{G}_{25} = \left[V'_{020(2)} + z'^{2}_{25} V'_{200(1)} + u'^{2}_{25} V'_{002(2)} - 2 z'_{25} V'_{110(1)} \right.$$

$$\left. - 2u'_{25} V'_{011(2)} + 2 z'_{25} u'_{25} V'_{101(1)} \right]$$

By substituting the optimum values of z_5' and u_5', we get $\lambda_5'^{opt}$ as,

$$\lambda_{25}'^{opt} = \frac{1}{1 + A_{25}'^{*}}$$

where

$$A_{25}'^{*} = \left(V_{020(2)}' - \frac{V_{110(1)}'^{2} V_{002(2)}' + V_{011(1)}'^{2} V_{200(1)}' - 2V_{110(1)}' V_{101(2)}' V_{011(1)}'}{V_{200(1)}' V_{002(2)}' - V_{101(2)}'^{2}} \right), \tag{5.2.12}$$

We obtain asymptotic variance of the proposed generalized estimator as the expression for $Var\left(t_{25}'^{G}\right)$ was considered up to 1st degree of error term, so minimum MSE may be written as,

$$MSE_{\min}\left(t_{25}'^{G}\right) = AsymtoticVar\left(t_{25}'^{G}\right) = \bar{Y}_s'^{2}\left(\frac{A_{25}'^{*}}{1 + A_{25}'^{*}}\right). \tag{5.2.13}$$

From (5.2.13), we observe that asymptotic variance of the proposed estimator is less than the Variance of **Usual Linear Regression Estimator.** We may observe from (5.2.13) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value $\lambda_{25}'^{opt}$ and $a_{25}'^{opt}, b_{25}'^{opt}$ in (5.2.5) , we get optimal estimator as,

$$\hat{t}_{25}'^{G} = \hat{\lambda}_{25}' \bar{y}_s' \exp\left(\alpha_{25}' \left(1 - \frac{\hat{a}_{25}' \bar{x}_{s(1)}'}{\left(\bar{X}_s' + (\hat{a}_{25}' - 1)\bar{x}_{s(1)}' \right)} \right) \right) \exp\left(\beta_{25}' \left(1 - \frac{\hat{b}_{25}' \bar{z}_{s(2)}'}{\left(\bar{Z}_s' + (\hat{b}_{25}' - 1)\bar{z}_{s(2)}' \right)} \right) \right), \quad 0 < \lambda_{25}' \le 1 \tag{5.2.14}$$

As described earlier in section (5.2.1.1.1) that in some practical situations, when it becomes difficult to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimates as,

$$\hat{a}'^{\,opt}_{25} = \frac{\alpha'_{25}\left(\hat{V}'_{200(1)}\hat{V}'_{002(2)} - \hat{V}'^{\,2}_{101(2)}\right)}{\hat{V}'_{110(1)}\hat{V}'_{002(2)} - \hat{V}'_{011(1)}\hat{V}'_{101(2)}},$$

and

$$\hat{b}'^{\,opt}_{25} = \frac{\beta'_{25}\left(\hat{V}'_{200(1)}\hat{V}'_{002(2)} - \hat{V}'^{\,2}_{101(2)}\right)}{\hat{V}'_{200(1)}\hat{V}'_{011(1)} - \hat{V}'_{110(1)}\hat{V}'_{101(2)}},$$

$$\hat{\lambda}'^{\,opt}_{25} = \frac{1}{1+\hat{A}'^{\,*}_{25}}$$

where

$$\hat{A}'^{\,*}_{25} = \left(\hat{V}'_{020(2)} - \frac{\hat{V}'^{\,2}_{110(1)}\hat{V}'_{002(2)} + \hat{V}'^{\,2}_{011(1)}\hat{V}'_{200(1)} - 2\hat{V}'_{110(1)}\hat{V}'_{101(2)}\hat{V}'_{011(1)}}{\hat{V}'_{200(1)}\hat{V}'_{002(2)} - \hat{V}'^{\,2}_{101(2)}}\right),$$

(5.2.15)

So (5.2.15) may be wriiten as,

$$\hat{t}'^{G}_{25} = \hat{\lambda}'^{\,opt}_{25}\ \overline{y}'_s \exp\left(\alpha'_{25}\left(1-\frac{\hat{a}'^{\,opt}_{25}\ \overline{x}'_{s(1)}}{\left(\overline{X}'_s + (\hat{a}'^{\,opt}_{25}-1)\overline{x}'_{s(1)}\right)}\right)\right)$$
$$\exp\left(\beta'_{25}\left(1-\frac{\hat{b}'^{\,opt}_{25}\overline{z}'_{s(2)}}{\left(\overline{Z}'_s + (\hat{b}'^{\,opt}_{25}-1)\overline{z}'_{s(2)}\right)}\right)\right),\quad 0<\lambda'_{25}\le 1 \qquad (5.2.16)$$

Also the minimum Variance may be written as,

$$Var_{\min}\left(\hat{t}'^{G}_{25}\right) = AsymtoticVar\left(\hat{t}'^{G}_{25}\right) = \overline{Y}'^{2}_s\left(\frac{\hat{A}'^{\,*}_{25}}{1+\hat{A}'^{\,*}_{25}}\right). \qquad (5.2.17)$$

Remark 5.1

i) For $\alpha'_{25}=1, \beta'_{25}=1$, we get exponential-type ratio-cum-ratio estimators given in Table 5.1. The variance of t'^{G}_{25} is expressed using (5.2.11) as,

$$
\mathrm{MSE}(t'^{j}_{25}) = \begin{cases} \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{25}\begin{pmatrix} V'_{020(2)} + V'_{200(1)} + V'_{002(2)} \\ -2V'_{110(1)} - 2V_{011(2)} + 2V_{101(1)} \end{pmatrix} + (\lambda'_{25}-1)^{2}\right) & j(\in G) = 1 \\ \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{25}\begin{pmatrix} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^{2}}_{25}}V'_{200(1)} + \dfrac{1}{b'^{\left(\frac{j-1}{2}\right)^{2}}_{25}}V'_{002(2)} \\ -2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{25}}V'_{110(1)} - 2\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{25}}V_{011(2)} \\ +2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{25}}\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{25}}V_{101(1)} \end{pmatrix} + (\lambda'_{25}-1)^{2}\right) & j(\in G) = 3,5,\ldots,11 \end{cases},
$$

(5.2.18)

The optimal values which lead to minimum MSE as,

$$
a'^{\left(\frac{j-1}{2}\right)}_{25} = \frac{V'_{200(1)}V'_{002(2)} - V'^{2}_{101(2)}}{V'_{110(1)}V'_{002(2)} - V'_{011(1)}V'_{101(2)}}
$$

and

$$
b'^{\left(\frac{j-1}{2}\right)}_{25} = \frac{V'_{200(1)}V'_{002(2)} - V'^{2}_{101(2)}}{V'_{200(1)}V'_{011(1)} - V'_{110(1)}V'_{101(2)}}, \lambda'_{25} = \frac{1}{1 + A'^{G}_{25}}
$$

where

$$
A'^{G}_{25} = \begin{bmatrix} V'_{020} + \dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^{2}}_{25}}V'_{200(1)} + \dfrac{1}{b'^{\left(\frac{j-1}{2}\right)^{2}}_{25}}V_{002(2)} - 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{25}}V_{110(1)} \\ -2\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{25}}V_{011(2)} + 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{25}}\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{25}}V_{101(1)} \end{bmatrix},
$$

ii) For $\alpha'_{25} = -1, \beta'_{25} = -1$, we get exponential-type product-cum-product estimators given in Table 5.1. The Variance of t'^{G}_{5} is expressed using (5.2.11) as,

$$
MSE(t_{25}'^{k}) = \begin{cases} \bar{Y}_s'^2 \left(\lambda_{25}'^2 \begin{pmatrix} V_{020(2)}' + V_{200(1)}' + V_{002(2)}' + 2V_{110(1)}' \\ +2V_{011(2)} - 2V_{101(1)} \end{pmatrix} + (\lambda_{25}' - 1)^2 \right) & k(\in G) = 2 \\ \bar{Y}_s'^2 \left(\lambda_{25}'^2 \begin{pmatrix} V_{020(2)}' + \frac{1}{a_{25}'^{\left(\frac{k}{2}\right)^2}} V_{200(1)}' + \frac{1}{b_{25}'^{\left(\frac{k}{2}\right)^2}} V_{002(2)}' \\ +2 \frac{1}{a_{25}'^{\left(\frac{k}{2}\right)}} V_{110(1)}' + 2 \frac{1}{b_{25}'^{\left(\frac{k}{2}\right)}} V_{011(2)}' \\ -2 \frac{1}{a_{25}'^{\left(\frac{k}{2}\right)}} \frac{1}{b_{25}'^{\left(\frac{k}{2}\right)}} V_{101(1)}' \end{pmatrix} + (\lambda_{25}' - 1)^2 \right) & k(\in G) = 4,6,\ldots,12 \end{cases},
$$

(5.2.19)

The optimal values which lead to minimum MSE as,

$$
a_{25}'^{\left(\frac{j-1}{2}\right)} = \frac{V_{200(1)}' V_{002(2)}' - V_{101(2)}'^{\,2}}{V_{110(1)}' V_{002(2)}' - V_{011(1)}' V_{101(2)}'}
$$

and

$$
b_{25}'^{\left(\frac{j-1}{2}\right)} = \frac{V_{200(1)}' V_{002(2)}' - V_{101(2)}'^{\,2}}{V_{200(1)}' V_{011(1)}' - V_{110(1)}' V_{101(2)}'}, \lambda_{25}' = \frac{1}{1 + A_{25}'^{\,G}}
$$

where

$$
A_{25}'^{G} = \begin{bmatrix} V_{020}' + \frac{1}{a_{25}'^{\left(\frac{j-1}{2}\right)^2}} V_{200(1)}' + \frac{1}{b_{25}'^{\left(\frac{j-1}{2}\right)^2}} V_{002(2)} - 2 \frac{1}{a_{25}'^{\left(\frac{j-1}{2}\right)}} V_{110(1)} \\ -2 \frac{1}{b_{25}'^{\left(\frac{j-1}{2}\right)}} V_{011(2)} + 2 \frac{1}{a_{25}'^{\left(\frac{j-1}{2}\right)}} \frac{1}{b_{25}'^{\left(\frac{j-1}{2}\right)}} V_{101(1)} \end{bmatrix},
$$

iii) For $\alpha_{25}' = -1, \beta_{25}' = 1$, we get exponential-type product-cum-ratio estimators given in Table 5.1. The Variance of $t_{25}'^{G}$ is expressed as,

$$Var(t'^{l}_{25}) = \begin{cases} \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{25}\begin{pmatrix} V'_{020(2)} + V'_{200(1)} + V'_{002(2)} + 2V'_{110(1)} \\ -2V'_{011(2)} - 2V'_{101(1)} \end{pmatrix} + (\lambda'_{25} - 1)^{2}\right) & l(\in G) = 13 \\ \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{25}\begin{pmatrix} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^{2}}_{25}}V'_{200(1)} + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^{2}}_{25}}V'_{002(2)} \\ +2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{25}}V'_{110(1)} - 2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{25}}V'_{011(2)} \\ -2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{25}}\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{25}}V'_{101(1)} \end{pmatrix} + (\lambda'_{25} - 1)^{2}\right) & l(\in G) = 15,17,\ldots,23 \end{cases},$$

(5.2.21)

The optimal values which lead to minimum Vaiance as,

$$a'^{\left(\frac{l-1}{2}\right)}_{25} = \frac{V'_{200(1)}V'_{002(2)} - V'^{2}_{101(2)}}{V'_{110(1)}V'_{002(2)} - V'_{011(1)}V'_{101(2)}}$$

and

$$b'^{\left(\frac{l-1}{2}\right)}_{25} = \frac{V'_{200(1)}V'_{002(2)} - V'^{2}_{101(2)}}{V'_{200(1)}V'_{011(1)} - V'_{110(1)}V'_{101(2)}}, \lambda'_{25} = \frac{1}{1 + A'^{G}_{25}}$$

where

$$A'^{G}_{25} = \begin{bmatrix} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^{2}}_{25}}V'_{200(1)} + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^{2}}_{25}}V'_{002(2)} - 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{25}}V'_{110(1)} \\ -2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{25}}V'_{011(2)} + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{25}}\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{25}}V'_{101(1)} \end{bmatrix},$$

iv) For $\alpha'_{25} = 1, \beta'_{25} = -1$, we get exponential-type ratio-cum-product estimators given in Table 5.1. The Variance of t'^{G}_{25} is expressed as,

$$
MSE(t'^{m}_{25}) = \begin{cases} \bar{Y}'^{2}_{s}\left[\lambda'^{2}_{25}\begin{pmatrix} V'_{020(2)} + V'_{200(1)} + V'_{002(2)} \\ -2V'_{110(1)} + 2V'_{011(2)} - 2V'_{101(1)} \end{pmatrix} + (\lambda'_{25}-1)^{2}\right] & m(\in G) = 14 \\ \bar{Y}'^{2}_{s}\left[\lambda'^{2}_{25}\begin{pmatrix} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{k}{2}\right)^{2}}_{25}} V'_{200(1)} + \dfrac{1}{b'^{\left(\frac{k}{2}\right)^{2}}_{25}} V'_{002(2)} \\ -2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{25}} V'_{110(1)} + 2\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{25}} V'_{011(2)} \\ -2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{25}}\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{25}} V'_{101(1)} \end{pmatrix} + (\lambda'_{25}-1)^{2}\right] & m(\in G) = 16,18,\ldots,24 \end{cases},
\tag{5.2.22}
$$

The optimal values which lead to minimum MSE as,

$$
a'^{\left(\frac{j-1}{2}\right)}_{25} = \frac{V'_{200(1)}V'_{002(2)} - V'^{2}_{101(2)}}{V'_{110(1)}V'_{002(2)} - V'_{011(1)}V'_{101(2)}}
$$

and

$$
b'^{\left(\frac{j-1}{2}\right)}_{25} = \frac{V'_{200(1)}V'_{002(2)} - V'^{2}_{101(2)}}{V'_{200(1)}V'_{011(1)} - V'_{110(1)}V'_{101(2)}}, \lambda'_{25} = \frac{1}{1 + A'^{G}_{25}}
$$

where

$$
A'^{G}_{25} = \begin{bmatrix} V'_{020} + \dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^{2}}_{25}} V'_{200(1)} + \dfrac{1}{b'^{\left(\frac{j-1}{2}\right)^{2}}_{25}} V_{002(2)} - 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{25}} V_{110(1)} \\ -2\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{25}} V_{011(2)} + 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{25}}\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{25}} V_{101(1)} \end{bmatrix},
$$

Case II:

The generalized estimator-I under case II may be proposed by following (5.2.1) as,

$$
t''^{G}_{25} = \lambda''_{25}\bar{y}''_{s}\exp\left(\alpha''_{25}\left(1 - \frac{a''_{25}\ \bar{x}''_{s(1)}}{\left(\bar{X}''_{s} + (a''_{25}-1)\bar{x}''_{s(1)}\right)}\right)\right)
$$

$$
\exp\left(\beta''_{25}\left(1 - \frac{b''_{25}\ \bar{z}'_{s(2)}}{\left(\bar{Z}''_{s} + (b''_{25}-1)\bar{z}''_{s(2)}\right)}\right)\right), \quad 0 < \lambda''_{25} \le 1 \tag{5.2.23}
$$

where (a''_{25}, b''_{25}, λ''_{25}) are constants to be determined such that the mean square error (MSE) is minimum. (α''_{25}, β''_{25}) are known constants takes the value (0,1,-1) to produce different ratio-type and product-type estimators.

The proposed estimator in (5.2.23) follows the same way along with the class of estimators given in Table 5.1, as that for case-I in Section 5.2.1. In addition, the relationships between a''_{25}, α''_{25}, λ''_{25} and b''_{25}, β''_{25} in case-II is also the same as that for case-I in Section 5.2.1. Finally, the same is true for the Variance and the Bias. It is therefore directly from Section 5.2.1, we may write $\mathrm{Bias}\,(t''^{G}_{25})$ and $\mathrm{Var}(t''^{G}_{25})$ following the same, and we may also produce a class of estimators for similar choices of a''_{25}, α''_{25} and λ''_{25} in case-II. The bias of (5.2.23) may be obtain directly using the notations and expectations presented in Section 5.1 for case II.

$$Bias(t''^{G}_{25}) = \lambda''_{25}\bar{Y}''_s\left[\frac{\alpha''^{2}_{25}}{a''^{2}_{25}}V''_{200(1)} + \frac{\beta''^{2}_{25}}{b''^{2}_{25}}V''_{002(2)} - \frac{\beta''_{25}}{b''_{25}}V''_{011(2)} - \frac{\alpha''_{25}}{a''_{25}}V'_{110(1)} + \frac{\alpha''_{25}}{a''_{25}}\frac{\beta''_{25}}{b''_{25}}V''_{101(1)}\right] + \bar{Y}''_s(\lambda''_{25} - 1), \tag{5.2.24}$$

Similarly the expression of Variance may also be given as,

$$Var\left(t''^{G}_{25}\right) = \bar{Y}''^{2}_s\left[\lambda''^{2}_{25}\left(V''_{020(2)} + z''^{2}_{25}V''_{200(1)} + u''^{2}_{25}V''_{002(2)} - 2z''_{25}V''_{110(1)} - 2u''_{25}V''_{011(2)} + 2z''_{25}u''_{25}V''_{101(1)}\right) + (\lambda''_{25} - 1)^2\right], \tag{5.2.25}$$

where $z''_{25} = \dfrac{\alpha''_{25}}{a''_{25}}$ and $u''_{25} = \dfrac{\beta''_{25}}{b''_{25}}$.

For the following optimal value of the constants z''_{25} and u''_{25}, we achieve the minimum Variance among the class of proposed generalized estimator.

$$z''_{25} = \frac{V''_{110(1)}V''_{002(2)} - V''_{011(1)}V''_{101(2)}}{V''_{200(1)}V''_{002(2)} - V''^{2}_{101(2)}} \text{ and } u''_{25} = \frac{V''_{200(1)}V''_{011(1)} - V''_{110(1)}V''_{101(2)}}{V''_{200(1)}V''_{002(2)} - V''^{2}_{101(2)}},$$

$$\lambda''_{25} = \frac{1}{1 + A''^{G}_{25}}$$

where

$$A''^{G}_{25}=\begin{bmatrix} V''_{020(2)}+z''^{2}_{25}V''_{200(1)}+u''^{2}_{25}V''_{002(2)} \\ -2z''_{25}V''_{110(1)}-2u''_{25}V''_{011(2)}+2z''_{25}u''_{25}V''_{101(1)} \end{bmatrix},$$

By substituting the optimum values of z''_{25} and u''_{25}, we get λ''^{opt}_{25} as,

$$\hat{\lambda}''^{opt}_{25}=\frac{1}{1+\hat{A}''^{*}_{25}}$$

where

$$\hat{A}''^{*}_{25}=\left(\hat{V}''_{020(2)}-\frac{\hat{V}''^{2}_{110(1)}\hat{V}''_{002(2)}+\hat{V}''^{2}_{011(1)}\hat{V}''_{200(1)}-2\hat{V}''_{110(1)}\hat{V}''_{101(2)}\hat{V}''_{011(1)}}{\hat{V}''_{200(1)}\hat{V}''_{002(2)}-\hat{V}''^{2}_{101(2)}}\right), \tag{5.2.26}$$

The minimum MSE may be obtained as,

.

$$Var_{\min}\left(\hat{t}'^{G}_{25}\right)=AsymtoticVar\left(\hat{t}'^{G}_{25}\right)=\bar{Y}'^{2}_{s}\left(\frac{\hat{A}'^{*}_{25}}{1+\hat{A}'^{*}_{25}}\right). \tag{5.2.27}$$

Remark 5.2

i) For $\alpha''_{25}=1,\beta''_{25}=1$, we get exponential-ratio type estimators given in Table 5.1. The Variance of t''^{G}_{25} is expressed using (5.2.25) as,

$$MSE(t''^{j}_{25})=\begin{cases} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{25}\left(\begin{aligned}&V''_{020(2)}+V''_{200(1)}+V''_{002(2)}\\&-2V''_{110(1)}-2V''_{011(2)}+2V''_{101(1)}\end{aligned}\right)+(\lambda''_{25}-1)^{2}\right) & j(\in G)=1 \\ \bar{Y}''^{2}_{s}\left(\lambda'^{2}_{25}\left(\begin{aligned}&V''_{020(2)}+\frac{1}{a''^{\left(\frac{j-1}{2}\right)^{2}}_{25}}V''_{200(1)}+\frac{1}{b''^{\left(\frac{j-1}{2}\right)^{2}}_{25}}V''_{002(2)}\\&-2\frac{1}{a''^{\left(\frac{j-1}{2}\right)}_{25}}V''_{110(2)}-2\frac{1}{b''^{\left(\frac{j-1}{2}\right)}_{25}}V''_{011(2)}\\&+2\frac{1}{a''^{\left(\frac{j-1}{2}\right)}_{25}}\frac{1}{b''^{\left(\frac{j-1}{2}\right)}_{25}}V''_{101(1)}\end{aligned}\right)+(\lambda''_{25}-1)^{2}\right) & j(\in G)=3,5,\ldots,11 \end{cases},$$

(5.2.28)

The optimal values which lead to minimum MSE as,

$$a_{25}''^{\left(\frac{j-1}{2}\right)} = \frac{V''_{200(1)}V''_{002(2)} - V''^{\,2}_{101(2)}}{V''_{110(1)}V''_{002(2)} - V''_{011(1)}V''_{101(2)}}$$

and

$$b_{25}''^{\left(\frac{j-1}{2}\right)} = \frac{V''_{200(1)}V''_{002(2)} - V''^{\,2}_{101(2)}}{V''_{200(1)}V''_{011(1)} - V''_{110(1)}V''_{101(2)}}, \lambda''_{25} = \frac{1}{1 + A''^{\,G}_{25}},$$

where

$$A''^{\,G}_{25} = \left[\begin{array}{l} V''_{020(2)} + \dfrac{1}{a_{25}''^{\left(\frac{j-1}{2}\right)^2}}V''_{200(1)} + \dfrac{1}{b_{25}''^{\left(\frac{j-1}{2}\right)^2}}V''_{002(2)} - 2\dfrac{1}{a_{25}''^{\left(\frac{j-1}{2}\right)}}V''_{110(1)} \\ -2\dfrac{1}{b_{25}''^{\left(\frac{j-1}{2}\right)}}V''_{011(2)} + 2\dfrac{1}{a_{25}''^{\left(\frac{j-1}{2}\right)}}\dfrac{1}{b_{25}''^{\left(\frac{j-1}{2}\right)}}V''_{101(1)} \end{array}\right],$$

ii) For $\alpha''_{25} = -1, \beta''_{25} = -1$, we get exponential-ratio product estimators given in Table 5.1. The Variance of t''^{G}_{25} is expressed using (5.2.25) as,

$$MSE(t''^{k}_{25}) = \left\{\begin{array}{ll} \bar{Y}_s''^2\left(\lambda''^{2}_{25}\left(\begin{array}{l} V''_{020(2)} + V''_{200(1)} + V''_{002(2)} \\ + 2V''_{110(1)} + 2V''_{011(2)} - 2V''_{101(1)} \end{array}\right) + (\lambda''_{25} - 1)^2\right) & k(\in G) = 2 \\ \bar{Y}_s''^2\left(\lambda''^{2}_{25}\left(\begin{array}{l} V''_{020(2)} + \dfrac{1}{a_{25}''^{\left(\frac{k}{2}\right)^2}}V''_{200(1)} + \dfrac{1}{b_{25}''^{\left(\frac{k}{2}\right)^2}}V''_{002(2)} \\ + 2\dfrac{1}{a_{25}''^{\left(\frac{k}{2}\right)}}V''_{110(1)} + 2\dfrac{1}{b_{25}''^{\left(\frac{k}{2}\right)}}V''_{011(2)} \\ - 2\dfrac{1}{a_{25}''^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{25}''^{\left(\frac{k}{2}\right)}}V''_{101(1)} \end{array}\right) + (\lambda''_{25} - 1)^2\right) & k(\in G) = 4,6,\ldots,12 \end{array}\right\}, \tag{5.2.29}$$

The optimal values which lead to minimum MSE as,

$$a_{25}''^{\left(\frac{k}{2}\right)} = \frac{V''_{200(1)}V''_{002(2)} - V''^{\,2}_{101(2)}}{V''_{110(1)}V''_{002(2)} - V''_{011(1)}V''_{101(2)}}$$

and

$$b''^{\left(\frac{k}{2}\right)}_{25} = \frac{V''_{200(1)}V''_{002(2)} - V''^{\,2}_{101(2)}}{V''_{200(1)}V''_{011(1)} - V''_{110(1)}V''_{101(2)}}, \lambda''_{25} = \frac{1}{1 + A''^{\;G}_{25}},$$

where

$$A''^{\;G}_{25} = \left[\begin{array}{c} V''_{020(2)} + \dfrac{1}{\left(a''^{\left(\frac{k}{2}\right)}_{25}\right)^2} V''_{200(1)} + \dfrac{1}{\left(b''^{\left(\frac{k}{2}\right)}_{25}\right)^2} V''_{002(2)} - 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{25}} V''_{110(1)} \\ -2\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{25}} V''_{011(2)} + 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{25}} \dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{25}} V''_{101(1)} \end{array}\right],$$

iii) For $\alpha''_{25} = -1, \beta''_{25} = 1$, we get exponential-type product cum ratio estimators given in Table 5.1. The Variance of t''^{G}_{25} is expressed using (5.2.25) as,

$$MSE(t''^{\,l}_{25}) = \left\{\begin{array}{ll} \bar{Y}''^{2}_{s}\left(\lambda''^{\,2}_{25}\left(\begin{array}{l} V''_{020(2)} + V''_{200(1)} + V''_{002(2)} \\ \quad + 2V''_{110(1)} - 2V''_{011(2)} - 2V''_{101(1)} \end{array}\right) + (\lambda'_{25}{}' - 1)^2\right) & l(\in G) = 13 \\ \bar{Y}''^{2}_{s}\left(\lambda''^{\,2}_{25}\left(\begin{array}{l} V''_{020(2)} + \dfrac{1}{\left(a'^{\left(\frac{l-1}{2}\right)}_{25}\right)^2} V''_{200(1)} + \dfrac{1}{\left(b'^{\left(\frac{l-1}{2}\right)}_{25}\right)^2} V''_{002(2)} \\ + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{25}} V''_{110(1)} - 2\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{25}} V''_{011(2)} \\ - 2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{25}} \dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{25}} V''_{101(1)} \end{array}\right) + (\lambda''_{25} - 1)^2\right) & l(\in G) = 15, 17, \ldots, 23 \end{array}\right\},$$

(5.2.30)

The optimal values which lead to minimum MSE as,

$$a''^{\left(\frac{l-1}{2}\right)}_{25} = \frac{V''_{200(1)}V''_{002(2)} - V''^{\,2}_{101(2)}}{V''_{110(1)}V''_{002(2)} - V''_{011(1)}V''_{101(2)}}$$

and

$$b''^{\left(\frac{l-1}{2}\right)}_{25} = \frac{V''_{200(1)}V''_{002(2)} - V''^{\,2}_{101(2)}}{V''_{200(1)}V''_{011(1)} - V''_{110(1)}V''_{101(2)}}, \lambda''_{25} = \frac{1}{1 + A''^{G}_{25}},$$

where

$$A_{25}''^{G}=\left[\begin{array}{l} V''_{020(2)}+\dfrac{1}{a_{25}''^{\left(\frac{l-1}{2}\right)^2}}V''_{200(1)}+\dfrac{1}{b_{25}''^{\left(\frac{l-1}{2}\right)^2}}V''_{002(2)}-2\dfrac{1}{a_{25}''^{\left(\frac{l-1}{2}\right)}}V''_{110(1)} \\ -2\dfrac{1}{b_{25}''^{\left(\frac{l-1}{2}\right)}}V''_{011(2)}+2\dfrac{1}{a_{25}''^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{25}''^{\left(\frac{l-1}{2}\right)}}V''_{101(1)} \end{array}\right],$$

iv) For $\alpha''_{25}=1,\beta''_{25}=-1$, we get exponential-ratio cum product type estimators given in Table 5. The Variance of t_{25}^{G} is expressed using (5.2.25) as,

$$\mathrm{MSE}(t_{25}''^{m})=\left\{\begin{array}{ll} \bar{Y}_s''^2\left(\lambda_{25}''^2\left(\begin{array}{l} V''_{020(2)}+V''_{200(1)}+V''_{002(2)} \\ -2V''_{110(1)}+2V''_{011(2)}-2V''_{101(1)} \end{array}\right)+(\lambda_5''-1)^2\right) & m(\in G)=14 \\ \bar{Y}_s''^2\left(\lambda_{25}''^2\left(\begin{array}{l} V''_{020(2)}+\dfrac{1}{a_{25}''^{\left(\frac{k}{2}\right)^2}}V''_{200(1)}+\dfrac{1}{b_{25}''^{\left(\frac{k}{2}\right)^2}}V''_{002(2)} \\ -2\dfrac{1}{a_{25}''^{\left(\frac{k}{2}\right)}}V''_{110(1)}+2\dfrac{1}{b_{25}''^{\left(\frac{k}{2}\right)}}V''_{011(2)} \\ -2\dfrac{1}{a_{25}''^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{25}''^{\left(\frac{k}{2}\right)}}V''_{101(1)} \end{array}\right)+(\lambda_{25}''-1)^2\right) & m(\in G)=16,18,\ldots,24 \end{array}\right\},$$

(5.2.31)

The optimal values which lead to minimum MSE as,

$$a_{25}''^{\left(\frac{m}{2}\right)}=\frac{V''_{200(1)}V''_{002(2)}-V''^{\,2}_{101(2)}}{V''_{110(1)}V''_{002(2)}-V''_{011(1)}V''_{101(2)}}$$

and

$$b_{25}''^{\left(\frac{m}{2}\right)}=\frac{V''_{200(1)}V''_{002(2)}-V''^{\,2}_{101}}{V''_{200(1)}V''_{011(1)}-V''_{110(1)}V''_{101(2)}},\lambda''_{25}=\frac{1}{1+A_{25}''^{G}},$$

where

$$A_{25}''^{\;G}=\left[\begin{array}{l} V_{020(2)}+\dfrac{1}{a_{25}''^{\left(\frac{m}{2}\right)^2}}V''_{200(1)}+\dfrac{1}{b_{25}''^{\left(\frac{m}{2}\right)^2}}V''_{002(2)}-2\dfrac{1}{a_{25}''^{\left(\frac{m}{2}\right)}}V''_{110(1)} \\ +2\dfrac{1}{b_{25}''^{\left(\frac{m}{2}\right)}}V''_{011(2)}-2\dfrac{1}{a_{25}''^{\left(\frac{m}{2}\right)}}\dfrac{1}{b_{25}''^{\left(\frac{m}{2}\right)}}V''_{101(1)} \end{array}\right],$$

Case III:

The generalized estimator under case III may be proposed following (5.2.5) as,

$$t_{25}^{G}=\lambda_{25}\bar{y}_s\exp\left(\alpha_5\left(1-\frac{a_{25}\bar{x}_{s(1)}}{\left(\bar{X}_s+(a_{25}-1)\bar{x}_{s(1)}\right)}\right)\right)$$

$$\exp\left(\beta_{25}\left(1-\frac{b_{25}\bar{z}_{s(2)}}{\left(\bar{Z}_s+(b_{25}-1)\bar{z}_{s(2)}\right)}\right)\right),\quad 0<\lambda_{25}\leq 1 \qquad (5.2.32)$$

where (a_{25}, b_{25}, λ_{25}) are constants to be determined such that the mean square error (MSE) is minimum. (α_{25}, β_{25}) are known constants takes the value (0,1,-1) to produce different ratio-type and product-type estimators.

The proposed estimator in (5.2.32) follows naturally in exactly the same fashion along with the class of estimator in Table 1, as that for case-I in Section 5.2.1. In addition, the relation between a_{25}, α_{25}, λ_{25} and b_{25}, β_{25} in case-II is also the same as that for case-I in Section 5.2.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 5.2.1, we may write $\mathrm{Bias}(t_5^G)$ and $\mathrm{MSE}(t_5^G)$ following the same, and we may also produce a class of estimators for similar choices of a_{25}, α_{25} and λ_{25} b_{25}, β_{25} in case-III.

$$Bias(t_{25}^{G})=\lambda_{25}\bar{Y}_s\left[\frac{\alpha_{25}{}^2}{a_{25}{}^2}V_{200(1)}+\frac{\beta_{25}{}^2}{b_{25}{}^2}V_{002(2)}-\frac{\beta_{25}}{b_{25}}V_{011(2)}\right.$$

$$\left.-\frac{\alpha_{25}}{a_{25}}V_{110(1)}+\frac{\alpha_{25}}{a_{25}}\frac{\beta_{25}}{b_{25}}V_{101(1)}\right]+\bar{Y}_s(\lambda_{25}-1), \qquad (5.2.33)$$

$$MSE\left(t_{25}^{G}\right)=\bar{Y}_s{}^2\left[\lambda_{25}{}^2\left(V_{020(2)}+z_{25}{}^2V_{200(1)}+u_{25}{}^2V_{002(2)}-2z_{25}V_{110(1)}\right.\right.$$

$$\left.\left.-2u_{25}V_{011(2)}+2z_{25}u_{25}V_{101(1)}\right)+(\lambda_{25}-1)^2\right], \qquad (5.2.34)$$

For the following optimal value of the constants z_5 and u_5, we achieve the minimum MSE among the class of proposed generalized estimator.

$$z_{25} = \frac{V_{110(1)}V_{002(2)} - V_{011(1)}V_{101(2)}}{V_{200(1)}V_{002(2)} - V''_{101(2)}{}^{2}}$$

and

$$u_{25} = \frac{V_{200(1)}V_{011(1)} - V_{110(1)}V_{101(2)}}{V_{200(1)}V_{002(2)} - V_{101(2)}{}^{2}}, \lambda_{25} = \frac{1}{1 + A_{25}{}^{G}},$$

where

$$A_{25}{}^{G} = \begin{bmatrix} V_{020(2)} + z_{25}{}^{2}V_{200(1)} + u_{25}{}^{2}V_{002(2)} - 2z_{25}V_{110(1)} \\ -2u_{25}V_{011(2)} + 2z_{25}u_{25}V_{101(1)} \end{bmatrix},$$

By substituting the optimum values of z_{25} and u_{25}, we get λ''^{opt}_{5} as,

$$\lambda_{25}{}^{opt} = \frac{1}{1 + A_{25}{}^{*}}$$

where

$$A_{25}{}^{*} = \left(V_{020(2)} - \frac{V_{110(1)}{}^{2}V_{002(2)} + V_{011(1)}{}^{2}V_{200(1)} - 2V_{110(1)}V_{101(2)}V_{011(1)}}{V_{200(1)}V_{002(2)} - V_{101(2)}{}^{2}} \right), \tag{5.2.35}$$

The minimum MSE may be obtained as,

$$MSE_{\min}\left(\hat{t}_{25}^{G}\right) = \bar{Y}_{s}^{2}\left(\frac{A_{25}{}^{*}}{1 + A_{25}{}^{*}}\right). \tag{5.2.36}$$

Remark 5.3

i) For $\alpha_{25} = 1, \beta_{25} = 1$, we get exponential-ratio type estimators given in Table 5.1. The MSE of t_{25}^{G} is expressed using (5.2.33) as

$$
MSE(t_{25}^{j}) = \begin{cases} \bar{Y}_s^2 \left\{ \lambda_{25}^2 \begin{pmatrix} V_{020(2)} + V_{200(1)} + V_{002(2)} \\ -2V_{110(1)} - 2V_{011(2)} + 2V_{101(1)} \end{pmatrix} + (\lambda_{25} - 1)^2 \right\} & j(\in G) = 1 \\ \bar{Y}_s^2 \left\{ \lambda'^{2}_{25} \begin{pmatrix} V_{020(2)} + \dfrac{1}{a_{25}^{\left(\frac{j-1}{2}\right)^2}} V_{200(1)} + \dfrac{1}{b_{25}^{\left(\frac{j-1}{2}\right)^2}} V_{002(2)} \\ -2\dfrac{1}{a_{25}^{\left(\frac{j-1}{2}\right)}} V_{110(2)} - 2\dfrac{1}{b_{25}^{\left(\frac{j-1}{2}\right)}} V_{011(2)} \\ +2\dfrac{1}{a_{25}^{\left(\frac{j-1}{2}\right)}} \dfrac{1}{b_{25}^{\left(\frac{j-1}{2}\right)}} V_{101(1)} \end{pmatrix} + (\lambda_{25} - 1)^2 \right\} & j(\in G) = 3,5,\ldots,11 \end{cases},
\tag{5.2.37}
$$

The optimal values which lead to minimum MSE as,

$$
a_{25}^{\left(\frac{j-1}{2}\right)} = \frac{V_{200(1)}V_{002(2)} - V_{101(2)}^{\;\;2}}{V_{110(1)}V_{002(2)} - V_{011(1)}V_{101(2)}}
$$

and

$$
b_{25}^{\left(\frac{j-1}{2}\right)} = \frac{V_{200(1)}V_{002(2)} - V_{101(2)}^{\;\;2}}{V_{200(1)}V_{011(1)} - V_{110(1)}V_{101(2)}}, \lambda_{25} = \frac{1}{1 + A_{25}^{\;\;G}},
$$

where

$$
A_{25}^{\;\;G} = \begin{bmatrix} V_{020(2)} + \dfrac{1}{a_{25}^{\left(\frac{j-1}{2}\right)^2}} V_{200(1)} + \dfrac{1}{b_{25}^{\left(\frac{j-1}{2}\right)^2}} V_{002(2)} - 2\dfrac{1}{a_{25}^{\left(\frac{j-1}{2}\right)}} V_{110(1)} \\ -2\dfrac{1}{b_{25}^{\left(\frac{j-1}{2}\right)}} V_{011(2)} + 2\dfrac{1}{a_{25}^{\left(\frac{j-1}{2}\right)}} \dfrac{1}{b_{25}^{\left(\frac{j-1}{2}\right)}} V_{101(1)} \end{bmatrix}.
$$

ii) For $\alpha_{25} = -1, \beta_{25} = -1$, we get exponential-ratio product estimators given in Table 5.1. The MSE of t_{25}^{G} is expressed (5.2.33) as

$$
MSE\left(t_{25}^{k}\right)=\begin{cases}\overline{Y}_s^{2}\left\{\lambda_{25}^{2}\begin{pmatrix}V_{020(2)}+V_{200(1)}+V_{002(2)}\\ +2V_{110(1)}+2V_{011(2)}-2V_{101(1)}\end{pmatrix}+\left(\lambda_{25}-1\right)^{2}\right\} & k(\in G)=2\\[2ex] \overline{Y}_s^{2}\left\{\lambda_{25}^{2}\begin{pmatrix}V_{020(2)}+\dfrac{1}{a_{25}^{\left(\frac{k}{2}\right)^{2}}}V_{200(1)}+\dfrac{1}{b_{25}^{\left(\frac{k}{2}\right)^{2}}}V_{002(2)}\\ +2\dfrac{1}{a_{25}^{\left(\frac{k}{2}\right)}}V_{110(1)}+2\dfrac{1}{b_{25}^{\left(\frac{k}{2}\right)}}V_{011(2)}\\ -2\dfrac{1}{a_{25}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{25}^{\left(\frac{k}{2}\right)}}V_{101(1)}\end{pmatrix}+\left(\lambda_{25}-1\right)^{2}\right\} & k(\in G)=4,6,\ldots,12\end{cases},
$$

(5.2.38)

The optimal values which lead to minimum MSE as,

$$
a_{25}^{\left(\frac{k}{2}\right)}=\frac{V_{200(1)}V_{002(2)}-V_{101(2)}{}^{2}}{V_{110(1)}V_{002(2)}-V_{011(1)}V_{101(2)}}
$$

and

$$
b_{25}^{\left(\frac{k}{2}\right)}=\frac{V_{200(1)}V_{002(2)}-V_{101(2)}{}^{2}}{V_{200(1)}V_{011(1)}-V_{110(1)}V_{101(2)}},\lambda_{25}=\frac{1}{1+A_{25}{}^{G}},
$$

where

$$
A_{25}{}^{G}=\begin{bmatrix}V_{020(2)}+\dfrac{1}{a_{25}^{\left(\frac{k}{2}\right)^{2}}}V_{200(1)}+\dfrac{1}{b_{25}^{\left(\frac{k}{2}\right)^{2}}}V_{002(2)}-2\dfrac{1}{a_{25}^{\left(\frac{k}{2}\right)}}V_{110(1)}\\ -2\dfrac{1}{b_{25}^{\left(\frac{k}{2}\right)}}V_{011(2)}+2\dfrac{1}{a_{25}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{25}^{\left(\frac{k}{2}\right)}}V_{101(1)}\end{bmatrix}.
$$

iii) For $\alpha_{25}=-1,\beta_{25}=1$, we get exponential-product cum ratio type estimators given in Table 5.1. The MSE of t_{25}^{G} is expressed (2.5.33) as,

$$MSE(t_{25}^{l}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{25}^2 \begin{pmatrix} V_{020(2)} + V_{200(1)} + V_{002(2)} \\ + 2V_{110(1)} - 2V_{011(2)} - 2V_{101(1)} \end{pmatrix} + (\lambda_{25} - 1)^2 \right) l(\in G) = 13 \\ \bar{Y}_s^2 \left(\lambda_{25}^2 \begin{pmatrix} V_{020(2)} + \frac{1}{a_{25}^{\left(\frac{l-1}{2}\right)^2}} V_{200(1)} + \frac{1}{b_{25}^{\left(\frac{l-1}{2}\right)^2}} V_{002(2)} \\ + 2\frac{1}{a_{25}^{\left(\frac{l-1}{2}\right)}} V_{110(1)} - 2\frac{1}{b_{25}^{\left(\frac{l-1}{2}\right)}} V_{011(2)} \\ - 2\frac{1}{a_{25}^{\left(\frac{l-1}{2}\right)}} \frac{1}{b_{25}^{\left(\frac{l-1}{2}\right)}} V_{101(1)} \end{pmatrix} + (\lambda_{25} - 1)^2 \right) l(\in G) = 15, 17, \ldots, 23 \end{cases},$$

(5.2.39)

The optimal values which lead to minimum MSE as

$$a_{25}^{\left(\frac{l-1}{2}\right)^2} = \frac{V_{200(1)}V_{002(2)} - V_{101(2)}{}^2}{V_{110(1)}V_{002(2)} - V_{011(1)}V_{101(2)}}$$

and

$$a_{25}^{\left(\frac{l-1}{2}\right)^2} = \frac{V_{200(1)}V_{002(2)} - V_{101(2)}{}^2}{V_{110(1)}V_{002(2)} - V_{011(1)}V_{101(2)}}$$

where

$$A^{G}{}_{25} = \begin{bmatrix} V_{020(2)} + \frac{1}{a_{25}^{\left(\frac{l-1}{2}\right)^2}} V_{200(1)} + \frac{1}{b_{25}^{\left(\frac{l-1}{2}\right)^2}} V_{002(2)} - 2\frac{1}{a_{25}^{\left(\frac{l-1}{2}\right)}} V_{110(1)} \\ -2\frac{1}{b_{25}^{\left(\frac{l-1}{2}\right)}} V_{011(2)} + 2\frac{1}{a_{25}^{\left(\frac{l-1}{2}\right)}} \frac{1}{b_{25}^{\left(\frac{l-1}{2}\right)}} V_{101(1)} \end{bmatrix}.$$

iv) For $\alpha_{25} = 1, \beta_{25} = -1$, we get exponential-ratio cum product type estimators given in Table 5.1. The MSE of t_{25}^{G} is expressed (5.2.33) as

$$MSE(t_{25}^{m}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{25}^2 \begin{pmatrix} V_{020(2)} + V_{200(1)} + V_{002(2)} \\ -2V_{110(1)} + 2V_{011(2)} - 2V_{101(1)} \end{pmatrix} + (\lambda_{25} - 1)^2 \right) & m (\in G) = 14 \\ \bar{Y}_s^2 \left(\lambda_{25}^2 \begin{pmatrix} V_{020(2)} + \frac{1}{a_{25}^{\left(\frac{k}{2}\right)^2}} V_{200(1)} + \frac{1}{b_{25}^{\left(\frac{k}{2}\right)^2}} V_{002(2)} \\ -2\frac{1}{a_{25}^{\left(\frac{k}{2}\right)}} V_{110(1)} + 2\frac{1}{b_{25}^{\left(\frac{k}{2}\right)}} V_{011(2)} \\ -2\frac{1}{a_{25}^{\left(\frac{k}{2}\right)}} \frac{1}{b_{25}^{\left(\frac{k}{2}\right)}} V_{101(1)} \end{pmatrix} + (\lambda_{25} - 1)^2 \right) & m (\in G) = 16, 18, \ldots, 24 \end{cases}, \tag{5.2.40}$$

The optimal values which lead to minimum MSE as

$$a_{25}^{\left(\frac{m}{2}\right)} = \frac{V_{200(1)} V_{002(2)} - V_{101(2)}^2}{V_{110(1)} V_{002(2)} - V_{011(1)} V_{101(2)}}$$

and

$$b_{25}^{\left(\frac{m}{2}\right)} = \frac{V_{200(1)} V_{002(2)} - V_{101}^2}{V_{200(1)} V_{011(1)} - V_{110(1)} V_{101(2)}}, \lambda_{25} = \frac{1}{1 + A^G{}_{25}},$$

where

$$A_{25}{}^{G} = \begin{bmatrix} V_{020(2)} + \frac{1}{a_{25}^{\left(\frac{m}{2}\right)^2}} V_{200(1)} + \frac{1}{b_{25}^{\left(\frac{m}{2}\right)^2}} V_{002(2)} - 2\frac{1}{a_{25}^{\left(\frac{m}{2}\right)}} V_{110(1)} \\ +2\frac{1}{b_{25}^{\left(\frac{m}{2}\right)}} V_{011(2)} - 2\frac{1}{a_{25}^{\left(\frac{m}{2}\right)}} \frac{1}{b_{25}^{\left(\frac{m}{2}\right)}} V_{101(1)} \end{bmatrix}.$$

5.2.2 Proposed Generalized Exponential Estimator for Situation II Using Two Auxiliary Variable

In this section generalized estimator VII has been developed by assuming an exponential relationship between study variable and the two auxiliary variable for three different cases (defined in chapter 4) under situation I.

