National College of Business Administration and Economics Lahore

GENERALIZED MEAN ESTIMATORS FOR SENSITIVE AND NON-SENSITIVE VARIABLES IN THE PRESENCE OF MEASUREMENT ERRORS

BY

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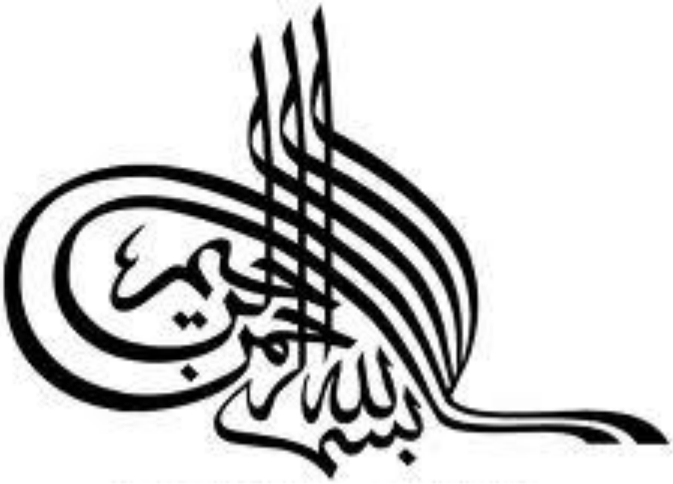

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The Most Beneficial,
The Most Merciful,

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No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Faculty of Social Sciences** in partial fulfillment of requirements for the degree of requirements for the degree of Doctor of Philosophy in the field of **Statistics**, National College of Business Administration and Economics, Lahore.

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Dedicated

to

My Sweet Mother

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SUMMARY

In survey research, accurate collection and recording of information is very critical. The researcher must deal with many potential problems. First the response rate may be poor due to various reasons such as poorly prepared survey instrument, poor execution of survey, survey questions being very personal in nature, and untrained field workers. Some of these issues lead to measurement errors which are the most common form of non-sampling errors. More formally, these errors are defined as the difference between the true value of a variable and its recorded value. It is for this reason these errors are also known as observational errors.

Measurement errors have been studied by various authors with Cochran (1963) drawing early attention to these errors. While mean estimation for non-sensitive variables has been studied extensively in the presence of measurement errors, no attempt has been made to study mean estimation for sensitive variables in the presence of measurement errors. By sensitive variable, we mean a variable for which there is a natural tendency on the part of survey respondent to either refuse to answer or to give a socially desirable answer as opposed to correct answer. Randomized Response Technique (RRT) introduced originally by Warner (1965), and later refined by many researchers, is a great tool to deal with the problem of social desirability bias in surveys involving sensitive questions. The main focus of this thesis is on introducing a generalized mean estimator for non-sensitive as well as sensitive quantitative variables in the presence of measurement errors, and on studying the impact of measurement errors on mean estimation.

In Chapter 1, we have provided a brief discussion about measurement errors, sensitive variables, and various versions of the Randomized Response Techniques (RRT). Furthermore, measurement errors under simple random sampling and stratified random sampling have been illustrated. Greater details on these two important topics, measurement errors and randomized response methodology, are provided as part of literature review in Chapter 2. In Chapter 3, we have reviewed some existing mean estimators for non-sensitive and sensitive study variables in the presence of measurement errors under both sampling designs.

The major contributions of this thesis start from Chapter 4. In this chapter, a generalized mean estimator for a non-sensitive study variable under simple random sampling design has been proposed to examine the impact of measurement errors on mean estimation. Some special cases for generalized mean estimator have also been discussed. In Chapter 5, we continue the study

undertaken in Chapter 4 but in the context of the stratified random sampling design. Chapters 6 and 7 are very important chapters where we have examined the impact of measurement errors on mean estimation of a sensitive study variable under the simple random sampling design and the stratified random sampling design respectively.

We have used extensive simulations and numerical examples to validate our theoretical findings. Finally some concluding remarks with some possible future directions are mentioned in Chapter 8.

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION AND BACKGROUND

This thesis is an attempt to examine the impact of measurement errors in mean estimation. We will study the problem in various contexts. For these contexts, we will use two major sampling designs:

(i) Simple Random Sampling without replacement (SRSWOR), and
(ii) Stratified Random Sampling.

Under each of these two designs, we will consider two cases:

(i) When the study variable is non-sensitive in nature, thereby allowing direct responses, and
(ii) When the study variable is sensitive in nature where direct responses are subject to serious social desirability bias (SDB).

1.2 MEASUREMENT ERRORS (ME)

In survey research, researchers very often make the fatal assumption that data have been collected without error. But this is not always true. When this assumption is violated, resulting estimates may be flawed. A common form of error is known as non-sampling error. And among non-sampling errors, the most common one is the measurement error which is defined as the difference between the true value of a variable and its recorded value. These errors are also known as *observational errors*. Measurement errors normally occur during the data acquisition or data recording stage as opposed to data analysis stage. These errors concern the accuracy of measurement at the level of individual units .The main sources of such errors are survey instrument, the data collection method, inadequate instructions to field staff, and the intentional or unintentional mistakes by the respondent. Some of the ways to reduce measurement errors are:

(i) Pre-testing of the survey instrument
(ii) Better training of data recorders
(iii) Double entry of the data followed by careful matching of the two spread-sheets.

Measurement errors may be random or may be systematic. As the term suggests, random errors are truly random in nature therefore we need to make only minimal distributional assumptions. These errors are indicated by inconsistent repeated readings on the same object. In social sciences research, even mood swings may cause a certain scale to give you different values of the variable being measured such as pain and stress level. In recollecting the amount of money spent last month on food and groceries, a respondent may not be able to recollect the correct amount spent. Similarly, the temperature in a room may not necessarily be equal to the level where the air-conditioning thermostat is set. These errors generally affect the variability, and not the mean value of the variable. In contrast to random errors, systematic errors follow a specific pattern, which may even be predictable. These errors generally result from *calibration* related issues which allow the reading to be systematically more (or less) than the actual value of the variable. Thus, these errors do cause a shift in the mean value also (known as bias).

Measurement errors have been studied by many researchers. One may find a good account of the history of measurement errors in Cochran (1963, 1968, and 1977). A review of various issues surrounding measurement errors may be found in Biemer et al. (1991), Fuller (1995), O'Muircheartaigh (1997), Shalabh (1997), Srivastava & Shalabh (2001), Manisha & Singh (2001), Manisha & Singh (2002), Wang (2002), Allen et al. (2003), Singh & Karpe (2007, 2008, 2009, 2010), Chambers (2008), Gregoire & Salas (2009), Salas & Gregoire (2010), Kumar et al. (2011), Shukla et al. (2012), Singh et al. (2014), Azeem & Hanif (2015), and Shalabh & Tsai (2016) etc.

If measurement errors are negligible in size, estimates may not be affected much. But large measurement errors can cause serious problem with the accuracy of the estimates, as we will see in this thesis. Following Singh and Karpe (2009), we have assumed that the measurement errors are normally distributed with zero mean, and are independent of the variable being measured.

1.3 SENSITIVE VARIABLES

Sensitive survey questions are those where the respondent feels uncomfortable answering the question in a face–to-face survey mode. We encounter such questions very commonly in social, behavioral and health sciences. For example, a question about whether a student has cheated on an exam is clearly sensitive. Similarly, questions relating to induced abortions, drug use, and assaulting someone are also sensitive in nature. Face–to-face responses to such questions are subject to serious social desirability bias (SDB) which in simple terms is the tendency in human beings to look good.

A quick search will reveal that even for surveys dealing with non-sensitive questions, a response rate of around 40 - 50 percent is considered excellent. Infact, it is much smaller. This is based on the articles such as Fan & Yan (2010) and Miller & Dillman (2011). The problem gets further complicated if the survey questions are sensitive in nature. High degree of sensitivity of a survey question leads to higher non-response rate.

Researchers have used various techniques for dealing with the problem of non-response or inaccurate response when dealing with sensitive questions. Actually non-response is a better evil than inaccurate response. Sociologists have used Reynold's (1982) 13-point social desirability bias (SDB) scale to quantify the degree of tendency in a respondent to look good to others. This score can later be used as a covariate to do group comparisons. Another method, popular with psychologists is the use of Bogus Pipeline (BPL). For a good review of BPL, refer to Jones and Sigall (1971). It is a fake lie detector test that connects respondent's fingers to a computer and the respondent is tricked into believing that physiological reflexes while giving inaccurate responses can be detected by the computer. This fear compels some respondents to tell the truth. However the most scientific approach is the Randomized Response Technique (RRT) introduced by Warner (1965) which is based on valid probabilistic considerations. The technique, to be described in the next sub-section in greater detail, offers respondents enough privacy to not give a false response to sensitive questions. While RRT models help in reducing the non-response rate by offering privacy to respondents, these models can not address the problem of intentional inaccurate response. They only decrease the need for inaccurate response. Hence a study of measurement errors is still important in the context of RRT models so that more accurate confidence intervals can be constructed for the unknown parameters like the population mean.

1.3.1 Randomized Response Technique (RRT)

The randomized response technique was first introduced by Warner (1965) in a very simple but very significant paper. Many variations of that original technique have since been introduced by authors that include Greenberg et al. (1969, 1971), Eichhorn and Hayre (1983), Gupta et al. (2002, 2006, 2010) and Huang (2010).

Warner's (1965) original approach dealt with a binary response situation. This was done by asking the research question directly to some of the randomly selected respondents. Other respondents are asked the indirect version of the same question. For example, some of the respondents face the question "Did you file a correct income tax return last year?" while others face the question "Did you file an incorrect income tax return last year?". The researcher does not know which respondent faced which question. Essentially the responses are randomized (or scrambled). These scrambled responses can later be unscrambled at aggregate level but not at the individual level. That is, one can estimate what proportion of tax payers cheated on taxes last year but there is no way of knowing who cheated and who did not.

1.3.2 Methods Dealing With Randomized Response Technique

RRT models can be characterized in various ways. A RRT model is called quantitative, binary or multi-category depending on the type of response the question carries. For quantitative RRT models, interest is in estimating the mean of the variable of interest. For binary RRT models we are interested in estimating the prevalence of certain behaviors in the population. A RRT model can also be called Full RRT, Partial RRT, or Optional RRT depending on whether all or only some of the respondents provide scrambled responses, and depending on whether a respondent has been forced to provide a true response, as in Mangat and Singh (1990) two-stage model, or the choice is left to the respondent as in Gupta (2002) optional model. RRT models can also be classified as additive, multiplicative or generalized depending on whether we use additive scrambling, multiplicative scrambling, or more general scrambling.

In this thesis, we will concentrate on the additive quantitative RRT model presented by Warner (1971). The respondent is asked to provide a scrambled response by adding a random number to the real answer. The researcher knows the probability distribution of the scrambling variable but does not know what specific number was used for scrambling that response. Again the scrambled responses can later be unscrambled at a group level but not at individual level. This model is used in this thesis when we estimate the mean of a sensitive variable in the presence of measurement errors.

A very important point to be noted is that both measurement errors and sensitivity of survey questions are major problems individually. However the problem becomes more acute when the two of them occur simultaneously. Handling this situation was the main motivation for undertaking this study.