5.2.2.1 Proposed Generalized Exponential Estimator VII

Case I

i) The exponential-type ratio cum ratio estimator may be defined as

$$t_{26}'^{RR} = \bar{y}'_{s(2)} exp\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(2)}}\right) exp\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}_s + \bar{z}'_{s(2)}}\right), \quad (5.2.41)$$

ii) The exponential-type product-cum-ratio estimator may be defined as,

$$t_{26}'^{PR} = \bar{y}'_{s(2)} exp\left(-\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(1)}}\right)\right) \exp\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}_s + \bar{z}'_{s(2)}}\right), \quad (5.2.42)$$

iii) The exponential-type ratio cum product estimator may be defined as,

$$t_{26}'^{RP} = \bar{y}'_{s(2)} \exp\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(1)}}\right) \exp\left(-\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}_s + \bar{z}'_{s(2)}}\right)\right), \quad (5.2.43)$$

iv) The exponential-type product cum product estimator may be defined as,

$$t_{26}'^{PP} = \bar{y}'_{s(2)} exp\left(-\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(1)}}\right)\right) \exp\left(-\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{Z}_s + \bar{z}'_{s(2)}}\right)\right), \quad (5.2.44)$$

From (5.2.41)-(5.2.44), we may write a generalized form by introducing real constants α'_{26}, β'_{26}, a'_{26} λ'_{26} and b'_{26}.

$$t_{26}'^{G} = \lambda'_{26} \bar{y}'_{26} \exp\left(\alpha'_{26}\left(1 - \frac{a'_6 \bar{x}'_{s(2)}}{\left(\bar{x}'_{s(1)} + (a'_{26} - 1)\bar{x}'_{s(2)}\right)}\right)\right)$$
$$\exp\left(\beta'_{26}\left(1 - \frac{b'_{26} \bar{z}'_{s(2)}}{\left(\bar{Z}_s + (b'_{26} - 1)\bar{z}'_{s(2)}\right)}\right)\right), \; 0 < \lambda'_{26} \leq 1 \quad (5.2.45)$$

where (a'_{26}, b'_{26}, λ'_{26}) are constants to be determined such that the mean square error (MSE) is minimum. (α'_{26}, β'_{26}) are known constants takes the value (0,1,-1) to produce different ratio-type and product-type estimators.

By substituting different values to the constants in (5.2.45), we get a class of estimators as given in Table 5.2. Some members of estimators t_{26}^{G}.

Table 5.2

Some Special Cases of the Generalized Estimator t'^{G}_{26}

Ratio-cum-product estimator $\alpha'_{26}=1, \beta'_{26}=1$	**Product -cum-product estimator** $\alpha'_{26}=-1, \beta'_{26}=-1$	a'_{26}	λ'_{26}	b'_{26}
$t'^{1}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+\bar{x}'_{s(2)}}\right)\exp\left(\frac{\bar{Z}'_{s}-\bar{z}'_{s(2)}}{\bar{Z}'_{s}+\bar{z}'_{s(2)}}\right)$	$t'^{2}_{26}=\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(2)}+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{z}'_{s(2)}+\bar{Z}'_{s}}\right)$	2	1	2
$t'^{3}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{Z}'_{s}-\bar{z}'_{s(2)}}{\bar{Z}'_{s}}\right)$	$t'^{4}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'}{\bar{Z}'_{s}}\right)$	1	1	1
$t'^{5}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+(a'_{6}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{Z}'_{s}-\bar{z}'_{s(2)}}{\bar{Z}'_{s}+(b'_{6}-1)\bar{z}'_{s(2)}}\right)$	$t'^{5}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}+(a'_{5}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+(b'_{5}-1)\bar{z}'_{s}}\right)$	a'_{26}	1	b'_{26}
$t'^{7}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+\bar{x}'_{s(2)}}\right)\exp\left(\frac{\bar{Z}'_{s}-\bar{z}'_{s(2)}}{\bar{Z}'_{s}+\bar{z}'_{s(2)}}\right)$	$t'^{7}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(2)}+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+\bar{z}'_{s(2)}}\right)$	2		2
$t'^{9}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{Z}'_{s}-\bar{z}'_{s(2)}}{\bar{Z}'_{s}}\right)$	$t'^{9}_{6}=\lambda'_{5}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}}\right)$	1	λ'_{26}	1
$t'^{11}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+(a'_{6}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{Z}'_{s}-\bar{z}'_{s(2)}}{\bar{Z}'_{s}+(b'_{6}-1)\bar{z}'_{s(2)}}\right)$	$t'^{12}_{6}=\lambda'_{6}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}+(a'_{6}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+(b'_{6}-1)\bar{z}'_{s(2)}}\right)$	a'_{26}	λ'_{26}	b'_{26}
Product -cum-ratio estimator $\alpha'_{6}=-1, \beta'_{6}=1$	**Ratio-cum-product estimator** $\alpha'_{6}=1, \beta'_{6}=-1$	a'_{26}	λ'_{26}	b'_{26}
$t'^{13}_{6}=\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+\bar{x}'_{s(2)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+\bar{z}'_{s(2)}}\right)$	$t'^{14}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(2)}+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{z}'_{s(2)}+\bar{Z}'_{s}}\right)$	2	1	2
$t'^{15}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(1)}-\bar{Z}'_{s}}{\bar{Z}'_{s}}\right)$	$t'^{16}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}}\right)$	1	1	1
$t'^{17}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+(a'_{26}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+(b'_{26}-1)\bar{z}'_{s(2)}}\right)$	$t'^{18}_{26}=\bar{y}'_{s(2)}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}+(a'_{26}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+(b'_{26}-1)\bar{z}'_{s(2)}}\right)$	a'_{26}	1	b'_{26}
$t'^{19}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+\bar{z}'_{s(2)}}\right)$	$t'^{20}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(2)}+\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+\bar{z}'_{s(2)}}\right)$	2	λ'_{26}	2
$t'^{21}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}}\right)$	$t'^{22}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}}\right)$	1	λ'_{26}	1
$t'^{23}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(1)}-\bar{x}'_{s(2)}}{\bar{x}'_{s(1)}+(a'_{26}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+(b'_{26}-1)\bar{z}'_{s(2)}}\right)$	$t'^{24}_{26}=\lambda'_{26}\bar{y}'_{s}\exp\left(\frac{\bar{x}'_{s(2)}-\bar{x}'_{s(1)}}{\bar{x}'_{s(1)}+(a'_{26}-1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)}-\bar{Z}'_{s}}{\bar{Z}'_{s}+(b'_{26}-1)\bar{z}'_{s(2)}}\right)$	a'_{26}	λ'_{26}	b'_{26}

The Bias and Mean Square Error of Generalized Estimator VII

To derive the bias and mean square error we proceed as follows:

Using (5.1.1) we can express (5.2.45) as,

$$t_{26}'^{G} = \lambda_{26}' \bar{Y}_s' \left(1 + e_{0(2)}'\right) \exp\left[-\frac{\alpha_{26}'}{a_{26}'} \left(e_{1(2)}' - e_{1(1)}'\right) \left(1 + \frac{e_{1(1)}'}{a_{26}'} + \frac{(a_6' - 1)}{a_{26}'} e_{1(2)}'\right)^{-1} \right]$$

$$\exp\left[-\frac{\beta_{26}'}{b_{26}'} e_{2(1)}' \left(1 + \frac{(b_{26}' - 1)}{b_{26}'} e_{2(2)}'\right)^{-1} \right], \qquad (5.2.46)$$

We assume that $\left|e_{1(1)}'\right| < 1$, we expand the series, $\left(1 + \dfrac{e_{1(2)}'}{a_{26}'} + \dfrac{(a_{26}' - 1)}{a_{26}'} e_{1(1)}'\right)^{-1}$

and $\left(1 + \dfrac{(b_{26}' - 1)}{b_{26}'} e_{2(2)}'\right)^{-1}$, we get,

$$t_{26}'^{G} = \lambda_{26}' \bar{Y}_s' \left(1 + e_{0(2)}'\right) \exp\left[-\frac{\alpha_{26}'}{a_{26}'} \left(e_{1(2)}' - e_{1(1)}'\right) \right] \exp\left[-\frac{\beta_{26}'}{b_{26}'} e_{2(2)}' \right], \qquad (5.2.47)$$

$$t_{26}'^{G} - \bar{Y}_s' = \lambda_{26}' \bar{Y}_s' \left[-\frac{\alpha_{26}'}{a_{26}'} \left(e_{1(2)}' - e_{1(1)}'\right) + \frac{\alpha_{26}'^2 \left(e_{1(2)}'^2 + e_{1(1)}'^2 - 2 e_{1(2)}' e_{1(1)}'\right)}{a_{26}'^2} \right.$$

$$- \frac{\beta_{26}'}{b_{26}'} e_{2(2)}' + \frac{\beta_{26}'^2}{2 b_{26}'^2} e_{2(2)}'^2 - \frac{\alpha_{26}'}{a_{26}'} \frac{\beta_{26}'}{b_{26}'} e_{2(2)}' \left(e_{1(2)}' - e_{1(1)}'\right)$$

$$\left. + e_{0(2)}' - \frac{\alpha_{26}'}{a_{26}'} e_{0(2)}' \left(e_{1(2)}' - e_{1(1)}'\right) - \frac{\beta_{26}'}{b_{26}'} e_{0(2)}' e_{2(2)}' \right] + \left(\lambda_{26}' - 1\right) \bar{Y}_s',$$

$$(5.2.48)$$

By taking expectation of (5.2.48) and using (5.1.1), we get the $Bias\left(t_{26}'^{G}\right)$ as,

$$Bias\left(t_{26}'^{G}\right) = \lambda_{26}' \bar{Y}' \left[\frac{\alpha_{26}'^2 \left(V_{200(2)}' - V_{200(1)}'\right)}{a_{26}'^2} + \frac{\beta_{26}'^2}{2 b_{26}'^2} V_{200(2)}' \right.$$

$$\left. - \frac{\alpha_{26}'}{a_{26}'} \left(V_{110(2)}' - V_{110(1)}'\right) - \frac{\beta_{26}'}{b_{26}'} V_{110(2)}' \right] + \left(\lambda_{26}' - 1\right) \bar{Y}_s', \qquad (5.2.49)$$

To get the MSE of the estimator, we take square and we take expectation of (5.2.8) and we obtain

$$\left(t_{26}^{\prime G} - \overline{Y}_s'\right)^2 = \lambda_{26}'^{\,2}\,\overline{Y}_s'^{\,2}\left[e_{0(2)}'^{\,2} + \frac{\alpha_{26}'^{\,2}}{a_{26}'^{\,2}}\left(e_{1(2)}' - e_{1(1)}'\right)^2 + \frac{\beta_{26}'^{\,2}}{b_{26}'^{\,2}} e_{2(1)}'^{\,2} \right.$$
$$-2\frac{\alpha_{26}'}{a_{26}'} e_{0(2)}'\left(e_{1(2)}' - e_{1(1)}'\right) - 2\frac{\beta_{26}'}{b_{26}'} e_{0(2)}' e_{2(2)}'$$
$$\left. +2\frac{\beta_{26}'}{b_{26}'}\frac{\alpha_{26}'}{a_{26}'} e_{2(2)}'\left(e_{1(2)}' - e_{1(1)}'\right)\right] + \left(\lambda_{26}' - 1\right)^2 \overline{Y}_s'^{\,2}, \qquad (5.2.50)$$

We take expectation on (5.2.50) by using (5.1.1) as,

$$MSE\left(t_{26}^{\prime G}\right) = \lambda_{26}'^{\,2}\,\overline{Y}_s'^{\,2}\left[V_{020(2)}' + z_{26}'^{\,2}\left(V_{200(2)}' - V_{200(1)}'\right) + u_{26}'^{\,2} V_{002(2)}' \right.$$
$$-2z_{26}'\left(V_{110(2)}' - V_{110(1)}'\right) - 2u_{26}' V_{011(2)}'$$
$$\left. +2u_{26}' z_{26}'\left(V_{101(2)}' - V_{101(1)}'\right)\right] + \left(\lambda_{26}' - 1\right)^2 \overline{Y}_s'^{\,2}, \qquad (5.2.51)$$

where $z_{26}' = \dfrac{\alpha_{26}'}{a_{26}'}, u_{26}' = \dfrac{\beta_{26}'}{b_{26}'}$.

For the following optimal value of the constants z_{26}' and u_{26}', we achieve the minimum MSE among the class of proposed generalized estimator.

$$\left.\begin{array}{l}
u_{26}' = \dfrac{\left(V_{200(2)}' - V_{200(1)}'\right)V_{011(2)}' - \left(V_{110(2)}' - V_{110(1)}'\right)\left(V_{101(2)}' - V_{101(1)}'\right)}{\left(V_{200(2)}' - V_{200(1)}'\right)V_{002(2)}' - \left(V_{101(2)}' - V_{101(1)}'\right)^2}, \\[2ex]
z_{26}' = \dfrac{\left(V_{110(2)}' - V_{110(1)}'\right)V_{002(2)}' - V_{011(2)}'\left(V_{101(2)}' - V_{101(1)}'\right)}{\left(V_{200(2)}' - V_{200(1)}'\right)V_{002(2)}' - \left(V_{101(2)}' - V_{101(1)}'\right)^2} \\[2ex]
\text{and} \quad \lambda_{26}' = \dfrac{1}{1 + A_{26}'^{\,G}}, \\[2ex]
\text{where} \\[1ex]
A_{26}'^{\,G} = \left(\begin{array}{l}
V_{020(2)}' + z_{26}'^{\,2}\left(V_{200(2)}' - V_{200(1)}'\right) + u_{26}'^{\,2} V_{002(2)}' \\
-\, 2z_{26}'\left(V_{110(2)}' - V_{110(1)}'\right) - 2u_{26}' V_{011(2)}' \\
+\, 2u_{26}' z_{26}'\left(V_{101(2)}' - V_{101(1)}'\right)
\end{array}\right),
\end{array}\right\}$$

(5.2.52)

By substituting the optimum values of z'_6 and u'_6, we get λ'^{opt}_6 as,

$$\lambda'^{opt}_{26} = \frac{1}{1 + A'^{*}_{26}}$$

where

$$A'^{*}_{26} = \left(V'_{020} - \frac{\left(V'_{110(2)} - V'_{110(1)}\right)^2 V'_{002(2)} + V'^{2}_{011(2)}\left(V'_{200(2)} - V'_{200(1)}\right) - 2\left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)V'_{011(2)}}{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^2} \right), \tag{5.2.53}$$

We obtain asymptotic variance of the proposed generalized estimator as the expression for $MSE\left(t'^{G}_6\right)$ was considered upto first degree of error term, so minimum MSE may be written as,

$$MSE_{\min}\left(t'^{G}_{26}\right) = AsymtoticVar\left(t'^{G}_{26}\right) = \bar{Y}'^{2}_{s}\left(\frac{A'^{*}_{26}}{1 + A'^{*}_{26}}\right). \tag{5.2.54}$$

From (5.2.54), we observe that asymptotic variance of the proposed estimator is less than **Usual Linear Regression Estimator**. We may also observe from (5.2.54) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value λ'^{opt}_{26} and $a'^{opt}_{26}, b'^{opt}_{26}$ in (5.2.54), we get optimal estimator as,

$$\hat{t}'^{G}_{26} = \hat{\lambda}'_{26}\bar{y}'_s \exp\left(\alpha'_{26}\left(1 - \frac{\hat{a}'_{26}\bar{x}'_s}{\left(\bar{X}'_s + (\hat{a}'_{26} - 1)\bar{x}'_s\right)}\right)\right)$$
$$\exp\left(\beta'_{26}\left(1 - \frac{\hat{b}'_{26}\bar{z}'_s}{\left(\bar{Z}'_s + (\hat{b}'_{26} - 1)\bar{z}'_s\right)}\right)\right), \quad 0 < \lambda'_{26} \le 1 \tag{5.2.55}$$

As described earlier in section (5.2.1.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as,

$$\lambda'^{opt}_{26} = \frac{1}{1 + A'^{*}_{26}}$$

where

$$A'^{*}_{26} = \left(V'_{020} - \frac{\left(V'_{110(2)} - V'_{110(1)}\right)^2 V'_{002(2)} + V'^{2}_{011(2)} \left(V'_{200(2)} - V'_{200(1)}\right) - 2\left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right) V'_{011(2)}}{\left(V'_{200(2)} - V'_{200(1)}\right) V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^2} \right), \tag{5.2.56}$$

So (5.2.55) may be wriiten as,

$$\hat{t}'^{G}_{26} = \hat{\lambda}'^{opt}_{26}\, \bar{y}'_s \exp\left(\alpha'_{26} \left(1 - \frac{a'^{opt}_{26}\, \bar{x}'_s}{\left(\bar{X}'_s + (a'^{opt}_{26} - 1)\bar{x}'_s \right)} \right) \right) \exp\left(\beta'_{26} \left(1 - \frac{b'^{opt}_{26}\, \bar{z}'_s}{\left(\bar{Z}'_s + (b'^{opt}_{26} - 1)\bar{z}'_s \right)} \right) \right), \quad 0 < \lambda'_{26} \le 1 \tag{5.2.57}$$

Also the minimum MSE may be written as,

$$MSE_{\min}\left(\hat{t}'^{G}_{26}\right) = AsymtoticVar\left(\hat{t}'^{G}_{26}\right) = \bar{Y}'^{2}_{s} \left(\frac{A'^{*}_{26}}{1 + A'^{*}_{26}} \right). \tag{5.2.58}$$

Remark 5.4

i) For $\alpha'_{26} = 1, \beta'_{26} = 1$, we get exponential-type ratio cum ratio type estimators given in Table 5.2. The MSE of t'^{G}_{26} is expressed as

$$MSE(t'^{j}_{26}) = \begin{cases} \bar{Y}'^2_s \left\{ \lambda'^2_{26} \begin{pmatrix} V'_{020(2)} + \left(V'_{200(2)} - V'_{200(2)}\right) + V'_{002(2)} \\ -2\left(V'_{110(2)} - V'_{110(1)}\right) + 2\left(V_{101(2)} - V'_{101(1)}\right) - 2V_{011(2)} \end{pmatrix} + (\lambda'_{26} - 1)^2 \right\} & j(\in G) = 1 \\ \bar{Y}'^2_s \left\{ \lambda'^2_{26} \begin{pmatrix} V'_{020} + \dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^2}_{6}}\left(V'_{200(2)} - V'_{200(2)}\right) + \dfrac{1}{b'^{\left(\frac{j-1}{2}\right)^2}_{6}} V'_{002(2)} \\ -2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{6}}\left(V'_{110(2)} - V'_{110(1)}\right) - 2\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{6}}\left(V_{101(2)} - V'_{101(1)}\right) \\ +2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{6}}\dfrac{1}{b'^{\left(\frac{j-1)}{2}\right)}_{6}} V'_{011(2)} \end{pmatrix} + (\lambda'_{26} - 1)^2 \right\} & j(\in G) = 3,5,\ldots,11 \end{cases}$$

(5.2.59)

The optimal values which lead to minimum MSE as,

$$a'^{\left(\frac{j-1}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(2)}\right)V'_{002(2)} - \left(V_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{110(2)} - V'_{110(1)}\right)V'_{002(2)} - V'_{011(2)}\left(V_{101(2)} - V'_{101(1)}\right)}$$

and

$$b'^{\left(\frac{j-1}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(2)}\right)V'_{002(2)} - \left(V_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{200(2)} - V'_{200(2)}\right)V'_{011(2)} - \left(V'_{110(2)} - V'_{110(1)}\right)\left(V_{101(2)} - V'_{101(1)}\right)},$$

$$\lambda'_{26} = \frac{1}{1 + A'^{G}_{26}},$$

where

$$A'^{G}_{26} = \begin{bmatrix} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^2}_{26}}\left(V'_{200(2)} - V'_{200(2)}\right) + \dfrac{1}{b'^{\left(\frac{j-1}{2}\right)^2}_{26}} V'_{002(2)} - 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{26}}\left(V'_{110(2)} - V'_{110(1)}\right) \\ -2\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{26}}\left(V_{101(2)} - V'_{101(1)}\right) + 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{26}}\dfrac{1}{b'^{\left(\frac{j-1}{2}\right)}_{26}} V'_{011(2)} \end{bmatrix}.$$

ii) For $\alpha'_{6} = -1, \beta'_{6} = -1$, we get exponential-product cum-product type estimators given in Table 5.2. The MSE of t'^{G}_{6} is expressed as,

$$MSE(t'^{k}_{26}) = \begin{cases} \overline{Y}'^{2}_{s} \left(\lambda'^{2}_{26} \left(\begin{aligned} & V'_{020(2)} + \left(V'_{200(2)} - V'_{200(2)}\right) + V'_{002(2)} \\ & + 2\left(V'_{110(2)} - V'_{110(1)}\right) + 2\left(V_{101(2)} - V'_{101(1)}\right) - 2V_{011(2)} \end{aligned} \right) + (\lambda'_{26} - 1)^{2} \right) & k(\in G) = 2 \\[2ex] \overline{Y}'^{2}_{s} \left(\lambda'^{2}_{26} \left(\begin{aligned} & V'_{020(2)} + \frac{1}{a'^{\left(\frac{k}{2}\right)^{2}}_{26}}\left(V'_{200(2)} - V'_{200(2)}\right) + \frac{1}{b'^{\left(\frac{k}{2}\right)^{2}}_{26}} V'_{002(2)} \\ & + 2\frac{1}{a'^{\left(\frac{k}{2}\right)}_{26}}\left(V'_{110(2)} - V'_{110(1)}\right) + 2\frac{1}{b'^{\left(\frac{k}{2}\right)}_{26}}\left(V_{101(2)} - V'_{101(1)}\right) \\ & - 2\frac{1}{a'^{\left(\frac{k}{2}\right)}_{26}}\frac{1}{b'^{\left(\frac{k}{2}\right)}_{26}} V_{011(2)} \end{aligned} \right) + (\lambda'_{26} - 1)^{2} \right) & k(\in G) = 4, 6, \ldots, 12 \end{cases}, \tag{5.2.59}$$

The optimal values which lead to minimum MSE as,

$$a'^{\left(\frac{k}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^{2}}{\left(V'_{110(2)} - V'_{110(1)}\right)V'_{002(2)} - V'_{011(2)}\left(V'_{101(2)} - V'_{101(1)}\right)}$$

and

$$b'^{\left(\frac{k}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^{2}}{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{011(2)} - \left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)},$$

$$\lambda'_{26} = \frac{1}{1 + A'^{G}_{26}},$$

where

$$A'^{G}_{26} = \left[\begin{aligned} & V_{020(2)} + \frac{1}{a'^{\left(\frac{k}{2}\right)^{2}}_{26}}\left(V'_{200(2)} - V'_{200(1)}\right) + \frac{1}{b'^{\left(\frac{k}{2}\right)^{2}}_{26}} V'_{002(2)} \\ & - 2\frac{1}{a'^{\left(\frac{k}{2}\right)}_{26}}\left(V'_{110(2)} - V'_{110(1)}\right) - 2\frac{1}{b'^{\left(\frac{k}{2}\right)}_{26}}\left(V'_{101(2)} - V'_{101(1)}\right) \\ & + 2\frac{1}{a'^{\left(\frac{k}{2}\right)}_{26}}\frac{1}{b'^{\left(\frac{k}{2}\right)}_{26}} V'_{011(2)} \end{aligned} \right],$$

iii) For $\alpha'_{26} = -1, \beta'_{26} = 1$, we get exponential-product cum ratio type estimators given in Table 5.2. The MSE of t'^{G}_{26} is expressed as,

$$MSE(t'^{l}_{26}) = \begin{cases} \bar{Y}'^{2}_{s}\left\{ \lambda'^{2}_{26}\left(\begin{array}{l} V'_{020(2)} + \left(V'_{200(2)} - V'_{200(1)}\right) + V'_{002(2)} \\ \quad + 2\left(V'_{110(2)} - V'_{110(1)}\right) - 2\left(V'_{101(2)} - V'_{101(1)}\right) - 2V'_{011(2)} \end{array} \right) + (\lambda'_{26} - 1)^{2} \right\} & l(\in G) = 13 \\ \bar{Y}'^{2}_{s}\left\{ \lambda'^{2}_{26}\left(\begin{array}{l} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^{2}}_{6}}\left(V'_{200(2)} - V'_{200(1)}\right) \\ \quad + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^{2}}_{6}} V'_{002(2)} + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{26}}\left(V'_{110(2)} - V'_{110(1)}\right) \\ \quad - 2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{26}} V_{101} - 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{26}}\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{26}} V_{011(2)} \end{array} \right) + (\lambda'_{26} - 1)^{2} \right\} & l(\in G) = 15, 17, \ldots, 23 \end{cases},$$

(5.2.60)

The optimal values which lead to minimum MSE as

$$a'^{\left(\frac{l-1}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^{2}}{\left(V'_{110(2)} - V'_{110(1)}\right)V'_{002(2)} - V'_{011(2)}\left(V'_{101(2)} - V'_{101(1)}\right)}$$

and

$$b'^{\left(\frac{l-1}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^{2}}{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{011(2)} - \left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)},$$

$$\lambda'_{26} = \frac{1}{1 + A'^{G}_{26}},$$

where

$$A'^{G}_{26} = \left[\begin{array}{l} V_{020(2)} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^{2}}_{26}}\left(V'_{200(2)} - V'_{200(1)}\right) + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^{2}}_{26}} V_{002(2)} \\ \quad - 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{26}}\left(V'_{110(2)} - V'_{110(1)}\right) - 2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{26}} V_{101} + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{26}}\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{26}} V_{011(2)} \end{array} \right],$$

iv) For $\alpha'_{26} = 1, \beta'_{26} = -1$, we get exponential-ratio type estimators given in Table 5.2. The MSE of t'^{G}_{26} is expressed as,

$$MSE(t'^{m}_{26}) = \left\{ \begin{array}{ll} \bar{Y}'^2_s \left\{ \lambda'^2_{26} \left(\begin{array}{l} V'_{020} + \left(V'_{200(2)} - V'_{200(1)}\right) \\ + V'_{002(2)} - 2\left(V'_{110(2)} - V'_{110(1)}\right) \\ + 2\left(V'_{101(2)} - V'_{101(1)}\right) - 2V'_{011(2)} \end{array} \right) + (\lambda'_{26} - 1)^2 \right\} & m(\in G) = 14 \\ \bar{Y}'^2_s \left\{ \lambda'^2_{26} \left(\begin{array}{l} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{k}{2}\right)^2}_{26}}\left(V'_{200(2)} - V'_{200(1)}\right) \\ + \dfrac{1}{b'^{\left(\frac{k}{2}\right)^2}_{26}} V'_{002(2)} - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{26}}\left(V'_{110(2)} - V'_{110(1)}\right) \\ + 2\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{26}}\left(V'_{101(2)} - V'_{101(1)}\right) - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{26}}\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{26}} V'_{011(2)} \end{array} \right) + (\lambda'_{26} - 1)^2 \right\} & m(\in G) = 16, 18, .., 24 \end{array} \right\},$$

(5.2.61)

The optimal values which lead to minimum MSE as

$$a'^{\left(\frac{m}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{110(2)} - V'_{110(1)}\right)V'_{002(2)} - V'_{011(2)}\left(V'_{101(2)} - V'_{101(1)}\right)},$$

and

$$b'^{\left(\frac{m}{2}\right)}_{26} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{002(2)} - \left(V'_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{200(2)} - V'_{200(1)}\right)V'_{011(2)} - \left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)},$$

$$\lambda'_{26} = \frac{1}{1 + A'^{G}_{26}},$$

where

$$A'^{G}_{26} = \left[\begin{array}{l} V'_{020} + \dfrac{1}{a'^{\left(\frac{m}{2}\right)^2}_{26}}\left(V'_{200(2)} - V'_{200(1)}\right) + \dfrac{1}{b'^{\left(\frac{m}{2}\right)^2}_{26}} V'_{002(2)} \\ - 2\dfrac{1}{a'^{\left(\frac{m}{2}\right)}_{26}}\left(V'_{110(2)} - V'_{110(1)}\right) \\ + 2\dfrac{1}{b'^{\left(\frac{m}{2}\right)}_{26}}\left(V'_{101(2)} - V'_{101(1)}\right) - 2\dfrac{1}{a'^{\left(\frac{m}{2}\right)}_{26}}\dfrac{1}{b'^{\left(\frac{m}{2}\right)}_{26}} V'_{011(2)} \end{array} \right].$$

Case II:

The generalized estimator under case II may be proposed following (5.2.) as,

$$t''^{G}_{26} = \lambda''_{26}\,\bar{y}''_{26}\,\exp\left(\alpha''_{26}\left(1-\frac{a''_{26}\,\bar{x}''_{s(2)}}{\left(\bar{x}''_{s(1)}+(a''_{26}-1)\bar{x}''_{s(2)}\right)}\right)\right)$$
$$\exp\left(\beta''_{26}\left(1-\frac{b''_{26}\;\bar{z}''_{s(2)}}{\left(\bar{Z}_{s}+(b''_{26}-1)\bar{z}''_{s(2)}\right)}\right)\right),\quad 0<\lambda''_{26}\le 1 \qquad (5.2.62)$$

where (a'_{26}, b'_{26}, λ'_{26}) are constants to be determined such that the mean square error (MSE) is minimum. (α'_{26}, β'_{26}) are known constants takes the value(0,1,-1) to produce different ratio-type and product-type estimators.

The proposed estimator (5.2.62) follow the same routine along with the class of estimator in Table 5.2 as that for case-I in Section 5.2.2.2 In addition, the relation between a''_{26}, α''_{26}, λ''_{26} and b''_{26}, β''_{26} in case-II is the same as that for case-I in Section 5.2.2.1.Finally, the same is true for the MSE and the Bias. It is therefore directly from Section 5.3.2.1, we may write $\mathrm{Bias}\,(t''^{G}_{26})$ and MSE(t''^{G}_{26}) following the same, and we may also produce a class of estimators for similar choices of a''_{26}, α''_{26}, λ''_{26} in case-II. The bias of(5.2.62) may be obtain by following the notations and expectations for case II presented in Section 5.1.

The bias and MSE expressions may be given as,

$$Bias\left(t''^{G}_{26}\right) = \lambda''_{26}\;\bar{Y}''_{s}\left[\frac{\alpha''^{2}_{26}\left(V''_{200(2)}-V''_{200(1)}\right)}{a''^{\;2}_{26}}+\frac{\beta''^{2}_{26}}{2b''^{\;2}_{26}}V''_{200(2)}\right.$$
$$\left.-\frac{\alpha''_{26}}{a''_{26}}\left(V''_{110(2)}-V''_{110(1)}\right)-\frac{\beta''_{26}}{b''_{26}}V''_{110(2)}\right]+\left(\lambda''_{26}-1\right)\bar{Y}''_{s}, \qquad (5.2.63)$$

$$MSE\,(t''^{G}_{26}) = \lambda''^{\;2}_{26}\,\bar{Y}''^{\;2}_{s}\left[V''_{020(2)}+z''^{2}_{26}\left(V''_{200(2)}-V''_{200(1)}\right)+u''^{2}_{26}\,V''_{002(2)}\right.$$
$$-2z''_{26}\left(V''_{110(2)}-V''_{110(1)}\right)-2u''_{26}V''_{011(2)}$$
$$\left.+2u''_{26}z''_{26}\left(V''_{101(2)}-V''_{101(1)}\right)\right]+\left(\lambda''_{26}-1\right)^{2}\bar{Y}''^{\;2}_{s}, \qquad (5.2.64)$$

Now by substituting the optimal values of the z'_{26} and u'_{26}, we achieve the minimum MSE among the class of proposed generalized estimator.

$$\left.\begin{aligned}
u''_{26} &= \frac{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{011(2)} - \left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)}{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)}\right)^2}, \\
z''_{26} &= \frac{\left(V''_{110(2)} - V''_{110(1)}\right)V''_{002(2)} - V''_{011(2)}\left(V''_{101(2)} - V''_{101(1)}\right)}{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)}\right)^2} \\
\text{and} \quad \lambda''_{26} &= \frac{1}{1 + A''^{\,G}_{26}}, \\
\text{where} \\
A''^{\,G}_{26} &= \Big[V''_{020(2)} + z''^2_{26}\left(V''_{200(2)} - V''_{200(1)}\right) + u''^2_{26} V''_{002(2)} \\
&\quad - 2z''_{26}\left(V''_{110(2)} - V''_{110(1)}\right) - 2u''_{26}V''_{011(2)} + 2u''_{26}z''_{26}\left(V''_{101(2)} - V''_{101(1)}\right)\Big]
\end{aligned}\right\}$$

(5.2.65)

By substituting the optimum values of z''_6 and u''_6, we get $\lambda'^{\,opt}_3$ as,

$$\lambda''^{\,opt}_{26} = \frac{1}{1 + A''^{\,*}_{26}}$$

where

$$A''^{\,*}_{26} = \left\{ V''_{020} - \frac{\begin{array}{c}\left(V''_{110(2)} - V''_{110(1)}\right)^2 V''_{002(2)} + V''^{\,2}_{011(2)}\left(V''_{200(2)} - V''_{200(1)}\right) \\ -2\left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)V''_{011(2)}\end{array}}{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)}\right)^2} \right\}.$$

(5.2.67)

We obtain asymptotic variance of the proposed generalized estimator as the expression for $MSE\left(t''^{G}_{26}\right)$ was considered upto first degree of error term, so minimum MSE may be written as

$$MSE_{\min}\left(t''^{G}_{26}\right) = AsymtoticVar\left(t''^{G}_{26}\right) = \bar{Y}''^{\,2}_{26}\left(\frac{A''^{\,*}_{26}}{1 + A''^{\,*}_{26}}\right). \tag{5.2.68}$$

It may be observed from (5.2.68), that asymptotic variance of the proposed estimator is less than **Usual Linear Regression Estimator**, We may observe from (5.2.68) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value $\lambda_{26}''^{\,opt}$ and $a_{26}''^{\,opt}, b_{26}''^{\,opt}$ in (5.2.62), we get optimal estimator as,

$$\hat{t}_{26}''^{G} = \hat{\lambda}_{26}'' \bar{y}_s' \exp\left(\alpha_{26}'' \left(1 - \frac{\hat{a}_{26}'' \bar{x}_s''}{\left(\bar{X}_s'' + (\hat{a}_{26}'' - 1)\bar{x}_s'' \right)} \right) \right)$$
$$\exp\left(\beta_{26}'' \left(1 - \frac{\hat{b}_{26}'' \bar{z}_s''}{\left(\bar{Z}_s'' + (\hat{b}_{26}'' - 1)\bar{z}_s'' \right)} \right) \right), \; 0 < \lambda_{26}'' \le 1 \qquad (5.2.69)$$

As described earlier in section (5.2.2.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as

$$\lambda_{26}''^{\,*} = \frac{1}{1 + A_{26}''^{\,*}}$$

where

$$A_{26}''^{\,*} = \left(V''_{020} - \frac{\left(V''_{110(2)} - V''_{110(1)} \right)^2 V''_{002(2)} + V''^{\,2}_{011(2)} \left(V''_{200(2)} - V''_{200(1)} \right) - 2\left(V''_{110(2)} - V''_{110(1)} \right)\left(V''_{101(2)} - V''_{101(1)} \right) V''_{011(2)}}{\left(V''_{200(2)} - V''_{200(1)} \right) V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)} \right)^2} \right)$$

(5.2.70)

So (5.2.70) may be wriiten as,

$$\hat{t}''^{G}_{26} = \hat{\lambda}''^{opt}_{26}\,\overline{y}''_s \exp\left(\alpha''_6\left(1-\frac{\hat{a}''^{opt}_{26}\,\overline{x}''_s}{\left(\overline{X}''_s+(\hat{a}''^{opt}_{26}-1)\overline{x}''_s\right)}\right)\right)$$

$$\exp\left(\beta''_{26}\left(1-\frac{\hat{b}''^{opt}_{26}\,\overline{z}''_s}{\left(\overline{Z}''_s+(\hat{b}''^{opt}_{26}-1)\overline{z}''_s\right)}\right)\right),\quad 0<\lambda''_{26}\le 1 \qquad (5.2.71)$$

Also the minimum MSE may be written as,

$$MSE_{\min}\left(\hat{t}''^{G}_{26}\right) = AsymtoticVar\left(\hat{t}''^{G}_{26}\right) = \overline{Y}''^{2}_s\left(\frac{\hat{A}''^{*}_{26}}{1+\hat{A}''^{*}_{26}}\right). \qquad (5.2.72)$$

Remark 5.5

i) For $\alpha''_{26}=1,\beta''_{26}=1$, we get exponential-ratio cum ratio type estimators given in Table 6. The MSE of t''^{G}_{26} is expressed as,

$$MSE(t''^{j}_{26}) = \begin{cases} \overline{Y}''^{2}_s\left(\lambda''^{2}_{26}\left(\begin{array}{l} V''_{020(2)}+\left(V''_{200(2)}-V''_{200(2)}\right)+V''_{002(2)} \\ -2\left(V''_{110(2)}-V''_{110(1)}\right)+2\left(V''_{101(2)}-V''_{101(1)}\right)-2V''_{011(2)} \end{array}\right)+(\lambda''_{26}-1)^2\right) & j(\in G)=1 \\ \overline{Y}''^{2}_s\left(\lambda''^{2}_{26}\left(\begin{array}{l} V''_{020}+\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)^2}_{26}}\left(V''_{200(2)}-V''_{200(2)}\right) \\ +\dfrac{1}{b''^{\left(\frac{j-1}{2}\right)^2}_{26}}V''_{002(2)}-2\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{26}}\left(V''_{110(2)}-V''_{110(1)}\right) \\ -2\dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{26}}\left(V''_{101(2)}-V''_{101(1)}\right)+2\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{26}}\dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{26}}V''_{011(2)} \end{array}\right)+(\lambda''_{26}-1)^2\right) & j(\in G)=3,5,\ldots,11 \end{cases},$$

(5.2.73)

The optimal values which lead to minimum MSE as,

$$a''^{\left(\frac{j-1}{2}\right)}_{26} = \frac{\left(V''_{200(2)}-V''_{200(2)}\right)V''_{002(2)}-\left(V''_{101(2)}-V''_{101(1)}\right)^2}{\left(V''_{110(2)}-V''_{110(1)}\right)V''_{002(2)}-V''_{011(2)}\left(V''_{101(2)}-V''_{101(1)}\right)}$$

and

$$b''^{\left(\frac{j-1}{2}\right)}_{26} = \frac{\left(V''_{200(2)}-V''_{200(2)}\right)V''_{011(2)}-\left(V''_{110(2)}-V''_{110(1)}\right)\left(V''_{101(2)}-V''_{101(1)}\right)}{\left(V''_{200(2)}-V''_{200(2)}\right)V''_{002(2)}-\left(V''_{101(2)}-V''_{101(1)}\right)^2},$$

$$\lambda''_{26} = \frac{1}{1 + A''^{G}_{26}},$$

where

$$A''^{G}_{26} = \left[\begin{array}{l} V''_{020(2)} + \dfrac{1}{a''^{\left(\frac{j-1}{2}\right)^2}_{26}}\left(V''_{200(2)} - V''_{200(2)}\right) + \dfrac{1}{b''^{\left(\frac{j-1}{2}\right)^2}_{26}} V''_{002(2)} \\ -\,2\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{26}}\left(V''_{110(2)} - V''_{110(1)}\right) - 2\dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{26}}\left(V''_{101(2)} - V''_{101(1)}\right) \\ +\,2\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{26}}\dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{26}} V''_{011(2)} \end{array} \right],$$

ii) For $\alpha''_{26} = -1, \beta''_{26} = -1,$ we get exponential-product cum product type estimators given in Table 5.2 The MSE of t''^{G}_{26} is expressed as,

$$MSE(t''^{k}_{26}) = \begin{cases} \bar{Y}''^{2}_{s}\left\{\lambda''^{2}_{26}\left(\begin{array}{l} V''_{020(2)} + \left(V''_{200(2)} - V''_{200(2)}\right) + V''_{002(2)} \\ +\,2\left(V''_{110(2)} - V''_{110(1)}\right) + 2\left(V''_{101(2)} - V''_{101(1)}\right) - 2V''_{011(2)} \end{array}\right) + (\lambda''_{26} - 1)^2\right\} & k(\in G) = 2 \\ \bar{Y}''^{2}_{s}\left\{\lambda''^{2}_{26}\left(\begin{array}{l} V''_{020(2)} + \dfrac{1}{a''^{\left(\frac{k}{2}\right)^2}_{26}}\left(V''_{200(2)} - V''_{200(2)}\right) \\ +\,\dfrac{1}{b''^{\left(\frac{k}{2}\right)^2}_{26}} V''_{002(2)} + 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{26}}\left(V''_{110(2)} - V''_{110(1)}\right) \\ +\,2\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{26}}\left(V''_{101(2)} - V''_{101(1)}\right) - 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{26}}\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{26}} V''_{011(2)} \end{array}\right) + (\lambda''_{26} - 1)^2\right\} & k(\in G) = 4, 6, \ldots, 12 \end{cases},$$

(5.2.74)

The optimal values which lead to minimum MSE as

$$a_{26}''^{\left(\frac{k}{2}\right)} = \frac{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{110(2)} - V''_{110(1)}\right)V''_{002(2)} - V''_{011(2)}\left(V''_{101(2)} - V''_{101(1)}\right)}$$

$$\text{and } b_{26}''^{\left(\frac{k}{2}\right)} = \frac{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{011(2)} - \left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)},$$

$$\lambda''_{26} = \frac{1}{1 + A_{26}''^{G}},$$

$$\text{where } A_{26}''^{G} = \left[\begin{array}{l} V''_{020(2)} + \dfrac{1}{a_{26}''^{\left(\frac{k}{2}\right)^2}}\left(V''_{200(2)} - V''_{200(1)}\right) + \dfrac{1}{b_{26}''^{\left(\frac{k}{2}\right)^2}}V''_{002(2)} - 2\dfrac{1}{a_{26}''^{\left(\frac{k}{2}\right)}}\left(V''_{110(2)} - V''_{110(1)}\right) \\ -2\dfrac{1}{b_{26}''^{\left(\frac{k}{2}\right)}}\left(V''_{101(2)} - V''_{101(1)}\right) + 2\dfrac{1}{a_{26}''^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{26}''^{\left(\frac{k}{2}\right)}}V''_{011(2)} \end{array}\right].$$

iii) For $\alpha''_{26} = -1, \beta''_{26} = -1,$ we get exponential-product cum ratio type estimators given in Table 6. The MSE of $t_{26}''^{G}$ is expressed as,

$$MSE(t_{26}''^{l}) = \begin{cases} \bar{Y}_s''^2\left\{\lambda_{26}''^2\left(\begin{array}{l} V''_{020(2)} + \left(V''_{200(2)} - V''_{200(1)}\right) + V''_{002(2)} \\ + 2\left(V''_{110(2)} - V''_{110(1)}\right) - 2\left(V''_{101(2)} - V''_{101(1)}\right) - 2V''_{011(2)} \end{array}\right) + (\lambda''_{26} - 1)^2\right\} & l(\in G) = 13 \\ \bar{Y}_s''^2\left\{\lambda_{26}''^2\left(\begin{array}{l} V''_{020(2)} + \dfrac{1}{a_{26}''^{\left(\frac{l-1}{2}\right)^2}}\left(V''_{200(2)} - V''_{200(2)}\right) \\ + \dfrac{1}{b_{26}''^{\left(\frac{l-1}{2}\right)^2}}V''_{002(2)} + 2\dfrac{1}{a_{26}''^{\left(\frac{l-1}{2}\right)}}\left(V''_{110(2)} - V''_{110(1)}\right) \\ - 2\dfrac{1}{b_{26}''^{\left(\frac{l-1}{2}\right)}} - 2\dfrac{1}{a_{26}''^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{26}''^{\left(\frac{l-1}{2}\right)}}V''_{011(2)} \end{array}\right) + (\lambda''_{26} - 1)^2\right\} & k(\in G) = 15, 17, \ldots, 23 \end{cases},$$

(5.2.75)

The optimal values which lead to minimum MSE as,

$$a_{26}''^{\left(\frac{l-1}{2}\right)} = \frac{\left(V''_{110(2)} - V''_{110(1)}\right)V''_{002(2)} - V''_{011(2)}\left(V''_{101(2)} - V''_{101(1)}\right)}{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)}\right)^2}$$

and

$$b_{26}''^{\left(\frac{l-1}{2}\right)} = \frac{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{002(2)} - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{200(2)} - V''_{200(1)}\right)V''_{011(2)} - \left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)},$$

$$\lambda''_{26} = \frac{1}{1 + A_{26}''^{G}},$$

where

$$A''^{G}_{26}=\left[\begin{array}{l} V''_{020(2)}+\dfrac{1}{\left(a''^{\left(\frac{l-1}{2}\right)}_{26}\right)^{2}}\left(V''_{200(2)}-V''_{200(1)}\right)+\dfrac{1}{\left(b''^{\left(\frac{l-1}{2}\right)}_{26}\right)^{2}}V''_{002(2)} \\ -2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{26}}\left(V''_{110(2)}-V''_{110(1)}\right) \\ -2\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{26}}V''_{101}+2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{26}}\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{26}}V''_{011(2)} \end{array}\right].$$

iv) For $\alpha''_{26}=1,\beta''_{26}=-1$, we get exponential-ratio type estimators given in Table 5.2. The MSE of t''^{G}_{26} is expressed as,

$$MSE\left(t''^{m}_{26}\right)=\left\{\begin{array}{ll} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{26}\left(\begin{array}{l} V''_{020}+\left(V''_{200(2)}-V''_{200(1)}\right)+V''_{002(2)}-2\left(V''_{110(2)}-V''_{110(1)}\right) \\ +2\left(V''_{101(2)}-V''_{101(1)}\right)-2V''_{011(2)} \end{array}\right)+\left(\lambda''_{6}-1\right)^{2}\right) & m(\in G)=14 \\ \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{26}\left(\begin{array}{l} V''_{020(2)}+\dfrac{1}{\left(a''^{\left(\frac{k}{2}\right)}_{26}\right)^{2}}\left(V''_{200(2)}-V''_{200(1)}\right) \\ +\dfrac{1}{\left(b''^{\left(\frac{k}{2}\right)}_{26}\right)^{2}}V''_{002(2)}-2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{26}}\left(V''_{110(2)}-V''_{110(1)}\right) \\ +2\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{26}}\left(V''_{101(2)}-V''_{101(1)}\right)-2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{26}}\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{26}}V''_{011(2)} \end{array}\right)+\left(\lambda''_{26}-1\right)^{2}\right) & m(\in G)=16,18,\ldots,24 \end{array}\right\},$$

(5.2.76)

The optimal values which lead to minimum MSE as

$$a''^{\left(\frac{m}{2}\right)}_{26}=\frac{\left(V''_{200(2)}-V''_{200(1)}\right)V''_{002(2)}-\left(V''_{101(2)}-V''_{101(1)}\right)^{2}}{\left(V''_{110(2)}-V''_{110(1)}\right)V''_{002(2)}-V''_{011(2)}\left(V''_{101(2)}-V''_{101(1)}\right)}$$

and

$$b''^{\left(\frac{m}{2}\right)}_{26}=\frac{\left(V''_{200(2)}-V''_{200(1)}\right)V''_{011(2)}-\left(V''_{110(2)}-V''_{110(1)}\right)\left(V''_{101(2)}-V''_{101(1)}\right)}{\left(V''_{200(2)}-V''_{200(1)}\right)V''_{002(2)}-\left(V''_{101(2)}-V''_{101(1)}\right)^{2}},$$

$$\lambda''_{26}=\frac{1}{1+A''^{G}_{26}},$$

where

$$A''^{G}_{26} = \left[\begin{array}{l} V''_{020} + \dfrac{1}{a''^{\left(\frac{m}{2}\right)^2}_{26}}\left(V''_{200(2)} - V''_{200(1)}\right) + \dfrac{1}{b''^{\left(\frac{m}{2}\right)^2}_{26}} V''_{002(2)} \\ \\ -\, 2\dfrac{1}{a''^{\left(\frac{m}{2}\right)}_{26}}\left(V''_{110(2)} - V''_{110(1)}\right) \\ \\ +\, 2\dfrac{1}{b''^{\left(\frac{m}{2}\right)}_{26}}\left(V''_{101(2)} - V''_{101(1)}\right) - 2\dfrac{1}{a''^{\left(\frac{m}{2}\right)}_{26}}\dfrac{1}{b''^{\left(\frac{m}{2}\right)}_{26}} V''_{011(2)} \end{array}\right].$$

Case III

The generalized estimator under case II may be proposed following (5.2.) as,

$$t^{G}_{26} = \lambda_{26}\bar{y}_s \exp\left(\alpha_{26}\left(1 - \frac{a_{26}\bar{x}_{s(2)}}{\left(\bar{x}_{s(1)} + (a_{26} - 1)\bar{x}_{s(2)}\right)}\right)\right)$$

$$\exp\left(\beta_{26}\left(1 - \frac{b_{26}\,\bar{z}_{s(2)}}{\left(\bar{Z}_s + (b_{26} - 1)\bar{z}_{s(2)}\right)}\right)\right), \quad 0 < \lambda_{26} \le 1 \qquad (5.2.77)$$

where $(a_{26}, b_{26}, \lambda_{26})$ are constants to be determined such that the mean square error (MSE) is minimum. $(\alpha_{26},\ \beta_{26})$ are known constants takes the value(0,1,-1) to produce different ratio-type and product-type estimators.

The proposed estimator (5.2.77) follow the same routine along with the class of estimator in Table 6, as that for case-I in Section 5.3.2. In addition, the relation between a_{26}, α_{26}, λ_{26} and b_{26}, β_{26} in case-II is the same as that for case-I in Section 5.3.2.1. Finally, the same is true for the MSE and the Bias. It is therefore directly from Section 5.3.2.1, we may write $\text{Bias}\,(t^{G}_{26})$ and MSE (t^{G}_{26}) following the same, and we may also produce a class of estimators for similar choices of a_{26}, α_{26}, λ_{26} and b_{26}, β_{26} in case-II. The bias of(5.2.77) may be obtain by following the notations and expectations for case II presented in Section 5.1,

The bias and mean square expressions may be given as

$$Bias\left(t_{26}^{G}\right)=\lambda_{26}\,\bar{Y}_s\left[\frac{\alpha^2{}_{26}\left(V_{200(2)}-V_{200(1)}\right)}{a_{26}{}^2}+\frac{\beta^2{}_{26}}{2b_{26}{}^2}V_{200(2)}\right.$$

$$\left.-\frac{\alpha_{26}}{a_{26}}\left(V_{110(2)}-V''_{110(1)}\right)-\frac{\beta_{26}}{b_{26}}V_{110(2)}\right]+\left(\lambda_{26}-1\right)\bar{Y}_s \quad (5.2.78)$$

$$MSE\left(t_{26}^{G}\right)=\lambda_{26}{}^2\bar{Y}_s{}^2\left[V_{020(2)}+z^2{}_{26}\left(V_{200(2)}-V_{200(1)}\right)+u^2{}_{26}\,V_{002(2)}\right.$$

$$-2z_{26}\left(V_{110(2)}-V_{110(1)}\right)-2u_{26}V_{011(2)}$$

$$\left.+2u_{26}z_{26}\left(V_{101(2)}-V_{101(1)}\right)\right]+\left(\lambda_{26}-1\right)^2\bar{Y}_s{}^2, \quad (5.2.79)$$

Now by substituting the optimal values of the z_6 and u_6, we achieve the minimum MSE among the class of proposed generalized estimator.

$$\left.\begin{array}{l}
u_{26}=\dfrac{\left(V_{200(2)}-V_{200(1)}\right)V_{011(2)}-\left(V_{110(2)}-V_{110(1)}\right)\left(V_{101(2)}-V_{101(1)}\right)}{\left(V_{200(2)}-V_{200(1)}\right)V_{002(2)}-\left(V_{101(2)}-V_{101(1)}\right)^2},\\[2ex]
z_{26}=\dfrac{\left(V_{110(2)}-V_{110(1)}\right)V_{002(2)}-V_{011(2)}\left(V_{101(2)}-V_{101(1)}\right)}{\left(V_{200(2)}-V_{200(1)}\right)V_{002(2)}-\left(V_{101(2)}-V_{101(1)}\right)^2}\\[2ex]
\text{and}\quad \lambda_6=\dfrac{1}{1+A_6{}^G}\\[2ex]
\text{where}\\[1ex]
A_{26}{}^G=\left(\begin{array}{l}V_{020(2)}+z^2{}_{26}\left(V_{200(2)}-V_{200(1)}\right)+u^2{}_{26}\,V_{002(2)}\\ -2z_{26}\left(V_{110(2)}-V_{110(1)}\right)-2u_{26}V_{011(2)}\\ +2u_{26}z_{26}\left(V_{101(2)}-V_{101(1)}\right)\end{array}\right)
\end{array}\right\} \quad (5.2.80)$$

By substituting the optimum values of z_6 and u_6, we get $\lambda_6{}^{opt}$ as,

$$\lambda_{26}{}^{opt}=\frac{1}{1+A_{26}{}^{*}}$$

where

$$A_{26}^{*} = \left(V_{020} - \frac{\left(V_{110(2)} - V_{110(1)}\right)^2 V_{002(2)} + V_{011(2)}^{\ 2}\left(V_{200(2)} - V_{200(1)}\right) - 2\left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)V_{011(2)}}{\left(V_{200(2)} - V_{200(1)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^2} \right).$$

(5.2.81)

We obtain asymptotic variance of the proposed generalized estimator as the expression for $MSE\left(t_6^G\right)$ was considered upto first degree of approximation of error term, so minimum MSE may be written as,

$$MSE_{\min}\left(t_{26}^G\right) = AsymtoticVar\left(t_{26}^G\right) = \bar{Y}_{26}^{\ 2}\left(\frac{A_{26}^{\ *}}{1 + A_{26}^{\ *}}\right). \tag{5.2.82}$$

It may be observed from (5.2.82), that asymptotic variance of the proposed estimator is less than **Usual Linear Regression Estimator**, We may observe from (5.2.82) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value $\lambda_{26}^{\ opt}$ and $a_{26}^{\ opt}, b_{26}^{\ opt}$ in (5.2.77) , we get optimal estimator as,

$$\hat{t}_{26}^G = \hat{\lambda}_{26}\bar{y}_s \exp\left(\alpha_{26}\left(1 - \frac{\hat{a}_{26}\bar{x}_s}{\left(\bar{X}_s + (\hat{a}_{26} - 1)\bar{x}_s\right)}\right)\right)$$
$$\exp\left(\beta_{26}\left(1 - \frac{\hat{b}_{26}\bar{z}_s}{\left(\bar{Z}_s + (\hat{b}_{26} - 1)\bar{z}_s\right)}\right)\right),\quad 0 < \lambda_{26} \le 1 \tag{5.2.83}$$

As described earlier in section (5.2.2.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as,

$$\hat{\lambda}_{26}^{\ opt} = \frac{1}{1 + \hat{A}_{26}^{\ *}}$$

where

$$\hat{A}_{26}^{*}=\left(V_{020}-\frac{\left(V_{110(2)}-V_{110(1)}\right)^{2}V_{002(2)}+V_{011(2)}^{2}\left(V_{200(2)}-V_{200(1)}\right)-2\left(V_{110(2)}-V_{110(1)}\right)\left(V_{101(2)}-V_{101(1)}\right)V_{011(2)}}{\left(V_{200(2)}-V_{200(1)}\right)V_{002(2)}-\left(V_{101(2)}-V_{101(1)}\right)^{2}}\right)$$

(5.2.84)

So (5.2.84) may be wriiten as,

$$\hat{t}_{26}^{G}=\hat{\lambda}_{26}^{opt}\,\bar{y}_s\exp\left(\alpha_{26}\left(1-\frac{\hat{a}_{26}^{opt}\,\bar{x}_s}{\left(\bar{X}_s+(\hat{a}_{26}^{opt}-1)\bar{x}_s\right)}\right)\right)$$

$$\exp\left(\beta_{26}\left(1-\frac{\hat{b}_{26}^{opt}\,\bar{z}_s}{\left(\bar{Z}_s+(\hat{b}_{26}^{opt}-1)\bar{z}_s\right)}\right)\right),\ 0<\lambda_{26}\leq 1 \qquad (5.2.85)$$

Also the minimum MSE may be written as,

$$MSE_{\min}\left(\hat{t}_{26}^{G}\right)=AsymtoticVar\left(\hat{t}_{26}^{G}\right)=\bar{Y}_s^{2}\left(\frac{\hat{A}_{26}^{*}}{1+\hat{A}_{26}^{*}}\right). \qquad (5.2.86)$$

Remark 5.6

i) For $\alpha_{26}=1, \beta_{26}=1$, we get exponential-ratio cum ratio type estimators given in Table 5.2. The MSE of t_{26}^{G} is expressed as,

$$MSE(t_{26}^{j})=\begin{cases}\bar{Y}_s^{2}\left(\lambda_{26}^{2}\left(\begin{array}{l}V_{020(2)}+\left(V_{200(2)}-V_{200(2)}\right)+V_{002(2)}\\-2\left(V_{110(2)}-V_{110(1)}\right)+2\left(V_{101(2)}-V_{101(1)}\right)-2V_{011(2)}\end{array}\right)+(\lambda_{26}-1)^{2}\right) & j(\in G)=1\\[2ex] \bar{Y}_s^{2}\left(\lambda_{26}^{2}\left(\begin{array}{l}V_{020}+\dfrac{1}{a_{26}^{\left(\frac{j-1}{2}\right)^{2}}}\left(V_{200(2)}-V_{200(2)}\right)+\dfrac{1}{b_{26}^{\left(\frac{j-1}{2}\right)^{2}}}V_{002(2)}\\-2\dfrac{1}{a_{26}^{\left(\frac{j-1}{2}\right)}}\left(V_{110(2)}-V_{110(1)}\right)-2\dfrac{1}{b_{26}^{\left(\frac{j-1}{2}\right)}}\left(V_{101(2)}-V_{101(1)}\right)\\+2\dfrac{1}{a_{26}^{\left(\frac{j-1}{2}\right)}}\dfrac{1}{b_{26}''^{\left(\frac{j-1)}{2}\right)}}V_{011(2)}\end{array}\right)+(\lambda_{26}-1)^{2}\right) & j(\in G)=3,5,\ldots,11\end{cases},$$

(5.2.87)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{26}^{\left(\frac{j-1}{2}\right)}} = \frac{\left(V_{110(2)} - V_{110(1)}\right)V_{002(2)} - V_{011(2)}\left(V_{101(2)} - V_{101(1)}\right)}{\left(V_{200(2)} - V_{200(2)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^2}$$

and

$$\frac{1}{b_{26}^{\left(\frac{j-1}{2}\right)}} = \frac{\left(V_{200(2)} - V_{200(2)}\right)V_{011(2)} - \left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)}{\left(V_{200(2)} - V_{200(2)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^2},$$

$$\lambda_{26} = \frac{1}{1 + A_{26}{}^{G}},$$

where

$$A_{26}{}^{G} = \left[\begin{array}{c} V_{020(2)} + \dfrac{1}{a_{26}^{\left(\frac{j-1}{2}\right)^2}}\left(V_{200(2)} - V_{200(2)}\right) + \dfrac{1}{b_{26}^{\left(\frac{j-1}{2}\right)^2}}V_{002(2)} \\ -2\dfrac{1}{a_{26}^{\left(\frac{j-1}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) \\ -2\dfrac{1}{b_{26}^{\left(\frac{j-1}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) + 2\dfrac{1}{a_{26}^{\left(\frac{j-1}{2}\right)}}\dfrac{1}{b_{26}^{\left(\frac{j-1}{2}\right)}}V_{011(2)} \end{array}\right].$$

ii) For $\alpha_{26} = -1, \beta_{26} = -1$, we get exponential-product cum product type estimators given in Table 5.2. The MSE of t_6^G is expressed as,

$$
MSE(t_{26}^{k}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{26}^2 \left(\begin{array}{l} V_{020(2)} + \left(V_{200(2)} - V_{200(2)}\right) + V_{002(2)} + 2\left(V_{110(2)} - V_{110(1)}\right) \\ \quad + 2\left(V_{101(2)} - V_{101(1)}\right) - 2V_{011(2)} \end{array} \right) + (\lambda_{26} - 1)^2 \right) & k(\in G) = 2 \\ \bar{Y}_s^2 \left(\lambda_{26}^2 \left(\begin{array}{l} V_{020(2)} + \dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)^2}}\left(V_{200(2)} - V_{200(2)}\right) \\ \quad + \dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)^2}} V_{002(2)} + 2\dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) \\ \quad + 2\dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) - 2\dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)}} V_{011(2)} \end{array} \right) + (\lambda_{26} - 1)^2 \right) & k(\in G) = 4, 6, \ldots, 12 \end{cases},
$$

(5.2.88)

The optimal values which lead to minimum MSE as

$$
a_{26}^{\left(\frac{k}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{110(2)} - V_{110(1)}\right)V_{002(2)} - V_{011(2)}\left(V_{101(2)} - V_{101(1)}\right)},
$$

and

$$
b_{26}^{\left(\frac{k}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{200(2)} - V_{200(1)}\right)V_{011(2)} - \left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)},
$$

$$
\lambda_{26} = \frac{1}{1 + A_{26}^{G}},
$$

where

$$
A_{26}^{G} = \left[\begin{array}{l} V_{020(2)} + \dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)^2}}\left(V_{200(2)} - V_{200(1)}\right) + \dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)^2}} V_{002(2)} \\ \quad - 2\dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) \\ \quad - 2\dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) + 2\dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)}} V_{011(2)} \end{array} \right].
$$

iii) For $\alpha_{26} = -1, \beta_{26} = 1$, we get exponential-type product cum ratio estimators given in Table 5.2. The MSE of t_{26}^{G} is expressed as

$$
MSE(t_{26}^{l}) = \begin{cases} \bar{Y}_s^{2}\left\{ \lambda_{26}^{2}\begin{pmatrix} V_{020(2)} + \left(V''_{200(2)} - V''_{200(1)}\right) + V''_{002(2)} \\ + 2\left(V''_{110(2)} - V''_{110(1)}\right) - 2\left(V''_{101(2)} - V''_{101(1)}\right) - 2V''_{011(2)} \end{pmatrix} + (\lambda''_{26} - 1)^{2} \right\} & l(\in G) = 13 \\ \bar{Y}_s^{2}\left\{ \lambda_{26}^{2}\begin{pmatrix} V_{020(2)} + \dfrac{1}{a''^{\left(\frac{l-1}{2}\right)^{2}}_{26}}\left(V_{200(2)} - V''_{200(1)}\right) \\ + \dfrac{1}{b''^{\left(\frac{l-1}{2}\right)^{2}}_{26}} V''_{002(2)} + 2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{26}}\left(V''_{110(2)} - V''_{110(1)}\right) \\ - 2\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{26}} V''_{101} - 2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{26}}\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{26}} V''_{011(2)} \end{pmatrix} + (\lambda''_{26} - 1)^{2} \right\} & l(\in G) = 15, 17, \ldots, 23 \end{cases}
$$

(5.2.89)

The optimal values which lead to minimum MSE as

$$
a_{26}^{\left(\frac{l-1}{2}\right)} = \frac{\left(V_{110(2)} - V_{110(1)}\right)V_{002(2)} - V_{011(2)}\left(V_{101(2)} - V_{101(1)}\right)}{\left(V_{200(2)} - V_{200(1)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^{2}},
$$

and

$$
b_{26}^{\left(\frac{l-1}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^{2}}{\left(V_{200(2)} - V_{200(1)}\right)V_{011(2)} - \left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)},
$$

$$
\lambda_{26} = \frac{1}{1 + A_{26}^{G}},
$$

where

$$
A_{26}^{G} = \begin{bmatrix} V_{020(2)} + \dfrac{1}{a_{26}^{\left(\frac{l-1}{2}\right)^{2}}}\left(V_{200(2)} - V_{200(1)}\right) + \dfrac{1}{b_{26}^{\left(\frac{l-1}{2}\right)^{2}}} V_{002(2)} \\ - 2\dfrac{1}{a_{26}^{\left(\frac{l-1}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) \\ - 2\dfrac{1}{b_{26}^{\left(\frac{l-1}{2}\right)}} V_{101} + 2\dfrac{1}{a_{26}^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{26}^{\left(\frac{l-1}{2}\right)}} V_{011(2)} \end{bmatrix}.
$$

iv) For $\alpha_{26} = 1, \beta_{26} = -1$, we get exponential-type-ratio cum product estimators given in Table 5.2. The MSE of t_{26}^{G} is expressed as,

$$MSE(t_{26}^{m}) = \begin{cases} \bar{Y}_s^2 \left\{ \lambda_{26}^2 \left(\begin{array}{l} V_{020} + \left(V_{200(2)} - V_{200(1)}\right) + V_{002(2)} \\ -2\left(V_{110(2)} - V_{110(1)}\right) + 2\left(V_{101(2)} - V_{101(1)}\right) - 2V_{011(2)} \end{array} \right) + (\lambda_6 - 1)^2 \right\} & m(\in G) = 14 \\ \bar{Y}_s^2 \left\{ \lambda_{26}^2 \left(\begin{array}{l} V_{020(2)} + \dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)^2}}\left(V_{200(2)} - V_{200(1)}\right) + \dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)^2}} V_{002(2)} \\ -2\dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) \\ +2\dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) - 2\dfrac{1}{a_{26}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{26}^{\left(\frac{k}{2}\right)}} V_{011(2)} \end{array} \right) + (\lambda_{26} - 1)^2 \right\} & m(\in G) = 16, 18, .., 24 \end{cases},$$

(5.2.90)

The optimal values which lead to minimum MSE as

$$a_{26}^{\left(\frac{m}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{110(2)} - V_{110(1)}\right)V_{002(2)} - V_{011(2)}\left(V_{101(2)} - V_{101(1)}\right)},$$

and

$$b_{26}^{\left(\frac{m}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)V_{002(2)} - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{200(2)} - V_{200(1)}\right)V_{011(2)} - \left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)},$$

$$\lambda_{26} = \frac{1}{1 + A_{26}^{\;G}},$$

where

$$A_{26}^{\;G} = \left[\begin{array}{l} V_{020} + \dfrac{1}{a_{26}^{\left(\frac{m}{2}\right)^2}}\left(V_{200(2)} - V_{200(1)}\right) + \dfrac{1}{b_{26}^{\left(\frac{m}{2}\right)^2}} V_{002(2)} \\ -2\dfrac{1}{a_{26}^{\left(\frac{m}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) \\ +2\dfrac{1}{b_{26}^{\left(\frac{m}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) - 2\dfrac{1}{a_{26}^{\left(\frac{m}{2}\right)}}\dfrac{1}{b_{26}^{\left(\frac{m}{2}\right)}} V_{011(2)} \end{array} \right].$$

5.2.3 Proposed Generalized Exponential Estimator for Situation III

In this section generalized estimator VI has been developed by assuming an exponential relationship between study variable and the two auxiliary variable for three different cases (defined in chapter 4) under situation I.

5.2.3.1 Proposed Generalized Exponential Estimator VIII

Case I

i) The exponential-type ratio cum ratio estimator may be defined as

$$t'^{RR}_{27} = \bar{y}'_{s(2)} exp\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(2)}}\right) exp\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{z}'_{s(1)} + \bar{z}'_{s(2)}}\right), \tag{5.2.91}$$

ii) The exponential-type product cum ratio-type estimator may be defined as

$$t'^{PR}_{27} = \bar{y}'_{s(2)} exp\left(-\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(1)}}\right)\right) \mathrm{exp}\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{z}'_{s(1)} + \bar{z}'_{s(2)}}\right), \tag{5.2.92}$$

iii) The exponential-type ratio cum product estimator may be defined as

$$t'^{RP}_{27} = \bar{y}'_{s(2)} \,\mathrm{exp}\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(1)}}\right) \mathrm{exp}\left(-\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{z}'_{s(1)} + \bar{z}'_{s(2)}}\right)\right),$$

(5.2.93)

iv) The exponential product cum product estimator may be defined as,

$$t'^{PP}_{27} = \bar{y}'_{s(2)} exp\left(-\left(1 - \frac{2\bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(1)}}\right)\right) \mathrm{exp}\left(-\left(1 - \frac{2\bar{z}'_{s(2)}}{\bar{z}'_{s(1)} + \bar{z}'_{s(2)}}\right)\right), \tag{5.2.94}$$

From (5.2.91)-(5.2.94), we may write a generalized form by introducing real constants α'_{27}, β'_{27}, a'_{27} λ'_{27} and b'_{27}.

$$t_{27}'^{G} = \lambda'_{27}\,\overline{y}'_{s(2)} \exp\left(\alpha'_{27}\left(1 - \frac{a'_{27}\,\overline{x}'_{s(2)}}{\left(\overline{x}'_{s(1)} + (a'_{27} - 1)\overline{x}'_{s(2)}\right)}\right)\right)$$

$$\exp\left(\beta'_{27}\left(1 - \frac{b'_{27}\,\overline{z}'_{s(2)}}{\left(\overline{z}'_{s(1)} + (b'_{27} - 1)\overline{z}'_{s(2)}\right)}\right)\right),\ 0 < \lambda'_{27} \le 1 \qquad (5.2.95)$$

where α'_{27}, β'_{27} are known constants. $a'_{27} \neq 0$ and $b'_{27} \neq 0$ are unknown constants whose values are to be estimated from large scale surveys. It is also assumed in advance that value of $0 < \lambda'_{7} \le 1$ to get more precise results.

By substituting different values to the constants in (5.2.95), we get a class of estimators as given in Table 5.3.

Table 5.3

Some Members of the Generalized Estimator t_{27}^{G}

Ratio-cum-product estimator $\alpha'_{27} = 1, \beta'_{27} = 1$	**Product-cum-product estimator** $\alpha'_{27} = -1, \beta'_{27} = -1$	a'_{27}	b'_{7}	λ'_{7}
$t_{27}'^{1} = \overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(1)} - \overline{x}'_{s(2)}}{\overline{x}'_{s(1)} + \overline{x}'_{s(2)}}\right) \exp\left(\frac{\overline{z}'_{s(1)} - \overline{z}'_{s(2)}}{\overline{z}'_{s(1)} + \overline{z}'_{s(2)}}\right)$	$t_{27}'^{2} = \overline{y}'_{s} \exp\left(\frac{\overline{x}'_{s(2)} - \overline{x}'_{s(1)}}{\overline{x}'_{s(2)} + \overline{x}'_{s(1)}}\right) \exp\left(\frac{\overline{z}'_{s(2)} - \overline{z}'_{s(1)}}{\overline{z}'_{s(2)} + \overline{z}'_{s(1)}}\right)$	2	2	1
$t_{27}'^{3} = \overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(1)} - \overline{x}'_{s(2)}}{\overline{x}'_{s(1)}}\right) \exp\left(\frac{\overline{z}'_{s(1)} - \overline{z}'_{s(2)}}{\overline{z}'_{s(1)}}\right)$	$t_{27}'^{4} = \overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(2)} - \overline{x}'_{s(1)}}{\overline{x}'_{s(1)}}\right) \exp\left(\frac{\overline{z}'_{s(2)} - \overline{z}'_{s(1)}}{\overline{z}'_{s(1)}}\right)$	1	1	1
$t_{27}'^{5} = \overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(1)} - \overline{x}'_{s(2)}}{\overline{x}'_{s(1)} + (a'_{7} - 1)\overline{x}'_{s(2)}}\right) \exp\left(\frac{\overline{z}'_{s(1)} - \overline{z}'_{s(2)}}{\overline{z}'_{s(1)} + \left(b'_{7} - 1\right)\overline{z}'_{s(2)}}\right)$	$t_{27}'^{6} = \overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(2)} - \overline{x}'_{s(1)}}{\overline{x}'_{s(1)} + (a'_{7} - 1)\overline{x}'_{s(2)}}\right) \exp\left(\frac{\overline{z}'_{s(2)} - \overline{z}'_{s(1)}}{\overline{z}'_{s(1)} + \left(b'_{7} - 1\right)\overline{z}'_{s(2)}}\right)$	a'_{27}	b'_{7}	1
$t_{27}'^{7} = \lambda'_{27}\,\overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(1)} - \overline{x}'_{s(2)}}{\overline{x}'_{s(1)} + \overline{x}'_{s(2)}}\right) \exp\left(\frac{\overline{z}'_{s(1)} - \overline{z}'_{s(2)}}{\overline{z}'_{s(1)} + \overline{z}'_{s(2)}}\right)$	$t_{27}'^{7} = \lambda'_{27}\,\overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(2)} - \overline{x}'_{s(1)}}{\overline{x}'_{s(2)} + \overline{x}'_{s(1)}}\right) \exp\left(\frac{\overline{z}'_{s(2)} - \overline{z}'_{s(1)}}{\overline{z}'_{s(1)} + \overline{z}'_{s(2)}}\right)$	2	2	
$t_{27}'^{9} = \lambda'_{27}\,\overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(1)} - \overline{x}'_{s(2)}}{\overline{x}'_{s(1)}}\right) \exp\left(\frac{\overline{z}'_{s(1)} - \overline{z}'_{s(2)}}{\overline{z}'_{s(1)}}\right)$	$t_{27}'^{9} = \lambda'_{27}\,\overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(2)} - \overline{x}'_{s(1)}}{\overline{x}'_{s(1)}}\right) \exp\left(\frac{\overline{z}'_{s(2)} - \overline{z}'_{s(1)}}{\overline{z}'_{s(1)}}\right)$	1	1	λ'_{7}
$t_{27}'^{11} = \lambda'_{27}\,\overline{y}'_{s(2)} \exp\left(\frac{\overline{x}'_{s(1)} - \overline{x}'_{s(2)}}{\overline{x}'_{s(1)} + (a'_{27} - 1)\overline{x}'_{s(2)}}\right) \exp\left(\frac{\overline{z}'_{s(1)} - \overline{z}'_{s(2)}}{\overline{Z}'_{s} + \left(b'_{27} - 1\right)\overline{z}'_{s(2)}}\right)$	$t_{27}'^{12} = \lambda'_{27}\,\overline{y}'_{s} \exp\left(\frac{\overline{x}'_{s(2)} - \overline{x}'_{s(1)}}{\overline{x}'_{s(1)} + (a'_{27} - 1)\overline{x}'_{s(2)}}\right) \exp\left(\frac{\overline{z}'_{s(2)} - \overline{z}'_{s(1)}}{\overline{z}'_{s(1)} + \left(b_{27}' - 1\right)\overline{z}'_{s(2)}}\right)$	a'_{27}	b'_{7}	λ'_{7}