1.4 SIMPLE RANDOM SAMPLING VS. STRATIFIED RANDOM SAMPLING

The most basic sampling design is the simple random sampling design, where we select a sample of size *n* from a population of size *N* by giving equal probability to all the ${}^{N}C_{n}$ possible samples. By contrast, in stratified random sampling, the target population is divided into a number of groups (strata) based on some criteria, and a simple random sample is drawn from each stratum. There are several reasons for stratification but the primary reason is to convert a highly heterogeneous population into non-overlapping subgroups of homogeneous sub-populations. This often results in big savings in terms of sampling cost. Need for sub-domain level estimates is another reason for stratification. Geographic considerations can also be a factor. If stratification is used, we first get estimates for each stratum and then combine the results using appropriate methodology.

1.5 MEASUREMENT ERRORS UNDER DIFFERENT SAMPLING DESIGNS

In all the sub-sections below, the measurement errors are assumed to be normally distributed with mean zero and known variances. It is also assumed that the measurement errors are uncorrelated even though the study variable and the auxiliary variable are correlated. Measurement errors are also assumed to be independent of the true values of the variables.

1.5.1 Simple Random Sampling

(i) Non-Sensitive Variables

A random sample of size *n* is selected from the population of size *N* by a simple random sampling without replacement (SRSWOR) method. Let $\left(y_i, x_i\right)$ be the observed values and $\left(Y_i, X_i\right)$ be the true values on two characteristics $\left(y, x\right)$ respectively associated with the i^{th} $(i = 1, 2, ... n)$ sample unit. The measurement errors are then given by

$$U_i = y_i - Y_i \tag{1.5.1}$$

$$V_i = x_i - X_i \tag{1.5.2}$$

These measurement errors (ME) are assumed to be random and uncorrelated with mean zero and variances S_U^2 and S_V^2 respectively. Let S_Y^2 and S_X^2 denote the population variances, C_Y and C_X be the population co-efficient of variation, and ρ_{YX} be the co-efficient of correlation for the study variable (Y) and the auxiliary variable (X).

(ii) Sensitive Study Variable and Non-Sensitive Auxiliary Variable

Let Y be the sensitive study variable, which cannot be observed directly due to potential respondent bias, and X be a non-sensitive auxiliary variable which has a positive correlation with Y. Let S be a scrambling variable which is independent of Y and X. The respondent is asked to report a scrambled response for study variable (Y) given by $Z=Y+S$, but is asked to provide a true response for the auxiliary variable (X) (as in Gupta et al. (2012)). Here additive scrambling is used as opposed to the multiplicative scrambling used in Gupta et al. (2002). A simple random sample of size n is drawn without replacement from the population of size N. Let $\left(y_i, x_i, z_i\right)$ respectively be the observed values (factoring in measurement errors) and $\left(Y_i, X_i, Z_i\right)$ be the true values for the study variable Y, the auxiliary variable X, and the scrambled response variable Z associated with the i^{th} $(i=1,2,...n)$ sample unit. The respective measurement errors (ME) associated with the scrambled response variable (Z) and the auxiliary variable (X) respectively be given by

$$T_i = z_i - Z_i, \qquad (1.5.3)$$

$$V_i = x_i - X_i.$$

These measurement errors are assumed to be random and uncorrelated with mean zero and variances S_T^2 and S_V^2 respectively. Let S_Z^2 and S_X^2 be the population variances, C_Z and C_X be the coefficients of variations for the scrambled response variable (Z) and the auxiliary variable (X) respectively. Let ρ_{ZX} denote the co-efficient of correlation between Z and X.

1.5.2 Stratified Random Sampling

(i) Non-Sensitive Variables

Consider a finite population of size N and let Y and X respectively be the study and auxiliary variables associated with each population unit. Let the whole population be divided into L strata with N_h units $\left(h=1,2,...L\right)$ in the h^{th}

stratum such that $\sum_{h=1}^{L} N_h = N$. A simple random sample of size n_h is drawn without replacement from the h^{th} stratum such that $\sum_{h=1}^{L} n_h = n$. Let $\left(y_{hi}, x_{hi}\right)$ be the observed pair of values instead of true pair of values $\left(Y_{hi}, X_{hi}\right)$ of the study variable *Y* and the auxiliary variable *X* on the i^{th} unit of the h^{th} stratum, where $(i = 1, 2, ... N_h)$. The measurement errors in h^{th} stratum are given by

$$U_{hi} = y_{hi} - Y_{hi}, \tag{1.5.4}$$

$$V_{hi} = x_{hi} - X_{hi}, \tag{1.5.5}$$

These measurement errors are stochastic in nature with mean zero and variances S_{Uh}^2 and S_{Vh}^2 respectively. Let S_{Yh}^2 and S_{Xh}^2 be the variances, C_{Yh} and C_{Xh} be the coefficients of variations and ρ_{YXh} be the coefficient of correlation, for the study variable (*Y*) and the auxiliary variable (*X*) respectively in the h^{th} stratum. It is assumed that $U_{hi}'s$ and $V_{hi}'s$ are uncorrelated although (Y_{hi}, X_{hi}) are correlated.

(ii) Sensitive Study Variable and Non-Sensitive Auxiliary Variable

Let *Y* be the sensitive study variable which is not directly observable. Let a non-sensitive auxiliary variable *X* be available, which is strongly correlated with *Y*. Let *S* be a zero-mean scrambling random variable with known distribution. The respondent is asked to report an additively scrambled response for the study variable *Y* given by $Z = Y + S$, and is also asked to provide a true response for auxiliary variable *X*. Let $\left(y_{hi}, x_{hi}, z_{hi}\right)$ be the observed values (factoring in measurement errors) and $\left(Y_{hi}, X_{hi}, Z_{hi}\right)$ be the true values for the sensitive study variable *Y*, the auxiliary variable *X* and the scrambled response variable *Z* associated with the i^{th} unit of the h^{th} stratum, $(i = 1, 2, ... N_h)$. The respective measurement errors associated with the scrambled response variable *Z* and the auxiliary variable *X* in the h^{th} stratum are given by

$$T_{hi} = z_{hi} - Z_{hi}, \tag{1.5.6}$$

$$V_{hi} = x_{hi} - X_{hi}$$

These measurement errors are stochastic in nature with mean zero and variances S_{Th}^2 and S_{Vh}^2 respectively. Let S_{Zh}^2 and S_{Xh}^2 be the variances, C_{Zh}

and C_{Xh} be the coefficients of variations and ρ_{ZXh} be the coefficient of correlation, for the scrambled response variable (Z) and the auxiliary variable (X) respectively in h^{th} stratum.

1.6 OBJECTIVES OF THE STUDY

The four major objectives of this thesis are the study of following issues:

- Impact of measurement errors on mean estimation of a non-sensitive study variable in Simple Random Sampling.
- Impact of measurement errors on mean estimation of a non-sensitive study variable in Stratified Random Sampling.
- Impact of measurement errors on mean estimation of a sensitive study variable in Simple Random Sampling.
- Impact of measurement errors on mean estimation of a sensitive study variable in Stratified Random Sampling.

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

We divide this literature review into two main parts: one for presenting the literature about the mean estimation for the situation when measurement errors occur, and the other for presenting the literature about randomized response models used to estimate the mean of sensitive study variable.

2.1.1 Measurement Errors

We give below a chronological attempt to deal with measurement errors by various researchers.

Berkson (1950) studied measurement errors and pointed a situation where measurement errors in auxiliary variables cause no harm, but in general these errors need to be handled carefully.

Measurement errors, if not handled properly, can introduce serious bias in statistical estimation. This was studied by Cochran (1968) in the context of ordinary least square estimates where these errors can lead to inconsistent estimates.

Non-response is also an equally serious issue. It increases when we deal with surveys that involve sensitive questions. This poses even more serious problems in conjunction with measurement errors. Hansen and Hurwitz (1946) and Cochran (1977) discussed these problems and introduced estimators by exploiting auxiliary information. Cochran (1977) concluded that:

> "*Measurement errors that are independent from unit to unit within the sample and average to zero over the whole population are properly taken into account in the usual formulae for computing the standard errors of estimates, provided that finite population correction term (fpc) is negligible. Such errors decrease the precision of the estimates and it is worthwhile to find out whether this decrease is serious*".

Srivenkataramana (1980) exploited the non-response group very cleverly and introduced a dual-to-ratio estimator for the population mean by using auxiliary information and conducted an empirical study to check the performance of proposed estimator with existing estimators.

Khare and Srivastava (1995) discussed the non-response in surveys and compared some ratio and product estimators with some regression type estimators by factoring in fixed survey cost. However, focus of this thesis is on measurement errors and not non-response.

Fuller (1995) highlighted the significance of measurement errors in the estimation of various population parameters, estimation of quantiles and estimation through regression model.

Shalabh (1997) discussed the impact of measurement errors on the ratio estimator and made comparison among the ratio estimator and the mean per unit estimator of population mean in the presence of measurement errors.

Srivastava and Shalabh (2001) examined the influence of measurement error on two estimators of regression co-efficient by using information on a single auxiliary variable. One of these estimators was arising from direct regression while the other was arising from reverse regression.

Manisha and Singh (2001), Manisha and Singh (2002) presented additional discussion on measurement errors in the context of ratio and regression estimation respectively. Comparative studies were made among the linear regression estimator, the ratio estimator and the mean per unit estimator in the presence of measurement errors.

Wang (2002) offered an adjustment for measurement errors in the context of various regression situations. The suggested correction is easy to implement in computing with minimal computing cost.

Allen et al. (2003) utilized multiple auxiliary variables to develop a family of estimators for the mean. They analyzed the properties of their suggested estimators for situations where measurement errors were present in case of complete response.

Singh and Karpe (2007, 2008) suggested new classes of estimators for the population mean and the population variance respectively in the presence of measurement errors.

Chambers (2008) also discussed the effect of measurement errors in auxiliary variables under two different situations. In the first situation, it was assumed that the auxiliary variables had known population means, and survey weights were constructed in such a way that for the auxiliary variables, the sample means equal to their population means. The second situation was that the marginal population information for use with regression estimation contained errors.

Singh and Karpe (2008a) analyzed ratio and product estimators of mean in the presence of measurement errors. Singh and Karpe (2009) proposed estimators for the ratio and product of two population means in the presence of measurement errors and analyzed their properties.

Gregoire and Salas (2009) examined the case of systematic measurement error as well as measurement error that varies according to a fixed distribution in auxiliary variable when estimating the population total.

Singh and Karpe (2010) introduced separate and combined ratio, product and difference estimators for population mean in stratified random sampling when the observations are contaminated with measurement errors. Generalized versions of those estimators were also given along with their properties.

Salas and Gregoire (2010) presented extensive simulation study to examine the impact of systematic and random measurement errors in estimating the ratio-of-means and the mean-of-ratios.

Kumar et al. (2011) studied the impact of measurement errors on exponential-type mean estimators. They compared their suggested estimators with the existing estimators for situations when measurement errors were present.