Product -cum-ratio estimator $\alpha'_7 = -1, \beta'_7 = 1$	Ratio-cum-product estimator $\alpha'_7 = 1, \beta'_7 = -1$	a'_{27}	b'_7	λ'_7
$t'^{13}_{27} = \bar{y}'_{s(2)} \exp\left(\frac{\bar{x}'_{s(1)} - \bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(2)}}\right)\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)} + \bar{z}'_{s(2)}}\right)$	$t'^{14}_{27} = \bar{y}'_{s(2)} \exp\left(\frac{\bar{x}'_{s(2)} - \bar{x}'_{s(1)}}{\bar{x}'_{s(2)} + \bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(2)} + \bar{z}'_{s(1)}}\right)$	2	2	1
$t'^{15}_{27} = \bar{y}'_{s(2)} \exp\left(\frac{\bar{x}'_{s(1)} - \bar{x}'_{s(2)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(1)} - \bar{Z}'_{s}}{\bar{Z}'_{s}}\right)$	$t'^{16}_{27} = \bar{y}'_{s(2)} \exp\left(\frac{\bar{x}'_{s(2)} - \bar{x}'_{s(1)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)}}\right)$	1	1	1
$t'^{17}_{27} = \bar{y}'_{s(2)} \exp\left(\frac{\bar{x}'_{s(1)} - \bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + (a'_{27} - 1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)} + (b'_{27} - 1)\bar{z}'_{s(2)}}\right)$	$t'^{18}_{27} = \bar{y}'_{s(2)} \exp\left(\frac{\bar{x}'_{s(2)} - \bar{x}'_{s(1)}}{\bar{x}'_{s(1)} + (a'_{27} - 1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)} + (b'_{27} - 1)\bar{z}'_{s(2)}}\right)$	a'_{27}	b'_7	1
$t'^{19}_{27} = \lambda'_{27}\bar{y}'_{s} \exp\left(\frac{\bar{x}'_{s(1)} - \bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + \bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)} + \bar{z}'_{s(2)}}\right)$	$t'^{20}_{27} = \lambda'_{27}\bar{y}'_{s} \exp\left(\frac{\bar{x}'_{s(2)} - \bar{x}'_{s(1)}}{\bar{x}'_{s(2)} + \bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)} + \bar{z}'_{s(2)}}\right)$	2	2	λ'_7
$t'^{21}_{27} = \lambda'_{27}\bar{y}'_{s} \exp\left(\frac{\bar{x}'_{s(1)} - \bar{x}'_{s(2)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)}}\right)$	$t'^{22}_{27} = \lambda'_{27}\bar{y}'_{s} \exp\left(\frac{\bar{x}'_{s(2)} - \bar{x}'_{s(1)}}{\bar{x}'_{s(1)}}\right)\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)}}\right)$	1	1	λ'_7
$t'^{23}_{27} = \lambda'_{27}\bar{y}'_{s} \exp\left(\frac{\bar{x}'_{s(1)} - \bar{x}'_{s(2)}}{\bar{x}'_{s(1)} + (a'_{27} - 1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)} + (b'_{27} - 1)\bar{z}'_{s(2)}}\right)$	$t'^{24}_{27} = \lambda'_{7}\bar{y}'_{s} \exp\left(\frac{\bar{x}'_{s(2)} - \bar{x}'_{s(1)}}{\bar{x}'_{s(1)} + (a'_{27} - 1)\bar{x}'_{s(2)}}\right)$ $\exp\left(\frac{\bar{z}'_{s(2)} - \bar{z}'_{s(1)}}{\bar{z}'_{s(1)} + (b'_{27} - 1)\bar{z}'_{s(2)}}\right)$	a'_{27}	b'_7	λ'_7

The Bias and Mean Square Error of Generalized Estimator

To derive the bias and mean square error we proceed as follows,
Using (5.1.1) we can express (5.2.95) as

$$t'^{G}_{27} = \lambda'_{27}\bar{Y}'_{s}\left(1 + e'_{0(2)}\right)\exp\left[-\frac{\alpha'_{27}}{a'_{27}}\left(e'_{1(2)} - e'_{1(1)}\right)\left(1 + \frac{e'_{1(1)}}{a'_{27}} + \frac{(a'_{27} - 1)}{a'_{27}}e'_{1(2)}\right)^{-1}\right],$$

$$\exp\left[-\frac{\beta'_{27}}{b'_{27}}\left(e'_{2(2)} - e'_{2(1)}\right)\left(1 + \frac{e'_{2(1)}}{b'_{27}} + \frac{(b'_{27} - 1)}{b'_{27}}e'_{2(2)}\right)^{-1}\right] \qquad (5.2.96)$$

We assume that $\left|e'_{1(1)}\right| < 1, \left|e'_{2(1)}\right| < 1$, we expand the series, $\left(1 + \frac{e'_{1(2)}}{a'_{27}} + \frac{(a'_{27} - 1)}{a'_{27}}e'_{1(1)}\right)^{-1}$ and $\left(1 + \frac{e'_{2(1)}}{b'_{27}} + \frac{(b'_{27} - 1)}{b'_{27}}e'_{2(2)}\right)^{-1}$, we get,

$$t'^{G}_{27} = \lambda'_{27}\bar{Y}'_{s}\left(1 + e'_{0(2)}\right)\exp\left[-\frac{\alpha'_{27}}{a'_{27}}\left(e'_{1(2)} - e'_{1(1)}\right)\right]\exp\left[-\frac{\beta'_{27}}{b'_{27}}\left(e'_{2(2)} - e'_{2(1)}\right)\right], \qquad (5.2.97)$$

$$t'^{G}_{27} - \bar{Y}'_s = \lambda'_{27}\bar{Y}'_s\left[-\frac{\alpha'_{27}}{a'_{27}}\left(e'_{1(2)} - e'_{1(1)}\right) + \frac{\alpha'^2_{27}\left(e'^2_{1(2)} + e'^2_{1(1)} - 2e'_{1(2)}e'_{1(1)}\right)}{2a'^2_{27}}\right.$$

$$-\frac{\beta'_{27}}{b'_{27}}\left(e'_{2(2)} - e'_{2(1)}\right) + \frac{\beta'^2_{27}}{b'^2_{27}}\left(e'^2_{2(2)} + e'^2_{2(1)} - 2e'_{2(2)}e'_{2(1)}\right)$$

$$-\frac{\alpha'_{27}}{a'_{27}}\frac{\beta'_{27}}{b'_{27}}\left(e'_{2(2)} - e'_{2(1)}\right)\left(e'_{1(2)} - e'_{1(1)}\right) + e'_{0(2)} - \frac{\alpha'_{27}}{a'_{27}}e'_{0(2)}\left(e'_{1(2)} - e'_{1(1)}\right)$$

$$\left. -\frac{\beta'_{27}}{b'_{27}}e'_{0(2)}\left(e'_{2(2)} - e'_{2(1)}\right)\right] + \left(\lambda'_{27} - 1\right)\bar{Y}'_s, \tag{5.2.98}$$

By taking expectation of (5.2.98) and using (5.1.1), we get the $Bias\left(t'^{G}_{7}\right)$ as,

$$Bias\left(t'^{G}_{27}\right) = \lambda'_{27}\bar{Y}'_s\left[\frac{\alpha'^2_{27}\left(V'_{200(2)} - V'_{200(1)}\right)}{2a'^2_{27}} + \frac{\beta'^2_{27}}{2b'^2_{27}}\left(V'_{002(2)} - V'_{002(1)}\right)\right.$$

$$\left. -\frac{\alpha'_{27}}{a'_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) - \frac{\beta'_{27}}{b'_{27}}\left(V'_{011(2)} - V'_{011(1)}\right)\right] + \left(\lambda'_{27} - 1\right)\bar{Y}'_s, \tag{5.2.99}$$

To get the MSE of the estimator, we take square and we take expectation of (5.2.98) and we obtain

$$\left(t'^{G}_{27} - \bar{Y}'_s\right)^2 = \lambda'^2_{27}\bar{Y}'^2_s\left[e'^2_{0(2)} + \frac{\alpha'^2_{27}}{a'^2_{27}}\left(e'_{1(2)} - e'_{1(1)}\right)^2 + \frac{\beta'^2_{27}}{b'^2_{27}}\left(e'_{2(2)} - e'_{2(1)}\right)^2\right.$$

$$-2\frac{\alpha'_{27}}{a'_{27}}e'_{0(2)}\left(e'_{1(2)} - e'_{1(1)}\right) - 2\frac{\beta'_{27}}{b'_{27}}e'_{0(2)}\left(e'_{2(2)} - e'_{2(1)}\right)$$

$$\left. +2\frac{\beta'_{27}}{b'_{27}}\frac{\alpha'_{27}}{a'_{27}}\left(e'_{2(2)} - e'_{2(1)}\right)\left(e'_{1(2)} - e'_{1(1)}\right)\right] + \left(\lambda'_{27} - 1\right)^2\bar{Y}'^2_s, \tag{5.2.100}$$

We take expectation on (5.2.100) by using (5.1.1) as

$$MSE\left(t'^{G}_{27}\right) = \lambda'^2_{27}\bar{Y}'^2_s\left[V'_{020(2)} + z'^2_{7}\left(V'_{200(2)} - V'_{200(1)}\right) + u'^2_{27}\left(V'_{002(2)} - V'_{002(1)}\right)\right.$$

$$-2z'_{27}\left(V'_{110(2)} - V'_{110(1)}\right) - 2u'_{27}\left(V'_{011(2)} - V'_{011(2)}\right)$$

$$\left. +2u'_{27}z'_{27}\left(V'_{101(2)} - V'_{101(1)}\right)\right] + \left(\lambda'_{27} - 1\right)^2\bar{Y}'^2_s, \tag{5.2.101}$$

where $z'_{27} = \dfrac{\alpha'_{27}}{a'_{27}}, u'_{27} = \dfrac{\beta'_{27}}{b'_{27}}$, For the following optimal value of the constants z'_{27} and u'_{27}, we achieve the minimum MSE among the class of proposed generalized estimator.

$$\left.\begin{aligned}
u'_{27} &= \frac{\left(V'_{200(2)}-V'_{200(1)}\right)\left(V'_{011(2)}-V'_{011(1)}\right)-\left(V'_{110(2)}-V'_{110(1)}\right)\left(V'_{101(2)}-V'_{101(1)}\right)}{\left(V'_{200(2)}-V'_{200(1)}\right)\left(V'_{002(2)}-V'_{002(1)}\right)-\left(V'_{101(2)}-V'_{101(1)}\right)^2},\\
z'_{27} &= \frac{\left(V'_{110(2)}-V'_{110(1)}\right)\left(V'_{002(2)}-V'_{002(1)}\right)-\left(V'_{011(2)}-V'_{011(1)}\right)\left(V'_{101(2)}-V'_{101(1)}\right)}{\left(V'_{200(2)}-V'_{200(1)}\right)\left(V'_{002(2)}-V'_{002(1)}\right)-\left(V'_{101(2)}-V'_{101(1)}\right)^2},\\
\text{and } \lambda'_{27} &= \frac{1}{1+A'^{\,G}_{27}}\\
\text{where}\\
A'^{\,G}_{27} &= \left(\begin{aligned}
&V'_{020(2)}+z'^2_{27}\left(V'_{200(2)}-V'_{200(1)}\right)\\
&+u'^2_{27}\left(V'_{002(2)}-V'_{002(1)}\right)-2z'_{27}\left(V'_{110(2)}-V'_{110(1)}\right)\\
&-2u'_{27}\left(V'_{011(2)}-V'_{011(1)}\right)+2u'_{27}z'_{27}\left(V'_{101(2)}-V'_{101(1)}\right)
\end{aligned}\right)
\end{aligned}\right\}$$

(5.2.103)

By substituting the optimum values of z'_6 and u'_6, we get $\lambda'^{\,opt}_{6}$ as,

$$\lambda'^{\,opt}_{27} = \frac{1}{1+A'^{\,*}_{27}}$$

where

$$A'^{\,*}_{27} = \left(V'_{020(2)} - \frac{\begin{aligned}&\left(V'_{110(2)}-V'_{110(1)}\right)^2\left(V'_{002(2)}-V'_{002(1)}\right)\\ &+\left(V'_{011(2)}-V'_{011(1)}\right)^2\left(V'_{200(2)}-V'_{200(1)}\right)\\ &-2\left(V'_{110(2)}-V'_{110(1)}\right)\left(V'_{101(2)}-V'_{101(1)}\right)\left(V'_{011(2)}-V'_{011(1)}\right)\end{aligned}}{\left(V'_{200(2)}-V'_{200(1)}\right)\left(V'_{002(2)}-V'_{002(1)}\right)-\left(V'_{101(2)}-V'_{101(1)}\right)^2}\right)$$

(5.2.104)

We obtain asymptotic variance of the proposed generalized estimator as the expression for $MSE\left(t'^{G}_{27}\right)$ was considered upto ist degree of error term , so minimum MSE may be written as ,

$$MSE_{\min}\left(t'^{G}_{27}\right) = AsymtoticVar\left(t'^{G}_{27}\right) = \overline{Y}'^{2}_{s}\left(\frac{A'^{*}_{27}}{1+A'^{*}_{27}}\right). \tag{5.2.105}$$

From (5.2.105), we observe that asymptotic variance of the proposed estimator is less than Usual Linear Regression Estimator, We may observe from (5.2.105) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value λ'^{opt}_{27} and $a'^{opt}_{7}, b'^{opt}_{27}$ in (5.2.105), we get optimal estimator as,

$$\hat{t}'^{G}_{27} = \hat{\lambda}'_{27}\overline{y}'_{s(2)}\exp\left(\alpha'_{7}\left(1-\frac{\hat{a}'_{27}\overline{x}'_{s(2)}}{\left(\overline{x}'_{s(1)}+(\hat{a}'_{27}-1)\overline{x}'_{s(2)}\right)}\right)\right)$$
$$\exp\left(\beta'_{27}\left(1-\frac{\hat{b}'_{27}\overline{z}'_{s(2)}}{\left(\overline{z}'_{s(1)}+(\hat{b}'_{27}-1)\overline{z}'_{s(2)}\right)}\right)\right),\ 0<\hat{\lambda}'_{27}\le 1 \tag{5.2.106}$$

As described earlier in section (5.2.3.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as,

$$\hat{\lambda}'^{*}_{27} = \frac{1}{1+\hat{A}'^{*}_{27}}$$

Where

$$\hat{A}'^{*}_{27} = \left(\begin{array}{l}\hat{V}'_{020(2)} - \\ \dfrac{\left(\begin{array}{l}\left(\hat{V}'_{110(2)}-\hat{V}'_{110(1)}\right)^{2}\left(\hat{V}'_{002(2)}-\hat{V}'_{002(1)}\right)+\left(\hat{V}'_{011(2)}-\hat{V}'_{011(1)}\right)^{2}\left(\hat{V}'_{200(2)}-\hat{V}'_{200(1)}\right)\\ -2\left(\hat{V}'_{110(2)}-\hat{V}'_{110(1)}\right)\left(\hat{V}'_{101(2)}-\hat{V}'_{101(1)}\right)\left(\hat{V}'_{011(2)}-\hat{V}'_{011(1)}\right)\end{array}\right)}{\left(\hat{V}'_{200(2)}-\hat{V}'_{200(1)}\right)\left(\hat{V}'_{002(2)}-\hat{V}'_{002(1)}\right)-\left(\hat{V}'_{101(2)}-\hat{V}'_{101(1)}\right)^{2}}\end{array}\right)$$

(5.2.107)

So (5.2.107) may be wriiten as,

$$\hat{t}'^{G}_{27} = \hat{\lambda}'^{opt}_{27}\ \overline{y}'_{s(2)} \exp\left(\alpha'_{27}\left(1 - \frac{\hat{a}'^{opt}_{27}\ \overline{x}'_{s(2)}}{\left(\overline{X}'_{s} + (\hat{a}'^{opt}_{27} - 1)\overline{x}'_{s(2)}\right)}\right)\right)$$

$$\exp\left(\beta'_{27}\left(1 - \frac{\hat{b}'^{\ opt}_{27}\overline{z}'_{s(2)}}{\left(\overline{z}'_{s(1)} + (\hat{b}'^{\ opt}_{27} - 1)\overline{z}'_{s(2)}\right)}\right)\right),\ 0 < \lambda'_{27} \leq 1 \qquad (5.2.108)$$

Also the minimum MSE may be written as,

$$MSE_{\min}\left(\hat{t}'^{G}_{27}\right) = Asymtotic Var\left(\hat{t}'^{G}_{27}\right) = \overline{Y}'^{2}_{s}\left(\frac{A'^{\ *}_{27}}{1 + A'^{\ *}_{27}}\right). \qquad (5.2.109)$$

Remark

i) For $\alpha'_{27} = 1, \beta'_{27} = 1$, we get exponential- type ratio cum ratio type estimators given in Table 5.3. The MSE of t'^{G}_{27} is expressed as,

$$MSE(t'^{j}_{27}) = \begin{cases} \bar{Y}'^{2}_{s}\left\{\lambda'^{2}_{27}\left(\begin{aligned} &V'_{020(2)} + \left(V'_{200(2)} - V'_{200(2)}\right) + \left(V'_{002(2)} - V'_{002(1)}\right) - 2\left(V'_{110(2)} - V'_{110(1)}\right) \\ &+2\left(V_{101(2)} - V'_{101(1)}\right) - 2\left(V'_{011(2)} - V'_{011(1)}\right) \end{aligned}\right) + (\lambda'_{7} - 1)^{2}\right\} & j(\in G) = 1 \\ \bar{Y}'^{2}_{s}\left\{\lambda'^{2}_{27}\left(\begin{aligned} &V'_{020(2)} + \frac{1}{a'^{\left(\frac{j-1}{2}\right)^{2}}_{27}}\left(V'_{200(2)} - V'_{200(2)}\right) + \frac{1}{b'^{\left(\frac{j-1}{2}\right)^{2}}_{27}}\left(V'_{002(2)} - V'_{002(1)}\right) \\ &- 2\frac{1}{a'^{\left(\frac{j-1}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) - 2\frac{1}{b'^{\left(\frac{j-1}{2}\right)}_{27}}\left(V_{101(2)} - V'_{101(1)}\right) \\ &+ 2\frac{1}{a'^{\left(\frac{j-1}{2}\right)}_{27}}\frac{1}{b'^{\left(\frac{j-1)}{2}\right)}_{27}}\left(V'_{011(2)} - V'_{011(1)}\right) \end{aligned}\right) + (\lambda'_{27} - 1)^{2}\right\} & j(\in G) = 3,5,..,11 \end{cases},$$

(5.2.110)

The optimal values which lead to minimum MSE as,

$$a'^{\left(\frac{j-1}{2}\right)}_{27} = \frac{\left(V'_{200(2)} - V'_{200(2)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V_{101(2)} - V'_{101(1)}\right)^{2}}{\left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{011(2)} - V'_{011(1)}\right)\left(V_{101(2)} - V'_{101(1)}\right)},$$

and

$$b'^{\left(\frac{j-1}{2}\right)}_{27} = \frac{\left(V'_{200(2)} - V'_{200(2)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V_{101(2)} - V'_{101(1)}\right)^{2}}{\left(V'_{200(2)} - V'_{200(2)}\right)\left(V'_{011(2)} - V'_{011(1)}\right)},$$

$$-\left(V'_{110(2)} - V'_{110(1)}\right)\left(V_{101(2)} - V'_{101(1)}\right)$$

$$\lambda'_{7} = \frac{1}{1 + A'^{G}_{7}}$$

where

$$A'^{G}_{27} = \left[\begin{aligned} &V'_{020(2)} + \frac{1}{a'^{\left(\frac{j-1}{2}\right)^{2}}_{27}}\left(V'_{200(2)} - V'_{200(2)}\right) + \frac{1}{b'^{\left(\frac{j-1}{2}\right)^{2}}_{27}}\left(V'_{002(2)} - V'_{002(1)}\right) \\ &- 2\frac{1}{a'^{\left(\frac{j-1}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) - 2\frac{1}{b'^{\left(\frac{j-1}{2}\right)}_{27}}\left(V_{101(2)} - V'_{101(1)}\right) \\ &+ 2\frac{1}{a'^{\left(\frac{j-1}{2}\right)}_{27}}\frac{1}{b'^{\left(\frac{j-1}{2}\right)}_{27}}\left(V'_{011(2)} - V'_{011(1)}\right) \end{aligned}\right]$$

ii) For $\alpha'_{27} = -1, \beta'_{27} = -1$, we get exponential-type product cum product type estimators given in Table 5.3. The MSE of t'^{G}_{27} is expressed as,

$$MSE(t'^{k}_{27}) = \begin{cases} \bar{Y}_s'^2 \left\{ \lambda'^{2}_{27} \left\{ \begin{array}{l} V'_{020(2)} + \left(V'_{200(2)} - V'_{200(2)}\right) + \left(V'_{002(2)} - V'_{002(1)}\right) \\ + 2\left(V'_{110(2)} - V'_{110(1)}\right) + 2\left(V'_{101(2)} - V'_{101(1)}\right) - 2\left(V'_{011(2)} - V'_{011(1)}\right) \end{array} \right\} \right. \\ \left. \quad + (\lambda'_{27} - 1)^2 \right\} & k (\in G) = 2 \\ \bar{Y}_s'^2 \left\{ \lambda'^{2}_{27} \left\{ \begin{array}{l} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{k}{2}\right)^2}_{27}}\left(V'_{200(2)} - V'_{200(2)}\right) + \dfrac{1}{b'^{\left(\frac{k}{2}\right)^2}_{27}} V'_{002(2)} \\ + 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) + 2\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{101(2)} - V'_{101(1)}\right) \\ - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{27}}\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{011(2)} - V'_{011(1)}\right) \end{array} \right\} + (\lambda'_{27} - 1)^2 \right\} & k (\in G) = 4, 6, .., 12 \end{cases},$$

(5.2.111)

The optimal values which lead to minimum MSE as

$$a'^{\left(\frac{k}{2}\right)}_{27} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{011(2)} - V'_{011(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)},$$

and

$$b'^{\left(\frac{k}{2}\right)}_{27} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{011(2)} - V'_{011(1)}\right) - \left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)},$$

$$\lambda'_{27} = \frac{1}{1 + A'^{G}_{27}}$$

where

$$A'^{G}_{27} = \left[\begin{array}{l} V_{020(2)} + \dfrac{1}{a'^{\left(\frac{k}{2}\right)^2}_{27}}\left(V'_{200(2)} - V'_{200(1)}\right) + \dfrac{1}{b'^{\left(\frac{k}{2}\right)^2}_{27}}\left(V'_{002(2)} - V'_{002(1)}\right) \\ - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) - 2\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{101(2)} - V'_{101(1)}\right) \\ + 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{27}}\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{011(2)} - V'_{011(1)}\right) \end{array} \right].$$

iii) For $\alpha'_{27} = -1, \beta'_{27} = 1$, we get exponential-type-product cum ratio type estimators given in Table 5.3. The MSE of t'^{G}_{27} is expressed as,

$$MSE(t'^{l}_{27}) = \begin{cases} \bar{Y}'^{2}_{s}\left\{\lambda'^{2}_{27}\left(\begin{array}{l} V'_{020(2)} + \left(V'_{200(2)} - V'_{200(1)}\right) + \left(V'_{002(2)} - V'_{002(1)}\right) \\ + 2\left(V'_{110(2)} - V'_{110(1)}\right) - 2\left(V'_{101(2)} - V'_{101(1)}\right) - 2V'_{011(2)} \end{array}\right) + (\lambda'_{27} - 1)^{2}\right\} & l(\in G) = 13 \\ \bar{Y}'^{2}_{s}\left\{\lambda'^{2}_{27}\left(\begin{array}{l} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^{2}}_{27}}\left(V'_{200(2)} - V'_{200(1)}\right) \\ + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^{2}}_{27}} V'_{002(2)} + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) \\ - 2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{27}} V_{101} - 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{27}}\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{27}} V_{011(2)} \end{array}\right) + (\lambda'_{27} - 1)^{2}\right\} & l(\in G) = 15, 17, \ldots, 23 \end{cases},$$

(5.2.112)

The optimal values which lead to minimum MSE as,

$$a'^{\left(\frac{l-1}{2}\right)}_{27} = \frac{\left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{011(2)} - V'_{011(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)}{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{101(2)} - V'_{101(1)}\right)^{2}}$$

and

$$b'^{\left(\frac{l-1}{2}\right)}_{27} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{101(2)} - V'_{101(1)}\right)^{2}}{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{011(2)} - V'_{011(1)}\right) - \left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)},$$

$$\lambda'_{27} = \frac{1}{1 + A'^{G}_{27}}$$

where

$$A'^{G}_{27} = \left[\begin{array}{l} V_{020(2)} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^{2}}_{27}}\left(V'_{200(2)} - V'_{200(1)}\right) + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^{2}}_{27}}\left(V'_{002(2)} - V'_{002(1)}\right) \\ - 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) - 2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{27}} V_{101} \\ + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{27}}\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{27}}\left(V'_{011(2)} - V'_{011(1)}\right) \end{array}\right].$$

iv) For $\alpha'_{27} = 1, \beta'_{27} = -1$, we get exponential- type ratio cum product estimators given in Table 5.3. The MSE of t'^{G}_{27} is expressed as

$$MSE(t'^{m}_{27}) = \begin{cases} \bar{Y}'^2_s \left\{ \lambda'^2_{27} \left(\begin{array}{l} V'_{020} + \left(V'_{200(2)} - V'_{200(1)}\right) + V'_{002(2)} - 2\left(V'_{110(2)} - V'_{110(1)}\right) \\ +2\left(V'_{101(2)} - V'_{101(1)}\right) - 2V'_{011(2)} \end{array} \right) + (\lambda'_{27} - 1)^2 \right\} & m(\in G) = 14 \\ \bar{Y}'^2_s \left\{ \lambda'^2_{27} \left(\begin{array}{l} V'_{020(2)} + \dfrac{1}{a'^{\left(\frac{k}{2}\right)^2}_{27}}\left(V'_{200(2)} - V'_{200(1)}\right) \\ + \dfrac{1}{b'^{\left(\frac{k}{2}\right)^2}_{27}} V'_{002(2)} - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) \\ + 2\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{27}}\left(V'_{101(2)} - V'_{101(1)}\right) - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{27}}\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{27}} V'_{011(2)} \end{array} \right) + (\lambda'_{27} - 1)^2 \right\} & m(\in G) = 16,18,..,24 \end{cases}$$

(5.2.113)

The optimal values which lead to minimum MSE as,

$$a'^{\left(\frac{m}{2}\right)}_{27} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{011(2)} - V'_{011(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)}$$

and

$$b'^{\left(\frac{m}{2}\right)}_{27} = \frac{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{002(2)} - V'_{002(1)}\right) - \left(V'_{101(2)} - V'_{101(1)}\right)^2}{\left(V'_{200(2)} - V'_{200(1)}\right)\left(V'_{011(2)} - V'_{011(1)}\right) - \left(V'_{110(2)} - V'_{110(1)}\right)\left(V'_{101(2)} - V'_{101(1)}\right)},$$

$$\lambda'_{27} = \frac{1}{1 + A'^{G}_{27}},$$

where

$$A'^{G}_{27} = \left[\begin{array}{l} V'_{020} + \dfrac{1}{a'^{\left(\frac{m}{2}\right)^2}_{27}}\left(V'_{200(2)} - V'_{200(1)}\right) + \dfrac{1}{b'^{\left(\frac{m}{2}\right)^2}_{27}}\left(V'_{002(2)} - V'_{002(1)}\right) \\ - 2\dfrac{1}{a'^{\left(\frac{m}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) + 2\dfrac{1}{b'^{\left(\frac{m}{2}\right)}_{27}}\left(V'_{101(2)} - V'_{101(1)}\right) \\ - 2\dfrac{1}{a'^{\left(\frac{m}{2}\right)}_{27}}\dfrac{1}{b'^{\left(\frac{m}{2}\right)}_{27}}\left(V'_{110(2)} - V'_{110(1)}\right) \end{array} \right].$$

Case II

The generalized estimator under case II may be proposed following (5.2.) as

$$t''^{G}_{27} = \lambda''_{27}\,\bar{y}''_{s(2)}\exp\left(\alpha''_{7}\left(1-\frac{a''_{27}\,\bar{x}''_{s(2)}}{\left(\bar{x}''_{s(1)}+(a''_{27}-1)\bar{x}''_{s(2)}\right)}\right)\right)$$

$$\exp\left(\beta''_{7}\left(1-\frac{b''_{27}\;\bar{z}''_{s(2)}}{\left(\bar{z}''_{s(1)}+(b''_{27}-1)\bar{z}''_{s(2)}\right)}\right)\right), 0<\lambda''_{27}\leq 1 \qquad (5.2.114)$$

where α''_{27}, β''_{27} are known constants. $a''_{27}\neq 0$ and $b''_{27}\neq 0$ are unknown constants whose values are to be estimated from large scale surveys. It is also assumed in advance that value of $0<\lambda'_{7}\leq 1$ to get more precise results.

The proposed estimator (5.2.114) follows the same fashion along with the class of estimator in Table 5.3, as that for case-I in Section 5.2.3.2. In addition, the relation between a''_{27}, α''_{27}, λ''_{27} and b''_{27}, β''_{27} in case-II is the same as that for case-I in Section 5.2.3.2.1. Finally, the same is true for the MSE and the Bias. It is therefore directly from Section 5.3.2.1, we may write $\text{Bias}\,(t''^{G}_{27})$ and $\text{MSE}(t''^{G}_{27})$ following the same, and we may also produce a class of estimators for similar choices of , a''_{27}, α''_{27}, λ''_{27} in case-II. The bias of (5.2.115) may be obtain by following the notations and expectations for case II presented in Section 5.1,

The bias and mean square expressions may be given as,

$$Bias\left(t''^{G}_{27}\right) = \lambda''_{27}\;\bar{Y}''_{s}\left[\frac{\alpha''^{2}_{27}\left(V''_{200(2)}-V''_{200(1)}\right)}{a''^{\;2}_{27}}+\frac{\beta''^{2}_{27}}{2b''^{\;2}_{27}}\left(V''_{200(2)}-V''_{200(1)}\right)\right.$$

$$\left.-\frac{\alpha''_{27}}{a''_{27}}\left(V''_{110(2)}-V''_{110(1)}\right)-\frac{\beta''_{27}}{b''_{27}}\left(V''_{110(2)}-V''_{110(1)}\right)\right]+\left(\lambda''_{27}-1\right)\bar{Y}''_{s} \qquad (5.2.116)$$

$$MSE\,(t''^{G}_{27}) = \lambda''^{\;2}_{27}\bar{Y}''^{\;2}_{s}\left[V''_{020(2)}+z''^{2}_{27}\left(V''_{200(2)}-V''_{200(1)}\right)+u''^{2}_{27}\left(V''_{002(2)}-V''_{002(1)}\right)\right.$$

$$-2z''_{27}\left(V''_{110(2)}-V''_{110(1)}\right)-2u''_{27}\left(V''_{011(2)}-V''_{011(1)}\right)$$

$$\left.+2u''_{27}z''_{27}\left(V''_{101(2)}-V''_{101(1)}\right)\right]+\left(\lambda''_{27}-1\right)^{2}\bar{Y}''^{\;2}_{s}, \qquad (5.2.117)$$

Now by substituting the optimal values of the z_7'' and u_7'', we achieve the minimum MSE among the class of proposed generalized estimator.

$$\left.\begin{array}{l}
u_{27}'' = \dfrac{\left(V_{200(2)}'' - V_{200(1)}''\right)\left(V_{011(2)}'' - V_{011(1)}''\right) - \left(V_{110(2)}'' - V_{110(1)}''\right)\left(V_{101(2)}'' - V_{101(1)}''\right)}{\left(V_{200(2)}'' - V_{200(1)}''\right)\left(V_{002(2)}'' - V_{002(1)}''\right) - \left(V_{101(2)}'' - V_{101(1)}''\right)^2}, \\[2ex]
z_{27}'' = \dfrac{\left(V_{110(2)}'' - V_{110(1)}''\right)\left(V_{002(2)}'' - V_{002(1)}''\right) - \left(V_{011(2)}'' - V_{011(1)}''\right)\left(V_{101(2)}'' - V_{101(1)}''\right)}{\left(V_{200(2)}'' - V_{200(1)}''\right)\left(V_{002(2)}'' - V_{002(1)}''\right) - \left(V_{101(2)}'' - V_{101(1)}''\right)^2} \\[2ex]
\text{and} \quad \lambda_{27}'' = \dfrac{1}{1 + A_{27}''^{\,G}} \\[2ex]
\text{where} \\[1ex]
A_{27}''^{\,G} = \left(\begin{array}{l}
V_{020(2)}'' + z_{27}''^2\left(V_{200(2)}'' - V_{200(1)}''\right) + u_{27}''^2\left(V_{002(2)}'' - V_{002(1)}''\right) \\
\quad - 2z_{27}''\left(V_{110(2)}'' - V_{110(1)}''\right) - 2u_{27}''\left(V_{011(2)}'' - V_{011(1)}''\right) \\
\quad + 2u_{27}''z_{27}''\left(V_{101(2)}'' - V_{101(1)}''\right)
\end{array}\right)
\end{array}\right\}$$

(5.2.118)

By substituting the optimum values of z_{27}'' and u_{27}'', we get $\lambda_3'^{\,opt}$ as,

$$\lambda_{27}''^{\,opt} = \frac{1}{1 + A_{27}''^{\,*}},$$

where

$$A_{27}''^{\,*} = \left(V_{020(2)}'' - \frac{\begin{array}{c}\left(V_{110(2)}'' - V_{110(1)}''\right)^2\left(V_{002(2)}'' - V_{002(1)}''\right) \\ + \left(V_{011(2)}'' - V_{011(1)}''\right)^2\left(V_{200(2)}'' - V_{200(1)}''\right) \\ - 2\left(V_{110(2)}'' - V_{110(1)}''\right)\left(V_{101(2)}'' - V_{101(1)}''\right)\left(V_{011(2)}'' - V_{011(1)}''\right)\end{array}}{\left(V_{200(2)}'' - V_{200(1)}''\right)\left(V_{002(2)}'' - V_{002(1)}''\right) - \left(V_{101(2)}'' - V_{101(1)}''\right)^2}\right).$$

(5.2.119)

We obtain asymptotic variance of the proposed generalized estimator as the expression for $MSE\left(t''^{G}_{27}\right)$ was considered upto first degree of approximation of error terms, so minimum MSE may be written as

$$MSE_{\min}\left(t''^{G}_{27}\right) = AsymtoticVar\left(t''^{G}_{27}\right) = \overline{Y}''^{2}_{s(2)}\left(\frac{A''^{*}_{27}}{1 + A''^{*}_{27}}\right). \tag{5.2.120}$$

It may be observed from (5.2.120), that asymptotic variance of the proposed estimator is less than **Usual Linear Regression Estimator**, We may observe from (5.2.68) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value λ''^{opt}_{27} and $a''^{opt}_{27}, b''^{opt}_{27}$ in (5.2.120), we get optimal estimator as:

$$\hat{t}''^{G}_{27} = \hat{\lambda}''_{27}\,\overline{y}'_{s(2)}\exp\left(\alpha''_{7}\left(1 - \frac{\hat{a}''_{27}\,\overline{x}''_{s(2)}}{\left(\overline{X}''_{s} + (\hat{a}''_{27} - 1)\overline{x}''_{s(2)}\right)}\right)\right)$$

$$\exp\left(\beta''_{27}\left(1 - \frac{\hat{b}''_{27}\,\overline{z}''_{s(2)}}{\left(\overline{Z}''_{s} + (\hat{b}''_{27} - 1)\overline{z}''_{s(2)}\right)}\right)\right),\ 0 < \lambda''_{27} \le 1 \tag{5.2.121}$$

As described earlier in section (5.2.3.1.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as described earlier in section (5.2.3.1.1)

$$\lambda''^{opt}_{27} = \frac{1}{1 + A''^{*}_{27}},\quad \text{where}$$

$$A''^{*}_{27} = \left(V''_{020(2)} - \frac{\left(V''_{110(2)} - V''_{110(1)}\right)^2\left(V''_{002(2)} - V''_{002(1)}\right) + \left(V''_{011(2)} - V''_{011(1)}\right)^2\left(V''_{200(2)} - V''_{200(1)}\right) - 2\left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)\left(V''_{011(2)} - V''_{011(1)}\right)}{\left(V''_{200(2)} - V''_{200(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{101(2)} - V''_{101(1)}\right)^2}\right).$$

(5.2.122)

So (5.2.122) may be written as,

$$\hat{t}_{27}''^{G} = \hat{\lambda}_{27}''^{opt}\,\overline{y}_{s(2)}'' \exp\left(\alpha_{27}''\left(1-\frac{a_{27}''^{\;opt}\,\overline{x}_{s(2)}''}{\left(\overline{x}_{s(1)}''+(a_{27}''^{\;opt}-1)\,\overline{x}_{s(2)}''\right)}\right)\right)$$

$$\exp\left(\beta_{27}''\left(1-\frac{b_{27}''^{\;opt}\,\overline{z}_{s(2)}''}{\left(\overline{z}_{s(1)}''+(b_{27}''^{\;opt}-1)\,\overline{z}_{s(2)}''\right)}\right)\right),\ 0<\lambda_{27}''\le 1 \qquad (5.2.123)$$

Also the minimum MSE may be written as,

$$MSE_{\min}\left(\hat{t}_{27}''^{G}\right) = AsymtoticVar\left(\hat{t}_{27}''^{G}\right) = \overline{Y}_s''^{2}\left(\frac{A_{27}''^{\;*}}{1+A_{27}''^{\;*}}\right). \qquad (5.2.124)$$

Remark

i) For $\alpha_{27}''=1, \beta_{27}''=1$, we get exponential-type ratio cum ratio type estimators given in Table 5.3. The MSE of $t_{27}''^{G}$ is expressed as,

$$MSE(t_{27}''^{j}) = \begin{cases} \overline{Y}_s''^{2}\left(\lambda_{27}''^{2}\left(\begin{array}{l} V_{020(2)}''+\left(V_{200(2)}''-V_{200(2)}''\right) \\ +\left(V_{002(2)}''-V_{002(1)}''\right)-2\left(V_{110(2)}''-V_{110(1)}''\right) \\ +2\left(V_{101(2)}''-V_{101(1)}''\right)-2\left(V_{011(2)}''-V_{011(1)}''\right) \end{array}\right)+(\lambda_{27}''-1)^{2}\right) & j(\in G)=1 \\ \overline{Y}_s''^{2}\left(\begin{array}{l} \lambda_{27}''^{2}\left(\begin{array}{l} V_{020}''+\dfrac{1}{a_{27}''^{\left(\frac{j-1}{2}\right)^{2}}}\left(V_{200(2)}''-V_{200(2)}''\right)+\dfrac{1}{b_{27}''^{\left(\frac{j-1}{2}\right)^{2}}}\left(V_{002(2)}''-V_{002(1)}''\right) \\ -2\dfrac{1}{a_{27}''^{\left(\frac{j-1}{2}\right)}}\left(V_{110(2)}''-V_{110(1)}''\right)-2\dfrac{1}{b_{27}''^{\left(\frac{j-1}{2}\right)}}\left(V_{101(2)}''-V_{101(1)}''\right) \\ +2\dfrac{1}{a_{27}''^{\left(\frac{j-1}{2}\right)}}\dfrac{1}{b_{27}''^{\left(\frac{j-1}{2}\right)}}\left(V_{011(2)}''-V_{011(1)}''\right) \end{array}\right) \\ +(\lambda_{27}''-1)^{2} \end{array}\right) & j(\in G)=3,5,..,11 \end{cases},$$

(5.2.125)

The optimal values which lead to minimum MSE as,

$$a_{27}''^{\left(\frac{j-1}{2}\right)} = \frac{\left(V''_{200(2)} - V''_{200(2)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{011(2)} - V''_{011(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)}$$

and

$$b_{27}''^{\left(\frac{j-1}{2}\right)} = \frac{\left(V''_{200(2)} - V''_{200(2)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{200(2)} - V''_{200(2)}\right)\left(V''_{011(2)} - V''_{011(1)}\right) - \left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)},$$

$$\lambda''_{27} = \frac{1}{1 + A_{27}''^{G}},$$

where

$$A_{27}''^{G} = \left[\begin{array}{l} V''_{020(2)} + \dfrac{1}{a_{27}''^{\left(\frac{j-1}{2}\right)^2}}\left(V''_{200(2)} - V''_{200(2)}\right) + \dfrac{1}{b_{27}''^{\left(\frac{j-1}{2}\right)^2}} V''_{002(2)} \\ -2\dfrac{1}{a_{27}''^{\left(\frac{j-1}{2}\right)}}\left(V''_{110(2)} - V''_{110(1)}\right) - 2\dfrac{1}{b_{27}''^{\left(\frac{j-1}{2}\right)}}\left(V''_{101(2)} - V''_{101(1)}\right) \\ +2\dfrac{1}{a_{27}''^{\left(\frac{j-1}{2}\right)}}\dfrac{1}{b_{27}''^{\left(\frac{j-1}{2}\right)}}\left(V''_{011(2)} - V''_{011(1)}\right) \end{array}\right].$$

ii) For $\alpha''_{27} = -1, \beta''_{27} = -1$, we get exponential-type product cum product type estimators given in Table 5.3. The MSE of $t_{27}''^{G}$ is expressed as,

$$MSE(t''^{k}_{27}) = \begin{cases} \bar{Y}_s''^2 \left\{ \lambda''^2_{27} \left(\begin{aligned} & V''_{020(2)} + \left(V''_{200(2)} - V''_{200(2)}\right) + \left(V''_{002(2)} - V''_{002(1)}\right) \\ & + 2\left(V''_{110(2)} - V''_{110(1)}\right) + 2\left(V''_{101(2)} - V''_{101(1)}\right) - 2\left(V''_{011(2)} - V''_{011(1)}\right) \end{aligned} \right) + (\lambda''_{27} - 1)^2 \right\} & k(\in G) = 2 \\ \bar{Y}_s''^2 \left\{ \lambda''^2_{27} \left(\begin{aligned} & V''_{020(2)} + \frac{1}{a''^{\left(\frac{k}{2}\right)^2}_{27}}\left(V''_{200(2)} - V''_{200(2)}\right) + \frac{1}{b''^{\left(\frac{k}{2}\right)^2}_{27}}\left(V''_{002(2)} - V''_{002(1)}\right) \\ & + 2\frac{1}{a''^{\left(\frac{k}{2}\right)}_{27}}\left(V''_{110(2)} - V''_{110(1)}\right) + 2\frac{1}{b''^{\left(\frac{k}{2}\right)}_{27}}\left(V''_{101(2)} - V''_{101(1)}\right) \\ & - 2\frac{1}{a''^{\left(\frac{k}{2}\right)}_{27}}\frac{1}{b''^{\left(\frac{k}{2}\right)}_{27}}\left(V''_{011(2)} - V''_{011(1)}\right) \end{aligned} \right) + (\lambda''_{27} - 1)^2 \right\} & k(\in G) = 4, 6, .., 12 \end{cases}, \tag{5.2.126}$$