Shukla et al. (2012) proposed an estimator of population mean in the presence of measurement errors that was a linear combination of Srivenkataramana’s (1980) dual-to-ratio estimator and the usual mean per unit estimator. They found the conditions under which their suggested estimator was more efficient than the Shalabh (1997) and the Manisha and Singh (2001) estimators.

Singh et al. (2014) proposed a mean estimator in the presence of measurement errors in the context of finance related modeling. Their investigation indicated that proposed estimator is the most suitable estimator with a smaller *MSE* relative to other estimators under measurement errors.

Singh, V.K., Singh, R., and Smarandache (2014) suggested difference-type mean estimator in the presence of measurement errors using auxiliary information. The merits of proposed method over other traditional methods were illustrated through an empirical study.

Azeem and Hanif (2015) studied mean estimation when measurement errors and non-response exist simultaneously and presented theoretical and empirical results. Shalabh and Jia-Ren Tsai (2016) studied mean estimation using ratio and product methods when the observations on both the study and the auxiliary variables are subject to correlated measurement errors.

As stated earlier, the problem of measurement errors gets complicated further when the study variable is sensitive in nature and direct observation on it is subject to serious response bias. This necessitates the use of randomized response methodology. With that in mind, we provide below some literature review on RRT models.

2.1.2 Randomized Response Models

Warner (1965) was the first to introduce the concept of randomized response methodology by using indirect questioning technique where based on a random selection, some respondents answer the direct version of a question and others answer an indirect version of the same question. For example, some respondents are asked if they filed correct tax return last year and some are asked if they filed an incorrect tax return last year. Clearly both questions reflect on cheating on taxes but in one case, a "yes" response makes the respondent honest, and in the other case a "no" response makes the respondent honest.

To avoid the situation where all the respondents are confronted with a direct or indirect version of the sensitive question, Greenberg et al. (1969) introduced the famous unrelated question model where some respondents, based on random selection, answer the real question and others answer an unrelated question such as "were you born in the month of January?"

Since the work in this thesis concerns quantitative response situations, we are more interested in quantitative response RRT models. Both Warner (1971) and Greenberg et al. (1971) have provided quantitative response RRT models. Greenberg et al. (1971) still relied on the unrelated question technique where some respondents answer the actual quantitative response question and some answer an unrelated quantitative response question. Warner (1971) on the other hand, suggested the use of additive and multiplicative scrambling. Since additive scrambling will be used extensively in this thesis, we provide below Warner (1971) idea briefly.

Suppose Y is the study variable which is sensitive in nature and S is a scrambling variable, preferably with zero-mean and non-negligible variance. The respondent is asked to provide a response Z given by $Z = Y + S$. Note that larger variance of S ensures greater privacy but dilutes the data quality more. Smaller variance of S does the exact opposite. The researcher must find the right balance just as one does in constructing confidence intervals where one must find the right balance between the confidence level and the margin of error.

Eichhron and Hayre (1983) proposed a multiplicative randomized response model given by $Z = YS$, but this model compromises respondent's anonymity since a non-zero response clearly suggests that for that respondent, Y could not have been zero. This is a major criticism of this model.

Gupta et al. (2002) proposed what they called "Optional RRT Model" where the sensitivity level of the survey question plays an important role. Clearly, some questions are more sensitive and some are less sensitive. In Gupta et al. (2002), a respondent is asked to provide a scrambled response if he/she considers the question sensitive and an accurate response otherwise. The model is given by $Z = Y.S^T$, where $T \sim Bernoulli(W)$ and W is the sensitivity level of the research question. That is, W is the proportion of respondents in the population who consider the question too sensitive and personal for a direct response. Although it was shown that this model is more efficient than the Eichhron and Hayre (1983) model, it introduces an additional parameter in the model in the form of W.

Gupta and Shabbir (2004) proposed another multiplicative "Optional" RRT model where simultaneous estimation of the sensitivity level and the mean are possible but some approximation is required. Ryu, J. -B, et al. (2005) proposed a quantitative randomized response model based on Mangat and Singh's (1990) two-stage randomized response model. They derived the estimator of the sensitive variable mean and showed that their method was more efficient than other randomized response models suggested by Greenberg et al. (1971) and Gupta et al. (2002) estimators. However, this model does not account for the sensitivity level of the survey question.

Arnab, R. and Dorffner (2006) discussed RRT models for complex surveys and presented theoretical and empirical results.

Hussain et al. (2007) relied on double responses from each respondent to get improved estimates. Efficiency comparison with several existing estimators is provided.

Saha (2008) presented a variation of the Eichhorn and Hayre (1983) model under unequal probability sampling.

Hussain and Shabbir (2009) provided improved estimates for the mean of a sensitive variable by modifying the Ryu et al. (2005) estimator using simple random sampling and stratified random sampling protocols.

Gupta et al. (2010) introduced a two-stage optional additive RRT model that allowed simultaneous estimation of the mean and sensitivity level without the need of any approximation. This was done by using the traditional split approach. It is similar to the additive model suggested by Warner (1971) but far more flexible. This model produces estimators that are unbiased and asymptotically normally distributed. This model has been used heavily in many studies involving ratio and regression mean estimation, and will be used in this thesis as well. Some of these studies are described below.

Sousa et al. (2010) was the first to use non-sensitive auxiliary information to estimate the mean of a sensitive variable by using ratio estimation. They show that there is hardly any difference in the first order and second order approximations for MSE even for small sample sizes, and also generalized the proposed estimator to the case of transformed ratio estimators but these transformations did not result in any significant reduction in MSE. An extensive simulation study was also presented to evaluate the performance of the proposed estimator. They also presented a numerical example using real data from Information and Communication Technologies (ICT) sector in Portugal.

Gupta et al. (2012) carried this study forward and introduced regression estimator of the mean that performed better than the ratio estimator of Sousa et al. (2010). This work was extended again in Koyuncu et al. (2014) by introducing exponential-type estimators using one and two auxiliary variables. It was shown that the proposed exponential-type estimators are more efficient than the existing estimators.

Gupta et al. (2014) extended further the work done in Sousa et al. (2010) and Gupta et al. (2012) by utilizing optional additive RRT models. The optional models, while being more complex, did result in further gains in efficiency.

Blattman et al. (2014) developed and tested a survey validation technique that uses intensive qualitative work to check for measurement error in potentially sensitive topics such as stealing, substance abuse, and gambling etc.

Kalucha et al. (2015) used optional RRT methodology to introduce another ratio estimator of the mean which they called an additive ratio estimator. It performed better than the ordinary RRT mean estimator.

Gupta et al. (2015) introduced further improvement in mean estimation through optional RRT models by introducing a regression estimator that performed better than the ratio estimator even for modest correlation between the study and the auxiliary variables. Gupta et al. (2016) extended this work further through an exponential mean estimator using optional RRT models.

Batool et al. (2016) used additive, multiplicative, and combination of both additive and multiplicative scrambling models to achieve further gains in mean estimation.

Mushtaq et al. (2016) introduced a ratio, a regression, and a general class of estimators for the mean of sensitive variable using non-sensitive auxiliary variable based on randomized response technique in stratified two-phase sampling. Efficiency comparison with several existing estimators is provided.

Mushtaq et al. (2017) proposed a combined general family of RRT estimators for estimating finite population mean of a sensitive study variable in stratified random sampling. A simulation study showed that the proposed class of estimators is more efficient than the existing estimators, i.e., usual sample mean estimator, Sousa et al (2014) ratio and regression estimators of the sensitive variable in the same design.

CHAPTER 3

SOME WELL KNOWN ESTIMATORS OF THE POPULATION MEAN

3.1 INTRODUCTION

In this chapter, we have reviewed some existing mean estimators of population mean along with their mean square errors (MSE) in the presence of measurement errors (ME) in simple random sampling and stratified random sampling designs. When the data are subject to measurement errors on both the study variable *Y* and the auxiliary variable *X*, these estimators are presented according to two situations:

(i) Both *Y* and *X* are non- sensitive variables
(ii) Study variable *Y* is sensitive and the auxiliary variable *X* is non-sensitive

3.2 MEAN ESTIMATORS IN SIMPLE RANDOM SAMPLING

3.2.1 Mean Estimators when both the Study Variable and the Auxiliary Variable are Non-Sensitive

(i) The Mean Per Unit Estimator

The sample mean per unit estimator is given by

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i , \tag{3.2.1}$$

The variance of the mean per unit estimator with measurement error is:

$$Var^*(\bar{y}) = \theta\left(S_Y^2 + S_U^2\right), \tag{3.2.2}$$

where $\theta = \dfrac{N-n}{Nn}$.

The variance of $\bar{y}$ without measurement error may be obtained by putting $S_U^2 = 0$ in (3.2.2).

(ii) Shalabh's (1997) Estimator

Shalabh (1997) introduced the following ratio-type estimator in the presence of measurement errors:

$$t_1 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right). \tag{3.2.3}$$

The mean square error of t_1 up to the first order of approximation is

$$MSE^*(t_1) = \theta\left(S_Y^2 + R^2 S_X^2 - 2R\rho_{XY} S_X S_Y\right) + \theta\left(S_U^2 + R^2 S_V^2\right), \tag{3.2.4}$$

where $R = \dfrac{\bar{Y}}{\bar{X}}$.

The mean square error of t_1 without measurement error may be obtained by putting $S_U^2 = S_V^2 = 0$ in (3.2.4).

(iii) Singh et al. (2010) Estimator

Singh et al. (2010) studied an exponential ratio type estimator i.e.

$$t_2 = \bar{y}\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right). \tag{3.2.5}$$

The mean square error of t_2 up to the first order of approximation is

$$MSE^*(t_2) = \frac{S_Y^2}{n}\left[1 - \frac{C_X}{C_Y}\left(\rho - \frac{C_X}{4C_Y}\right)\right] + \frac{1}{n}\left[\frac{\bar{Y}^2}{4\bar{X}^2} S_V^2 + S_U^2\right]. \tag{3.2.6}$$

The mean square error of t_2 without measurement error may be obtained by putting $S_U^2 = S_V^2 = 0$ in (3.2.6).

(iv) Shukla et al. (2012) Estimator

Shukla et al. (2012) proposed the following estimator

$$t_3 = \alpha \bar{y} \frac{\bar{x}'}{\bar{X}} + (1-\alpha)\bar{y}. \qquad (3.2.7)$$

The mean square error of t_3 up to the first order of approximation in the presence of measurement error is

$$MSE^*(t_3) = \theta \bar{Y}^2 \left[C_Y^2 + \eta^2 C_X^2 - 2\eta \rho_{YX} C_Y C_X \right] + \theta \bar{Y}^2 \left[\frac{S_U^2}{\bar{Y}^2} + \eta^2 \frac{S_V^2}{\bar{X}^2} \right], \qquad (3.2.8)$$

where $\eta = \dfrac{\alpha n}{N-n}$.

The optimum value of η is:

$$\eta_{(opt)} = \frac{S_X^2}{S_X^2 + S_V^2} \rho_{YX} \frac{C_Y}{C_X} = \eta_0,$$

The optimum mean square error of t_3 up to the first order of approximation is

$$MSE^*_{opt}(t_3) = \theta \bar{Y}^2 \left[C_Y^2 + \eta_0^2 C_X^2 - 2\eta_0 \rho_{YX} C_Y C_X \right] + \theta \bar{Y}^2 \left[\frac{S_U^2}{\bar{Y}^2} + \eta_0^2 \frac{S_V^2}{\bar{X}^2} \right]. \qquad (3.2.9)$$

The mean square error of t_3 without measurement error may be obtained by putting $S_U^2 = S_V^2 = 0$ in (3.2.9).

(v) Azeem and Hanif (2015) Estimator

Azeem and Hanif (2015) developed the following estimator

$$t_4 = \bar{y}^* \left[\alpha \exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*} \right) + (1-\alpha) \exp\left(\frac{\bar{x}'^* - \bar{X}}{\bar{x}'^* + \bar{X}} \right) \right]. \qquad (3.2.10)$$

where $\bar{x}'^* = \dfrac{N\bar{X} - n\bar{x}^*}{N-n}$, The mean square error of t_4 up to the first order of approximation is

$$MSE^*(t_4) = \lambda \bar{Y}^2 \left(C_Y^2 + \frac{1}{4}\mu^2 C_X^2 - \mu\rho C_Y C_X \right)$$
$$+\theta \bar{Y}^2 \left(C_{Y(2)}^2 + \frac{1}{4}\mu^2 C_{X(2)}^2 - \mu\rho_{(2)} C_{Y(2)} C_{X(2)} \right)$$
$$+\lambda \bar{Y}^2 \left(\frac{S_U^2}{\bar{Y}^2} + \frac{1}{4}\mu^2 \frac{S_V^2}{\bar{X}^2} \right) + \theta \bar{Y}^2 \left(\frac{S_{U(2)}^2}{\bar{Y}^2} + \frac{1}{4}\mu^2 \frac{S_{V(2)}^2}{\bar{X}^2} \right). \quad (3.2.11)$$

The mean square error of t_4 without measurement error may be obtained by putting $S_U^2 = S_V^2 = 0$ in (3.2.11).

3.2.2 Mean Estimators when Study Variable is Sensitive and the Auxiliary Variable is Non-Sensitive

Here we discuss some existing mean estimators by using Randomized Response Technique (RRT). In these estimators, an additive quantitative RRT model is used to get the scrambled responses.

(i) Ordinary RRT Mean Estimator

The RRT mean estimator for sensitive study variable Y of a finite population is given by

$$\bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i. \quad (3.2.12)$$

Here $Z = Y + S$, where S is a scrambling variable with zero mean and known variance.

The variance of $\bar{z}$ in the presence of measurement error is

$$Var^*(\bar{z}) = \theta\left(S_Z^2 + S_T^2 \right), \quad (3.2.13)$$

The mean square error of $\bar{z}$ without measurement error may be obtained by putting $S_T^2 = 0$ in eq. (3.2.13).

(ii) Sousa et al. (2010) Estimator

Sousa et al. (2010) proposed a ratio estimator for the mean of sensitive variable (*Y*) utilizing information from a non-sensitive auxiliary variable (*X*). This estimator is given by

$$t_5 = \bar{z}\left(\frac{\bar{X}}{\bar{x}}\right). \tag{3.2.14}$$

If measurement error is considered, the mean square error of t_5 up to the first order of approximation is

$$MSE^*(t_5) = \theta\left(S_Z^2 + R^2 S_X^2 - 2R\rho_{ZX} S_X S_Z\right) + \theta\left(S_T^2 + R^2 S_V^2\right), \tag{3.2.15}$$

where $R = \dfrac{\bar{Z}}{\bar{X}}$.

The mean square error of t_5 without measurement error may be obtained by putting $S_T^2 = S_V^2 = 0$ in (3.2.15).

(iii) Gupta et al. (2012) Estimator

A regression estimator was proposed by Gupta et al. (2012), for estimating the mean of a sensitive variable. It is given by

$$t_6 = \bar{z} + \hat{\beta}_{ZX}\left(\bar{X} - \bar{x}\right). \tag{3.2.16}$$

The expression of mean square error of t_6 up to the first order of approximation with measurement error is

$$MSE^*(t_6) = \theta S_Z^2\left(1 - \rho_{ZX}^2\right) + \theta\left(S_T^2 + \beta^2 S_V^2\right), \tag{3.2.17}$$

The mean square error of t_6 without measurement error may be obtained by putting $S_T^2 = S_V^2 = 0$ in (3.2.17).

3.3 MEAN ESTIMATORS IN STRATIFIED RANDOM SAMPLING

3.3.1 Mean Estimators when both the Study Variable and the Auxiliary Variable are Non- Sensitive

Here we discuss some existing mean estimators for a non-sensitive study variable when auxiliary variable is also non-sensitive. We do this in stratified random sampling.

(i) Combined Sample Mean

The traditional unbiased mean estimator (i.e. stratified sample mean) is defined as:

$$\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h \ . \tag{3.3.1}$$

The variance of $\bar{y}_{st}$ in the presence of measurement error is

$$Var^*(\bar{y}_{st}) = \sum_{h=1}^{L} \frac{W_h^2 S_{Yh}^2}{n_h \theta_{Yh}}, \tag{3.3.2}$$

where $S_{Yh}^2 = \sum_{i=1}^{N_h} \frac{(y_{hi} - \bar{Y}_h)^2}{N_h - 1}$ and $\theta_{Yh} = S_{Yh}^2 \big/ (S_{Uh}^2 + S_{Yh}^2)$.

The variance of $\bar{y}_{st}$ without measurement error may be obtained by putting $S_{Uh}^2 = 0$ in (3.3.2).

(ii) Singh and Karpe (2010) Ratio Estimator

Singh and Karpe (2010), presented a combined ratio estimator of the population mean given by

$$t_7 = \bar{y}_{st} \left(\frac{\bar{X}}{\bar{x}_{st}} \right). \tag{3.3.3}$$

The mean square error of t_7 up to the first order of approximation in the presence of measurement error is

$$MSE^*(t_7)=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R\frac{S_{Xh}^2}{\theta_{Xh}}\left(R-2\beta_{YXh}\theta_{Xh}\right)\right), \tag{3.3.4}$$

where

$$R=\frac{\bar{Y}}{\bar{X}},\ S_{Xh}^2=\sum_{i=1}^{N_h}\frac{(x_{hi}-\bar{X}_h)^2}{N_h-1},\ S_{YXh}=\sum_{i=1}^{N_h}\frac{(y_{hi}-\bar{Y}_h)(x_{hi}-\bar{X}_h)}{N_h-1},$$

$$\theta_{Xh}=S_{Xh}^2\Big/(S_{Vh}^2+S_{Xh}^2),\ \text{and}\ \beta_{YXh}=\frac{S_{YXh}}{S_{Xh}^2}.$$

The mean square error of t_7 without measurement error may be obtained by putting $S_{Uh}^2=S_{Vh}^2=0$ in (3.3.4).

(iii) Singh and Karpe (2010) Product Estimator

Singh and Karpe (2010) presented a combined product estimator which is defined as:

$$t_8=\bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}}\right). \tag{3.3.5}$$

The mean square error of t_8 in the presence of measurement error is

$$MSE^*(t_8)=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R\frac{S_{Xh}^2}{\theta_{Xh}}\left(R+2\beta_{YXh}\theta_{Xh}\right)\right). \tag{3.3.6}$$

The mean square error of t_8 without measurement error may be obtained by putting $S_{Uh}^2=S_{Vh}^2=0$ in (3.3.6).

(iv) Singh and Karpe (2010) Difference Estimator

Singh and Karpe (2010) presented a combined difference estimator which is defined as:

$$t_9=\bar{y}_{st}+d\left(\bar{X}-\bar{x}_{st}\right). \tag{3.3.7}$$

The minimum variance of t_9 in the presence of measurement error is

$$Var^*_{\min}(t_9) = \sum_{h=1}^{L} \frac{W_h^2}{n_h}\left(\frac{S_{Yh}^2}{\theta_{Yh}}\right) - \frac{\left(\sum_{h=1}^{L} \frac{W_h^2}{n_h}\beta_{YXh}S_{Xh}^2\right)^2}{\sum_{h=1}^{L} \frac{W_h^2}{n_h}\left(\frac{S_{Xh}^2}{\theta_{Xh}}\right)}. \qquad (3.3.8)$$

The mean square error of t_9 without measurement error may be obtained by putting $S_{Uh}^2 = S_{Vh}^2 = 0$ in (3.3.8).

3.3.2 Mean Estimators when Study Variable is Sensitive and the Auxiliary Variable is Non-Sensitive

Here we discuss some existing mean estimators by using Randomized Response Technique (RRT). In these estimators, an additive quantitative RRT model is used to get the scrambled responses.

(i) Combined Sample Mean (using RRT)

The usual combined sample mean, ignoring the auxiliary information, is given by

$$\bar{z}_{st} = \sum_{h=1}^{L} W_h \bar{z}_h \qquad (3.3.9)$$

The variance of $\bar{z}_{st}$ in the presence of measurement error is:

$$Var^*(\bar{z}_{st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S_{Zh}^2}{\theta_{Zh}}\right), \qquad (3.3.10)$$

where

$$S_{Zh}^2 = (S_{Yh}^2 + S_{Sh}^2),\ S_{Yh}^2 = \frac{\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)^2}{(N_h - 1)},\ S_{sh}^2 = \frac{\sum_{i=1}^{N_h}(s_{hi} - \bar{S}_h)^2}{(N_h - 1)},$$

$$\gamma_h = \left(n_h^{-1} - N_h^{-1}\right) \text{ and } \theta_{zh} = \frac{S_{zh}^2}{S_{Th}^2 + S_{zh}^2}.$$

The variance of $\bar{z}_{st}$ without measurement error may be obtained by putting $S_{Th}^2 = 0$ in (3.3.10).

(ii) Sousa et al. (2014) Ratio Estimator

Sousa et al. (2014) proposed a combined ratio estimator, defined as:

$$t_{10} = \bar{z}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right). \tag{3.3.11}$$

The mean square error of t_{10} up to the first order of approximation in the presence of measurement error is:

$$MSE^*(t_{10}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S_{Zh}^2}{\theta_{Zh}} + R\frac{S_{Xh}^2}{\theta_{Xh}}\left(R - 2\beta_{ZXh}\theta_{Xh}\right)\right), \tag{3.3.12}$$

where

$$R = \frac{\bar{Z}}{\bar{X}}, \beta_{ZXh} = \frac{S_{YXh}}{S_{Xh}^2} = \frac{S_{ZXh}}{S_{Xh}^2}, S_{YXh} = \sum_{i=1}^{N_h}\frac{(y_{hi} - \bar{Y}_h)(x_{hi} - \bar{X}_h)}{N_h - 1},$$

$$S_{xh}^2 = \sum_{i=1}^{N_h}\frac{(x_{hi} - \bar{X}_h)^2}{N_h - 1} \text{ and } \theta_{xh} = \frac{S_{xh}^2}{S_{vh}^2 + S_{xh}^2}.$$

The mean square error of t_{10} without measurement error may be obtained by putting $S_{Th}^2 = S_{vh}^2 = 0$ in (3.3.12).