The optimal values which lead to minimum MSE as,

$$a''^{\left(\frac{k}{2}\right)}_{27} = \frac{\left(V''_{200(2)} - V''_{200(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{011(2)} - V''_{011(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)}$$

and

$$b''^{\left(\frac{k}{2}\right)}_{27} = \frac{\left(V''_{200(2)} - V''_{200(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{200(2)} - V''_{200(1)}\right)\left(V''_{011(2)} - V''_{011(1)}\right) - \left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)},$$

$$\lambda''_{27} = \frac{1}{1 + A''^{G}_{27}}$$

where

$$A''^{\ G}_{27} = \left[\begin{aligned} & V''_{020(2)} + \frac{1}{a''^{\left(\frac{k}{2}\right)^2}_{27}}\left(V''_{200(2)} - V''_{200(1)}\right) + \frac{1}{b''^{\left(\frac{k}{2}\right)^2}_{27}} V''_{002(2)} \\ & -2\frac{1}{a''^{\left(\frac{k}{2}\right)}_{27}}\left(V''_{110(2)} - V''_{110(1)}\right) - 2\frac{1}{b''^{\left(\frac{k}{2}\right)}_{27}}\left(V''_{101(2)} - V''_{101(1)}\right) \\ & +2\frac{1}{a''^{\left(\frac{k}{2}\right)}_{27}}\frac{1}{b''^{\left(\frac{k}{2}\right)}_{27}}\left(V''_{011(2)} - V''_{011(1)}\right) \end{aligned} \right].$$

iii) For $\alpha''_{27} = -1, \beta''_{27} = 1$, we get exponential-type product cum ratio type estimators given in Table 5.3. The MSE of t''^{G}_{27} is expressed as,

$$MSE(t''^{l}_{27}) = \begin{cases} \bar{Y}''^2_s \left\{ \lambda''^2_{27} \left(\begin{array}{l} V''_{020(2)} + \left(V''_{200(2)} - V''_{200(1)}\right) + \left(V''_{002(2)} - V''_{002(1)}\right) \\ + 2\left(V''_{110(2)} - V''_{110(1)}\right) - 2\left(V''_{101(2)} - V''_{101(1)}\right) - 2\left(V''_{011(2)} - V''_{011(1)}\right) \end{array} \right) + (\lambda''_{27} - 1)^2 \right\} l(\in G) = 13 \\ \bar{Y}''^2_s \left\{ \lambda''^2_{27} \left(\begin{array}{l} V''_{020(2)} + \dfrac{1}{a''^{\left(\frac{l-1}{2}\right)^2}_{27}}\left(V''_{200(2)} - V''_{200(1)}\right) + \dfrac{1}{b''^{\left(\frac{l-1}{2}\right)^2}_{27}}\left(V''_{002(2)} - V''_{002(1)}\right) \\ + 2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{27}}\left(V''_{110(2)} - V''_{110(1)}\right) - 2\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{27}}\left(V''_{101(2)} - V''_{101(1)}\right) \\ - 2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{27}}\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{27}}\left(V''_{011(2)} - V''_{011(1)}\right) \end{array} \right) + (\lambda''_{27} - 1)^2 \right\} l(\in G) = 15, 17, \ldots, 23 \end{cases},$$

(5.2.127)

The optimal values which lead to minimum MSE as,

$$a''^{\left(\frac{l-1}{2}\right)}_{27} = \frac{\left(V''_{200(2)} - V''_{200(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{011(2)} - V''_{011(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)}$$

and

$$b''^{\left(\frac{l-1}{2}\right)}_{27} = \frac{\left(V''_{200(2)} - V''_{200(1)}\right)\left(V''_{002(2)} - V''_{002(1)}\right) - \left(V''_{101(2)} - V''_{101(1)}\right)^2}{\left(V''_{200(2)} - V''_{200(1)}\right)\left(V''_{011(2)} - V''_{011(1)}\right) - \left(V''_{110(2)} - V''_{110(1)}\right)\left(V''_{101(2)} - V''_{101(1)}\right)},$$

$$\lambda''_{27} = \frac{1}{1 + A''^{G}_{27}},$$

where

$$A''^{G}_{27} = \left[\begin{array}{l} V''_{020(2)} + \dfrac{1}{a''^{\left(\frac{l-1}{2}\right)^2}_{27}}\left(V''_{200(2)} - V''_{200(1)}\right) + \dfrac{1}{b''^{\left(\frac{l-1}{2}\right)^2}_{27}} V''_{002(2)} \\ \qquad -2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{27}}\left(V''_{110(2)} - V''_{110(1)}\right) - 2\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{27}}\left(V''_{101(2)} - V''_{101(1)}\right) \\ \qquad +2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{27}}\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{27}}\left(V''_{011(2)} - V''_{011(1)}\right) \end{array} \right].$$

iv) For $\alpha_7'' = 1, \beta_7'' = -1$, we get exponential-type ratio cum product estimators given in Table 5.3. The MSE of $t_{27}''^{G}$ is expressed as

$$MSE(t_{27}''^{m}) = \begin{cases} \bar{Y}_s''^{2}\left\{ \lambda_{27}''^{2} \begin{pmatrix} V_{020}'' + \left(V_{200(2)}'' - V_{200(1)}''\right) + V_{002(2)}'' \\ -2\left(V_{110(2)}'' - V_{110(1)}''\right) \\ +2\left(V_{101(2)}'' - V_{101(1)}''\right) - 2V_{011(2)}'' \end{pmatrix} + (\lambda_{27}'' - 1)^2 \right\} & m(\in G) = 14 \\ \bar{Y}_s''^{2}\left\{ \lambda_{7}''^{2} \begin{pmatrix} V_{020(2)}'' + \dfrac{1}{a_{27}''^{\left(\frac{k}{2}\right)^2}}\left(V_{200(2)}'' - V_{200(1)}''\right) \\ + \dfrac{1}{b_{27}''^{\left(\frac{k}{2}\right)^2}} V_{002(2)}'' - 2\dfrac{1}{a_{27}''^{\left(\frac{k}{2}\right)}}\left(V_{110(2)}'' - V_{110(1)}''\right) \\ + 2\dfrac{1}{b_{27}''^{\left(\frac{k}{2}\right)}}\left(V_{101(2)}'' - V_{101(1)}''\right) - 2\dfrac{1}{a_{27}''^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{27}''^{\left(\frac{k}{2}\right)}} V_{011(2)}'' \end{pmatrix} + (\lambda_{27}'' - 1)^2 \right\} & m(\in G) = 16, 18, .., 24 \end{cases},$$

(5.2.128)

The optimal values which lead to minimum MSE as

$$a_{27}''^{\left(\frac{m}{2}\right)} = \frac{\left(V_{200(2)}'' - V_{200(1)}''\right)\left(V_{002(2)}'' - V_{002(1)}''\right) - \left(V_{101(2)}'' - V_{101(1)}''\right)^2}{\begin{array}{c}\left(V_{110(2)}'' - V_{110(1)}''\right)\left(V_{002(2)}'' - V_{002(1)}''\right) \\ -\left(V_{011(2)}'' - V_{011(1)}''\right)\left(V_{101(2)}'' - V_{101(1)}''\right)\end{array}}$$

and $$b_{27}''^{\left(\frac{m}{2}\right)} = \frac{\left(V_{200(2)}'' - V_{200(1)}''\right)\left(V_{002(2)}'' - V_{002(1)}''\right) - \left(V_{101(2)}'' - V_{101(1)}''\right)^2}{\begin{array}{c}\left(V_{200(2)}'' - V_{200(1)}''\right)\left(V_{011(2)}'' - V_{011(1)}''\right) \\ -\left(V_{110(2)}'' - V_{110(1)}''\right)\left(V_{101(2)}'' - V_{101(1)}''\right)\end{array}},$$

$$\lambda_{27}'' = \frac{1}{1 + A_{27}''^{G}},$$

where

$$A''^{\,G}_{27} = \left[\begin{array}{l} V''_{020} + \dfrac{1}{a''^{\left(\frac{m}{2}\right)^2}_{27}}\left(V''_{200(2)} - V''_{200(1)}\right) + \dfrac{1}{b''^{\left(\frac{m}{2}\right)^2}_{27}}\left(V''_{002(2)} - V''_{002(1)}\right) \\ \qquad -2\dfrac{1}{a''^{\left(\frac{m}{2}\right)}_{27}}\left(V''_{110(2)} - V''_{110(1)}\right) + 2\dfrac{1}{b''^{\left(\frac{m}{2}\right)}_{27}}\left(V''_{101(2)} - V''_{101(1)}\right) \\ \qquad -2\dfrac{1}{a''^{\left(\frac{m}{2}\right)}_{27}}\dfrac{1}{b'^{\left(\frac{m}{2}\right)}_{27}}\left(V''_{011(2)} - V''_{011(1)}\right) \end{array}\right].$$

Case III

The generalized estimator under case II may be proposed following (5.2.) as,

$$t^{G}_{27} = \lambda_{27}\,\overline{y}_{s(2)}\exp\left(\alpha_{27}\left(1 - \frac{a_{27}\,\overline{x}_{s(2)}}{\left(\overline{x}_{s(1)} + (a_{27}-1)\overline{x}_{s(2)}\right)}\right)\right)$$

$$\exp\left(\beta_{27}\left(1 - \frac{b_{27}\,\overline{z}_{s(2)}}{\left(\overline{z}_{s(1)} + (b_{27}-1)\overline{z}_{s(2)}\right)}\right)\right), 0 < \lambda_{27} \le 1 \qquad (5.2.129)$$

Where ($a_{27}, b_{27}, \lambda_{27}$) are constants to be determined such that the mean square error (MSE) is minimum. (α_{27}, β_{27}) are known constants takes the value (0,1,-1) to produce different ratio-type and product-type estimators.

The proposed estimator (5.2.129) follows the same routine along with the class of estimator in Table 5.3, as that for case-I in Section 5.3.2. In addition, the relation between a_{27}, α_{27}, λ_{27} and b_{27}, β_{27} in case-II is the same as that for case-I in Section 5.3.2.1. Finally, the same is true for the MSE and the Bias. It is therefore directly from Section 5.3.2.1, we may write $\text{Bias}\,(t^{G}_{27})$ and $\text{MSE}(t^{G}_{27})$ following the same, and we may also produce a class of estimators for similar choices of a_{27}, α_{27}, λ_{27} in case-II. The bias of (5.2.129) may be obtain by following the notations and expectations for case II presented in Section 5.1,

The bias and MSE expressions may be given as,

$$Bias\left(t_{27}^{G}\right)=\lambda_{27}\,\overline{Y}_s\left[\frac{\alpha^2{}_{27}\left(V_{200(2)}-V_{200(1)}\right)}{a_{27}{}^2}+\frac{\beta_{27}{}^2}{2b_{27}{}^2}\left(V_{200(2)}-V_{200(1)}\right)\right.$$
$$\left.-\frac{\alpha_{27}}{a_{27}}\left(V_{110(2)}-V_{110(1)}\right)-\frac{\beta_{27}}{b_{27}}\left(V_{110(2)}-V_{110(1)}\right)\right]+(\lambda_{27}-1)\overline{Y}_s,$$
(5.2.130)

$$MSE(t_{27}^{G})=\lambda_{27}{}^2\overline{Y}_s{}^2\left[V_{020(2)}+z_{27}{}^2\left(V_{200(2)}-V_{200(1)}\right)+u_{27}{}^2\left(V_{002(2)}-V_{002(1)}\right)\right.$$
$$-2z_{27}\left(V_{110(2)}-V_{110(1)}\right)-2u_{27}\left(V_{011(2)}-V_{011(1)}\right)$$
$$\left.+2u_{27}z_{27}\left(V_{101(2)}-V_{101(1)}\right)\right]+(\lambda_{27}-1)^2\,\overline{Y}_s{}^2,\qquad(5.2.131)$$

Now by substituting the optimal values of the z_{27} and u_{27}, we achieve the minimum MSE among the class of proposed generalized estimator.

$$\left.\begin{array}{l}
u_{27}=\dfrac{\left(V_{200(2)}-V_{200(1)}\right)\left(V_{011(2)}-V_{011(1)}\right)-\left(V_{110(2)}-V_{110(1)}\right)\left(V_{101(2)}-V_{101(1)}\right)}{\left(V_{200(2)}-V_{200(1)}\right)\left(V_{002(2)}-V_{002(1)}\right)-\left(V_{101(2)}-V_{101(1)}\right)^2},\\[2ex]
z_{27}=\dfrac{\left(V_{110(2)}-V_{110(1)}\right)\left(V_{002(2)}-V_{002(1)}\right)-\left(V_{011(2)}-V_{011(1)}\right)\left(V_{101(2)}-V_{101(1)}\right)}{\left(V''_{200(2)}-V_{200(1)}\right)\left(V_{002(2)}-V_{002(1)}\right)-\left(V_{101(2)}-V_{101(1)}\right)^2}\\[2ex]
\text{and}\quad\lambda_{27}=\dfrac{1}{1+A_{27}{}^{G}}\\
\text{where}\\
A_{27}{}^{G}=\left(\begin{array}{c}V_{020(2)}+z^2{}_{27}\left(V_{200(2)}-V_{200(1)}\right)+u^2{}_{27}\left(V_{002(2)}-V_{002(1)}\right)\\-2z_{27}\left(V_{110(2)}-V_{110(1)}\right)-2u_{27}\left(V_{011(2)}-V_{011(1)}\right)\\+2u_{27}z_{27}\left(V_{101(2)}-V_{101(1)}\right)\end{array}\right)
\end{array}\right\}.$$
(5.2.132)

By substituting the optimum values of z_{27} and u_{27}, we get $\lambda_{27}{}^{opt}$ as,

$$\lambda_{27}{}^{opt}=\frac{1}{1+A_{27}{}^{*}}$$

where

$$A_{27}^{*} = \left(V_{020(2)} - \frac{\left(V_{110(2)} - V_{110(1)}\right)^2 \left(V_{002(2)} - V_{002(1)}\right) + \left(V_{011(2)} - V_{011(1)}\right)^2 \left(V_{200(2)} - V_{200(1)}\right) - 2\left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)\left(V_{011(2)} - V_{011(1)}\right)}{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{101(2)} - V_{101(1)}\right)^2} \right)$$

(5.2.133)

We obtain asymptotic variance of the proposed generalized estimator as the expression for $MSE\left(t_{27}^{G}\right)$ was considered upto first degree of error term, so minimum MSE may be written as

$$MSE_{\min}\left(t_{27}^{G}\right) = AsymtoticVar\left(t_{27}^{G}\right) = \overline{Y}_s^{\,2}\left(\frac{A_{27}^{\ *}}{1 + A^{*}_{\ 27}}\right). \qquad (5.2.134)$$

asymptotic variance of the proposed estimator is less than **Usual Linear Regression Estimator**, We may observe from (5.2.68) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value $\lambda_7^{\ opt}$ and $a_7^{\ opt}, b_7^{\ opt}$ in (5.2.129), we get optimal estimator as:

$$\hat{t}_{27}^{G} = \hat{\lambda}_{27}\overline{y}_s \exp\left(\alpha_{27}\left(1 - \frac{\hat{a}_{27}\overline{x}_{s(2)}}{\left(\overline{x}_{s(1)} + (\hat{a}_{27} - 1)\overline{x}_s\right)}\right)\right) \exp\left(\beta_{27}\left(1 - \frac{\hat{b}_{27}\overline{z}_{s(2)}}{\left(\overline{z}_{s(1)} + (\hat{b}_{27} - 1)\overline{z}_{s(2)}\right)}\right)\right), 0 < \lambda_{27} \le 1 \qquad (5.2.135)$$

As described earlier in section (5.2.1.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as described earlier in section (5.2.1.1)

$$\lambda_{27}^{opt} = \frac{1}{1 + A_{27}^{*}},$$

where

$$A_{27}^{*} = \left(V_{020(2)} - \frac{\begin{array}{c}\left(V_{110(2)} - V_{110(1)}\right)^2 \left(V_{002(2)} - V_{002(1)}\right) + \left(V_{011(2)} - V_{011(1)}\right)^2 \left(V_{200(2)} - V_{200(1)}\right) \\ -2\left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)\left(V_{011(2)} - V_{011(1)}\right)\end{array}}{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{101(2)} - V_{101(1)}\right)^2} \right)$$

(5.2.136)

So (5.2.136) may be wriiten as,

$$\hat{t}_{27}^{G} = \hat{\lambda}_{27}^{opt} \ \bar{y}_s \exp\left(\alpha_{27} \left(1 - \frac{\hat{a}_{27}^{opt} \ \bar{x}_{s(2)}}{\left(\bar{x}_{s(1)} + (\hat{a}_{27}^{opt} - 1)\bar{x}_{s(2)} \right)} \right) \right)$$

$$\exp\left(\beta_{27}'' \left(1 - \frac{\hat{b}_{27}^{opt} \bar{z}_{s(2)}}{\left(\bar{z}_{s(1)} + (\hat{b}_{27}^{opt} - 1)\bar{z}_{s(2)} \right)} \right) \right), \ 0 < \lambda_{27} \le 1 \qquad (5.2.137)$$

Also the minimum MSE may be written as,

$$MSE_{\min}\left(\hat{t}_{27}^{G}\right) = Asymtotic Var\left(\hat{t}_{27}^{G}\right) = \bar{Y}_s^{2} \left(\frac{A_{27}^{*}}{1 + A_{27}^{*}} \right). \qquad (5.2.138)$$

Remark

i) For $\alpha_{27} = 1, \beta_{27} = 1$, we get exponential-ratio cum ratio type estimators given in Table 5.3. The MSE of $t_{27}''^{G}$ is expressed as

$$MSE(t_{27}^{j})=\begin{cases}\bar{Y}_s^2\left\{\lambda_{27}^2\begin{pmatrix}V_{020(2)}+\left(V_{200(2)}-V_{200(2)}\right)\\+\left(V_{002(2)}-V_{002(1)}\right)-2\left(V_{110(2)}-V_{110(1)}\right)\\+2\left(V_{101(2)}-V_{101(1)}\right)-2\left(V_{011(2)}-V_{011(1)}\right)\end{pmatrix}+(\lambda_{27}-1)^2\right\} & j(\in G)=1\\ \bar{Y}_s^2\left\{\begin{array}{l}\lambda_{27}^2\begin{pmatrix}V_{020}+\dfrac{1}{a_{27}^{\left(\frac{j-1}{2}\right)^2}}\left(V_{200(2)}-V_{200(2)}\right)+\dfrac{1}{b_{27}^{\left(\frac{j-1}{2}\right)^2}}\left(V_{002(2)}-V_{002(1)}\right)\\-2\dfrac{1}{a_{27}^{\left(\frac{j-1}{2}\right)}}\left(V_{110(2)}-V_{110(1)}\right)-2\dfrac{1}{b_{27}^{\left(\frac{j-1}{2}\right)}}\left(V_{101(2)}-V_{101(1)}\right)\\+2\dfrac{1}{a_{27}^{\left(\frac{j-1}{2}\right)}}\dfrac{1}{b_{27}^{\left(\frac{j-1}{2}\right)}}\left(V_{011(2)}-V_{011(1)}\right)\end{pmatrix}\\+(\lambda_{27}-1)^2\end{array}\right\} & j(\in G)=3,5,\ldots,11\end{cases}$$

(5.2.125)

The optimal values which lead to minimum MSE as,

$$a_{27}^{\left(\frac{j-1}{2}\right)}=\frac{\left(V_{200(2)}-V_{200(2)}\right)\left(V_{002(2)}-V_{002(1)}\right)-\left(V_{101(2)}-V_{101(1)}\right)^2}{\left(V_{110(2)}-V_{110(1)}\right)\left(V_{002(2)}-V_{002(1)}\right)-\left(V_{011(2)}-V_{011(1)}\right)\left(V_{101(2)}-V_{101(1)}\right)}$$

and

$$b_{27}^{\left(\frac{j-1}{2}\right)}=\frac{\left(V_{200(2)}-V_{200(2)}\right)\left(V_{002(2)}-V_{002(1)}\right)-\left(V_{101(2)}-V_{101(1)}\right)^2}{\left(V_{200(2)}-V_{200(2)}\right)\left(V_{011(2)}-V_{011(1)}\right)-\left(V_{110(2)}-V_{110(1)}\right)\left(V_{101(2)}-V_{101(1)}\right)},$$

$$\lambda_{27}=\frac{1}{1+A_{27}{}^{G}}$$

where

$$A^{G}{}_{27}=\begin{bmatrix}V_{020(2)}+\dfrac{1}{a_{27}^{\left(\frac{j-1}{2}\right)^2}}\left(V_{200(2)}-V_{200(2)}\right)+\dfrac{1}{b_{27}^{\left(\frac{j-1}{2}\right)^2}}V_{002(2)}\\-2\dfrac{1}{a_{27}^{\left(\frac{j-1}{2}\right)}}\left(V_{110(2)}-V_{110(1)}\right)-2\dfrac{1}{b_{27}^{\left(\frac{j-1}{2}\right)}}\left(V_{101(2)}-V_{101(1)}\right)\\+2\dfrac{1}{a_{27}^{\left(\frac{j-1}{2}\right)}}\dfrac{1}{b_{27}^{\left(\frac{j-1}{2}\right)}}\left(V_{011(2)}-V_{011(1)}\right)\end{bmatrix}.$$

ii) For $\alpha_{27} = -1, \beta_{27} = -1$, we get exponential-product cum product type estimators given in Table 7. The MSE of t_{27}^{G} is expressed as

$$MSE(t_{27}^{k}) = \begin{cases} \bar{Y}_s^2 \left\{ \lambda_{27}^2 \left\{ \begin{array}{l} V_{020(2)} + \left(V_{200(2)} - V_{200(2)}\right) + \left(V_{002(2)} - V_{002(1)}\right) \\ + 2\left(V_{110(2)} - V_{110(1)}\right) + 2\left(V_{101(2)} - V_{101(1)}\right) \\ - 2\left(V_{011(2)} - V_{011(1)}\right) \end{array} \right\} + (\lambda_{27} - 1)^2 \right\} & k(\in G) = 2 \\ \bar{Y}_s^2 \left\{ \lambda_{27}^2 \left\{ \begin{array}{l} V_{020(2)} + \dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)^2}}\left(V_{200(2)} - V_{200(2)}\right) + \dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)^2}}\left(V_{002(2)} - V_{002(1)}\right) \\ + 2\dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) + 2\dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) \\ - 2\dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)}}\left(V_{011(2)} - V_{011(1)}\right) \end{array} \right\} + (\lambda_{27} - 1)^2 \right\} & k(\in G) = 4,6,..,12 \end{cases},$$

(5.2.126)

The optimal values which lead to minimum MSE as,

$$a_{27}^{\left(\frac{k}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{110(2)} - V_{110(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{011(2)} - V_{011(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)}$$

and

$$b_{27}^{\left(\frac{k}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{011(2)} - V_{011(1)}\right) - \left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)},$$

$$\lambda_{27} = \frac{1}{1 + A^{G}_{27}}$$

where

$$A_{27}{}^{G} = \left[\begin{array}{l} V_{020(2)} + \dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)^2}}\left(V_{200(2)} - V_{200(1)}\right) + \dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)^2}}\left(V_{002(2)} - V_{002(1)}\right) \\ \qquad -2\dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) - 2\dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) \\ \qquad +2\dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)}}\left(V_{011(2)} - V_{011(1)}\right) \end{array} \right].$$

iii) For $\alpha_{27} = -1, \beta_{27} = 1$, we get exponential-type product cum ratio estimators given in Table 7. The MSE of t_{27}^{G} is expressed as

$$MSE(t_{27}^{l}) = \left\{ \begin{array}{ll} \bar{Y}_s^2 \left(\lambda_{27}{}^2 \left(\begin{array}{l} V_{020(2)} + \left(V_{200(2)} - V_{200(1)}\right) \\ + \left(V_{002(2)} - V_{002(1)}\right) + 2\left(V_{110(2)} - V_{110(1)}\right) \\ - 2\left(V''_{101(2)} - V''_{101(1)}\right) - 2\left(V_{011(2)} - V_{011(1)}\right) \end{array} \right) + (\lambda_{27} - 1)^2 \right) & l(\in G) = 13 \\ \bar{Y}_s^2 \left(\lambda_{27}{}^2 \left(\begin{array}{l} V_{020(2)} + \dfrac{1}{a_{27}^{\left(\frac{l-1}{2}\right)^2}}\left(V_{200(2)} - V_{200(1)}\right) \\ + \dfrac{1}{b_{27}^{\left(\frac{l-1}{2}\right)^2}}\left(V_{002(2)} - V''_{002(1)}\right) + 2\dfrac{1}{a_{27}^{\left(\frac{l-1}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) \\ - 2\dfrac{1}{b_{27}^{\left(\frac{l-1}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) \\ -2\dfrac{1}{a_{27}^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{27}^{\left(\frac{l-1}{2}\right)}}\left(V_{011(2)} - V_{011(1)}\right) \end{array} \right) + (\lambda_{27} - 1)^2 \right) & l(\in G) = 15,17,..,23 \end{array} \right\}, \tag{5.2.127}$$

The optimal values which lead to minimum MSE as,

$$a_{27}^{\left(\frac{l-1}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{110(2)} - V_{110(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{011(2)} - V_{011(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)}$$

and

$$b_{27}^{\left(\frac{l-1}{2}\right)} = \frac{\left(V_{200(2)}-V_{200(1)}\right)\left(V_{002(2)}-V_{002(1)}\right)-\left(V_{101(2)}-V_{101(1)}\right)^2}{\left(V_{200(2)}-V_{200(1)}\right)\left(V_{011(2)}-V_{011(1)}\right) - \left(V_{110(2)}-V_{110(1)}\right)\left(V_{101(2)}-V_{101(1)}\right)},$$

$$\lambda_{27} = \frac{1}{1+A_{27}{}^{G}}$$

where

$$A_{27}{}^{G} = \left[\begin{array}{l} V_{020(2)} + \dfrac{1}{a_{27}^{\left(\frac{l-1}{2}\right)^2}}\left(V_{200(2)}-V_{200(1)}\right) + \dfrac{1}{b_{27}^{\left(\frac{l-1}{2}\right)^2}}V_{002(2)} \\ \quad -2\dfrac{1}{a_{27}^{\left(\frac{l-1}{2}\right)}}\left(V_{110(2)}-V_{110(1)}\right) - 2\dfrac{1}{b_{27}^{\left(\frac{l-1}{2}\right)}}\left(V_{101(2)}-V_{101(1)}\right) \\ \quad +2\dfrac{1}{a_{27}^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{27}^{\left(\frac{l-1}{2}\right)}}\left(V_{011(2)}-V_{011(1)}\right) \end{array}\right].$$

iv) For $\alpha_{27}=1, \beta_{27}=-1$, we get exponential type ratio cum product type estimators given in Table 7. The MSE of t_{27}^{G} is expressed as,

$$MSE(t_{27}^{m}) = \left\{\begin{array}{ll} \bar{Y}_s^2\left(\lambda_{27}{}^2\left(\begin{array}{l} V_{020}+\left(V_{200(2)}-V_{200(1)}\right) \\ +V_{002(2)}-2\left(V_{110(2)}-V_{110(1)}\right) \\ +2\left(V_{101(2)}-V_{101(1)}\right)-2V_{011(2)} \end{array}\right)+(\lambda_{27}-1)^2\right) & m(\in G)=14 \\ \bar{Y}_s^2\left(\lambda_{27}{}^2\left(\begin{array}{l} V_{020(2)}+\dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)^2}}\left(V_{200(2)}-V_{200(1)}\right) \\ +\dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)^2}}V_{002(2)}-2\dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)}}\left(V_{110(2)}-V_{110(1)}\right) \\ +2\dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)}}\left(V_{101(2)}-V_{101(1)}\right)-2\dfrac{1}{a_{27}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{27}^{\left(\frac{k}{2}\right)}}V_{011(2)} \end{array}\right)+(\lambda_{27}-1)^2\right) & m(\in G)=16,18,..,24 \end{array}\right\}, \tag{5.2.128}$$

The optimal values which lead to minimum MSE as,

$$a_{27}^{\left(\frac{m}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{110(2)} - V_{110(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{011(2)} - V_{011(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)}$$

and

$$b_{27}^{\left(\frac{m}{2}\right)} = \frac{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{002(2)} - V_{002(1)}\right) - \left(V_{101(2)} - V_{101(1)}\right)^2}{\left(V_{200(2)} - V_{200(1)}\right)\left(V_{011(2)} - V_{011(1)}\right) - \left(V_{110(2)} - V_{110(1)}\right)\left(V_{101(2)} - V_{101(1)}\right)},$$

$$\lambda_{27} = \frac{1}{1 + A^G{}_{27}},$$

where

$$A_{27}{}^G = \left[\begin{array}{l} V_{020} + \frac{1}{\left(a_{27}^{\left(\frac{m}{2}\right)}\right)^2}\left(V_{200(2)} - V_{200(1)}\right) + \frac{1}{\left(b_{27}^{\left(\frac{m}{2}\right)}\right)^2}\left(V_{002(2)} - V_{002(1)}\right) \\ \qquad -2\frac{1}{a_{27}^{\left(\frac{m}{2}\right)}}\left(V_{110(2)} - V_{110(1)}\right) + 2\frac{1}{b_{27}^{\left(\frac{m}{2}\right)}}\left(V_{101(2)} - V_{101(1)}\right) \\ \qquad -2\frac{1}{a_{27}^{\left(\frac{m}{2}\right)}}\frac{1}{b_{27}^{\left(\frac{m}{2}\right)}}\left(V_{011(2)} - V_{011(1)}\right) \end{array}\right].$$

CHAPTER 6

GENERALIZED EXPONENTIAL ESTIMATORS FOR POPULATION MEAN IN TWO-STAGE ADAPTIVE CLUSTER SAMPLING

6.1 INTRODUCTION

Adaptive cluster sampling provides more efficient estimates when nature of the population is rare and clustered. In adaptive cluster sampling, we take an the initial sample (say s_o) of size n units from N units that are labeled u=(1,2,…N) by using simple random sample without replacement and consider that the variable of interest y_i is highly correlated with auxiliary variable x_i. If any of the initially selected units satisfy the condition of interest *C*, its neighboring units will be added to the sample and observed. If any of the newly added units satisfies *C*, then the same procedure goes on until no new unit satisfies *C*, then all neighboring units which satisfy the condition *C* are called "*Network*" and the neighboring units that do not meet the condition (*C*) are called "*Edge units*". The network together with associated edge units is called "*Cluster*". The final sample consists of initially selected sample and the units added adaptively. ACS may be considered as simple random sample without replacement when the averages of networks are taken into consideration (Thompson, 1990; Thompson, 1991; Chao 2004; Dryver and Chao, 2007, and Chutiman, 2010).

In this chapter, generalized exponential estimators has been developed under following three cases by utilizing the information of single and two auxiliary variable under two-stage adaptive cluster sampling utilizing the different cases(as mentioned in chapter 4).

In order to obtain the bias and mean square error under two-stage sampling design, we follow the notations given as:

Notations:

Case-I: When first stage units are unequal and weighted mean is used.

Let a population consists of N first stage units (fsu's) and each fsu consists of M_i second stage units (ssu's). Let a sample of n fsu's is selected and an initial sample of m_i ssu's from each of n fsu's is selected by assigning weights $\eta_i = \frac{M_i}{\bar{M}}$ to the ssu's and then the neighboring unit for each of the m_i initially selected units is studied for a pre-defined condition (say y>0) .if any of the neighboring units meet the condition of interest, its neighborhood will also be studied and the process continues until the condition met. Let $\bar{M}$ be the average number of ssu's belonging to each fsu. u_{yji} is the mean of the study variable y in the j^{th} network with which i^{th} fsu belongs in initial sample of i^{th} first-stage unit. i.e. $u_{(y)ji} = \frac{y_{ji}}{t_{ji}}$ and t_{ji} is the number of units in j^{th} network belonging to i^{th} fsu. The same can be defined for the two auxiliary variables x and z.

Let $(\bar{u}_{(y)i})$, and $(\bar{u}_{(x)i}, \bar{u}_{(z)i})$ are sample means respectively for study variable and two auxiliary variables in i^{th} first-stage unit, based on transformed population.

ACS may be considered as simple random sample without replacement when the averages of networks are taken into consideration.

The following notations have been considered for the derivation of mean square error and bias.

Let,

$$\bar{u}'_{ys} = \frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{u}_{(y)i}, \bar{u}'_{xs} = \frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{u}_{(x)i}, \bar{w}'_{zs} = \frac{1}{n}\sum_{i=1}^{n}\eta_i \bar{u}_{(z)i},$$

$$\bar{u}_{(y)i} = \frac{1}{m_i}\sum_{j=1}^{m_i}\bar{u}_{(y)ji}, \bar{u}_{xi} = \frac{1}{m_i}\sum_{j=1}^{m_i}\bar{u}_{xji}, \bar{u}_{zi} = \frac{1}{m_i}\sum_{j=1}^{m_i}\bar{u}_{zji},$$

$$e'_{0(u)} = \frac{\bar{u}'_{ys} - \bar{Y}'_s}{\bar{Y}'_s}, e'_{1(u)} = \frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s}, e'_{2(u)} = \frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s}$$

ii) Expectation

$$\bar{u}'_{ys} = \bar{Y}'_s\left(1 + e'_{0(u)}\right), \bar{u}'_{xs} = \bar{X}'_s\left(1 + e'_{1(u)}\right), \bar{u}'_{ys} = \bar{Z}'_s\left(1 + e'_{2(u)}\right) \qquad (6.1.1)$$

$$\left.\begin{aligned}
&E(e'_{0(u)}) = E(e'_{1(u)}) = E(e'_{2(u)}) = 0 \\
&E(e'^{2}_{0(u)}) = V'_{020(u)},\ E(e'^{2}_{1(u)}) = V'_{200(u)},\ E(e'^{2}_{2(u)}) = V'_{002(u)}, \\
&E(e'_{0(u)}e'_{1(u)}) = V'_{110(u)},\ E(e'_{1(u)}e'_{2(u)}) = V'_{101(u)},\ E(e'_{0(u)}e'_{2(u)}) = V'_{011(u)} \\
&\text{where} \\
&V'_{020(u)} = \frac{1}{\bar{Y}'^{2}_{s}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{(uy)b} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)S'^{2}_{(uy)wi},\right\}, \\
&V'_{200(u)} = \frac{1}{\bar{X}'^{2}_{s}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{(ux)b} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)S'^{2}_{(ux)wi}\right\} \\
&V'_{002(u)} = \frac{1}{\bar{Z}'^{2}_{s}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S'^{2}_{(uz)b} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)S'^{2}_{(uz)wi}\right\} \\
&V'_{110(u)} = \frac{1}{\bar{Y}'_{s}\bar{X}'_{s}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{(uxy)b}S'_{(uy)b}S'_{(ux)b} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\rho_{(uxy)wi}S'_{(uy)wi}S'_{(ux)wi}\right\}, \\
&V'_{101(u)} = \frac{1}{\bar{Z}'_{s}\bar{X}'_{s}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{(uxz)b}S'_{(uz)b}S'_{(ux)b} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\rho_{(uxz)wi}S'_{(uz)wi}S'_{(ux)wi}\right\}, \\
&V'_{011(u)} = \frac{1}{\bar{Z}'_{s}\bar{Y}'_{s}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{(uyz)b}S'_{(uz)b}S'_{(uy)b} + \frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_i}-\frac{1}{M_i}\right)\rho_{(uyz)wi}S'_{(uz)wi}S'_{(uy)wi}\right\}
\end{aligned}\right\} \quad (6.1.2)$$

Case-II: When first stage units are unequal and un weighted mean is used

For case-I, in order to obtain a weighted and unbiased estimator for population mean we use a weighting constant $\eta_i = \frac{M_i}{\bar{M}}$ in two-stage sampling design. If we assume equal weights for all unequal first-stage units, it gives us an un-weighted and biased estimator of population mean in two-stage sampling design and this situation will be considered as case-II i.e. for $\eta_i = \frac{M_i}{\bar{M}} = 1$, an estimator defined in case-I may be turned into the estimator for case-II. So the procedure of two stage sampling design for case II will be same as described in Case I. The notations and expectations may be derived for case II as

$$\bar{u}''_{ys} = \frac{1}{n}\sum_{i=1}^{n}\bar{u}_{(y)i},\ \bar{u}''_{xs} = \frac{1}{n}\sum_{i=1}^{n}\bar{u}_{(x)i},\ \bar{w}''_{zs} = \frac{1}{n}\sum_{i=1}^{n}\bar{u}_{(z)i},$$

$$\bar{u}_{(y)i} = \frac{1}{m_i}\sum_{j=1}^{m_i}\bar{u}_{(y)ji},\ \bar{u}_{(x)i} = \frac{1}{m_i}\sum_{j=1}^{m_i}\bar{u}_{(x)ji},\ \bar{u}_{(z)i} = \frac{1}{m_i}\sum_{j=1}^{m_i}\bar{u}_{(z)ji},$$

$$e''_{u_y} = \frac{\bar{u}''_{ys} - \bar{Y}''_s}{\bar{Y}''_s},\ e''_{u_x} = \frac{\bar{u}''_{xs} - \bar{X}''_s}{\bar{X}''_s},\ e''_{u_z} = \frac{\bar{u}''_{zs} - \bar{Z}''_s}{\bar{Z}''_s}$$

ii) Expectation

$$\bar{u}''_{ys} = \bar{Y}'_s\left(1+e''_{\bar{u}_y}\right), \bar{u}'_{xs} = \bar{X}'_s\left(1+e''_{\bar{u}_x}\right), \bar{u}'_{zs} = \bar{Z}'_s\left(1+e''_{\bar{u}_z}\right) \tag{6.1.1}$$

$$\left.\begin{array}{l}
E(e''_{0(u)}) = E(e''_{1(u)}) = E(e''_{2(u)}) = 0 \\
E(e''^{2}_{0(u)}) = V''_{020(u)}, E(e''^{2}_{1(u)}) = V''_{200(u)}, E(e''^{2}_{2(u)}) = V''_{002(u)}, E(e''_{0(u)}e''_{\bar{u}_x}) = V''_{110(u)}, \\
\quad E(e''_{1(u)}e''_{2(u)}) = V''_{101(u)}, E(e''_{0(u)}e''_{2(u)}) = V''_{011(u)} \\
\text{where} \\
V''_{020(u)} = \dfrac{1}{\bar{Y}''^{2}_{s}}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^{2}_{(uy)b} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S''^{2}_{(uy)wi},\right\}, \\
V''_{200(u)} = \dfrac{1}{\bar{X}'^{2}_{s}}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^{2}_{(ux)b} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S''^{2}_{(ux)wi}\right\} \\
V''_{002(u)} = \dfrac{1}{\bar{Z}'^{2}_{s}}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)S''^{2}_{(uz)b} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)S''^{2}_{(uz)wi}\right\} \\
V''_{110(u)} = \dfrac{1}{\bar{Y}'_s\bar{X}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{(uxy)b}S''_{(uy)b}S''_{(ux)b} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{(uxy)wi}S''_{(uy)wi}S''_{(ux)wi}\right\}, \\
V''_{101(u)} = \dfrac{1}{\bar{Z}''_s\bar{X}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{(uxz)b}S''_{(uz)b}S''_{(ux)b} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{(uxz)wi}S''_{(uz)wi}S''_{(ux)wi}\right\}, \\
V''_{011(u)} = \dfrac{1}{\bar{Z}'_s\bar{Y}''_s}\left\{\left(\dfrac{1}{n}-\dfrac{1}{N}\right)\rho_{(uyz)b}S''_{(uz)b}S''_{(uy)b} + \dfrac{1}{nN}\sum_{i=1}^{N}\left(\dfrac{1}{m_i}-\dfrac{1}{M_i}\right)\rho_{(uyz)wi}S''_{(uz)wi}S''_{(uy)wi}\right\}
\end{array}\right\} \tag{6.1.4}$$

Case-III: When first stage units are of equal sizes.