(iii) Sousa et al. (2014) Regression Estimator

Sousa et al. (2014) proposed combined regression estimator for the population mean of a sensitive study variable (*Y*), which is given by

$$t_{11} = \bar{z}_{st} + \hat{\beta}_c\left(\bar{X} - \bar{x}_{st}\right). \tag{3.3.13}$$

The mean square error of t_{11} in the presence of measurement error is

$$MSE^*(t_{11}) = \sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Zh}^2}{\theta_{Zh}}\left(1 - \rho_c^2\right), \tag{3.3.14}$$

where

$$\rho_c = \frac{\sum_{h=1}^{L} W_h^2 \gamma_h S_{zxh}}{\sqrt{\sum_{h=1}^{L} W_h^2 \gamma_h S_{Zh}^2 / \theta_{Zh}} \sqrt{\sum_{h=1}^{L} W_h^2 \gamma_h S_{Xh}^2 / \theta_{Xh}}}, \; S_{zxh} = \rho_{zxh} S_{zh} S_{xh}$$

and $\rho_{zxh} = \dfrac{\rho_{yxh}}{\sqrt{1+\left(S_{sh}^2 / S_{yh}^2\right)}}$.

The mean square error of t_{11} without measurement error may be obtained by putting $S_{Th}^2 = S_{vh}^2 = 0$ in (3.3.14).

CHAPTER 4

MEAN ESTIMATION FOR A NON-SENSITIVE STUDY VARIABLE IN THE PRESENCE OF MEASUREMENT ERRORS UNDER SIMPLE RANDOM SAMPLING

4.1 INTRODUCTION

In this Chapter, a class of generalized estimators of population mean for a non-sensitive study variable has been proposed when both the study variable and the auxiliary variable are contaminated by measurement errors, and the sampling design is simple random sampling without replacement. The focus is on studying the impact of measurement errors on the efficiency of mean estimators. We have also compared our proposed estimator with some existing estimators.

4.2 SAMPLING PROCEDURE AND NOTATIONS

Consider a finite population of size N having identifiable units $M=(M_1, M_2, ...M_N)$. A sample of size n is selected from the population by simple random sampling without replacement (SRSWOR) method. Let (y_i, x_i) be the observed values (factoring in measurement errors) and (Y_i, X_i) be the true values on two characteristics Y and X for the ith $(i=1,2,3,...n)$ sample unit. We recall that the measurement errors associated with the non-sensitive study variable and the auxiliary variable, as defined in Equations 1.5.1 and 1.5.2 in Chapter-1 are given by:

$$U_i=(y_i-Y_i),$$

$$V_i=(x_i-X_i),$$

We will use the following notations:

$$\psi_Y=\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right), \tag{4.2.1}$$

$$\psi_U=\sum_{i=1}^{n}U_i, \tag{4.2.2}$$

$$\psi_X = \sum_{i=1}^{n} \left(X_i - \bar{X} \right), \tag{4.2.3}$$

and

$$\psi_V = \sum_{i=1}^{n} V_i. \tag{4.2.4}$$

In (4.2.1), $\left(Y_i - \bar{Y} \right)$ is the deviation of the true Y - values from the population mean of the non- sensitive study variable for the *i*th unit and these deviations are summed over the entire sample.

Adding Equations (4.2.1) and (4.2.2), we have

$$\psi_Y + \psi_U = \sum_{i=1}^{n} \left(Y_i - \bar{Y} \right) + \sum_{i=1}^{n} U_i. \tag{4.2.5}$$

Using Equation (1.5.1) in Equation (4.2.5), we have

$$\psi_Y + \psi_U = \sum_{i=1}^{n} \left(Y_i - \bar{Y} \right) + \sum_{i=1}^{n} \left(y_i - Y_i \right).$$

Dividing both sides by n, we have

$$\frac{1}{n}(\psi_Y + \psi_U) = \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - \bar{Y} \right) + \frac{1}{n} \sum_{i=1}^{n} \left(y_i - Y_i \right),$$

or

$$\frac{1}{n}(\psi_Y + \psi_U) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \bar{Y} \right).$$

After simplification, we get

$$\bar{y} = \bar{Y} + \bar{Y}', \text{ where } \bar{Y}' = \frac{1}{n}\left(\psi_Y + \psi_U \right). \tag{4.2.6}$$

Similarly, from Equations (4.2.3) and (4.2.4), we have

$$\bar{x} = \bar{X} + \bar{X}', \text{ where } \bar{X}' = \frac{1}{n}\left(\psi_X + \psi_V \right). \tag{4.2.7}$$

Squaring both sides of (4.2.5), we have

$$(\psi_Y + \psi_U)^2 = \left[\sum_{i=1}^{n} \left(Y_i - \bar{Y}\right) + \sum_{i=1}^{n} U_i \right]^2,$$

or

$$(\psi_Y + \psi_U)^2 = \left[\left\{ \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right) \right\}^2 + \left\{ \sum_{i=1}^{n} U_i \right\}^2 + 2\left\{ \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right) \right\} \left\{ \sum_{i=1}^{n} U_i \right\} \right].$$

Taking expectation on both sides, we have

$$E(\psi_Y + \psi_U)^2 = E\left[\sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)^2 + \sum_{i=1}^{n} \left(U_i - \bar{U}\right)^2 + 2\left\{ \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right) \right\} \left\{ \sum_{i=1}^{n} U_i \right\} \right].$$

Applying expectation, the cross-product term becomes zero since the true values of non-sensitive study variable Y are independent of measurement error, thus

$$E(\psi_Y + \psi_U)^2 = \sum_{i=1}^{n} E\left(Y_i - \bar{Y}\right)^2 + \sum_{i=1}^{n} E(U_i - \bar{U})^2,$$

or

$$E(\psi_Y + \psi_U)^2 = \sum_{i=1}^{n} S_Y^2 + \sum_{i=1}^{n} S_U^2 = n(S_Y^2 + S_U^2).$$

If the finite population correction factor is used,

$$E(\psi_Y + \psi_U)^2 = n(1 - f)(S_Y^2 + S_U^2),$$

where $f = \frac{n}{N}$. In this situation,

$$E(\psi_Y + \psi_U)^2 = n^2 \frac{(1 - f)}{n}(S_Y^2 + S_U^2),$$

or

$$E(\psi_Y + \psi_U)^2 = n^2 \theta (S_Y^2 + S_U^2),$$

where

$$\theta = \frac{(1 - f)}{n} = \frac{1 - n/N}{n} = \frac{(N - n)}{Nn}.$$

Dividing both sides by n^2, we get

$$E\left(\frac{\Psi_Y+\Psi_U}{n}\right)^2 = E(\bar{Y}')^2 = \theta\left(S_Y^2+S_U^2\right).$$

$$E\left(\frac{\Psi_X+\Psi_V}{n}\right)^2 = E(\bar{X}')^2 = \theta\left(S_X^2+S_V^2\right), \qquad (4.2.8)$$

$$E\left\{\left(\frac{\Psi_Y+\Psi_U}{n}\right)\left(\frac{\Psi_X+\Psi_V}{n}\right)\right\} = E(\bar{Y}'\bar{X}') = \theta\rho_{YX}S_YS_X.$$

4.3 PROPOSED GENERALIZED ESTIMATOR IN THE PRESENCE OF MEASUREMENT ERROR

In this section, we propose a generalized estimator for population mean in the presence of measurement error on both the non-sensitive study variable (*Y*) and the auxiliary variable (*X*) in Simple random sampling without replacement.

The proposed estimator is:

$$\hat{\bar{Y}}_{Pi} = \left[\bar{y}+k\left(\bar{X}-\bar{x}\right)\right]\left[\frac{\bar{D}}{\bar{d}}\right]^g, \qquad (4.3.1)$$

where

$\bar{d} = \lambda\left(a_i\bar{x}+b_i\right)+\left(1-\lambda\right)\left(a_i\bar{X}+b_i\right)$ and $\bar{D} = a_i\bar{X}+b_i$.

Here k and g are suitable constants, and λ is assumed to be an unknown constant whose value is to be determined from optimality considerations. Also a_i and b_i are the parameters or functions of parameters of the auxiliary variable (*X*). Many estimators can be deduced from the proposed class of estimators. For example, with $g=1$, we get various ratio estimators and with $g=-1$, we get various product estimators.

Remark 1:

For $g=1$, $\hat{\bar{Y}}_{Pi}$ can take the following form:

$$\hat{\hat{Y}}_{Pi} = \left[\bar{y} + k\left(\bar{X} - \bar{x}\right)\right]\left[\frac{(a_i\bar{X} + b_i)}{\lambda\left(a_i\bar{x} + b_i\right) + \left(1-\lambda\right)\left(a_i\bar{X} + b_i\right)}\right]. \quad (4.3.2)$$

By setting different values of unknown constants in Equation (4.3.2), various ratio estimators based on single auxiliary variable may be obtained as a family of $\hat{\hat{Y}}_{Pi}$. For example,

(i) By putting $k = 0$ and $\lambda = 1$, we have

$$\hat{\hat{Y}}_{P1} = \bar{y}\left[\frac{(a_i\bar{X} + b_i)}{(a_i\bar{x} + b_i)}\right]. \quad (4.3.3)$$

(ii) By putting $k = 1$ and $\lambda = 1$, we have

$$\hat{\hat{Y}}_{P2} = \left[\bar{y} + (\bar{X} - \bar{x})\right]\left[\frac{(a_i\bar{X} + b_i)}{(a_i\bar{x} + b_i)}\right]. \quad (4.3.4)$$

(iii) By putting $k = b_{YX}$ (the slope term in regression *Y* on *X*) and $\lambda = 1$, we have

$$\hat{\hat{Y}}_{P3} = \left[\bar{y} + b_{YX}(\bar{X} - \bar{x})\right]\left[\frac{(a_i\bar{X} + b_i)}{(a_i\bar{x} + b_i)}\right]. \quad (4.3.5)$$

(iv) By putting $k = 0$ and $\lambda = \lambda_{opt}$ (optimized value of λ relative to the *MSE* of the proposed estimator), we have

$$\hat{\hat{Y}}_{P4} = \bar{y}\left[\frac{(a_i\bar{X} + b_i)}{\lambda_{opt}\left(a_i\bar{x} + b_i\right) + \left(1-\lambda_{opt}\right)\left(a_i\bar{X} + b_i\right)}\right]. \quad (4.3.6)$$

(v) By putting $k = 1$ and $\lambda = \lambda_{opt}$, we have

$$\hat{\hat{Y}}_{P5} = \left[\bar{y} + (\bar{X} - \bar{x})\right]\left[\frac{(a_i\bar{X} + b_i)}{\lambda_{opt}\left(a_i\bar{x} + b_i\right) + \left(1-\lambda_{opt}\right)\left(a_i\bar{X} + b_i\right)}\right]. \quad (4.3.7)$$

(vi) By putting $k = b_{YX}$ and $\lambda = \lambda_{opt}$, we have

$$\hat{\hat{Y}}_{P6} = \left[\bar{y} + b_{YX}(\bar{X} - \bar{x})\right]\left[\frac{(a_i\bar{X} + b_i)}{\lambda_{opt}\left(a_i\bar{x} + b_i\right) + \left(1-\lambda_{opt}\right)\left(a_i\bar{X} + b_i\right)}\right]. \quad (4.3.8)$$

Remark 2:

For $g=-1$, $\hat{\hat{\bar{Y}}}_{Pi}$ can take the following form:

$$\hat{\hat{\bar{Y}}}_{Pi}=\left[\bar{y}+k\left(\bar{X}-\bar{x}\right)\right]\left[\frac{\lambda\left(a_i\bar{x}+b_i\right)+\left(1-\lambda\right)\left(a_i\bar{X}+b_i\right)}{(a_i\bar{X}+b_i)}\right]. \qquad (4.3.9)$$

By setting different values of unknown constants in Equation (4.3.9), various product estimators based on single auxiliary variable may be obtained as a family of $\hat{\hat{\bar{Y}}}_{Pi}$. For example,

(i) By putting $k=0$ and $\lambda=1$, we have

$$\hat{\hat{\bar{Y}}}_{P7}=\bar{y}\left[\frac{(a_i\bar{x}+b_i)}{(a_i\bar{X}+b_i)}\right]. \qquad (4.3.10)$$

(ii) By putting $k=1$ and $\lambda=1$, we have

$$\hat{\hat{\bar{Y}}}_{P8}=\left[\bar{y}+(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{x}+b_i)}{(a_i\bar{X}+b_i)}\right]. \qquad (4.3.11)$$

(iii) By putting $k=b_{YX}$ and $\lambda=1$, we have

$$\hat{\hat{\bar{Y}}}_{P9}=\left[\bar{y}+b_{YX}(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{x}+b_i)}{(a_i\bar{X}+b_i)}\right]. \qquad (4.3.12)$$

(iv) By putting $k=0$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{\bar{Y}}}_{P10}=\bar{y}\left[\frac{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}{(a_i\bar{X}+b_i)}\right]. \qquad (4.3.13)$$

(v) By putting $k=1$and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{\bar{Y}}}_{P11}=\left[\bar{y}+(\bar{X}-\bar{x})\right]\left[\frac{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}{(a_i\bar{X}+b_i)}\right]. \qquad (4.3.14)$$

(vi) By putting $k = b_{YX}$ and $\lambda = \lambda_{opt}$, we have

$$\hat{\hat{Y}}_{P12} = \left[\bar{y} + b_{YX}(\bar{X} - \bar{x})\right]\left[\frac{\lambda_{opt}\left(a_i\bar{x} + b_i\right) + \left(1-\lambda_{opt}\right)\left(a_i\bar{X} + b_i\right)}{(a_i\bar{X} + b_i)}\right]. \quad (4.3.15)$$

4.3.1 The Bias and Mean Square Error of the Proposed Generalized Estimator

Using Equations (4.2.6) and (4.2.7) in Equation (4.3.1), we have

$$\hat{\hat{Y}}_{Pi} = \left[\left(\bar{Y} + \bar{Y}'\right) + k\left(\bar{X} - \left(\bar{X} + \bar{X}'\right)\right)\right]$$
$$\left[\frac{\left(a_i\bar{X} + b_i\right)}{\lambda\left(a_i(\bar{X} + \bar{X}') + b_i\right) + (1-\lambda)\left(a_i\bar{X} + b_i\right)}\right]^g,$$

or

$$\hat{\hat{Y}}_{Pi} = \left[\bar{Y} + \bar{Y}' - k\bar{X}'\right]\left[\frac{\left(a_i\bar{X} + b_i\right)}{\left(a_i\bar{X} + b_i + \lambda a_i\bar{X}'\right)}\right]^g,$$

or

$$\hat{\hat{Y}}_{Pi} = \left[\bar{Y} + \bar{Y}' - k\bar{X}'\right]\left[1 + \frac{\lambda a_i\bar{X}'}{\left(a_i\bar{X} + b_i\right)}\right]^{-g}.$$

By using Taylor series expansion

$$\left(1+x\right)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \ldots,$$

we get

$$\hat{\hat{Y}}_{Pi} \approx \left[\bar{Y} + \bar{Y}' - k\bar{X}'\right]\left[1 - g\frac{\lambda a_i\bar{X}'}{(a_i\bar{X} + b_i)} + \frac{g(g+1)}{2}\left(\frac{\lambda a_i\bar{X}'}{(a_i\bar{X} + b_i)}\right)^2\right]. \quad (4.3.16)$$

In order to derive the expression of bias, using second order approximation, we have

$$\hat{\bar{Y}}_{Pi} - \bar{Y} \approx \frac{1}{\bar{Y}} \left[\frac{g\left(g+1\right)}{2} \left(\frac{\lambda a_i \overline{YX}'}{(a_i \bar{X} + b_i)} \right)^2 - g \frac{\lambda a_i \overline{YY}' \bar{X}'}{(a_i \bar{X} + b_i)} + g \frac{\lambda k a_i \overline{YX}'^2}{(a_i \bar{X} + b_i)} \right].$$

Taking expectation and using Equation (4.2.8), we get

$$E(\hat{\bar{Y}}_{Pi} - \bar{Y}) \approx \frac{\theta}{\bar{Y}} \left[\begin{array}{l} \frac{g\left(g+1\right)}{2} \lambda^2 R_i^2 \left(S_X^2 + S_V^2\right) \\ \qquad + g\lambda k R_i \left(S_X^2 + S_V^2\right) - g\lambda R_i \rho_{YX} S_Y S_X \end{array} \right],$$

where $R_i = \dfrac{a_i \bar{Y}}{(a_i \bar{X} + b_i)}$.

On simplification, we get

$$\begin{aligned} Bias^*(\hat{\bar{Y}}_{Pi}) \approx & \frac{\theta}{\bar{Y}} \left[\frac{g\left(g+1\right)}{2} \lambda^2 R_i^2 S_X^2 + g\lambda k R_i S_X^2 - g\lambda R_i \rho_{YX} S_Y S_X \right] \\ & + \frac{\theta}{\bar{Y}} \left[\frac{g\left(g+1\right)}{2} \lambda^2 R_i^2 S_V^2 + g\lambda k R_i S_V^2 \right], \end{aligned} \quad (4.3.17)$$

or

$$Bias^*(\hat{\bar{Y}}_{Pi}) \approx Bias(\hat{\bar{Y}}_{Pi}) + ME'(\hat{\bar{Y}}_{Pi}), \quad (4.3.18)$$

where

$$Bias(\hat{\bar{Y}}_{Pi}) \approx \frac{\theta}{\bar{Y}} \left[\frac{g\left(g+1\right)}{2} \lambda^2 R_i^2 S_X^2 + g\lambda k R_i S_X^2 - g\lambda R_i \rho_{YX} S_Y S_X \right], \quad (4.3.19)$$

is the bias of proposed estimator ($\hat{\bar{Y}}_{Pi}$) without measurement error, and

$$ME'(\hat{\bar{Y}}_{Pi}) \approx \frac{\theta}{\bar{Y}} \left[\frac{g\left(g+1\right)}{2} \lambda^2 R_i^2 S_V^2 + g\lambda k R_i S_V^2 \right], \quad (4.3.20)$$

is the term expressing the contribution of measurement error.

In order to obtain the mean square error of the proposed estimator, up to the first order approximation, we note that Equation (4.3.16) reduces to

$$\left(\hat{\bar{\bar{Y}}}_{Pi}-\bar{Y}\right)\approx\left[\bar{Y}'-g\lambda\frac{a_i\bar{Y}\bar{X}'}{(a_i\bar{X}+b_i)}-k\bar{X}'\right].$$

Squaring both sides and taking expectation, we have

$$E\left(\hat{\bar{\bar{Y}}}_{Pi}-\bar{Y}\right)^2\approx\left[\begin{array}{r}E(\bar{Y}'^2)+g^2\lambda^2R_i^{\,2}E(\bar{X}'^2)+k^2E(\bar{X}'^2)\\ -2g\lambda R_iE(\bar{Y}'\bar{X}')-2kE(\bar{Y}'\bar{X}')+2g\lambda kR_iE(\bar{X}'^2)\end{array}\right].$$

After simplification, the MSE of the proposed estimator is:

$$\begin{aligned}MSE^*(\hat{\bar{\bar{Y}}}_{Pi})\approx\theta\Big[&S_Y^2+g^2\lambda^2R_i^2S_X^2+k^2S_X^2-2g\lambda R_i\rho_{XY}S_XS_Y\\ &-2k\rho_{XY}S_XS_Y+2g\lambda kR_iS_X^2\Big]\\ &+\theta\Big[S_U^2+g^2\lambda^2R_i^2S_V^2+k^2S_V^2+2g\lambda kR_iS_V^2\Big],\end{aligned}\qquad(4.3.21)$$

or

$$MSE^*(\hat{\bar{\bar{Y}}}_{Pi})\approx MSE(\hat{\bar{\bar{Y}}}_{Pi})+ME(\hat{\bar{\bar{Y}}}_{Pi}),\qquad(4.3.22)$$

where

$$MSE(\hat{\bar{\bar{Y}}}_{Pi})\approx\theta\left[\begin{array}{l}S_Y^2+g^2\lambda^2R_i^2S_X^2+k^2S_X^2\\ \quad-2g\lambda R_i\rho_{XY}S_XS_Y-2k\rho_{XY}S_XS_Y+2g\lambda kR_iS_X^2\end{array}\right],\qquad(4.3.23)$$

is the MSE of the proposed estimator without measurement error, and

$$ME(\hat{\bar{\bar{Y}}}_{Pi})\approx\theta\Big[S_U^2+g^2\lambda^2R_i^2S_V^2+k^2S_V^2+2g\lambda kR_iS_V^2\Big],\qquad(4.3.24)$$

is the term due to measurement error.

To find the optimal value of λ, we differentiate the expression in Equation (4.3.21) with respect to λ, and then equate to zero, to get

$$\lambda=\frac{\rho_{YX}S_YS_X-k(S_X^2+S_V^2)}{gR_i(S_X^2+S_V^2)}=\lambda_{opt}\qquad(4.3.25)$$

Substitution of (4.3.25) in (4.3.21) yields the minimum MSE of $(\hat{\bar{\bar{Y}}}_{Pi})$ as:

$$MSE^*_{\min}(\hat{\bar{Y}}_{Pi}) \approx \theta\left(S_Y^2 + S_U^2 - \frac{\rho_{YX}^2 S_Y^2 S_X^2}{\left(S_X^2 + S_V^2\right)} \right). \tag{4.3.26}$$

The expression of minimized MSE of proposed estimator without measurement error can be obtained by putting $S_U^2 = S_V^2 = 0$ in Equation (4.3.26). It is given by,

$$MSE_{\min}(\hat{\bar{Y}}_{Pi}) = \theta S_Y^2 \left(1 - \rho_{YX}^2\right). \tag{4.3.27}$$

Note that this expression of $MSE_{\min}(\hat{\bar{Y}}_{Pi})$ is same as that of the approximate variance of the usual linear regression estimator.