Let a population consists of N first stage units (fsu's) and each fsu consists of M second stage units (ssu's). Let a sample of n fsu's is selected by using simple random sampling (WOR)and an initial sample of m ssu's from each of n fsu's by using Simple random sampling (WOR) and then the neighboring unit for each of the initially selected sample unit is studied for a specified condition (say y>0) .if any of the neighboring unit meets the condition of interest, its neighborhood will also be studied and the process continued until the condition met. The notations and expectations may be derived for case III as

$$\bar{u}_{ys} = \frac{1}{n}\sum_{i=1}^{n} \bar{u}_{(y)i}, \bar{u}_{(x)s} = \frac{1}{n}\sum_{i=1}^{n} \bar{u}_{(x)i}, \bar{w}_{(z)s} = \frac{1}{n}\sum_{i=1}^{n} \bar{u}_{(z)i},$$

$$\bar{u}_{(y)i} = \frac{1}{m}\sum_{j=1}^{m} \bar{u}_{(y)ji}, \bar{u}_{(x)i} = \frac{1}{m}\sum_{j=1}^{m} \bar{u}_{(x)ji}, \bar{u}_{(z)i} = \frac{1}{m}\sum_{j=1}^{m} \bar{u}_{(z)ji},$$

$$e_{0(u)} = \frac{\bar{u}_{ys} - \bar{Y}_s}{\bar{Y}_s}, e_{1(u)} = \frac{\bar{u}_{xs} - \bar{X}_s}{\bar{X}_s}, e_{2(u)} = \frac{\bar{u}_{zs} - \bar{Z}_s}{\bar{Z}_s}$$

ii) Expectation

$$\bar{u}_{ys} = \bar{Y}_s\left(1 + e_{0(u)}\right), \bar{u}_{xs} = \bar{X}_s\left(1 + e_{1(u)}\right), \bar{u}_{zs} = \bar{Z}_s\left(1 + e_{2(u)}\right) \tag{6.1.5}$$

$$\left.\begin{aligned}
&E(e_{0(u)}) = E(e_{1(u)}) = E(e_{2(u)}) = 0\\
&E(e_{0(u)}{}^2) = V_{020(u)}, E(e_{1(u)}{}^2) = V_{200(u)}, E(e_{2(u)}{}^2) = V_{002(u)}, E(e_{0(u)}e_{1(u)}) = V_{110(u)},\\
&\qquad E(e_{0(u)}e_{2(u)}) = V_{101(u)}, E(e_{0(u)}e_{2(u)}) = V_{011(u)}\\
&\text{where}\\
&V_{020(u)} = \frac{1}{\bar{Y}_s^2}\left\{\left(\frac{1}{n} - \frac{1}{N}\right)S^2_{(uy)b} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m} - \frac{1}{M}\right)S^2_{(uy)wi},\right\},\\
&V_{200(u)} = \frac{1}{\bar{X}_s^2}\left\{\left(\frac{1}{n} - \frac{1}{N}\right)S^2_{(ux)b} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m} - \frac{1}{M}\right)S^2_{(ux)wi}\right\}\\
&V_{002(u)} = \frac{1}{\bar{Z}_s^2}\left\{\left(\frac{1}{n} - \frac{1}{N}\right)S^2_{(uz)b} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m} - \frac{1}{M}\right)S^2_{(uz)wi}\right\}\\
&V_{110(u)} = \frac{1}{\bar{Y}_s\bar{X}_s}\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho_{(uxy)b}S_{(uy)b}S_{(ux)b} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m} - \frac{1}{M}\right)\rho_{(uxy)wi}S_{(uy)wi}S_{(ux)wi}\right\},\\
&V_{101(u)} = \frac{1}{\bar{Z}_s\bar{X}_s}\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho_{(uxz)b}S_{(uz)b}S_{(ux)b} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m} - \frac{1}{M}\right)\rho_{(uxz)wi}S_{(uz)wi}S_{(ux)wi}\right\},\\
&V_{011(u)} = \frac{1}{\bar{Z}_s\bar{Y}_s}\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho_{(uyz)b}S_{(uz)b}S_{(uy)b} + \frac{1}{nN}\sum_{i=1}^{N}\left(\frac{1}{m} - \frac{1}{M}\right)\rho_{(uyz)wi}S_{(uz)wi}S_{(uy)wi}\right\}
\end{aligned}\right\} \tag{6.1.6}$$

6.2 PROPOSED GENERALIZED ESTIMATOR-IX IN TWO STAGE ADAPTIVE CLUSTER SAMPLING USING SINGLE AUXILIARY VARIABLE

In this section, we propose generalized estimators using two auxiliary variables under two-stage sampling design by using the transformed population means for study variable and auxiliary variable. The estimators are constructed by assuming an exponential relationship between the study variable and auxiliary variable for different three cases (mentioned in chapter 4) under two- stage sampling.

6.2.1 Proposed Generalized Estimator-XI

Case I

The exponential ratio-type estimator may be defined under two-stage ACS as,

$$t_{28}'^{R} = \overline{u}_{ys}' \exp\left\{1 - \frac{2\overline{u}_{xs}'}{\overline{X}_s' + \overline{u}_{xs}'}\right\}, \tag{6.2.1}$$

ii) The exponential product -type estimator may also be defined under two-stage ACS as,

$$t_{28}'^{P} = \overline{u}_{ys}' \exp\left\{-\left(1 - \frac{2\overline{u}_{xs}'}{\overline{X}_s' + \overline{u}_{xs}'}\right)\right\}, \tag{6.2.2}$$

The estimators in (6.2.1) and (6.2.2) lead to the generalized form as by introducing constants λ_{28}' and a_{28}' as,

$$t_{28}'^{G} = \lambda_{28}'\overline{u}_{ys}' \exp\left\{\alpha_8'\left(1 - \frac{a_{28}'\overline{u}_{xs}'}{\overline{X}_s' + (a_{28}' - 1)\overline{u}_{xs}'}\right)\right\},\quad 0 < \lambda_{28} \le 1 \tag{6.2.3}$$

where $a_{28}'(\neq 0)$ and $\lambda_{28}'(\neq 0)$ in (6.2.3) are suitably chosen constants to be determined such as mean square error (MSE) of $t_{28}'^{G}$ is minimum and α_{20}' being constant takes the values (0,-1,1)for designing different ratio-type and product- type estimators. Also it is to be mentioned that for a different choice of a_{28}', λ_{28}' and α_{28}' we get different estimators under two-stage sampling design.

By choosing suitable values of the constants in (6.2.3), various estimators can be derived as a class of estimators of $t_{28}'^{G}$, i.e.

Table 6.1
Some Special Cases of the Generalized Estimator $t_{28}^{\prime G}$

Ratio-type Estimator $\alpha'_{28} = 1$	**Product-type Estimators** $\alpha'_{28} = -1$	a'_{28}	λ'_{28}
$t_{28}^{\prime 1} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + \bar{u}'_{xs}}\right)$	$t_{28}^{\prime 2} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}_s}{\bar{u}'_{xs} + \bar{X}_s}\right)$	2	1
$t_{28}^{\prime 3} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s}\right)$	$t_{28}^{\prime 4} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s}\right)$	1	1
$t_{28}^{\prime 5} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + (a'_{28} - 1)\bar{u}'_{xs}}\right)$	$t_{28}^{\prime 6} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{u}'_{xs} + (a'_{28} - 1)\bar{X}'_s}\right)$	a'_{28}	1
$t_{28}^{\prime 7} = \lambda'_{28}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + \bar{u}'_{xs}}\right)$	$t_{28}^{\prime 8} = \lambda'_{28}\bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{u}'_{xs} + \bar{X}_s}\right)$	2	λ'_{28}
$t_{28}^{\prime 9} = \lambda'_{28}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}_s}\right)$	$t_{28}^{\prime 10} = \lambda'_{28}\bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s}\right)$	1	λ'_{28}
$t_{28}^{\prime 11} = \lambda'_{28}\bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + (a'_8 - 1)\bar{u}'_{xs}}\right)$	$t_8^{\prime 12} = \lambda'_8\bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{u}'_{xs} + (a'_1 - 1)\bar{X}'_s}\right)$	a'_{28}	λ'_{28}

The Bias and Mean Square Error of Generalized Estimator-IX

Using notations from (4.1.1) the estimator given in (4.2.5) may be expressed in form of e'_1 as:

$$t_{28}^{\prime G} = \lambda'_{28}\bar{Y}'_s\left(1 + e'_{0(u)}\right)\exp\left[-\frac{\alpha'_{28}}{a'_{28}}e'_{1(u)}\left(1 + \frac{(a'_{28} - 1)}{a'_{28}}e'_{1(u)}\right)^{-1}\right], \qquad (6.2.4)$$

We assume that $|e'_{28}| < 1$, we expand the series, $\left(1 + \frac{(a'_{28} - 1)}{a'_{28}}e'_{1(u)}\right)^{-1}$, we get,

$$t_{28}^{\prime G} = \lambda'_{28}\bar{Y}'_s\left(1 + e'_{0(u)}\right)\exp\left[-\frac{\alpha'_{28}}{a'_{28}}e'_{1(u)}\left(1 - \frac{(a'_{28} - 1)}{a'_{28}}e'_{1(u)} + \frac{(a'_{28} - 1)^2}{a'^{2}_{28}}e'^{2}_{1(u)} + \ldots\right)\right],$$

(6.2.5)

It is assumed that the contribution of terms involving powers in $e'_{0(u)}$, and $e'_{1(u)}$ higher than one is negligible. It is therefore expanding the exponentials and ignoring terms in $e'_{0(u)}$, and $e'_{1(u)}$ of order higher than one, we have,

$$t'^{G}_{28} - \bar{Y}'_s = \lambda'_{28}\bar{Y}'_s\left[e'_{0(u)} - \frac{\alpha'_{28}}{a'_{28}}e'_{1(u)} + \frac{\alpha'_{28}(a'_{28}-1)}{a'^{2}_{28}}e'^{2}_{1(u)} - \frac{\alpha'_{28}}{a'_{28}}e'_{1(u)}e'_{0(u)}\right] + (\lambda'_{28}-1)\bar{Y}'_s, \tag{6.2.6}$$

In order to obtain the Bias (t'^{G}_{28}), we take expectation of (6.2.6) and using (6.1.2), the bias of t'^{G}_{28} will be as,

$$Bias(t'^{G}_{28}) = \lambda'_{28}\bar{Y}'_s\left[\frac{\alpha'_{28}(a'_{28}-1)}{a'^{2}_{28}}V_{20(u)} - \frac{\alpha'_{28}}{a'_{28}}V_{11(u)}\right] + (\lambda'_{28}-1)\bar{Y}'_s, \tag{6.2.7}$$

In order to derive the MSE of t'^{G}_{28}, we take square of (6.2.6) and retain terms in $e'_{0(u)}$, and $e'_{1(u)}$ up to the order one.

$$\left(t'^{G}_{28} - \bar{Y}'_s\right)^2 = \lambda'^{2}_{28}\bar{Y}'^{2}_s\left[e'^{2}_{0(u)} + \frac{\alpha'^{2}_{28}}{a'^{2}_{28}}e'^{2}_{1(u)} - \frac{2\alpha'_{28}}{a'_{28}}e'_{1(u)}e'_{0(u)}\right] + (\lambda'_{28}-1)^2\bar{Y}'^{2}_s, \tag{6.2.8}$$

On taking expectation and using (6.1.2), we have MSE (t'^{G}_{28}) as,

$$MSE\left(t'^{G}_{28}\right) \approx \bar{Y}'^{2}_s\left\{\lambda'^{2}_{28}\left(V'_{020(u)} + \left(\frac{\alpha'_{28}}{a'_{28}}\right)^2 V'_{200(u)} - 2\left(\frac{\alpha'_{28}}{a'_{28}}\right)V'_{110(u)}\right) + (\lambda'_{28}-1)^2\right\}, \tag{6.2.9}$$

In order to find the optimal value of λ'_{28} and a'_{28}, we differentiate (6.2.9) with respect to λ'_{28} and a'_{28}, then equate to zero, we will get,

$$\lambda'_{28} = \frac{1}{1 + V_{020(u)} + \left(\frac{\alpha'_{28}}{a'_{28}}\right)^2 V_{200(u)} - 2\left(\frac{\alpha'_{28}}{a'_{28}}\right) V_{110(u)}}, \tag{6.2.10}$$

and

$$a'^{\,opt}_{28} = \frac{\alpha'_{28} V'_{110(u)}}{V'_{200(u)}} \tag{6.2.11}$$

By substituting (6.2.11) in (6.2.10), we obtain

$$\lambda'^{\,opt}_{8} = \frac{1}{1 + \left(V'_{020(u)} - \frac{V'^{\,2}_{110(u)}}{V'_{200(u)}}\right)}, \tag{6.2.12}$$

Now by substituting (6.2.11) and (6.2.12) in (6.2.9), we get minimum MSE as the MSE given in (6.2.10) is derived upto the first order in e's, so it will be unbiased, and we may call it asymptotic variance as,

$$MSE_{\min}\left(t'^{G}_{28}\right) = AsymtoticVar\left(t'^{G}_{28}\right)$$

$$= \frac{\bar{Y}_s^2 V'_{020(u)}\left(1 - \rho'^2\right)}{\bar{Y}_s^2 + \bar{Y}_s^2 V'_{020(u)}\left(1 - \rho'^2\right)} = \frac{MSE(\mathrm{lr})}{\bar{Y}_s^2 + MSE(\mathrm{lr})}. \tag{6.2.13}$$

From (6.2.13) it is clearly observed that minimum MSE of t'^{G}_{28} is less than the MSE of **usual regression estimator in two-stage sampling**. We may observe from (6.2.13) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value $\lambda'^{\,opt}_{28}$ and $a'^{\,opt}_{28}$ in (6.2.3), we get optimal estimator as,

$$t'^{G}_{28} = \lambda_{28}'^{\,opt}\bar{u}'_{ys}\exp\left\{\alpha'_{8}\left(1 - \frac{a'^{\,opt}_{28}\bar{u}'_{xs}}{\bar{X}'_s + (a'^{\,opt}_{28} - 1)\bar{u}'_{xs}}\right)\right\}, 0 < \lambda'_{28} \le 1 \tag{6.2.14}$$

In real life situations, it is not possible for the researcher to presume the value of beta and lambda by employ all the resources e.g. see Horvitz and

Thompson (1952), Murthy (1967), Singh and Vishwakarma (2008), Singh and Kumar (2008), Singh and Karpe (2010), Upadhyaya et al. (2011), Yadav and Kadilar (2013) and Sanaullah et al. (2014), so it is better to replace these by their consistent estimates as,

$$\hat{\lambda}'^{\,opt}_{28} = \frac{1}{1 + \hat{V}_{020(u)} + \left(\dfrac{\alpha'_{28}}{\hat{a}'^{\,opt}_{28}}\right)^2 \hat{V}_{200(u)} - 2\left(\dfrac{\alpha'_{28}}{\hat{a}'^{\,opt}_{28}}\right)\hat{V}_{110(u)}}$$

and

$$\hat{a}'^{\,opt}_{8} = \frac{\alpha'_8 \hat{V}_{110(u)}}{\hat{V}_{200(u)}} \tag{6.2.15}$$

So (4.2.15) may be wriiten as,

$$t'^{G}_{28} = \lambda'^{opt}_{28}\bar{u}'_{ys}\exp\left\{\alpha'_8\left(1 - \frac{a'^{\,opt}_{28}\bar{u}'_{xs}}{\bar{X}'_s + (a'^{\,opt}_{28} - 1)\bar{u}'_{xs}}\right)\right\}, \qquad 0 < \lambda'^{opt}_{28} \le 1 \tag{6.2.16}$$

Also an estimator of the minimum MSE may be written as,

$$MSE_{\min}\left(t'^{G}_{28}\right) \approx MSE_{\min}\left(t'^{G}_{28}\right) = \frac{\bar{Y}_s^{2}\hat{V}_{020(u)}\left(1 - \hat{\rho}'^{2}\right)}{\bar{Y}_s^{2} + \bar{Y}_s^{2}\hat{V}_{020(u)}\left(1 - \hat{\rho}'^{2}\right)}. \tag{6.2.17}$$

Remark 6.1

i) For $\alpha'_{28} = 1$, we get exponential-ratio type estimators given in Table 6.1. The MSE of t'^{G}_{28} is expressed as

$$MSE(t'^{j}_{28}) = \begin{cases} \bar{Y}'^{2}_s\left(\lambda'^{2}_{28}\left(V'_{020(u)} + V'_{200(u)} - 2V'_{110(u)}\right) + (\lambda'_{28} - 1)^2\right) & j(\in G) = 1 \\ \bar{Y}'^{2}_s\left(\lambda'^{2}_{28}\left(V'_{020(u)} + \dfrac{1}{a'^{\left(\frac{j-1}{2}\right)^2}_{28}}V'_{200(u)} - 2\dfrac{1}{a'^{\left(\frac{j-1}{2}\right)}_{28}}V'_{110(u)}\right) + (\lambda'_{28} - 1)^2\right) & \\ & j(\in G) = 3,5,\ldots,11 \end{cases}$$

(6.2.18)

The optimal values which lead to minimum MSE as,

$$\lambda_{28}'^{opt} = \frac{1}{1 + V_{020(u)}' + \left(\frac{\alpha_{28}'}{a_{28}'^{\left(\frac{j-1}{2}\right)^{opt}}}\right)^2 V_{200(u)}' - 2\left(\frac{\alpha_{28}'}{a_{28}'^{\left(\frac{j-1}{2}\right)^{opt}}}\right) V_{110(u)}'},$$

and

$$a_{28}'^{\left(\frac{j-1}{2}\right)^{opt}} = \frac{V_{110(u)}'}{V_{200(u)}'}$$

ii) For $\alpha_{28}' = -1$, we get exponential-product type estimators given in Table 6.1. The MSE of t_{28}^{G} is expressed as,

$$MSE(t_{28}'^{k}) = \begin{cases} \bar{Y}_s'^2 \left(\lambda_{28}'^2 \left(V_{020(u)}' + a_{28}'^{opt2} V_{200(u)}' + 2a_{28}'^{opt} V_{110(u)}'\right) + (\lambda_{28}' - 1)^2\right) & k(\in G) = 2 \\ \bar{Y}_s'^2 \left(\lambda_{28}'^2 \left(V_{020(u)}' + \frac{1}{a_{28}'^{\left(\frac{k}{2}\right)^2}} V_{200(u)}' - 2\frac{1}{a_{28}'^{\left(\frac{k}{2}\right)}} V_{110(u)}'\right) + (\lambda_{28}' - 1)^2\right) & k(\in G) = 2,4,\ldots,12 \end{cases}$$

(6.2.19)

$$\lambda_{28}'^{opt} = \frac{1}{1 + V_{020(u)}' + \left(\frac{\alpha_{8}'}{a_{28}'^{\left(\frac{k}{2}\right)^{opt}}}\right)^2 V_{200(u)}' - 2\left(\frac{\alpha_{28}'}{a_{28}'^{\left(\frac{k}{2}\right)^{opt}}}\right) V_{110(u)}'},$$

and

$$a_{28}'^{\left(\frac{k}{2}\right)^{opt}} = \frac{V_{110(u)}'}{V_{200(u)}'}$$

Case II

The estimators in (6.2.3) may be adopted for case II as,

$$t_{28}''^{G} = \lambda_{28}'' \bar{u}_{ys}'' \exp\left\{\alpha_{28}'\left(1 - \frac{a_{28}''\bar{u}_{xs}''}{\bar{X}_s'' + (a_{28}'' - 1)\bar{u}_{xs}''}\right)\right\},\ 0 < \lambda_{28}'' \le 1 \qquad (6.2.20)$$

where $a_{28}''(\neq 0)$ and $\lambda_{28}''(\neq 0)$ in (6.2.3) are suitably chosen constants to be determined such as mean square error (MSE) of $t_{28}''^{G}$ is minimum and α_{20}' being constant takes the values (0,-1,1)for designing different ratio-type and product- type estimators. Also it is to be mentioned that for a different choice of a_{28}'', λ_{28}'' and α_{28}'' we get different estimators under two-stage sampling design.

The proposed estimator in (6.2.20) follows naturally in exactly the same fashion along with the class of estimator in Table 6.1, as that for case-I in Section 6.2.1. In addition, the relation between a_{28}'', α_{28}'' and λ_{28}'' in case-II is the same as that for case-I in Section 6.2.1.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 6.2.1.1, we may write $\mathrm{Bias}\,(t_{28}''^{G})$ and MSE$(t_8''^{G})$following the same, and we may also produce a class of estimators for similar choices of a_8'' α_8'' and λ_8'' in case-II.

Following the notations and expectations for case II presented in Section 6.1, the bias of (6.2.20) may be written directly from (6.2.7),

$$Bias(t_{28}''^{G}) = \lambda_{28}''\bar{Y}_s''\left[\frac{\alpha_{28}''(a_8''-1)}{a_{28}''^{2}}V_{200(u)} - \frac{\alpha_{28}''}{a_{28}''}V_{110(u)}\right] + \left(\lambda_{28}'' - 1\right)\bar{Y}_s''.$$

(6.2.21)

Also the expression for MSE ($t_{28}''^{G}$) may be directly written from (6.2.9) as,

$$MSE\left(t_{28}''^{G}\right) \approx \bar{Y}_s''^{2}\left(\lambda_{28}''^{2}\left(V_{020(u)}'' + \left(\frac{\alpha_{28}''}{a_{28}''}\right)^2 V_{200(u)}'' - 2\left(\frac{\alpha_{28}''}{a_{28}''}\right)V_{110(u)}''\right) + \left(\lambda_{28}'' - 1\right)^2\right)$$

(6.2.22)

In order to find the minimum MSE of (6.2.20) we will have optimal values of λ_{28}'' and a_{28}'' as,

$$\lambda''_{28} = \frac{1}{1 + V''_{020(u)} + \left(\frac{\alpha''_{28}}{a''^{\,opt}_{28}}\right)^2 V''_{200(u)} - 2\left(\frac{\alpha''_{28}}{a''^{\,opt}_{28}}\right) V''_{110(u)}},$$

where

$$a''_{28} = \frac{\alpha'''_{28} V''_{110(u)}}{V''_{200(u)}} \qquad (6.2.23)$$

The minimum MSE may also be given as,

$$MSE_{\min}\left(t''^{G}_{28}\right) = AsymtoticVar\left(t''^{G}_{28}\right) = \frac{\bar{Y}^2_s V''_{020(u)}\left(1-\rho^2\right)}{\bar{Y}^2_s + \bar{Y}^2_s V''_{020(u)}\left(1-\rho^2\right)} = \frac{MSE(lr)}{\bar{Y}^2_s + MSE(lr)}$$

(6.2.24)

Remark 6.2

i) For $\alpha''_{28} = 1$, we get exponential-ratio type estimators given in Table 6.1. The MSE of t''^{G}_{28} is expressed as,

$$MSE(t''^{j}_{28}) = \begin{cases} \bar{Y}''^2_s \left(\lambda''^2_{28}\left(V''_{020(u)} + V''_{200(u)} - 2V''_{110(u)}\right) + (\lambda''_{28}-1)^2\right) & j(\in G) = 1 \\ \bar{Y}''^2_s \left\{\lambda''^2_{28}\left\{V''_{020(u)} + \frac{1}{a''^{\left(\frac{j-1}{2}\right)^2}_{28}} V''_{200(u)} - 2\frac{1}{a''^{\left(\frac{j-1}{2}\right)}_{28}} V''_{110(u)}\right\} + (\lambda''_{28}-1)^2\right\} & j(\in G) = 3,5,\ldots,11 \end{cases}$$

(6.2.25)

The optimal values which lead to minimum MSE as,

$$\lambda''^{opt}_{28} = \frac{1}{1 + V''_{020(u)} + \left(\frac{1}{a''^{\left(\frac{j-1}{2}\right)^{opt}}_{28}}\right)^2 V''_{200(u)} - 2\left(\frac{1}{a''^{\left(\frac{j-1}{2}\right)^{opt}}_{28}}\right) V''_{110(u)}},$$

and

$$a_{28}''^{\left(\frac{j-1}{2}\right)^{opt}} = \frac{V''_{110(u)}}{V''_{200(u)}}$$

ii) For $\alpha''_{28} = -1$, we get exponential-ratio type estimators given in Table 6.1. The MSE of t''^{G}_{28} is expressed as,

$$MSE(t''^{k}_{28}) = \begin{cases} \bar{Y}_s''^2 \left(\lambda''^{\,2}_{28}\left(V''_{020(u)} + a''^{\,opt2}_{28} V'_{200(u)} + 2a''^{\,opt}_{28} V''_{110(u)}\right) + (\lambda''_{28} - 1)^2\right) & k(\in G) = 2 \\ \bar{Y}_s''^2 \left(\lambda''^{\,2}_{28}\left(V'_{020(u)} + \frac{1}{a''^{\left(\frac{k}{2}\right)^2}_{28}} V'_{200(u)} + 2\frac{1}{a''^{\left(\frac{k}{2}\right)}_{28}} V'_{110(u)}\right) + (\lambda''_{28} - 1)^2\right) & k(\in G) = 2,4,\ldots,12 \end{cases}$$

(6.2.26)

$$\lambda''^{opt}_{28} = \frac{1}{1 + V''_{020(u)} + \left(\frac{1}{a''^{\left(\frac{k}{2}\right)^{opt}}_{28}}\right)^2 V''_{200(u)} + 2\left(\frac{1}{a''^{\left(\frac{k}{2}\right)^{opt}}_{28}}\right) V''_{110(u)}},$$

and

$$a_{28}''^{\left(\frac{k}{2}\right)^{opt}} = \frac{V''_{110(u)}}{V''_{200(u)}}$$

Case III

The estimator for Case III may be written as by following (6.2.3),

$$t^{G}_{28} = \lambda_{28}\bar{u}_{ys} \exp\left\{\alpha_{28}\left(1 - \frac{a_{28}\bar{u}_{xs}}{\bar{X}_s + (a_{28} - 1)\bar{u}_{xs}}\right)\right\}, \quad 0 < \lambda_{28} \le 1 \qquad (6.2.27)$$

where $a_{28}(\neq 0)$ and $\lambda_{28}(\neq 0)$ in (6.2.3) are suitably chosen constants to be determined such as mean square error (MSE) of t^{G}_{28} is minimum and α_{28} being constant takes the values (0,-1,1)for designing different ratio-type and product- type estimators. Also it is to be mentioned that for a different choice of a_{28}, λ_{28} and α_{28} we get different estimators under two-stage sampling design.

The proposed estimator in (6.2.27) follows naturally in exactly the same fashion along with the class of estimator in Table 6.1, as that for case-I in Section 6.2.1. In addition, the relation between a, α and λ in case-II is the same as that for case-I in Section 6.2.1.1. Finally, the same is true for the MSE and the bias. It is therefore directly from Section 6.2.1.1, we may write $\text{Bias}\,(t_{28}^G)$ and $\text{MSE}(t_{28}^G)$ following the same, and we may also produce a class of estimators for similar choices of a, α_{28} and λ_{28} in case-III.

Following the notations and expectations for case II presented in Section 4.1, the bias of (6.2.27) may be obtained,

$$Bias(t_{28}^G) = \lambda_{28}\overline{Y}_s\left[\frac{\alpha_{28}(a_{28}-1)}{a_{28}{}^2}V_{200(u)} - \frac{\alpha_{28}}{a_{28}}V_{110(u)}\right] + (\lambda_{28}-1)\overline{Y}_s\,. \quad (6.2.28)$$

The expression for MSE of t_{28}^G may be written as,

$$MSE\left(t_{28}^G\right) \approx \overline{Y}_s^{\,2}\left(\lambda_{28}{}^2\left(V_{020(u)} + \left(\frac{\alpha_{28}}{a_{28}}\right)^2 V_{200(u)} - 2\left(\frac{\alpha_{28}}{a_{28}}\right)V_{110(u)}\right) + (\lambda_{28}-1)^2\right),$$

(6.2.29)

The optimal value which leads to the minimum MSE may also be derived exactly in the same manner as given in section 6.2.1.1 as,

$$\lambda_{28}{}^{opt} = \frac{1}{1 + V_{020(u)} + \left(\dfrac{\alpha_{28}}{a_{28}{}^{opt}}\right)^2 V_{200(u)} - 2\left(\dfrac{\alpha_{28}}{a_{28}{}^{opt}}\right)V_{110(u)}},$$

where

$$a_{28}{}^{opt} = \frac{\alpha_{28}V_{110(u)}}{V_{200(u)}} \quad (6.2.30)$$

Remark 6.3

i) For $\alpha_{28} = 1$, we get exponential-ratio type estimators given in Table 6.1. The MSE of t_{28}^G is expressed as,

$$MSE(t_{28}^{j}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{28}^2 \left(V_{020(u)} + V_{200(u)} - 2V_{110(u)}\right) + (\lambda_{28} - 1)^2\right) j(\in G) = 1 \\ \bar{Y}_s^2 \left(\lambda_{28}^2 \left(V_{020(u)} + \dfrac{1}{a_{28}^{\left(\frac{j-1}{2}\right)^2}} V_{200(u)} - 2\dfrac{1}{a_{28}^{\left(\frac{j-1}{2}\right)}} V_{110(u)}\right) + (\lambda_{28} - 1)^2\right) \\ \qquad j(\in G) = 3,5,\ldots,11 \end{cases}$$

(6.2.31)

The optimal values which lead to minimum MSE as,

$$\lambda_{28}^{opt} = \frac{1}{1 + V_{020(u)} + \left(\dfrac{1}{a_{28}^{\left(\frac{j-1}{2}\right)^{opt}}}\right)^2 V_{200(u)} - 2\left(\dfrac{1}{a_{28}^{\left(\frac{j-1}{2}\right)^{opt}}}\right) V_{110(u)}},$$

and

$$a_{28}^{\left(\frac{j-1}{2}\right)^{opt}} = \frac{V_{110(u)}}{V_{200(u)}}$$

ii) For $\alpha_{28} = -1$, we get exponential-ratio type estimators given in Table 6.1. The MSE of t_{28}^{G} is expressed as

$$MSE(t_{28}^{k}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{28}^2 \left(V_{020(u)} + a_{28}^{opt2} V_{200(u)} + 2a_{28}^{opt} V_{110(u)}\right) + (\lambda_{28} - 1)^2\right) \\ \qquad k(\in G) = 2 \\ \bar{Y}_s^2 \left(\lambda_{28}^2 \left(V_{020(u)} + \dfrac{1}{a_{28}^{\left(\frac{k}{2}\right)^2}} V_{200(u)} - 2\dfrac{1}{a_{28}^{\left(\frac{k}{2}\right)}} V_{110(u)}\right) + (\lambda_{8} - 1)^2\right) \\ \qquad k(\in G) = 2,4,\ldots,12 \end{cases}$$

(6.2.32)

$$\lambda_{28}^{opt} = \frac{1}{1 + V_{020(u)} + \left(\frac{1}{a_{28}^{\left(\frac{k}{2}\right)^{opt}}} \right)^2 V_{200(u)} - 2\left(\frac{1}{a_{28}^{\left(\frac{k}{2}\right)^{opt}}} \right) V_{110(u)}},$$

and

$$a_{28}^{\left(\frac{k}{2}\right)^{opt}} = \frac{V_{110(u)}}{V_{200(u)}}.$$

6.3 PROPOSED GENERALIZED ESTIMATOR-X IN TWO-STAGE ADAPTIVE CLUSTER SAMPLING USING TWO AUXILIARY VARIABLE

In this section, we propose two generalized estimators using two auxiliary variables under two-stage sampling design. By using the transformed population. The estimator are constructed by assuming an exponential relationship between the study variable and two auxiliary variable for different three cases (mentioned in chapter 4) under two- stage sampling.

6.3.1 Proposed Generalized Estimator-X for Case I

i. Let exponential-type ratio-cum-ratio estimator follows as,

$$t_{29}'^{1} = \bar{u}'_{ys} \exp\left(1 - \frac{\bar{u}'_{xs}}{\left(\bar{X}'_s + \bar{u}'_{xs} \right)} \right) \exp\left(1 - \frac{\bar{u}'_{zs}}{\left(\bar{Z}'_s + \bar{u}'_{zs} \right)} \right), \tag{6.3.1}$$

ii. Let exponential-type product-cum-product estimator follows as,

$$t_{29}'^{2} = \bar{u}'_{ys} \exp\left(-\left(1 - \frac{\bar{u}'_{xs}}{\left(\bar{X}'_s + \bar{u}'_{xs} \right)} \right) \right) \exp\left(-\left(1 - \frac{\bar{u}'_{zs}}{\left(\bar{Z}'_s + \bar{u}'_{zs} \right)} \right) \right) \tag{6.3.2}$$

iii. Let exponential-type ratio-cum-product estimator follows as,

$$t_{29}'^{3} = \bar{u}'_{ys} \exp\left(\left(1 - \frac{\bar{u}'_{xs}}{\left(\bar{X}'_s + \bar{u}'_{xs} \right)} \right) \right) \exp\left(-\left(1 - \frac{\bar{u}'_{zs}}{\left(\bar{Z}'_s + \bar{u}'_{zs} \right)} \right) \right) \tag{6.3.3}$$

iv. Let exponential-type product-cum-ratio estimator follows as,

$$t_{29}'^{4} = \overline{u}_{ys}' \exp\left(-\left(1 - \frac{\overline{u}_{xs}'}{\left(\overline{X}_s' + \overline{u}_{xs}' \right)} \right) \right) \exp\left(\left(1 - \frac{\overline{u}_{zs}'}{\left(\overline{Z}_s' + \overline{u}_{zs}' \right)} \right) \right) \qquad (6.3.4)$$

We may generalize (6.3.1)-(6.3.4) by introducing two real constants α_{29}' and β_{29}' whose values are known in advance as,

$$t_{29}'^{G} = \lambda_{29}' \overline{u}_{ys}' \exp\left(\alpha_{29}' \left(1 - \frac{a_{29}' \overline{u}_{xs}'}{\left(\overline{X}_s' + (a_{29}' - 1)\overline{u}_{xs}' \right)} \right) \right)$$

$$\exp\left(\beta_{29}' \left(1 - \frac{b_{29}' \overline{u}_{zs}'}{\left(\overline{Z}_s' + (b_{29}' - 1)\overline{u}_{zs}' \right)} \right) \right), \quad 0 < \lambda_{29}' \leq 1 \qquad (6.3.5)$$

where $(a_{29}',\ b_{29}',\ \lambda_{29}')$ are constants to be determined such that the mean square error (MSE) is minimum. $(\alpha_{29}',\ \beta_{29}')$ are known constants takes the value(0,1,-1) to produce different ratio-type and product-type estimators.

By substituting different values to the constants in (6.3.5), we get a class of estimators as given in Table 6.2.

Table 6.2
Some Special Cases of the Generalized Estimator t'^{G}_{29}

Ratio-cum-Product Estimator $\alpha'_{29} = 1, \beta'_{29} = 1$	**Product -cum-Product Estimator** $\alpha'_{29} = -1, \beta'_{29} = -1$	a'_{29}	λ'_{29}
$t'^{1}_{29} = \bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + \bar{u}'_{xs}}\right)\exp\left(\frac{\bar{Z}'_s - \bar{u}'_{zs}}{\bar{Z}'_s + \bar{u}'_{zs}}\right)$	$t'^{1}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{u}'_{xs} + \bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{u}'_{zs} + \bar{Z}'_s}\right)$	2	1
$t'^{3}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{Z}'_s - \bar{u}'_{zs}}{\bar{Z}'_s}\right)$	$t'^{4}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'}{\bar{Z}'_s}\right)$	1	1
$t'^{5}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{Z}'_s - \bar{u}'_{zs}}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$	$t'^{5}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$		1
$t'^{7}_{29} = \lambda'_{29}\bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + \bar{u}'_{xs}}\right)\exp\left(\frac{\bar{Z}'_s - \bar{u}'_{zs}}{\bar{Z}'_s + \bar{u}'_{zs}}\right)$	$t'^{7}_{29} = \lambda'_{29}\bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s + \bar{u}'_{xs}}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + \bar{u}'_{zs}}\right)$	2	λ'_{29}
$t'^{9}_{29} = \lambda'_{29}\bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{Z}'_s - \bar{u}'_{zs}}{\bar{Z}'_s}\right)$	$t'^{9}_{29} = \lambda'_{29}\bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s}\right)$	1	λ'_{29}
$t'^{11}_{29} = \lambda'_{29}\bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{Z}'_s - \bar{u}'_{zs}}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$	$t'^{12}_{29} = \lambda'_{29}\bar{y}'_s \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$	a'_{29}	λ'_{29}
Product -cum-Ratio Estimator $\alpha'_{29} = -1, \beta'_{29} = 1$	**Ratio-cum-Product Estimator** $\alpha'_{29} = 1, \beta'_{29} = -1$	a'_{29}	λ'_{29}
$t'^{13}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + \bar{u}'_{xs}}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + \bar{u}'_{zs}}\right)$	$t'^{14}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{u}'_{xs} + \bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{u}'_{zs} + \bar{Z}'_s}\right)$	2	1
$t'^{15}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s}\right)$	$t'^{16}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'}{\bar{Z}'_s}\right)$	1	1
$t'^{17}_{29} = \bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$	$t'^{18}_{29} = \bar{u}'_{xs} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$	a'_{29}	1
$t'^{19}_{29} = \lambda'_{29}\bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + \bar{u}'_{zs}}\right)$	$t'^{20}_{29} = \lambda'_{29}\bar{y}'_s \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s + \bar{x}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + \bar{u}'_{zs}}\right)$	2	λ'_{29}
$t'^{21}_{29} = \lambda'_{29}\bar{u}'_{ys} \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s}\right)$	$t'^{22}_{29} = \lambda'_{29}\bar{y}'_s \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'_s}{\bar{X}'_s}\right)\exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s}\right)$	1	λ'_{29}
$t'^{23}_{29} = \lambda'_{29}\bar{y}'_s \exp\left(\frac{\bar{X}'_s - \bar{u}'_{xs}}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$	$t'^{24}_{29} = \lambda'_{29}\bar{u}'_{xs} \exp\left(\frac{\bar{u}'_{xs} - \bar{X}'}{\bar{X}'_s + (a'_{29}-1)\bar{u}'_{xs}}\right) \exp\left(\frac{\bar{u}'_{zs} - \bar{Z}'_s}{\bar{Z}'_s + (b'_{29}-1)\bar{u}'_{zs}}\right)$	a'_{29}	λ'_{29}

The Bias and Mean Square Error of Generalized Estimator-X

To derive the bias and mean square error we proceed as follows,
Using (6.1.1) we can express (6.3.5) as,

$$t_{29}'^{G} = \lambda_{29}'\bar{Y}_s'\left(1+e_{0(u)}'\right)\exp\left[-\frac{\alpha_{29}'}{a_{29}'}e_{1(u)}'\left(1+\frac{(a_{29}'-1)}{a_{29}'}e_{1(u)}'\right)^{-1}\right]$$
$$\exp\left[-\frac{\beta_{29}'}{b_{29}'}e_{2(u)}'\left(1+\frac{(b_{29}'-1)}{b_{29}'}e_{2(u)}'\right)^{-1}\right] \qquad (6.3.6)$$

We assume that $\left|e_{1(u)}'\right| < 1$, we expand the series, $\left(1+\frac{(a_{29}'-1)}{a_{29}'}e_{1(u)}'\right)^{-1}$ and $\left(1+\frac{(b_{29}'-1)}{b_{29}'}e_{2(u)}'\right)^{-1}$, we get,

$$t_{29}'^{G} = \lambda_{29}'\bar{Y}_s'\left(1+e_{0(u)}'\right)$$
$$\exp\left[-\frac{\alpha_{29}'}{a_{29}'}e_{1(u)}'\left(1-\frac{(a_{29}'-1)}{a_{29}'}e_{1(u)}'+\frac{(a_{29}'-1)^2}{a_{29}'^{2}}e_{1(u)}'^{2}+\ldots\right)\right]$$
$$\exp\left[-\frac{\beta_{29}'}{b_{29}'}e_{2(u)}'\left(1-\frac{(b_{29}'-1)}{b_{29}'}e_{2(u)}'+\frac{(b_{29}'-1)^2}{b_{29}'^{2}}e_{2(u)}'^{2}+\ldots\right)\right], \qquad (6.3.7)$$