4.4 ADDITIONAL SPECIAL CASES OF THE GENERALIZED RATIO ESTIMATOR

Many additional estimators can be deduced from the generalized ratio estimator $\hat{\bar{Y}}_{P1}$ given in Equation (4.3.3). We denote the generalized ratio estimator by,

$$\hat{\bar{Y}}_{p1}^{j} = \bar{y}\left[\frac{(a_j \bar{X} + b_j)}{(a_j \bar{x} + b_j)} \right].$$

Various choices of a_j, b_j are given in the table below. The general expressions for the MSE and bias respectively with measurement error for the generalized ratio estimator $\hat{\bar{Y}}_{p1}^{j}$ are given by

$$\left.\begin{aligned} MSE^*(\hat{\bar{Y}}_{p1}^{j}) &\approx \theta\left(S_Y^2 + R_j^2 S_X^2 - 2R_j \rho_{YX} S_X S_Y\right) + \theta\left(S_U^2 + R_j^2 S_V^2\right), \\ Bias^*(\hat{\bar{Y}}_{p1}^{j}) &\approx \frac{\theta}{\bar{Y}}\left(R_j^2 S_x^2 - R_j \rho_{YX} S_x S_y\right) + \frac{\theta}{\bar{Y}}\left(R_j^2 S_v^2\right), \\ R_j &= \frac{a_j \bar{Y}}{(a_j \bar{X} + b_j)}. \end{aligned}\right] \tag{4.4.1}$$

Table 4.1

Additional Special Cases of the Generalized Ratio Estimator ($\hat{\bar{Y}}_{P1}^{j}$)

Proposed Estimators	a_j	b_j	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{p1}^{1} = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$	1	0	$MSE^*(\hat{\bar{Y}}_{p1}^{1}) = \theta\left(S_y^2 + R_1^2 S_x^2 - 2R_1\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_1^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{1}) \approx \frac{\theta}{\bar{Y}}\left(R_1^2 S_x^2 - R_1\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_1^2 S_v^2\right)$	$R_1 = \frac{\bar{Y}}{\bar{X}}$
$\hat{\bar{Y}}_{p1}^{2} = \bar{y}\left(\frac{\bar{X} + C_x}{\bar{x} + C_x}\right)$	1	C_x	$MSE^*(\hat{\bar{Y}}_{p1}^{2}) = \theta\left(S_y^2 + R_2^2 S_x^2 - 2R_2\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_2^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{2}) \approx \frac{\theta}{\bar{Y}}\left(R_2^2 S_x^2 - R_2\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_2^2 S_v^2\right)$	$R_2 = \frac{\bar{Y}}{\bar{X} + C_x}$
$\hat{\bar{Y}}_{p1}^{3} = \bar{y}\left(\frac{\bar{X} + \beta_2(x)}{\bar{x} + \beta_2(x)}\right)$	1	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{p1}^{3}) = \theta\left(S_y^2 + R_3^2 S_x^2 - 2R_3\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_3^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{3}) \approx \frac{\theta}{\bar{Y}}\left(R_3^2 S_x^2 - R_3\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_3^2 S_v^2\right)$	$R_3 = \frac{\bar{Y}}{\bar{X} + \beta_2(x)}$
$\hat{\bar{Y}}_{p1}^{4} = \bar{y}\left(\frac{\bar{X}\beta_2(x) + C_x}{\bar{x}\beta_2(x) + C_x}\right)$	$\beta_2(x)$	C_x	$MSE^*(\hat{\bar{Y}}_{p1}^{4}) = \theta\left(S_y^2 + R_4^2 S_x^2 - 2R_4\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_4^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{4}) \approx \frac{\theta}{\bar{Y}}\left(R_4^2 S_x^2 - R_4\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_4^2 S_v^2\right)$	$R_4 = \frac{\bar{Y}\beta_2(x)}{\bar{X}\beta_2(x) + C_x}$
$\hat{\bar{Y}}_{p1}^{5} = \bar{y}\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}C_x + \beta_2(x)}\right)$	C_x	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{p1}^{5}) = \theta\left(S_y^2 + R_5^2 S_x^2 - 2R_5\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_5^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{5}) \approx \frac{\theta}{\bar{Y}}\left(R_5^2 S_x^2 - R_5\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_5^2 S_v^2\right)$	$R_5 = \frac{\bar{Y}C_x}{\bar{X}C_x + \beta_2(x)}$
$\hat{\bar{Y}}_{p1}^{6} = \bar{y}\left(\frac{\bar{X} + \rho_{XY}}{\bar{x} + \rho_{XY}}\right)$	1	ρ_{XY}	$MSE^*(\hat{\bar{Y}}_{p1}^{6}) = \theta\left(S_y^2 + R_6^2 S_x^2 - 2R_6\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_6^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{6}) \approx \frac{\theta}{\bar{Y}}\left(R_6^2 S_x^2 - R_6\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_6^2 S_v^2\right)$	$R_6 = \frac{\bar{Y}}{\bar{X} + \rho_{xy}}$

Proposed Estimators	a_j	b_j	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{p1}^{7} = \bar{y}\left(\frac{\bar{X}C_x + \rho_{XY}}{\bar{x}C_x + \rho_{XY}}\right)$	C_x	ρ_{XY}	$MSE^*(\hat{\bar{Y}}_{p1}^{7}) = \theta\left(S_y^2 + R_7^2 S_x^2 - 2R_7\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_7^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{7}) \approx \frac{\theta}{\bar{Y}}\left(R_7^2 S_x^2 - R_7\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_7^2 S_v^2\right)$	$R_7 = \frac{\bar{Y}C_x}{\bar{X}C_x + \rho_{xy}}$
$\hat{\bar{Y}}_{p1}^{8} = \bar{y}\left(\frac{\bar{X}\rho_{XY} + C_x}{\bar{x}\rho_{XY} + C_x}\right)$	ρ_{XY}	C_x	$MSE^*(\hat{\bar{Y}}_{p1}^{8}) = \theta\left(S_y^2 + R_8^2 S_x^2 - 2R_8\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_8^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{8}) \approx \frac{\theta}{\bar{Y}}\left(R_8^2 S_x^2 - R_8\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_8^2 S_v^2\right)$	$R_8 = \frac{\bar{Y}\rho_{xy}}{\bar{X}\rho_{xy} + C_x}$
$\hat{\bar{Y}}_{p1}^{9} = \bar{y}\left(\frac{\bar{X}\beta_2(x) + \rho_{XY}}{\bar{x}\beta_2(x) + \rho_{XY}}\right)$	$\beta_2(x)$	ρ_{XY}	$MSE^*(\hat{\bar{Y}}_{p1}^{9}) = \theta\left(S_y^2 + R_9^2 S_x^2 - 2R_9\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_9^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{9}) \approx \frac{\theta}{\bar{Y}}\left(R_9^2 S_x^2 - R_9\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_9^2 S_v^2\right)$	$R_9 = \frac{\bar{Y}\beta_2(x)}{\bar{X}\beta_2(x) + \rho_{xy}}$
$\hat{\bar{Y}}_{p1}^{10} = \bar{y}\left(\frac{\bar{X}\rho_{XY} + \beta_2(x)}{\bar{x}\rho_{XY} + \beta_2(x)}\right)$	ρ_{XY}	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{p1}^{10}) = \theta\left(S_y^2 + R_{10}^2 S_x^2 - 2R_{10}\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_{10}^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{10}) \approx \frac{\theta}{\bar{Y}}\left(R_{10}^2 S_x^2 - R_{10}\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_{10}^2 S_v^2\right)$	$R_{10} = \frac{\bar{Y}\rho_{xy}}{\bar{X}\rho_{xy} + \beta_2(x)}$
$\hat{\bar{Y}}_{p1}^{11} = \bar{y}\left(\frac{\bar{X} + \beta_1(x)}{\bar{x} + \beta 1(x)}\right)$	1	$\beta_1(x)$	$MSE^*(\hat{\bar{Y}}_{p1}^{11}) = \theta\left(S_y^2 + R_{11}^2 S_x^2 - 2R_{11}\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_{11}^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{11}) \approx \frac{\theta}{\bar{Y}}\left(R_{11}^2 S_x^2 - R_{11}\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_{11}^2 S_v^2\right)$	$R_{11} = \frac{\bar{Y}}{\bar{X} + \beta_1(x)}$
$\hat{\bar{Y}}_{p1}^{12} = \bar{y}\left(\frac{\bar{X}\beta_1(x) + \beta_2(x)}{\bar{x}\beta 1(x) + \beta_2(x)}\right)$	$\beta_1(x)$	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{p1}^{12}) = \theta\left(S_y^2 + R_{12}^2 S_x^2 - 2R_{12}\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_{12}^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{12}) \approx \frac{\theta}{\bar{Y}}\left(R_{12}^2 S_x^2 - R_{12}\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_{12}^2 S_v^2\right)$	$R_{12} = \frac{\bar{Y}\beta_1(x)}{\bar{X}\beta_1(x) + \beta_2(x)}$
$\hat{\bar{Y}}_{p1}^{13} = \bar{y}\left(\frac{\bar{X} + Q_2}{\bar{x} + Q_2}\right)$	1	Q_2	$MSE^*(\hat{\bar{Y}}_{p1}^{13}) = \theta\left(S_y^2 + R_{13}^2 S_x^2 - 2R_{13}\rho_{xy}S_xS_y\right) + \theta\left(S_u^2 + R_{13}^2 S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{13}) \approx \frac{\theta}{\bar{Y}}\left(R_{13}^2 S_x^2 - R_{13}\rho_{xy}S_xS_y\right) + \frac{\theta}{\bar{Y}}\left(R_{13}^2 S_v^2\right)$	$R_{13} = \frac{\bar{Y}}{\bar{X} + Q_2}$