It is assumed that the contribution of terms involving powers in $e_{0(u)}'$, and $e_{1(u)}'$ and $e_{2(u)}'$ higher than one is negligible. It is therefore expanding the exponentials and ignoring terms in $e_{0(u)}'$ and $e_{1(u)}'$ of order higher than one, we have,

$$t_{29}'^{G} = \lambda_{29}'\bar{Y}_s'\left(1+e_{0(u)}'\right)\left[1-\frac{\alpha_{29}'}{a_{29}'}e_{1(u)}'+\frac{\alpha_{29}'^{2}}{a_{29}'^{2}}e_{1(u)}'^{2}\right]\left[1-\frac{\beta_{29}'}{b_{29}'}e_{2(u)}'+\frac{\beta_{29}'^{2}}{b_{29}'^{2}}e_{2(u)}'^{2}\right],$$
$$(6.3.8)$$

$$t_{29}^{\prime G}-\bar{Y}_s'=\lambda_{29}'\bar{Y}_s'\begin{bmatrix} e_{0(u)}'-\dfrac{\alpha_{29}'}{a_{29}'}e_{1(u)}'+\dfrac{\alpha_{29}'^2}{a_{29}'^2}e_{1(u)}'^2-\dfrac{\beta_{29}'}{b_{29}'}e_{2(u)}' \\ +\dfrac{\alpha_{29}'}{a_{29}'}\dfrac{\beta_{29}'}{b_{29}'}e_{2(u)}'e_{1(u)}'+\dfrac{\beta_{29}'^2}{b_{29}'^2}e_{2(u)}'^2 \\ -\dfrac{\beta_{29}'}{b_{29}'}e_{2(u)}'e_{0(u)}'-\dfrac{\alpha_{29}'}{a_{29}'}e_{1(u)}'e_{0(u)}' \end{bmatrix}+\bar{Y}_s'(\lambda_{29}'-1) \tag{6.3.9}$$

In order to get the bias , we take expectation on (6.3.9) and get,

$$Bias(t_{29}^{\prime G})=\lambda_{29}'\bar{Y}_s'\left[\frac{\alpha_{29}'^2}{a_{29}'^2}V_{200(u)}'+\frac{\beta_{29}'^2}{b_{29}'^2}V_{002(u)}'-\frac{\beta_{29}'}{b_{29}'}V_{101(u)}'\right.$$
$$\left.-\frac{\alpha_{29}'}{a_{29}'}V_{110(u)}'+\frac{\alpha_{29}'}{a_{29}'}\frac{\beta_{29}'}{b_{29}'}V_{011(u)}'\right]+\bar{Y}_s'(\lambda_{29}'-1), \tag{6.3.10}$$

To get the MSE of the estimator, we take square and retain terms upto second order of e’s then we take expectation of (6.3.9) and we obtain,

$$MSE\left(t_{29}^{\prime G}\right)=\bar{Y}_s'^2\left[\lambda_{29}'^2\left(V_{020(u)}'+z_{29}'^2V_{200(u)}'+u_{29}'^2V_{002(u)}'-2z_{29}'V_{110(u)}'\right.\right.$$
$$\left.\left.-2u_{29}'V_{011(u)}'+2z_{29}'u_{29}'V_{101(u)}'\right)+(\lambda_{29}'-1)^2\right], \tag{6.3.11}$$

where $z_{29}'=\dfrac{\alpha_{29}'}{a_{29}'}$ and $u_{29}'=\dfrac{\beta_{29}'}{b_{29}'}$

For the following optimal value of the constants z_{29}' and u_{29}', we achieve the minimum MSE among the class of proposed generalized estimator,

$$z_{29}'=\frac{V_{110(u)}'V_{002(u)}'-V_{011(u)}'V_{101(u)}'}{V_{200(u)}'V_{002(u)}'-V_{101(u)}'^2}$$

and

$$u_{29}'=\frac{V_{200(u)}'V_{011(u)}'-V_{110(u)}'V_{101(u)}'}{V_{200(u)}'V_{002(u)}'-V_{101(u)}'^2},\lambda_{29}'=\frac{1}{1+A_{29}'^G}$$

where

$$A'^{G}_{29} = \begin{bmatrix} V_{020(u)} + z'^{2}_{29} V_{200(u)} + u'^{2}_{29} V_{002(u)} \\ -2 z'_{29} V_{110(u)} - 2 u'_{29} V_{011(u)} + 2 z'_{29} u'_{29} V_{101(u)} \end{bmatrix} \tag{6.3.12}$$

By substituting the optimum values of z'_{29} and u'_{29}, we get λ'^{opt}_{29} as,

$$\lambda'^{opt}_{29} = \frac{1}{1 + A'^{*}_{29}}$$

where

$$A'^{*}_{29} = \left(V'_{020(u)} - \frac{V'^{2}_{110(u)} V'_{002(u)} + V'^{2}_{011(u)} V'_{200(u)} - 2 V'_{110(u)} V'_{101(u)} V'_{011(u)}}{V'_{200(u)} V'_{002(u)} - V'^{2}_{101(u)}} \right) \tag{6.3.13}$$

The minimum MSE may be written as,

$$MSE_{\min}\left(t'^{G}_{29}\right) = AsymtoticVar\left(t'^{G}_{29}\right) = \bar{Y}'^{2}_{s} \left(\frac{A'^{*}_{29}}{1 + A'^{*}_{29}} \right). \tag{6.3.14}$$

From (6.3.14), we observe that asymptotic variance of the proposed estimator is less than **Usual Linear Regression Estimator** , We may observe from (6.3.14) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

On substituting the optimal value λ'^{opt}_{29} and $a'^{opt}_{29}, b'^{opt}_{29}$ in (6.3.5), we get optimal estimator as,

$$\hat{t}'^{G}_{29} = \hat{\lambda}'_{29} \bar{u}'_{ys} \exp\left(\alpha'_{29} \left(1 - \frac{\hat{a}'_{29} \bar{x}'_{s}}{\left(\bar{X}'_{s} + (\hat{a}'_{29} - 1) \bar{x}'_{s} \right)} \right) \right) \exp\left(\beta'_{29} \left(1 - \frac{\hat{b}'_{29} \bar{z}'_{s}}{\left(\bar{Z}'_{s} + (\hat{b}'_{29} - 1) \bar{z}'_{s} \right)} \right) \right), 0 < \lambda'_{29} \le 1 \tag{6.3.15}$$

As described earlier in section (6.2.1.1) that in some practical situations, when it becomes impossible to collect information on some of the population characteristics, it is valuable to replace them with their consistent estimators as,

$$\hat{\lambda}_{29}'^{\,opt} = \frac{1}{1 + \hat{A}_{29}'^{*}}$$

where

$$\hat{A}_{29}'^{*} = \left(\hat{V}_{020(u)}' - \frac{\hat{V}_{110(u)}'^{2}\hat{V}_{002(u)}' + \hat{V}_{011(u)}'^{2}\hat{V}_{200(u)}' - 2\hat{V}_{110(u)}'\hat{V}_{101(u)}'\hat{V}_{011(u)}'}{\hat{V}_{200(u)}'\hat{V}_{002(u)}' - \hat{V}_{101(u)}'^{2}} \right) \qquad (6.3.16)$$

So (6.3.16) may be wriiten as,

$$\hat{t}_{29}'^{G} = \hat{\lambda}_{29}'^{opt}\,\bar{u}_{ys}'\exp\left(\alpha_{29}'\left(1 - \frac{\hat{a}_{29}'^{opt}\bar{u}_{xs}'}{\left(\bar{X}_s' + (\hat{a}_{29}'^{opt} - 1)\bar{u}_{xs}'\right)}\right)\right)$$

$$\exp\left(\beta_{29}'\left(1 - \frac{\hat{b}_{29}'^{\,opt}\bar{u}_{zs}'}{\left(\bar{Z}_s' + (\hat{b}_{29}'^{\,opt} - 1)\bar{u}_{zs}'\right)}\right)\right), 0 < \lambda_{29}' \le 1 \qquad (6.3.17)$$

Also the minimum MSE may be written as,

$$MSE_{\min}\left(\hat{t}_{29}'^{G}\right) = AsymtoticVar\left(\hat{t}_{29}'^{G}\right) = \bar{Y}_s'^{2}\left(\frac{\hat{A}_{29}'^{*}}{1 + \hat{A}_{29}'^{*}}\right). \qquad (6.3.18)$$

Remark 6.4

i) For $\alpha_{29}' = 1, \beta_{29}' = 1$, we get exponential-ratio type estimators given in Table 6.2. The MSE of $t_{29}'^{G}$ is expressed as,

$$MSE(t_{29}'^{j}) = \begin{cases} \bar{Y}_s'^{2}\left(\lambda_{29}'^{2}\begin{pmatrix} V_{020(u)}' + V_{200(u)}' + V_{002(u)}' \\ -2V_{110(u)}' - 2V_{011(u)} + 2V_{101(u)} \end{pmatrix} + (\lambda_{29}' - 1)^2\right) & j(\in G) = 1 \\ \bar{Y}_s'^{2}\left(\lambda_{29}'^{2}\begin{pmatrix} V_{020(u)}' + \frac{1}{a_{29}'^{\left(\frac{j-1}{2}\right)^2}}V_{200(u)}' + \frac{1}{b_{29}'^{\left(\frac{j-1}{2}\right)^2}}V_{002(u)}' \\ -2\frac{1}{a_{29}'^{\left(\frac{j-1}{2}\right)}}V_{110(u)}' - 2\frac{1}{b_{29}'^{\left(\frac{j-1}{2}\right)}}V_{011(u)} \\ +2\frac{1}{a_{29}'^{\left(\frac{j-1}{2}\right)}}\frac{1}{b_{29}'^{\left(\frac{j-1}{2}\right)}}V_{101(u)} \end{pmatrix} + (\lambda_{29}' - 1)^2\right) & j(\in G) = 3,5,\ldots,11 \end{cases}$$

(6.3.19)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}'^{\left(\frac{j-1}{2}\right)}}=\frac{V_{110(u)}'V_{002(u)}'-V_{011(u)}'V_{101(u)}'}{V_{200(u)}'V_{002(u)}'-V_{101(u)}'^{2}}$$

and

$$\frac{1}{b_{29}'^{\left(\frac{j-1}{2}\right)}}=\frac{V_{200(u)}'V_{011(u)}'-V_{110(u)}'V_{101(u)}'}{V_{200(u)}'V_{002(u)}'-V_{101(u)}'^{2}},\lambda_{29}'=\frac{1}{1+A_{29}'^{G}}$$

where

$$A_{29}'^{G}=\left[\begin{array}{l} V_{020(u)}+\frac{1}{a_{29}'^{\left(\frac{j-1}{2}\right)^{2}}}V_{200(u)}+\frac{1}{b_{29}'^{\left(\frac{j-1}{2}\right)^{2}}}V_{002(u)}-2\frac{1}{a_{29}'^{\left(\frac{j-1}{2}\right)}}V_{110(u)} \\ -2\frac{1}{b_{29}'^{\left(\frac{j-1}{2}\right)}}V_{011(u)}+2\frac{1}{a_{29}'^{\left(\frac{j-1}{2}\right)}}\frac{1}{b_{29}'^{\left(\frac{j-1}{2}\right)}}V_{101(u)} \end{array}\right]$$

ii) For $\alpha_{29}'=-1,\beta_{29}'=-1$, we get exponential-ratio type estimators given in Table 6.2. The MSE of $t_{29}'^{G}$ is expressed as,

$$MSE(t_{29}'^{k})=\begin{cases} \bar{Y}_s'^{2}\left(\lambda_{29}'^{2}\left(\begin{array}{l} V_{020(u)}'+V_{200(u)}'+V_{002(u)}' \\ +2V_{110(u)}'+2V_{011(u)}-2V_{101(u)} \end{array}\right)+(\lambda_{29}'-1)^{2}\right) k(\in G)=2 \\ \bar{Y}_s'^{2}\left(\lambda_{29}'^{2}\left(\begin{array}{l} V_{020(u)}'+\frac{1}{a_{29}'^{\left(\frac{k}{2}\right)^{2}}}V_{200(u)}'+\frac{1}{b_{29}'^{\left(\frac{k}{2}\right)^{2}}}V_{002(u)}' \\ +2\frac{1}{a_{29}'^{\left(\frac{k}{2}\right)}}V_{110(u)}'+2\frac{1}{b_{29}'^{\left(\frac{k}{2}\right)}}V_{011(u)} \\ -2\frac{1}{a_{29}'^{\left(\frac{k}{2}\right)}}\frac{1}{b_{29}'^{\left(\frac{k}{2}\right)}}V_{101(u)} \end{array}\right)+(\lambda_{29}'-1)^{2}\right) k(\in G)=4,6,\ldots,12 \end{cases}$$

(6.3.20)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}'^{\left(\frac{k}{2}\right)}}=\frac{V_{110(u)}'V_{002(u)}'-V_{011(u)}'V_{101(u)}'}{V_{200(u)}'V_{002(u)}'-V_{101(u)}'^{2}}$$

and

$$\frac{1}{b_{29}'^{\left(\frac{k}{2}\right)}} = \frac{V'_{200(u)}V'_{011(u)} - V'_{110(u)}V'_{101(u)}}{V'_{200(u)}V'_{002(u)} - V'^{2}_{101(u)}}, \lambda'_{29} = \frac{1}{1 + A'^{G}_{29}}$$

where

$$A'^{G}_{29} = \begin{bmatrix} V_{020(u)} + \frac{1}{a_{29}'^{\left(\frac{k}{2}\right)^2}} V_{200(u)} + \frac{1}{b_{29}'^{\left(\frac{k}{2}\right)^2}} V_{002(u)} - 2\frac{1}{a_{29}'^{\left(\frac{k}{2}\right)}} V_{110(u)} \\ -2\frac{1}{b_{29}'^{\left(\frac{k}{2}\right)}} V_{011(u)} + 2\frac{1}{a_{29}'^{\left(\frac{k}{2}\right)}}\frac{1}{b_{29}'^{\left(\frac{k}{2}\right)}} V_{101(u)} \end{bmatrix}$$

i) For $\alpha'_{29} = -1, \beta'_{29} = 1$, we get exponential-ratio type estimators given in Table 9. The MSE of t'^{G}_{29} is expressed as,

$$MSE(t'^{l}_{29}) = \begin{cases} \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{29}\begin{pmatrix} V'_{020(u)} + V'_{200(u)} + V'_{002(u)} \\ + 2V'_{110(u)} - 2V_{011(u)} - 2V_{101(u)} \end{pmatrix} + (\lambda'_{29} - 1)^2\right) & l(\in G) = 13 \\ \bar{Y}'^{2}_{s}\left(\lambda'^{2}_{29}\begin{pmatrix} V'_{020(u)} + \frac{1}{a_{29}'^{\left(\frac{l-1}{2}\right)^2}} V'_{200(u)} + \frac{1}{b_{29}'^{\left(\frac{l-1}{2}\right)^2}} V'_{002(u)} \\ + 2\frac{1}{a_{29}'^{\left(\frac{l-1}{2}\right)}} V'_{110(u)} - 2\frac{1}{b_{29}'^{\left(\frac{l-1}{2}\right)}} V_{011(u)} \\ - 2\frac{1}{a_{29}'^{\left(\frac{l-1}{2}\right)}}\frac{1}{b_{29}'^{\left(\frac{l-1}{2}\right)}} V_{101(u)} \end{pmatrix} + (\lambda'_{29} - 1)^2\right) & l(\in G) = 15, 17, \ldots, 23 \end{cases}$$

(6.3.21)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}'^{\left(\frac{l-1}{2}\right)}} = \frac{V'_{110(u)}V'_{002(u)} - V'_{011(u)}V'_{101(u)}}{V'_{200(u)}V'_{002(u)} - V'^{2}_{101(u)}}$$

and

$$\frac{1}{b_{29}'^{\left(\frac{l-1}{2}\right)}} = \frac{V'_{200(u)}V'_{011(u)} - V'_{110(u)}V'_{101(u)}}{V'_{200(u)}V'_{002(u)} - V'^{2}_{101(u)}}, \lambda'_{29} = \frac{1}{1 + A'^{G}_{29}}$$

where

$$A'^{G}_{29} = \begin{bmatrix} V_{020(u)} + \dfrac{1}{a'^{\left(\frac{l-1}{2}\right)^2}_{29}} V_{200(u)} + \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)^2}_{29}} V_{002(u)} - 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{29}} V_{110(u)} \\ -2\dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{29}} V_{011(u)} + 2\dfrac{1}{a'^{\left(\frac{l-1}{2}\right)}_{29}} \dfrac{1}{b'^{\left(\frac{l-1}{2}\right)}_{29}} V_{101(u)} \end{bmatrix}$$

ii) For $\alpha'_{29} = 1, \beta'_{29} = -1$, we get exponential-ratio type estimators given in Table 6.2. The MSE of t'^{G}_{29} is expressed as,

$$MSE(t'^{m}_{29}) = \begin{cases} \bar{Y}'^2_s \left(\lambda'^2_{29} \begin{pmatrix} V'_{020} + V'_{200} + V'_{002} \\ - 2V'_{110} + 2V'_{011} - 2V'_{101} \end{pmatrix} + (\lambda'_{29} - 1)^2 \right) & m (\in G) = 14 \\ \bar{Y}'^2_s \left(\lambda'^2_{29} \begin{pmatrix} V'_{020(u)} + \dfrac{1}{a'^{\left(\frac{k}{2}\right)^2}_{29}} V'_{200(u)} + \dfrac{1}{b'^{\left(\frac{k}{2}\right)^2}_{29}} V'_{002(u)} \\ - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{29}} V'_{110(u)} + 2\dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{29}} V'_{011(u)} \\ - 2\dfrac{1}{a'^{\left(\frac{k}{2}\right)}_{29}} \dfrac{1}{b'^{\left(\frac{k}{2}\right)}_{29}} V'_{101(u)} \end{pmatrix} + (\lambda'_{29} - 1)^2 \right) & m (\in G) = 16, 18, \ldots, 24 \end{cases}$$

(6.3.22)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a'^{\left(\frac{m}{2}\right)}_{29}} = \frac{V'_{110(u)} V'_{002(u)} - V'_{011(u)} V'_{101(u)}}{V'_{200(u)} V'_{002(u)} - V'^{2}_{101(u)}}$$

and

$$\frac{1}{b'^{\left(\frac{m}{2}\right)}_{29}} = \frac{V'_{200(u)} V'_{011(u)} - V'_{110(u)} V'_{101(u)}}{V'_{200(u)} V'_{002(u)} - V'^{2}_{101(u)}}, \lambda'_{29} = \frac{1}{1 + A'^{G}_{29}}$$

where

$$A'^{G}_{29} = \begin{bmatrix} V'_{020(u)} + \dfrac{1}{a'^{\left(\frac{m}{2}\right)^2}_{29}} V'_{200(u)} + \dfrac{1}{b'^{\left(\frac{m}{2}\right)^2}_{29}} V'_{002(u)} - 2\dfrac{1}{a'^{\left(\frac{m}{2}\right)}_{29}} V'_{110(u)} \\ +2\dfrac{1}{b'^{\left(\frac{m}{2}\right)}_{29}} V'_{011(u)} - 2\dfrac{1}{a'^{\left(\frac{m}{2}\right)}_{29}} \dfrac{1}{b'^{\left(\frac{m}{2}\right)}_{29}} V'_{101(u)} \end{bmatrix}$$

Case II

The generalized estimator under case II may be proposed following (4.3.1) as,

$$t_{29}''^{G} = \lambda_{29}''\bar{u}_{ys}'' \exp\left(\alpha_{29}''\left(1-\frac{a_{29}''\bar{u}_{xs}''}{\left(\bar{X}_s''+(a_{29}''-1)\bar{u}_{xs}''\right)}\right)\right) \exp\left(\beta_{29}''\left(1-\frac{b_{29}''\bar{u}_{zs}''}{\left(\bar{Z}_s''+(b_{29}''-1)\bar{u}_{zs}''\right)}\right)\right),\ 0<\lambda_{29}''\le 1 \qquad (6.3.23)$$

where (a_{29}'', b_{29}'', λ_{29}'') are constants to be determined such that the mean square error (MSE) is minimum. (α_{29}'', β_{29}'') are known constants takes the value(0,1,-1) to produce different ratio-type and product-type estimators.

We follow the same routine along with the class of estimator in Table 9 for proposed estimator in (6.3.23), as that for case-I in Section 6.3.1. In addition, the relation between a_{29}'', α_{29}'', λ_{29}'' and b_{29}'', β_{29}'' in case-II is the same as that for case-I in Section 6.3.1.1. Finally, the same is true for the MSE and the Bias. It is therefore directly from Section 6.3.1, we may write $\text{Bias}(t_{29}''^{G})$ and $\text{MSE}(t_{29}''^{G})$ following the same, and we may also produce a class of estimators for similar choices of a_{29}'', α_{29}'', λ_{29}'' in case-II. The bias of (6.3.23) may be obtain by following the notations and expectations for case II presented in Section 6.1,

The bias of (6.3.23) may be obtain by following the notations and expectations for case II presented in Section 6.1 as,

$$Bias(t_{29}''^{G}) = \lambda_{29}''^{2}\,\bar{Y}_s''\Bigg[\frac{\alpha_{29}''^{2}}{a_{29}''^{2}}V_{200(u)}'' - \frac{\beta_{29}''}{b_{29}''}V_{002(u)}'' + \frac{\beta_{29}''^{2}}{b_{29}''^{2}}V_{002(u)}'' - \frac{\beta_{29}''}{b_{29}''}V_{101(u)}'' - \frac{\alpha_{29}''}{a_{29}''}V_{110(u)}'' + \frac{\alpha_{29}''\beta_{29}''}{a_{29}''b_{29}''}V_{011(u)}''\Bigg] + \bar{Y}_s''(\lambda_{29}''-1). \qquad (6.3.24)$$

Similarly the expression of MSE may also be given as,

$$MSE\left(t_{29}''^{G}\right) = \lambda_{29}''^{2}\bar{Y}_s''^{2}\Big[V_{020(u)} + z_{29}''^{2}V_{200(u)}'' + u_{29}''^{2}V_{002(u)}'' - 2z_{29}''V_{110(u)}''$$

$$-2u''_{29}V''_{011(u)} + 2z''_{29}u''_{29}V''_{101(u)}\Big] + \bar{Y}_s''^{\,2}(\lambda''_{29} - 1)^2, \tag{6.3.25}$$

By substituting the optimum values of z''_{29} and u''_{29}, we get $\lambda''^{\,opt}_{29}$ as,

$$\lambda''^{\,opt}_{29} = \frac{1}{1 + A''^{\,*}_{29}}$$

where

$$A''^{*}_{29} = \left(V''_{020(u)} - \frac{V''^{\,2}_{110(u)}V''_{002(u)} + V''^{\,2}_{011(u)}V''_{200(u)} - 2V''_{110(u)}V''_{101(u)}V''_{011(u)}}{V''_{200(u)}V''_{002(u)} - V''^{\,2}_{101(u)}} \right) \tag{6.3.26}$$

The minimum MSE may be obtained as,

$$MSE_{\min}\left(t''^{G}_{29}\right) = \bar{Y}_s''^{2}\left(\frac{A''^{\,*}_{29}}{1 + A''^{\,*}_{29}}\right). \tag{6.3.27}$$

Remark 6.5

i) For $\alpha''_{29} = 1, \beta''_{29} = 1$, we get exponential-ratio type estimators given in Table 6.2. The MSE of t''^{G}_{29} is expressed as,

$$MSE(t''^{j}_{29}) = \begin{cases} \bar{Y}_s''^{2}\left(\lambda''^{\,2}_{29}\begin{pmatrix} V''_{020(u)} + V''_{200(u)} + V''_{002(u)} \\ -2V''_{110(u)} - 2V''_{101(u)} + 2V''_{011(u)} \end{pmatrix} + (\lambda''_{29} - 1)^2\right) & j(\in G) = 1 \\ \bar{Y}_s''^{2}\left(\lambda''^{\,2}_{29}\begin{pmatrix} V''_{020(u)} + \dfrac{1}{a''^{\left(\frac{j-1}{2}\right)^2}_{29}}V''_{200(u)} + \dfrac{1}{b''^{\left(\frac{j-1}{2}\right)^2}_{29}}V''_{002(u)} \\ -2\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{29}}V''_{110(u)} - 2\dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{29}}V''_{011(u)} \\ +2\dfrac{1}{a''^{\left(\frac{j-1}{2}\right)}_{29}}\dfrac{1}{b''^{\left(\frac{j-1}{2}\right)}_{29}}V''_{101(u)} \end{pmatrix} + (\lambda''_{29} - 1)^2\right) & j(\in G) = 3,5,\ldots,11 \end{cases}$$

(6.3.28)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}''^{\left(\frac{j-1}{2}\right)}} = \frac{V_{110(u)}''V_{002(u)}'' - V_{011(u)}''V_{101(u)}''}{V_{200(u)}''V_{002(u)}'' - V_{101(u)}''^{2}}$$

and

$$\frac{1}{b_{29}''^{\left(\frac{j-1}{2}\right)}} = \frac{V_{200(u)}''V_{011(u)}'' - V_{110(u)}''V_{101(u)}''}{V_{200(u)}''V_{002(u)}'' - V_{101(u)}''^{2}}, \lambda_{29}'' = \frac{1}{1 + A_{29}''^{G}}$$

where

$$A_{29}''^{G} = \left[\begin{array}{l} V_{020(u)}'' + \frac{1}{a_{29}''^{\left(\frac{j-1}{2}\right)^2}}V_{200(u)}'' + \frac{1}{b_{29}''^{\left(\frac{j-1}{2}\right)^2}}V_{002(u)}'' - 2\frac{1}{a_{29}''^{\left(\frac{j-1}{2}\right)}}V_{110(u)}'' \\ -2\frac{1}{b_{29}''^{\left(\frac{j-1}{2}\right)}}V_{011(u)}'' + 2\frac{1}{a_{29}''^{\left(\frac{j-1}{2}\right)}}\frac{1}{b_{29}''^{\left(\frac{j-1}{2}\right)}}V_{101(u)}'' \end{array}\right]$$

ii) For $\alpha_{29}'' = -1, \beta_{29}'' = -1$, we get exponential-ratio product estimators given in Table 6.2. The MSE of $t_{29}''^{G}$ is expressed as,

$$MSE(t_{29}''^{k}) = \begin{cases} \bar{Y}_s''^{2}\left(\lambda_{29}''^{2}\left(\begin{array}{c} V_{020(u)}'' + V_{200(u)}'' + V_{002(u)}'' + 2V_{110(u)}'' \\ +2V_{011(u)}'' - 2V_{101(u)}'' \end{array}\right) + (\lambda_{29}'' - 1)^2\right) k(\in G) = 2 \\ \bar{Y}_s''^{2}\left(\lambda_{29}''^{2}\left(\begin{array}{l} V_{020(u)}'' + \frac{1}{a_{29}''^{\left(\frac{k}{2}\right)^2}}V_{200(u)}'' + \frac{1}{b_{29}''^{\left(\frac{k}{2}\right)^2}}V_{002(u)}'' \\ +2\frac{1}{a_{29}''^{\left(\frac{k}{2}\right)}}V_{110(u)}'' + 2\frac{1}{b_{29}''^{\left(\frac{k}{2}\right)}}V_{011(u)}'' \\ -2\frac{1}{a_{29}''^{\left(\frac{k}{2}\right)}}\frac{1}{b_{29}''^{\left(\frac{k}{2}\right)}}V_{101(u)}'' \end{array}\right) + (\lambda_{29}'' - 1)^2\right) k(\in G) = 4, 6, \ldots, 12 \end{cases}$$

(6.3.29)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}''^{\left(\frac{k}{2}\right)}} = \frac{V_{110(u)}''V_{002(u)}'' - V_{011(u)}''V_{101(u)}''}{V_{200(u)}''V_{002(u)}'' - V_{101(u)}''^{2}}$$

and

$$\frac{1}{b_{29}''^{\left(\frac{k}{2}\right)}} = \frac{V''_{200(u)}V''_{011(u)} - V''_{110(u)}V''_{101(u)}}{V''_{200(u)}V''_{002(u)} - V''^{\,2}_{101(u)}}, \lambda''_{29} = \frac{1}{1 + A''^{\,G}_{29}}$$

where

$$A''^{\,G}_{29} = \left[\begin{array}{l} V''_{020(u)} + \dfrac{1}{a_{29}''^{\left(\frac{k}{2}\right)^2}}V''_{200(u)} + \dfrac{1}{b_{29}''^{\left(\frac{k}{2}\right)^2}}V''_{002(u)} - 2\dfrac{1}{a_{29}''^{\left(\frac{k}{2}\right)}}V''_{110(u)} \\ \quad -2\dfrac{1}{b_{29}''^{\left(\frac{k}{2}\right)}}V''_{011(u)} + 2\dfrac{1}{a_{29}''^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{29}''^{\left(\frac{k}{2}\right)}}V''_{101(u)} \end{array}\right]$$

iii) For $\alpha''_{29} = -1, \beta''_{29} = 1$, we get exponential-product cum ratio type estimators given in Table 9. The MSE of $t_9''^{G}$ is expressed as,

$$MSE(t_{29}''^{l}) = \begin{cases} \bar{Y}_s''^2\left(\lambda''^{\,2}_{29}\begin{pmatrix} V''_{020(u)} + V''_{200(u)} + V''_{002(u)} \\ + 2V''_{110(u)} - 2V''_{011(u)} - 2V''_{101(u)} \end{pmatrix} + (\lambda''_{29}-1)^2\right) & l(\in G) = 13 \\ \bar{Y}_s''^2\left(\lambda''^{\,2}_{29}\begin{pmatrix} V''_{020(u)} + \dfrac{1}{a_{29}''^{\left(\frac{l-1}{2}\right)^2}}V''_{200(u)} + \dfrac{1}{b_{29}''^{\left(\frac{l-1}{2}\right)^2}}V''_{002(u)} \\ + 2\dfrac{1}{a_{29}''^{\left(\frac{l-1}{2}\right)}}V''_{110(u)} - 2\dfrac{1}{b_{29}''^{\left(\frac{l-1}{2}\right)}}V''_{011(u)} \\ - 2\dfrac{1}{a_{29}''^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{29}''^{\left(\frac{l-1}{2}\right)}}V''_{101(u)} \end{pmatrix} + (\lambda''_{29}-1)^2\right) & \\ & l(\in G) = 15,17,\ldots,23 \end{cases}$$

(6.3.30)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}''^{\left(\frac{l-1}{2}\right)}} = \frac{V''_{110(u)}V''_{002(u)} - V''_{011(u)}V''_{101(u)}}{V''_{200(u)}V''_{002(u)} - V''^{\,2}_{101(u)}}$$

and

$$\frac{1}{b''^{\left(\frac{l-1}{2}\right)}_{29}} = \frac{V''_{200(u)}V''_{011(u)} - V''_{110(u)}V''_{101(u)}}{V''_{200(u)}V''_{002(u)} - V''^{\,2}_{101(u)}}, \lambda''_{29} = \frac{1}{1 + A''^{G}_{29}}$$

where

$$A''^{G}_{29} = \left[\begin{array}{l} V''_{020(u)} + \dfrac{1}{a''^{\left(\frac{l-1}{2}\right)^2}_{29}}V''_{200(u)} + \dfrac{1}{b''^{\left(\frac{l-1}{2}\right)^2}_{29}}V''_{002(u)} - 2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{29}}V''_{110(u)} \\ \qquad -2\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{29}}V''_{011(u)} + 2\dfrac{1}{a''^{\left(\frac{l-1}{2}\right)}_{29}}\dfrac{1}{b''^{\left(\frac{l-1}{2}\right)}_{29}}V''_{101(u)} \end{array}\right]$$

iv) For $\alpha''_{29} = 1, \beta''_{29} = -1$, we get exponential-ratio cum product type estimators given in Table 6.2. The MSE of t^{G}_{29} is expressed as,

$$MSE(t''^{m}_{29}) = \begin{cases} \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{29}\begin{pmatrix} V''_{020(u)} + V''_{200(u)} + V''_{002(u)} \\ - 2V''_{110(u)} + 2V''_{011(u)} - 2V''_{101(u)} \end{pmatrix} + (\lambda''_{29} - 1)^2\right) & m(\in G) = 14 \\ \bar{Y}''^{2}_{s}\left(\lambda''^{2}_{29}\begin{pmatrix} V''_{020(u)} + \dfrac{1}{a''^{\left(\frac{k}{2}\right)^2}_{29}}V''_{200(u)} + \dfrac{1}{b''^{\left(\frac{k}{2}\right)^2}_{29}}V''_{002(u)} \\ - 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{29}}V''_{110(u)} + 2\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{29}}V''_{011(u)} \\ - 2\dfrac{1}{a''^{\left(\frac{k}{2}\right)}_{29}}\dfrac{1}{b''^{\left(\frac{k}{2}\right)}_{29}}V''_{101(u)} \end{pmatrix} + (\lambda''_{29} - 1)^2\right) & \\ & m(\in G) = 16,18,\ldots,24 \end{cases}$$

(6.3.31)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a''^{\left(\frac{m}{2}\right)}_{29}} = \frac{V''_{110(u)}V''_{002(u)} - V''_{011(u)}V''_{101(u)}}{V''_{200(u)}V''_{002(u)} - V''^{\,2}_{101(u)}}$$

and

$$\frac{1}{b''^{\left(\frac{m}{2}\right)}_{29}} = \frac{V''_{200(u)}V''_{011(u)} - V''_{110(u)}V''_{101(u)}}{V''_{200(u)}V''_{002(u)} - V''^{\,2}_{101(u)}}, \lambda''_{29} = \frac{1}{1 + A''^{G}_{29}}$$

where

$$A''^{\,G}_{29} = \left[\begin{array}{l} V_{020(u)} + \dfrac{1}{a''^{\left(\frac{m}{2}\right)^2}_{29}} V''_{200(u)} + \dfrac{1}{b''^{\left(\frac{m}{2}\right)^2}_{29}} V''_{002(u)} - 2\dfrac{1}{a''^{\left(\frac{m}{2}\right)}_{29}} V''_{110(u)} \\ +2\dfrac{1}{b''^{\left(\frac{m}{2}\right)(u)}_{29}} V''_{011(u)} - 2\dfrac{1}{a''^{\left(\frac{m}{2}\right)}_{29}} \dfrac{1}{b''^{\left(\frac{m}{2}\right)}_{29}} V''_{101(u)} \end{array}\right]$$

Case III

The generalized estimator under case II may be proposed following (6.3.1) as

$$t^{G}_{29} = \lambda_{29}\bar{u}_{ys}\exp\left(\alpha_{29}\left(1 - \frac{a_{29}\bar{u}_{xs}}{\left(\bar{X}_s + (a_{29}-1)\bar{u}_{xs}\right)}\right)\right)$$
$$\exp\left(\beta_{29}\left(1 - \frac{b_{29}\bar{u}_{zs}}{\left(\bar{Z}_s + (b_{29}-1)\bar{u}_{zs}\right)}\right)\right), \quad 0 < \lambda_{29} \le 1 \qquad (6.3.32)$$

where (a_{29}, b_{29}, λ_{29}) are constants to be determined such that the mean square error (MSE) is minimum. (α_{29}, β'_{29}) are known constants takes the value(0,1,-1) to produce different ratio-type and product-type estimators.
The proposed estimator in (6.3.32) follows the same routine along with the class of estimator in Table 9, as that for case-I in Section 6.3.1. In addition, the relation between the constants a_{29}, α_{29}, b_{29}, β_{29} and λ_{29} in case-III is the same as that for case-I in Section 6.3.1.1. Also, the same is true for the MSE and the bias. It is therefore directly from Section 4.3.3.1, we may write $\mathrm{Bias}\,(t^{G}_{29})$ and $\mathrm{MSE}(t^{G}_{29})$ following the same, and we may also produce a class of estimators for similar choices of a_{29}, α_{29}, b_{29}, β_{29} and λ_{29} in case-III. The bias of (6.3.18) may be obtain by following the notations and expectations for case III presented in Section 6.1,

$$Bias(t_{29}^{G}) = \lambda_{29}\bar{Y}_s\left[\frac{\alpha_{29}}{a_{29}^{2}}V_{200(u)} - \frac{\beta_{29}}{b_{29}}V_{002(u)} + \frac{\beta_{29}^{2}}{b_{29}^{2}}V_{002(u)} - \frac{\beta_{29}}{b_{29}}V_{101(u)} - \frac{\alpha_{29}}{a_{29}}V_{110(u)} + \frac{\alpha_{29}}{a_{29}}\frac{\beta_{29}}{b_{29}}V_{011(u)}\right] + \bar{Y}_s(\lambda_{29}-1), \quad (6.3.33)$$

Similarly The expression of MSE is also given as,

$$MSE\left(t_{29}^{G}\right) = \lambda_{29}\bar{Y}_s^{2}\left[V_{020(u)} + t_{29}^{2}V_{200(u)} + u_{29}^{2}V_{002(u)} - 2t_{29}V_{110(u)} - 2u_{29}V_{011(u)} + 2t_{29}u_{29}V_{101(u)}\right] + \bar{Y}_s^{2}(\lambda_{29}-1)^{2}\ . \quad (6.3.34)$$

By substituting the optimum values of t_{29} and u_{29}, we get λ_{29}^{opt} as,

$$\lambda_{29}^{opt} = \frac{1}{1+A_{29}^{*}} \text{ where } A_{29}^{*} = \left(V_{020(u)} - \frac{V_{110(u)}^{2}V_{002(u)} + V_{011(u)}^{2}V_{200(u)} - 2V_{110(u)}V_{101(u)}V_{011(u)}}{V_{200(u)}V_{002(u)} - V_{101(u)}^{2}}\right) \quad (6.3.35)$$

The minimum MSE may be obtained as,

$$MSE_{\min}\left(t_{29}^{G}\right) = \bar{Y}_s^{2}\left(\frac{A_{29}^{opt}}{1+A_{29}^{opt}}\right). \quad (6.3.36)$$

Remark 6.6

i) For $\alpha_{29} = 1, \beta_{29} = 1$, we get exponential-ratio type estimators given in Table 9. The MSE of $t_{29}'^{G}$ is expressed as,

$$
MSE(t_{29}^{j}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{29}^2 \begin{pmatrix} V_{020(u)} + V_{200(u)} + V_{002(u)} \\ -2V_{110(u)} - 2V_{011(u)} + 2V_{101(u)} \end{pmatrix} + (\lambda_{29} - 1)^2 \right) j(\in G) = 1 \\ \bar{Y}_s^2 \left(\lambda_{29}^2 \begin{pmatrix} V_{020(u)} + \frac{1}{a_{29}^{\left(\frac{j-1}{2}\right)^2}} V_{200(u)} + \frac{1}{b_{29}^{\left(\frac{j-1}{2}\right)^2}} V_{002(u)} \\ -2\frac{1}{a_{29}^{\left(\frac{j-1}{2}\right)}} V_{110(u)} - 2\frac{1}{b_{29}^{\left(\frac{j-1}{2}\right)}} V_{011(u)} \\ +2\frac{1}{a_{29}^{\left(\frac{j-1}{2}\right)}} \frac{1}{b_{29}^{\left(\frac{j-1}{2}\right)}} V_{101(u)} \end{pmatrix} + (\lambda_{29} - 1)^2 \right) j(\in G) = 3,5,\ldots,11 \end{cases}
$$

(6.3.37)

The optimal values which lead to minimum MSE as,

$$
\frac{1}{a_{29}^{\left(\frac{j-1}{2}\right)}} = \frac{V_{110(u)}V_{002(u)} - V_{011(u)}V_{101(u)}}{V_{200(u)}V_{002(u)} - V_{101(u)}^2}
$$

and

$$
\frac{1}{b_{29}^{\left(\frac{j-1}{2}\right)}} = \frac{V_{200(u)}V_{011(u)} - V_{110(u)}V_{101(u)}}{V_{200(u)}V_{002(u)} - V_{101(u)}^2}, \lambda_{29} = \frac{1}{1 + A_{29}^{G}}
$$

where

$$
A_{29}^{G} = \begin{bmatrix} V_{020(u)} + \frac{1}{a_{29}^{\left(\frac{j-1}{2}\right)^2}} V_{200(u)} + \frac{1}{b_{29}^{\left(\frac{j-1}{2}\right)^2}} V_{002(u)} \\ -2\frac{1}{a_{29}^{\left(\frac{j-1}{2}\right)}} V_{110(u} - 2\frac{1}{b_{29}^{\left(\frac{j-1}{2}\right)}} V_{011(u)} + 2\frac{1}{a_{29}^{\left(\frac{j-1}{2}\right)}} \frac{1}{b_{29}^{\left(\frac{j-1}{2}\right)}} V_{101(u)} \end{bmatrix}
$$

ii) For $\alpha_{29} = -1, \beta_{29} = -1$, we get exponential-product type estimators given in Table 6.2. The MSE of t_{29}^{G} is expressed as,

$$MSE\left(t_{29}^{k}\right)=\begin{cases}\bar{Y}_s^{2}\left(\lambda_{29}^{2}\begin{pmatrix}V_{02(u)}+V_{200(u)}+V_{002(u)}\\ +2V_{110(u)}+2V_{011(u)}-2V_{101(u)}\end{pmatrix}+(\lambda_{29}-1)^2\right) & k(\in G)=2\\ \\ \bar{Y}_s^{2}\left(\lambda_{29}^{2}\begin{pmatrix}V_{020(u)}+\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)^2}}V'_{200}+\dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)^2}}V'_{002}\\ +2\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)}}V_{110}+2\dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)}}V_{011(u)}\\ -2\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)}}V_{101(u)}\end{pmatrix}+(\lambda_{29}-1)^2\right) & k(\in G)=4,6,\ldots,12\end{cases}$$

(6.3.38)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}^{\left(\frac{k}{2}\right)}}=\frac{V_{110(u)}V_{002(u)}-V_{011(u)}V_{101(u)}}{V_{200(u)}V_{002(u)}-V_{101(u)}^{2}}$$

and

$$\frac{1}{b_{29}^{\left(\frac{k}{2}\right)}}=\frac{V_{200(u)}V_{011(u)}-V_{110(u)}V_{101(u)}}{V_{200(u)}V_{002(u)}-V_{101(u)}^{2}},\ \lambda_{29}=\frac{1}{1+A_{29}^{G}}$$

where

$$A_{29}^{G}=\begin{bmatrix}V_{020(u)}+\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)^2}}V_{200(u)}+\dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)^2}}V_{002(u)}\\ \\ -2\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)}}V_{110(u)}-2\dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)}}V_{011(u)}+2\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)}}\dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)}}V_{101(u)}\end{bmatrix}$$

iii) For $\alpha_{29}=-1,\beta_{29}=1$, we get exponential-product cum ratio type estimators given in Table 6.2. The MSE of t_{29}^{G} is expressed as,

$$MSE(t_{29}^{l})=\begin{cases}\bar{Y}_s^{2}\left(\lambda_{29}^{2}\begin{pmatrix}V_{020(u)}+V_{200(u)}+V_{002(u)}\\ +2V_{110(u)}-2V_{101(u)}-2V_{011(u)}\end{pmatrix}+(\lambda_{29}-1)^{2}\right) & l(\in G)=13\\[2ex] \bar{Y}_s^{2}\left(\lambda_{29}^{2}\begin{pmatrix}V_{020(u)}+\dfrac{1}{\left(a_{29}^{\left(\frac{l-1}{2}\right)}\right)^{2}}V_{200(u)}+\dfrac{1}{\left(b_{29}^{\left(\frac{l-1}{2}\right)}\right)^{2}}V_{002(u)}\\ +2\dfrac{1}{a_{29}^{\left(\frac{l-1}{2}\right)}}V_{110(u)}-2\dfrac{1}{b_{29}^{\left(\frac{l-1}{2}\right)}}V_{011(u)}\\ -2\dfrac{1}{a_{29}^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{29}^{\left(\frac{l-1}{2}\right)}}V_{101(u)}\end{pmatrix}+(\lambda_{29}-1)^{2}\right) & l(\in G)=15,17,\ldots,23\end{cases}$$

(6.3.39)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}^{\left(\frac{l-1}{2}\right)}}=\frac{V_{110(u)}V_{002(u)}-V_{011(u)}V_{101(u)}}{V_{200(u)}V_{002(u)}-V_{101(u)}^{2}}$$

and

$$\frac{1}{b_{29}^{\left(\frac{l-1}{2}\right)}}=\frac{V_{200(u)}V_{011(u)}-V_{110(u)}V_{101(u)}}{V_{200(u)}V_{002(u)}-V_{101(u)}^{2}},\lambda_{29}=\frac{1}{1+A_{29}^{G}}$$

where

$$A_{29}^{G}=\begin{bmatrix}V_{020(u)}+\dfrac{1}{\left(a_{29}^{\left(\frac{l-1}{2}\right)}\right)^{2}}V_{200(u)}+\dfrac{1}{\left(b_{29}^{\left(\frac{l-1}{2}\right)}\right)^{2}}V_{002(u)}\\ -2\dfrac{1}{a_{29}^{\left(\frac{l-1}{2}\right)}}V_{110(u)}-2\dfrac{1}{b_{29}^{\left(\frac{l-1}{2}\right)}}V_{011(u)}+2\dfrac{1}{a_{29}^{\left(\frac{l-1}{2}\right)}}\dfrac{1}{b_{29}^{\left(\frac{l-1}{2}\right)}}V_{101(u)}\end{bmatrix}$$

iv) For $\alpha_{29}=1,\beta_{29}=-1$, we get exponential-ratio cum product type estimators given in Table 6.2. The MSE of t_{29}^{G} is expressed as,

$$MSE(t_{29}^{m}) = \begin{cases} \bar{Y}_s^2 \left(\lambda_{29}^2 \begin{pmatrix} V_{020(u)} + V_{200(u)} + V_{002(u)} \\ -2V_{110(u)} + 2V_{101(u)} - 2V_{011(u)} \end{pmatrix} + (\lambda_{29} - 1)^2 \right) & m(\in G) = 14 \\ \bar{Y}_s^2 \left(\lambda_{29}^2 \begin{pmatrix} V_{020(u)} + \dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)^2}} V_{200(u)} + \dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)^2}} V_{002(u)} \\ -2\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)}} V_{110(u)} + 2\dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)}} V_{011(u)} \\ -2\dfrac{1}{a_{29}^{\left(\frac{k}{2}\right)}} \dfrac{1}{b_{29}^{\left(\frac{k}{2}\right)}} V_{101(u)} \end{pmatrix} + (\lambda_{29} - 1)^2 \right) & m(\in G) = 16,18,\ldots,24 \end{cases}$$

(6.3.40)

The optimal values which lead to minimum MSE as,

$$\frac{1}{a_{29}^{\left(\frac{m}{2}\right)}} = \frac{V_{110(u)}V_{002(u)} - V_{011(u)}V_{101(u)}}{V_{200(u)}V_{002(u)} - V_{101(u)}^2}$$

and

$$\frac{1}{b_{29}^{\left(\frac{m}{2}\right)}} = \frac{V_{200(u)}V_{011(u)} - V_{110(u)}V_{101(u)}}{V_{200(u)}V_{002(u)} - V_{101(u)}^2}, \lambda_{29} = \frac{1}{1 + A_{29}^{G}}$$

where

$$A_{29}^{G} = \begin{bmatrix} V_{020(u)} + \dfrac{1}{a_{29}^{\left(\frac{m}{2}\right)^2}} V_{200(u)} + \dfrac{1}{b_{29}^{\left(\frac{m}{2}\right)^2}} V_{002(u)} - 2\dfrac{1}{a_{29}^{\left(\frac{m}{2}\right)}} V_{110(u)} \\ +2\dfrac{1}{b_{29}^{\left(\frac{m}{2}\right)}} V_{011(u)} - 2\dfrac{1}{a_{29}^{\left(\frac{m}{2}\right)}} \dfrac{1}{b_{29}^{\left(\frac{m}{2}\right)}} V_{101(u)} \end{bmatrix}$$

6.4 EFFICIENCY COMPARISON

6.4.1 Efficiency Comparison for Single Auxiliary Variable

To compare the efficiency among the generalized exponential estimator t_{28}^{G} with the other estimators of the class of proposed generalized estimator, we use the following

$$L' = V'_{020(u)} + \left(\frac{\alpha'_{28}}{a'_{28}}\right)^2 V'_{200(u)} - 2\left(\frac{\alpha'_{28}}{a'_{28}}\right) V'_{110(\mathrm{u})},$$

$$M' = V'_{020(u)} + V'_{200(u)} - 2V'_{110(u)},\ N' = V'_{020(u)} + \frac{1}{4}V'_{200(u)} - V'_{110(u)}$$

$$P' = V'_{020(u)} + \frac{1}{{a'_{28}}^2}V'_{200(u)} - \frac{2V'_{110(u)}}{a'_{28}},\ L'^{opt} = V'_{020(u)} - \frac{V'^2_{110(\mathrm{u})}}{V'_{200(u)}}$$

6.4.1.1 When a' is known and λ' is unknown

The efficiency conditions may be written as:

i) $MSE(t'^{G}_{28}) - MSE(\bar{y}'_s) \le 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1 + L')\left(1 - V'_{020(u)}\right)\right)}}{(1 + L')}\right) \le \lambda' \le \max\left(\frac{1 \pm \sqrt{\left(1 - (1 + L')\left(1 - V'_{020(u)}\right)\right)}}{(1 + L')}\right) \tag{6.4.1}$$

ii) $MSE(t'^{G}_{28}) - MSE(t'^{1}_{28}) \le 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1 + L')(1 - N')\right)}}{(1 + L')}\right) \le \lambda' \le \max\left(\frac{1 \pm \sqrt{\left(1 - (1 + L')(1 - N')\right)}}{(1 + L')}\right) \tag{6.4.2}$$

iii) $MSE(t'^{G}_{28}) - MSE(t'^{3}_{28}) \le 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1 + L')(1 - M')\right)}}{(1 + L')}\right) \le \lambda' \le \max\left(\frac{1 \pm \sqrt{\left(1 - (1 + L')(1 - M')\right)}}{(1 + L')}\right) \tag{6.4.3}$$

iv) $MSE(t'^{G}_{28}) - MSE(t'^{5}_{28}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1+L')(1-P')\right)}}{(1+L')}\right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1 - (1+L')(1-P')\right)}}{(1+L')}\right) \quad (6.4.4)$$

v) $MSE(t'^{G}_{28}) - \min MSE(t'^{11}_{28}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1+L')\left(\frac{1}{\left(1+L'^{opt}\right)}\right)\right)}}{(1+L')}\right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1 - (1+L')\left(\frac{1}{\left(1+L'^{opt}\right)}\right)\right)}}{(1+L')}\right) \quad (6.4.5$$

vi) $MSE(t'^{G}_{28}) - \min MSE(t'^{5}_{28}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1+L')\left(\frac{1}{(1+M')}\right)\right)}}{(1+L')}\right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1 - (1+L')\left(\frac{1}{(1+M')}\right)\right)}}{(1+L')}\right) \quad (6.4.6)$$

vii) $MSE(t'^{G}_{28}) - \min MSE(t'^{9}_{28}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1-(1+L')\left(\frac{1}{(1+N')}\right)\right)}}{(1+L')}\right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1-(1+L')\left(\frac{1}{(1+N')}\right)\right)}}{(1+L')}\right) \qquad (6.4.7)$$

6.4.1.2 When a' is unknown and λ' is known

i. $MSE(t'^{\ G}_{28}) - MSE(\overline{y}'_s) \leq 0$ If

$$\min\left(\frac{V'_{110(u)} \pm \sqrt{\left(V'^{\ 2}_{110(u)} - V'_{200(u)}\left(1-\frac{1}{\lambda'^2}\right)V'_{020(u)}\left(1-\frac{1}{\lambda'}\right)^2\right)}}{V'_{200(u)}}\right) \leq \frac{\alpha'_{28}}{a'_{28}} \leq \max\left(\frac{V'_{110(u)} \pm \sqrt{\left(V'^{\ 2}_{110(u)} - V'_{200(u)}\left(1-\frac{1}{\lambda'^2}\right)V'_{020(u)}\left(1-\frac{1}{\lambda'}\right)^2\right)}}{V'_{200(u)}}\right)$$

(6.4.8)

ii) $\quad MSE(t'^{G}_{28}) - \min MSE(t'^{1}_{28}) \le 0$ If

$$\min\left\{\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left(\left(1-\frac{1}{\lambda'}\right)^{2} + V'_{020(u)} - \frac{N'}{\lambda'^{2}}\right)\right)}}{V'_{200(u)}}\right\}$$

$$\le \frac{\alpha'_{28}}{a'_{28}} \le \max\left\{\frac{V'_{11} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left(\left(1-\frac{1}{\lambda'}\right)^{2} + V'_{020(u)} - \frac{N'}{\lambda'^{2}}\right)\right)}}{V'_{200(u)}}\right\}$$

(6.4.9)

iii) $\quad MSE(t'^{G}_{28}) - \min MSE(t'^{9}_{28}) \le 0$ If

$$\min\left\{\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left((\lambda'-1)^{2} + \lambda'^{2}V'_{020(u)} - \frac{N'}{1+N'}\right)\right)}}{V'_{200(u)}}\right\}$$

$$\le \frac{\alpha'_{28}}{a'_{28}} \le \max\left\{\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left((\lambda'-1)^{2} + \lambda'^{2}V'_{020(u)} - \frac{N'}{1+N'}\right)\right)}}{V'_{200(u)}}\right\}$$

(6.4.10)

6.4.2 Efficiency Comparison for Two Auxiliary Variable

To compare the efficiency among the generalized exponential estimator t^{G}_{29} with the other estimators of the class of proposed generalized estimator, we use the following

$$A' = V'_{020(u)} + \left(\frac{\alpha'_{28}}{a'_{28}}\right)^2 V'_{200(u)} + \left(\frac{\beta'_{29}}{b'_{29}}\right)^2 V'_{002(u)}$$

$$-2\left(\frac{\alpha'_{28}}{a'_{28}}\right)V'_{110(u)} - 2\left(\frac{\beta'_{29}}{b'_{29}}\right)V'_{011(u)} + 2\left(\frac{\alpha'_{28}}{a'_{28}}\right)\left(\frac{\beta'_{29}}{b'_{29}}\right)V'_{101(u)},$$

$$C' = V'_{020(u)} + V'_{200(u)} - 2V'_{110(u)} - 2V'_{011(u)} + 2V'_{101(u)},$$

$$D' = V'_{020(u)} + \frac{1}{4}V'_{200(u)} - V'_{110(u)} - V'_{011(u)} + \frac{1}{2}V'_{101(u)}$$

$$S' = V'_{020(u)} + \frac{1}{a'^{\,2}_{28}}V'_{200(u)} + \frac{1}{b'^{\,2}_{28}}V'_{002(u)} - \frac{2V'_{110(u)}}{a'_{28}} - \frac{2V'_{011u)}}{b'_{28}} + \frac{2V'_{101u)}}{a'_{28}b'_{28}},$$

$$A'^{opt} = \left(V'_{020(u)} - \frac{V'^{\,2}_{110(u)}V'_{002(u)} + V'^{\,2}_{011(u)}V'_{200(u)} - 2V'_{110(u)}V'_{101(u)}V'_{011(u)}}{V'_{200(u)}V'_{002(u)} - V'^{\,2}_{101(u)}}\right)$$

6.4.2.1 When a' is known and λ' is unknown

The efficiency conditions may be written as:

i. $MSE(t'^{\,G}_{29}) - MSE(\overline{y}'_s) \le 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1 + A')\left(1 - V'_{020(u)}\right)\right)}}{(1 + A')}\right) \le \lambda' \le \max\left(\frac{1 \pm \sqrt{\left(1 - (1 + A')\left(1 - V'_{020(u)}\right)\right)}}{(1 + A')}\right) \qquad (6.4.11)$$

ii) $MSE(t'^{G}_{29}) - MSE(t'^{1}_{29}) \le 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1 + A')(1 - D')\right)}}{(1 + A')}\right) \le \lambda' \le \max\left(\frac{1 \pm \sqrt{\left(1 - (1 + A')(1 - D')\right)}}{(1 + A')}\right) \qquad (6.4.12)$$

iii) $MSE(t'^{G}_{29}) - MSE(t'^{3}_{29}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1-(1+A')(1-C')\right)}}{(1+A')}\right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1-(1+A')(1-C')\right)}}{(1+A')}\right) \tag{6.4.13}$$

iv $MSE(t'^{G}_{29}) - MSE(t'^{5}_{29}) \leq 0$ if

$$\min\left(\frac{1 \pm \sqrt{\left(1-(1+A')(1-S')\right)}}{(1+A')}\right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1-(1+A')(1-S')\right)}}{(1+A')}\right) \tag{6.4.14}$$

v) $MSE(t'^{G}_{29}) - \min MSE(t'^{11}_{29}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1-(1+A')\left(\frac{1}{\left(1+A'^{opt}\right)}\right)\right)}}{(1+A')}\right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1-(1+A')\left(\frac{1}{\left(1+A'^{opt}\right)}\right)\right)}}{(1+A')}\right) \tag{6.4.15}$$

vi) $MSE(t'^{G}_{29}) - \min MSE(t'^{5}_{29}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1+A')\left(\frac{1}{(1+C')}\right)\right)}}{(1+A')} \right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1 - (1+A')\left(\frac{1}{(1+C')}\right)\right)}}{(1+A')} \right) \tag{6.4.16}$$

vii) $MSE(t'^{G}_{29}) - \min MSE(t'^{9}_{29}) \leq 0$ If

$$\min\left(\frac{1 \pm \sqrt{\left(1 - (1+A')\left(\frac{1}{(1+D')}\right)\right)}}{(1+A')} \right) \leq \lambda' \leq \max\left(\frac{1 \pm \sqrt{\left(1 - (1+A')\left(\frac{1}{(1+D')}\right)\right)}}{(1+A')} \right) \tag{6.4.17}$$

6.4.2.2 When a' is Unknown and λ' is Known

i) $MSE(t_{29}'^{G}) - MSE(\bar{y}_s') \leq 0$ If

$$\min\left(\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left(1 - \frac{1}{\lambda'^2}\right) V'_{020(u)}\left(1 - \frac{1}{\lambda'}\right)^2 \right)}}{V'_{200(u)}} \right)$$

$$\leq \frac{\alpha'_{29}}{a'_{29}} \leq \max\left(\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left(1 - \frac{1}{\lambda'^2}\right) V'_{020(u)}\left(1 - \frac{1}{\lambda'}\right)^2 \right)}}{V'_{200(u)}} \right) \tag{6.4.18}$$

ii) $MSE(t_{29}'^{G}) - \min MSE(t_{29}'^{1}) \leq 0$ If

$$\min\left(\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left(\left(1 - \frac{1}{\lambda'}\right)^2 + V'_{020(u)} - \frac{D'}{\lambda'^2}\right)\right)}}{V'_{200(u)}} \right)$$

$$\leq \frac{\alpha'_{29}}{a'_{29}} \leq \max\left(\frac{V'_{11} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left(\left(1 - \frac{1}{\lambda'}\right)^2 + V'_{020(u)} - \frac{D'}{\lambda'^2}\right)\right)}}{V'_{200(u)}} \right) \tag{6.4.19}$$

iii) $MSE(t_{28}'^{G}) - \min MSE(t_{28}'^{9}) \le 0$ If

$$\min\left(\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left((\lambda'-1)^2 + \lambda'^2 V'_{020(u)} - \frac{D'}{1+D'}\right)\right)}}{V'_{200(u)}}\right)$$

$$\le \frac{\alpha'_{28}}{a'_{28}} \le \max\left(\frac{V'_{110(u)} \pm \sqrt{\left(V'^{2}_{110(u)} - V'_{200(u)}\left((\lambda'-1)^2 + \lambda'^2 V'_{020(u)} - \frac{D'}{1+D'}\right)\right)}}{V'_{200(u)}}\right) \tag{6.4.20}$$

CHAPTER 7

EMPIRICAL STUDY

7.1 INTRODUCTION

The performance of the estimators developed under two- stage single phase sampling d and two-stage two-phase sampling has been demonstrated here by conducting an empirical study for the proposed estimators (chapter 4-chapter 6) in this chapter.

The proposed generalized estimators have been compared with usual two stage estimator and the class of proposed generalized estimators using two populations, Population –I has first stage units of unequal sizes and Population –II consists of first stage units are of equal size (See Appendix A).

The following three cases have been discussed for the both population I & population II.

Case-I: when first stage units are unequal and weighted mean is used.
Case-II: when first stage units are unequal and un-weighted mean is used.
Case-III: when first stage units are of equal size.

To illustrate the efficiency of the proposed estimators over there class of estimators an empirical study has been done. We have used Population-I, for the estimators proposed and its class of estimators under Case I and Case II. The Population-II has been utilized to illustrate the performance of the proposed estimators along with its class of estimators developed under Case-III. Moreover as the correlation among the study variable and auxiliary information is high so only ratio type estimators are considered to see their performance.

The percentage relative efficiencies (PRE) have been computed using the following formula:

$$PRE = \frac{Var(\overline{y})}{MSE(t_i^G)} \times 100, \quad i = 1, 2, \ldots, 29. \qquad (7.1.1)$$

where $MSE(t_i^G)$ is the mean square error (MSE) of the class of the estimators.

7.2 NUMERICAL COMPARISON UNDER TWO-STAGE SINGLE-PHASE SAMPLING DESIGN

In this section, the estimator proposed ,under two- stage sampling using single and two auxiliary variables (see chapter4) has been compared to their class of estimators which are also discussed in chapter 4 and the percentage relative efficiencies has been computed and are given in Appendix B1 ,B3 and B2,B4 respectively.

7.3 NUMERICAL COMPARISON UNDER TWO-PHASE WITHIN TWO STAGE SAMPLING

In this section, the efficiency of the proposed estimators has been done in terms of MSE and PRE under two phase within two stage sampling in the following sub-sections(7.3.1-7.3.3) for three different situations regarding the availability of the auxiliary information for for equal and unequal size of the clusters considering weighted and un-weighted mean for first stage units (see section-5.1)

7.3.1 Numerical Results for Situation I

The results are presented in Tables B9 and Table B10 respectively under Situation I for equal and unequal size of the clusters considering weighted and un-weighted mean for first stage units.

7.3.2 Numerical Results for Situation II

The MSE and PRE's computed for the estimators under Situation II are presented in Tables B11 and Table B12 respectively for all three cases under Situation II.

7.3.2 Numerical Results for Situation III

The MSE's and PRE's computed under Situation III for the estimators proposed in chapter 5 are presented in Tables B13 and Table B14 respectively.

CHAPTER 8

CONCLUSION

We have proposed some generalized ratio and exponential estimators under two-stage single phase sampling design by making use of the information of single and two auxiliary variables. The generalized exponential estimators have been developed in two-stage sampling two-phase sampling. Also the generalized exponential estimators under two-stage adaptive cluster sampling have been fashioned using information single and two auxiliary variables.

The MSE's of the estimators proposed in (4.2.5), (4.2.22), (4.2.27) has been compared with the MSE's of their class of estimators and usual two-stage estimator. From Table B1, we observe that our proposed generalized exponential estimator is efficient as compare to its class of estimators and usual two stage estimator. The PRE values are also calculated in Table B2, which also shows that among the proposed class of estimator t_{20}^{G} perform efficiently. Furthermore the performance of the proposed generalized estimator at optimum value of the constants t_{20}^{opt} is the same as t_{20}^{11} for all three cases discussed in chapter 4. We may conclude that the proposed estimator t_{20}^{G} perform well in case III as compare to case I & II.

The estimator given in (4.2.33), (4.2.48) and (4.2.55) are compared with their class of estimators as well as with usual two-stage estimator in terms of MSE and PRE values in Table B3-B4. The results demonstrate that the generalized estimator t_{21}^{G} performs better than its proposed class of estimator in Case III. Among case I and case II , the estimator performs better for Case II.

The MSE's and PRE's of the estimators proposed in (4.3.5), (4.3.23) and (4.3.32) and their proposed class of the estimators has been computed in Table B5 and Table B6 respectively. We may see that the MSE of the estimator t_{22}^{G} is minimum among the proposed class of the estimator for all three cases. The estimator is efficient for case II.

The estimators produced using the information of two auxiliary variables are given in (4.3.40), (4.3.57) and (4.3.66). The MSE and PRE values of the class of the proposed estimators have been shown in Table B7 and Table

B8. From Table B7 we observe that our proposed generalized estimator has less MSE at its optimum values as compare to other members of the same class. Similarly the PRE (=279%) when we have equal fsu's. the performance of our proposed estimator is good when unweighted mean is used for unequal fsu's.

We have introduced separate-type estimator in (4.4.3),(4.4.33) and (4.4.47), and are compared with their class of estimators in terms of MSE and PRE value. The results are presented in Table B9 and Table B10. From Table B9, we may observe that the MSE for the proposed generalized estimator provide us less MSE as compare to its class of estiamtors.

The PRE value(=281%) for proposed generalized estimator when we have equal fsu's. The PRE value(=246%) for proposed generalized estimator when unweighted mean is used for unequal fsu's which is better than the unequal fsu's when weighted mean is used.

We have introduced the concept of two-stage two-phase sampling by making use of three situations as full-information (FI), partial-information (PI) and no-information (NI).

The performance of the exponential estimators proposed in (5.2.3), (5.2.22), (5.2.24) using the information of two auxiliary variables has been expressed in form of their MSE's and PRE's values in Table B9 and Table B10 respectively for situation I for all three cases. From table B5 we may see that the minimum MSE suggested generalized estimator t_5^G that is equal to the MSE of t_5^{11}. The PRE's demonstrate the same result as the PRE of the suggested estimator is high among the class of proposed estimator and t_5^{11} is the same efficient as t_5^G for all three cases. The PRE value when we have equal size of first stage units (=216%) which is higher than both the cases when we have a population with unequal fsu's. When we assign weights to the second stage units i.e. $\eta_i = \frac{M_i}{\bar{M}}$ we may also observe that the PRE value for Case I (=207%) higher than the caseII (=196%) so it may be concluded that for this particular population the use of two phase sampling design under situation I at the second stage of the sampling design will be better to be used either for equal fsu's or by using weighted mean in case of unequal fsu's.

The estimators mentioned in (5.2.45), (5.2.62) and (5.2.77) have been compared to their class of estimators respectively under situation II. The MSE's and PRE's are given in Table B11 and Table B12. We may observe from Table B9 that the suggested estimator has minimum MSE among the class of estimator and maximum PRE value for all three cases. We also may conclude from Table B10 that the generalized estimator is equally efficient as t_6^{11} and we get PRE (=190%) maximum for equal sizes of the first stage units as compare to the unequal first stage units, for case II PRE (=156%) which shows that if we have population of unequal fsu's, it will be better to use weighted mean either giving equal weights to all unequal fsu's.

The performance of the estimators (5.2.106), (5.2.114) and (5.2.129) has been illustrated through the comparison of the MSE and PRE values with class of proposed estimators and usual two-stage estimator. The MSE and PRE's are given in Table B11 and Table B12 respectively. We may observe from Table B11 that the MSE of the suggested estimator t_7^G is less than the MSE's of all other class of estimator and it is equal to t_7^{11}. The PRE also shows that suggested estimator perform more efficiently than the class of estimator. The PRE of the suggested estimator (=197%) for case II (when first stage units are unequal and un-weighted mean is used) are highest among all three cases under Situation III. The PRE when equal fsu's are taken (=150%) is higher than Case I (when first stage units are unequal and weighted mean is used). The above results shows that under situation III , when first stage units are unequal and un-weighted mean is used perform more efficiently.

At the end, we may conclude that proposed generalized ratio and exponential estimators under two-stage single phase sampling, two-stage sampling two-phase sampling design performs efficiently under all three cases (mentioned in chapter 4). So these estimators can be used for practical purposes in practical life.

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APPENDIX-A

Table A1
Data Statistics for Population-I and Population-II

Cluster	Population-I (equal fsu's)				Population-II (unequal fsu's)			
	1	**2**	**3**	**4**	**1**	**2**	**3**	**4**
M_i	16	16	16	16	18	14	12	20
m_i	8	8	8	8	9	7	6	10
$\overline{Y}_i$	26.20625	24.12313	26.68875	22.11438	25.77722	22.79286	28.43500	23.0905
$\overline{X}_i$	50.96019	50.35994	62.70413	55.75731	51.06389	46.49700	67.00217	57.11855
$\overline{Z}_i$	35.71519	41.85756	39.68550	48.71470	35.84517	39.86467	39.86467	48.95286
$C_{y_i}^2$	0.62364	0.33905	0.32637	0.36886	0.58025	0.39297	0.34783	0.31545
$C_{x_i}^2$	0.47888	0.28038	0.38836	0.49081	0.43322	0.29984	0.41947	0.40689
$C_{z_i}^2$	0.53798	0.24680	0.28155	0.10532	0.29194	0.28803	0.28366	0.15534
ρ_{i1}	0.88451	0.85254	0.84212	0.80242	0.88373	0.83895	0.82425	0.82113
ρ_{i2}	0.79978	0.71317	0.87276	0.79080	0.79943	0.67443	0.90076	0.80311

APPENDIX-B

Table B1

***MSEs* of the Class of t_{20}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		**MSE using Population-II (unequal fsu's)**	
		$\eta_i = 1$ **(CASE-II)**	$\eta_i = \frac{M_i}{\bar{M}}$ **(CASE-I)**
$\bar{y}_s$	9.817826	10.22425	9.214122
t_{20}^{1}	4.152578	4.328521	4.578562
t_{20}^{3}	4.399081	4.669515	4.72486
t_{20}^{5}	3.531721	3.710106	3.98566
t_{20}^{7}	4.125222	4.298806	4.653287
t_{20}^{9}	4.368393	4.634952	4.688791
t_{20}^{11}	3.511914	3.688253	4.025188
t_{20}^{opt}	3.511914	3.688253	4.025188

Table B2

***PREs* of the Class of t_{20}^{G} for Population I and Population II for Equal and Unequal First Stage Units**

MSE using Population-I (equal fsu's) (CASE-III)		**MSE using Population-II (unequal fsu's)**	
		$\eta_i = 1$ **(CASE-II)**	$\eta_i = \frac{M_i}{\bar{M}}$ **(CASE-I)**
$\bar{y}_s$	100	100	100
t_{20}^{1}	236.4273	236.2065	201.2449
t_{20}^{3}	223.179	218.9574	195.0137
t_{20}^{5}	277.9899	275.5784	227.4114
t_{20}^{7}	237.9951	237.8393	198.0132
t_{20}^{9}	224.7469	220.5902	196.5138
t_{20}^{11}	279.5577	277.2112	228.9116
t_{20}^{opt}	279.5577	277.2112	228.9116

Table B3

***MSEs* of the Class of t_{21}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ (CASE-II)	$\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	9.817826	10.22425	9.214122
t_{21}^{1}	4.399081	4.669515	4.72486
t_{21}^{3}	4.329363	4.594165	4.668019
t_{21}^{5}	4.372123	4.640388	4.703417
t_{21}^{7}	4.003967	4.241189	4.409587
t_{21}^{9}	3.60534	3.80107	4.513339
t_{21}^{11}	4.305518	4.568376	4.650162
t_{21}^{13}	4.132491	4.380916	4.650162
t_{21}^{15}	4.397379	4.669139	4.724201
t_{21}^{17}	4.394168	4.667079	4.724382
t_{21}^{19}	4.380089	4.660068	4.723077
t_{21}^{21}	4.029842	4.26936	4.440704
t_{21}^{23}	4.369764	4.637839	4.694795
t_{21}^{25}	4.2704	4.530377	4.622165
t_{21}^{27}	4.39701	4.667278	4.724441
t_{21}^{opt}	3.531721	3.688253	3.985495

Table B4

***PREs* of the Class of t_{21}^{G} for Population I and Population II for Equal and Unequal First Stage Units**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's) $\eta_i = 1$ (CASE-II)	MSE using Population-II (unequal fsu's) $\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	100	100	
t_{21}^{1}	67.17357	67.29165	62.98315
t_{21}^{3}	69.31618	69.46606	64.81901
t_{21}^{5}	68.05673	68.18668	63.75667
t_{21}^{7}	82.38184	82.76132	75.54396
t_{21}^{9}	112.6066	113.8529	184.3926
t_{21}^{11}	70.10892	70.27073	65.48817
t_{21}^{13}	76.07729	76.34427	65.48817
t_{21}^{15}	67.19271	67.31097	63.01192
t_{21}^{17}	67.2368	67.35584	62.99796
t_{21}^{19}	67.49213	67.61401	63.05025
t_{21}^{21}	86.445	86.81885	77.39186
t_{21}^{23}	67.90199	68.03278	63.74145
t_{21}^{25}	71.62743	71.80838	66.79081
t_{21}^{27}	67.25246	67.37142	63.00016
t_{21}^{opt}	279.5578	277.2125	231.1914

Table B5

***MSEs* of the Class of t_{22}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ (CASE-II)	$\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	9.817826	10.22425	9.214122
t_{22}^{1}	6.957403	6.770836	7.338497
t_{22}^{3}	14.61561	14.78759	14.62934
t_{22}^{5}	3.414652	3.541619	3.985658
t_{22}^{7}	6.880951	6.698408	7.251852
t_{22}^{9}	14.28226	14.44643	14.289
t_{22}^{11}	3.511914	3.688253	3.959961
t_{22}^{opt}	3.511914	3.688253	3.959961

Table B6

***PREs* of the Class of t_{22}^{G} for Population I and Population II for Equal and Unequal First Stage Units.**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ (CASE-II)	$\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	100	100	100
t_{22}^{1}	141.1134	151.0042	125.5587
t_{22}^{3}	67.17356	69.14075	62.98385
t_{22}^{5}	287.5205	288.6886	231.182
t_{22}^{7}	142.6812	152.637	127.0589
t_{22}^{9}	68.7414	70.77354	64.48402
t_{22}^{11}	279.5577	277.2112	232.6821
t_{22}^{opt}	279.5577	277.2112	232.6821

Table B7

***MSEs* of the Class of t_{23}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's) $\eta_i = 1$ (CASE-II)	MSE using Population-II (unequal fsu's) $\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	9.817826	10.22425	9.214122
t_{23}^{1}	14.6156142	15.19401	14.6295
t_{23}^{3}	14.16383663	14.71841	14.21515
t_{23}^{5}	14.42594931	14.99457	14.45201
t_{23}^{7}	11.91746809	12.35396	12.19703
t_{23}^{9}	8.718698191	8.980271	4.997013
t_{23}^{11}	14.00368261	14.54987	14.0699
t_{23}^{13}	12.90507396	13.39236	14.0699
t_{23}^{15}	14.61145063	15.18965	14.62282
t_{23}^{17}	14.60186961	15.17953	14.62606
t_{23}^{19}	14.54662898	15.12157	14.61393
t_{23}^{21}	11.357314	11.77659	11.9058
t_{23}^{23}	14.4588261	15.02849	14.45546
t_{23}^{25}	13.70680203	14.23831	13.79549
t_{23}^{27}	14.59846947	15.17602	14.62555
t_{23}^{opt}	3.511914	3.688253	3.985495

Table B8

***PREs* of the Class of t_{23}^{G} for Population I and Population II for Equal and Unequal First Stage Units.**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's) $\eta_i = 1$ (CASE-II)	MSE using Population-II (unequal fsu's) $\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	100	100	100
t_{23}^{1}	67.17357	67.29165	62.98315
t_{23}^{3}	69.31618	69.46606	64.81901
t_{23}^{5}	68.05673	68.18668	63.75667
t_{23}^{7}	82.38184	82.76132	75.54396
t_{23}^{9}	112.6066	113.8529	184.3926
t_{23}^{11}	70.10892	70.27073	65.48817
t_{23}^{13}	76.07729	76.34427	65.48817
t_{23}^{15}	67.19271	67.31097	63.01192
t_{23}^{17}	67.2368	67.35584	62.99796
t_{23}^{19}	67.49213	67.61401	63.05025
t_{23}^{21}	86.445	86.81885	77.39186
t_{23}^{23}	67.90199	68.03278	63.74145
t_{23}^{25}	71.62743	71.80838	66.79081
t_{23}^{27}	67.25246	67.37142	63.00016
t_{23}^{opt}	279.5578	277.2125	231.1914

Table B9

***MSEs* of the Class of t_{24}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\alpha_i = 1$ (CASE-II)	$\alpha_i = \frac{M_i}{\bar{M}}$ (CASE-I)
t_{24}^{0}	9.21412	10.22425	13.89066
t_{24}^{1}	3.52483	4.57177	8.30537
t_{24}^{3}	3.445658	4.453917	8.303117
t_{24}^{5}	**3.399092**	4.469444	8.317577
t_{24}^{7}	3.4032	**4.4273914**	**8.191568**
t_{24}^{9}	3.461715	4.396992	8.303359
t_{24}^{11}	**3.392318**	4.53724	8.297662
t_{24}^{13}	**3.39339**	4.466713	8.22723
t_{24}^{15}	3.403285	4.570755	8.304419
t_{24}^{17}	3.403445	4.570942	8.304183
t_{24}^{19}	3.402799	4.568796	8.320907
t_{24}^{21}	3.558739	**4.473296**	**8.320907**
t_{24}^{23}	**3.399232**	4.559929	8.24252
t_{24}^{25}	**3.388666**	4.52186	8.264454
t_{24}^{27}	3.40345	4.571116	8.323517
t_{24}^{opt}	**3.2806**	**4.15116**	**8.034922**

Table B10

PRE's of the Estimator w.r.t. t_{24}^{G} for Population-I and Population-II

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\alpha_i = 1$ **(CASE-II)**	$\alpha_i = \frac{M_i}{\overline{M}}$ **(CASE-I)**
t_{24}^{0}	100	100	100
t_{24}^{1}	261.4061	223.6388	167.2491
t_{24}^{3}	267.4125	229.5564	167.2945
t_{24}^{5}	**271.0759**	228.7589	167.0037
t_{24}^{7}	270.7487	**230.9317**	**169.5727**
t_{24}^{9}	266.1721	**232.5283**	167.2896
t_{24}^{11}	**271.6172**	225.3407	167.4045
t_{24}^{13}	**271.5314**	**228.8987**	**168.8376**
t_{24}^{15}	270.7419	223.6884	167.2683
t_{24}^{17}	270.7292	223.6793	167.273
t_{24}^{19}	270.7806	223.7843	166.9368
t_{24}^{21}	258.9153	228.5619	166.9368
t_{24}^{23}	**271.0648**	224.2195	168.5244
t_{24}^{25}	**271.9099**	226.1072	168.0772
t_{24}^{27}	270.7288	223.6708	166.8845
t_{24}^{opt}	**280.8669**	**246.2986**	**172.8786**

Table B11

***MSEs* of the Class of t_{25}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ (CASE-II)	$\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	16.37238	18.14872	10.8367
t_{25}^{1}	7.697119	8.324106	5.838223
t_{25}^{3}	17.25092	18.25624	15.2364
t_{25}^{5}	7.664832	9.381903	5.267307
t_{25}^{7}	7.603869	8.214904	5.783813
t_{25}^{9}	16.78842	18.24159	14.86759
t_{25}^{11}	7.572147	9.243414	5.22252
t_{25}^{opt}	7.572147	9.243414	5.22252

Table B12

***PREs* of the Class of t_{25}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ (CASE-II)	$\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	100	100	100
t_{25}^{1}	212.7079	218.0261	185.6164
t_{25}^{3}	94.90729	99.41105	71.12376
t_{25}^{5}	213.6039	193.4439	205.7351
t_{25}^{7}	215.3164	220.9243	187.3626
t_{25}^{9}	97.52186	99.49089	72.88807
t_{25}^{11}	216.2185	196.3422	207.4994
t_{25}^{opt}	216.2185	196.3422	207.4994

Table B13

***MSEs* of the Class of t_{26}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ **(CASE-II)**	$\eta_i = \frac{M_i}{\overline{M}}$ **(CASE-I)**
$\overline{y}_s$	16.37238	18.14872	10.8367
t_{26}^{1}	8.775882	9.108777556	7.664782
t_{26}^{3}	14.60792	16.06868874	11.15029
t_{26}^{5}	8.727055	11.79063514	7.647826
t_{26}^{7}	8.654592	8.978178952	7.570311
t_{26}^{9}	14.27491	15.66666909	10.95147
t_{26}^{11}	8.607101	11.5727321	7.55377
t_{26}^{opt}	8.607101	11.5727321	7.55377

Table B14

***PREs* of the Class of t_{26}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ **(CASE-II)**	$\eta_i = \frac{M_i}{\overline{M}}$ **(CASE-I)**
$\overline{y}_s$	100	100	100
t_{26}^{1}	186.5599	199.2443	141.383
t_{26}^{3}	112.0781	112.9446	97.18761
t_{26}^{5}	187.6037	153.9249	141.6965
t_{26}^{7}	189.1745	202.1426	143.1474
t_{26}^{9}	114.6927	115.8429	98.95201
t_{26}^{11}	190.2183	156.8231	143.4608
t_{26}^{opt}	190.2183	156.8231	143.4608

Table B15

***MSEs* of the Class of t_{27}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ (CASE-II)	$\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	16.37238	18.14872	10.8367
t_{27}^{1}	11.15116922	11.79914	9.52239
t_{27}^{3}	14.75160896	16.62239	10.48083
t_{27}^{5}	11.10218594	9.366685	9.515283
t_{27}^{7}	10.95606565	11.58092	9.377012
t_{27}^{9}	14.41209572	16.19256	10.30498
t_{27}^{11}	10.90877779	9.228642	9.370121
t_{27}^{opt}	10.90877779	9.228642	9.370121

Table B16

***PREs* of the Class of t_{27}^{G} for Population I and Population II for Equal and Unequal fsu's**

MSE using Population-I (equal fsu's) (CASE-III)		MSE using Population-II (unequal fsu's)	
		$\eta_i = 1$ (CASE-II)	$\eta_i = \frac{M_i}{\overline{M}}$ (CASE-I)
$\overline{y}_s$	100	100	100
t_{27}^{1}	146.8221	153.8139	113.8023
t_{27}^{3}	110.9871	109.1824	103.3954
t_{27}^{5}	147.4699	193.7582	113.8873
t_{27}^{7}	149.4367	156.7122	115.5667
t_{27}^{9}	113.6017	112.0806	105.1598
t_{27}^{11}	150.0845	196.6565	115.6517
t_{27}^{opt}	150.0845	196.6565	115.6517