Proposed Estimators	a_j	b_j	**Mean Square Errors & Biases**	**Ratio's**
$\hat{\bar{Y}}_{p1}^{14}=\bar{y}\left(\frac{\bar{X}C_x+Q_2}{\bar{x}C_x+Q_2}\right)$	C_x	Q_2	$MSE^*(\hat{\bar{Y}}_{p1}^{14})=\theta\left(S_y^2+R_{14}^2S_x^2-2R_{14}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{14}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{14})\approx\frac{\theta}{\bar{Y}}\left(R_{14}^2S_x^2-R_{14}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{14}^2S_v^2\right)$	$R_{14}=\frac{\bar{Y}C_x}{\bar{X}C_x+Q_2}$
$\hat{\bar{Y}}_{p1}^{15}=\bar{y}\left(\frac{\bar{X}\beta_1(x)+Q_2}{\bar{x}\beta_1(x)+Q_2}\right)$	$\beta_1(x)$	Q_2	$MSE^*(\hat{\bar{Y}}_{p1}^{15})=\theta\left(S_y^2+R_{15}^2S_x^2-2R_{15}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{15}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{15})\approx\frac{\theta}{\bar{Y}}\left(R_{15}^2S_x^2-R_{15}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{15}^2S_v^2\right)$	$R_{15}=\frac{\bar{Y}\beta_1(x)}{\bar{X}\beta_1(x)+Q_2}$
$\hat{\bar{Y}}_{p1}^{16}=\bar{y}\left(\frac{\bar{X}\beta_2(x)+Q_2}{\bar{x}\beta_2(x)+Q_2}\right)$	$\beta_2(x)$	Q_2	$MSE^*(\hat{\bar{Y}}_{p1}^{16})=\theta\left(S_y^2+R_{16}^2S_x^2-2R_{16}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{16}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{16})\approx\frac{\theta}{\bar{Y}}\left(R_{16}^2S_x^2-R_{16}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{16}^2S_v^2\right)$	$R_{16}=\frac{\bar{Y}\beta_2(x)}{\bar{X}\beta_2(x)+Q_2}$
$\hat{\bar{Y}}_{p1}^{17}=\bar{y}\left(\frac{\bar{X}\beta_1(x)+QD}{\bar{x}\beta_1(x)+QD}\right)$	$\beta_1(x)$	QD	$MSE^*(\hat{\bar{Y}}_{p1}^{17})=\theta\left(S_y^2+R_{17}^2S_x^2-2R_{17}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{17}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{17})\approx\frac{\theta}{\bar{Y}}\left(R_{17}^2S_x^2-R_{17}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{17}^2S_v^2\right)$	$R_{17}=\frac{\bar{Y}\beta_1(x)}{\bar{X}\beta_1(x)+QD}$
$\hat{\bar{Y}}_{p1}^{18}=\bar{y}\left(\frac{\bar{X}\beta_2(x)+QD}{\bar{x}\beta_2(x)+QD}\right)$	$\beta_2(x)$	QD	$MSE^*(\hat{\bar{Y}}_{p1}^{18})=\theta\left(S_y^2+R_{18}^2S_x^2-2R_{18}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{18}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{18})\approx\frac{\theta}{\bar{Y}}\left(R_{18}^2S_x^2-R_{18}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{18}^2S_v^2\right)$	$R_{18}=\frac{\bar{Y}\beta_2(x)}{\bar{X}\beta_2(x)+QD}$
$\hat{\bar{Y}}_{p1}^{19}=\bar{y}\left(\frac{\bar{X}+TM}{\bar{x}+TM}\right)$	1	TM	$MSE^*(\hat{\bar{Y}}_{p1}^{19})=\theta\left(S_y^2+R_{19}^2S_x^2-2R_{19}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{19}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{19})\approx\frac{\theta}{\bar{Y}}\left(R_{19}^2S_x^2-R_{19}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{19}^2S_v^2\right)$	$R_{19}=\frac{\bar{Y}}{\bar{X}+TM}$
$\hat{\bar{Y}}_{p1}^{20}=\bar{y}\left(\frac{\bar{X}C_x+TM}{\bar{x}C_x+TM}\right)$	C_x	TM	$MSE^*(\hat{\bar{Y}}_{p1}^{20})=\theta\left(S_y^2+R_{20}^2S_x^2-2R_{20}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{20}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{20})\approx\frac{\theta}{\bar{Y}}\left(R_{20}^2S_x^2-R_{20}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{20}^2S_v^2\right)$	$R_{20}=\frac{\bar{Y}C_x}{\bar{X}C_x+TM}$

Proposed Estimators	a_j	b_j	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{p1}^{21}=\bar{y}\left(\frac{\bar{X}\rho_{XY}+TM}{\bar{x}\rho_{XY}+TM}\right)$	ρ_{XY}	TM	$MSE^*(\hat{\bar{Y}}_{p1}^{21})=\theta\left(S_y^2+R_{21}^2S_x^2-2R_{21}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{21}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{21})\approx\frac{\theta}{\bar{Y}}\left(R_{21}^2S_x^2-R_{21}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{21}^2S_v^2\right)$	$R_{21}=\frac{\bar{Y}\rho_{xy}}{\bar{X}\rho_{xy}+TM}$
$\hat{\bar{Y}}_{p1}^{22}=\bar{y}\left(\frac{\bar{X}+MR}{\bar{x}+MR}\right)$	1	MR	$MSE^*(\hat{\bar{Y}}_{p1}^{22})=\theta\left(S_y^2+R_{22}^2S_x^2-2R_{22}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{22}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{22})\approx\frac{\theta}{\bar{Y}}\left(R_{22}^2S_x^2-R_{22}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{22}^2S_v^2\right)$	$R_{22}=\frac{\bar{Y}}{\bar{X}+MR}$
$\hat{\bar{Y}}_{p1}^{23}=\bar{y}\left(\frac{\bar{X}C_x+MR}{\bar{x}C_x+MR}\right)$	C_x	MR	$MSE^*(\hat{\bar{Y}}_{p1}^{23})=\theta\left(S_y^2+R_{23}^2S_x^2-2R_{23}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{23}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{23})\approx\frac{\theta}{\bar{Y}}\left(R_{23}^2S_x^2-R_{23}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{23}^2S_v^2\right)$	$R_{23}=\frac{\bar{Y}C_x}{\bar{X}C_x+MR}$
$\hat{\bar{Y}}_{p1}^{24}=\bar{y}\left(\frac{\bar{X}\rho_{XY}+MR}{\bar{x}\rho_{XY}+MR}\right)$	ρ_{XY}	MR	$MSE^*(\hat{\bar{Y}}_{p1}^{24})=\theta\left(S_y^2+R_{24}^2S_x^2-2R_{24}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{24}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{24})\approx\frac{\theta}{\bar{Y}}\left(R_{24}^2S_x^2-R_{24}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{24}^2S_v^2\right)$	$R_{24}=\frac{\bar{Y}\rho_{xy}}{\bar{X}\rho_{xy}+MR}$
$\hat{\bar{Y}}_{p1}^{25}=\bar{y}\left(\frac{\bar{X}+HL}{\bar{x}+HL}\right)$	1	HL	$MSE^*(\hat{\bar{Y}}_{p1}^{25})=\theta\left(S_y^2+R_{25}^2S_x^2-2R_{25}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{25}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{25})\approx\frac{\theta}{\bar{Y}}\left(R_{25}^2S_x^2-R_{25}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{25}^2S_v^2\right)$	$R_{25}=\frac{\bar{Y}}{\bar{X}+HL}$
$\hat{\bar{Y}}_{p1}^{26}=\bar{y}\left(\frac{\bar{X}C_x+HL}{\bar{x}C_x+HL}\right)$	C_x	HL	$MSE^*(\hat{\bar{Y}}_{p1}^{26})=\theta\left(S_y^2+R_{26}^2S_x^2-2R_{26}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{26}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{26})\approx\frac{\theta}{\bar{Y}}\left(R_{26}^2S_x^2-R_{26}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{26}^2S_v^2\right)$	$R_{26}=\frac{\bar{Y}C_x}{\bar{X}C_x+HL}$
$\hat{\bar{Y}}_{p1}^{27}=\bar{y}\left(\frac{\bar{X}\rho_{XY}+HL}{\bar{x}\rho_{XY}+HL}\right)$	ρ_{XY}	HL	$MSE^*(\hat{\bar{Y}}_{p1}^{27})=\theta\left(S_y^2+R_{27}^2S_x^2-2R_{27}\rho_{xy}S_xS_y\right)+\theta\left(S_u^2+R_{27}^2S_v^2\right)$ $Bias^*(\hat{\bar{Y}}_{p1}^{27})\approx\frac{\theta}{\bar{Y}}\left(R_{27}^2S_x^2-R_{27}\rho_{xy}S_xS_y\right)+\frac{\theta}{\bar{Y}}\left(R_{27}^2S_v^2\right)$	$R_{27}=\frac{\bar{Y}\rho_{xy}}{\bar{X}\rho_{xy}+HL}$

Various terms used in Table 4.1 are described in the Appendix-A.

4.5 EFFICIENCY COMPARISON

To check the efficiency of the proposed generalized ratio estimator ($\hat{\bar{Y}}_{p1}^{j}$) against mean per unit estimator ($\bar{y}$), the mathematical conditions have been derived by using MSE formulas with measurement errors. The algebraic expressions are:

$$MSE^{*}(\hat{\bar{Y}}_{p1}^{j}) \leq Var^{*}(\bar{y})$$

if

$$\theta\left(S_Y^2 + R_j^2 S_X^2 - 2R_j \rho_{YX} S_X S_Y\right) + \theta\left(S_U^2 + R_j^2 S_V^2\right) \leq \theta\left(S_Y^2 + S_U^2\right),$$

or if

$$R_j S_X^2 + R_j S_V^2 \leq 2\rho_{XY} S_X S_Y,$$

or if

$$\rho_{XY} \geq \frac{R_j}{2}\left(\frac{S_X^2 + S_V^2}{S_X S_Y}\right) \tag{4.5.1}$$

When observations are recorded without measurement errors (ME), Condition (4.5.1) reduces to

$$\rho_{XY} \geq \frac{R_j}{2}\frac{S_X}{S_Y}. \tag{4.5.2}$$

4.6 SIMULATION RESULTS

In this section, we conduct a simulation study with particular focus on the following two issues:

a. How does the generalized estimator ($\hat{\bar{Y}}_{pi}$) and the generalized ratio estimator ($\hat{\bar{Y}}_{p1}^{j}$) compare with the mean per unit estimator ($\bar{y}$) in the presence and absence of measurement errors?
b. How are the MSE, PRE and bias, influenced with the contribution of measurement errors?

We have generated two populations from bivariate normal distribution with different choices of parameters by using R-Program. The data statistics for each population are given below.

POPULATION-I:

True values: $X = N(5,10),\ Y = X + N(0,1)$,

Observed values: $y = Y + N(1,3), x = X + N(1,3)$

POPULATION-II:

True values: $X = N(5,10), Y = X + N(0,1)$,

Observed values: $y = Y + N(1,5), x = X + N(1,5)$

From both the generated populations, we consider three different sample sizes $n = 500, 750$ and 1000. The following steps summarize the simulation procedures used to find the empirical MSE's of any specific estimator.

Step-I: Fifty thousand samples of size *n* were selected from generated populations by using simple random sampling without replacement.

Step-II: Using the data from Step-I, estimates $(\hat{\bar{Y}}^*)$ are obtained for each sample size.

Step-III: The empirical MSE of the estimators is computed by

$$EMSE(\hat{\bar{Y}}^*) = \frac{1}{50000} \sum_{i=1}^{50000} (\hat{\bar{Y}}^* - \bar{Y})^2,$$

where $\hat{\bar{Y}}^*$ is the estimator, deduced from Equation (4.3.1) and $\bar{Y}$ is the population mean of the study variable. The percent relative efficiency (PRE) of the estimators under study is calculated by using following equation:

$$PRE = \frac{VAR(\bar{y})}{MSE(\hat{\bar{Y}}^*)} \times 100,$$

The MSE's, PRE's and biases of the estimators for generated populations on different sampling fractions are presented in Tables (4.2-4.7).

Table 4.2
Theoretical (boldface) and Empirical MSE's, PRE's (without ME) of the Estimators Relative to Mean Per Unit Estimator in Simple Random Sampling for Population-I

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f = n / N$			$f = n / N$		
	500/5000	750/5000	1000/5000	500/5000	750/5000	1000/5000
$\bar{y}$	**0.1779** 0.1783	**0.1179** 0.1181	**0.0787** 0.0780	100.00	100.00	100.00
$\hat{\bar{Y}}_{pi}$	**0.0016** 0.0017	**0.0011** 0.0011	**0.0007** 0.0006	11118.75	10718.1818	11242.86
$\hat{\bar{Y}}_{p1}^{1}$	**0.0018** 0.0018	**0.0012** 0.0012	**0.0008** 0.0008	9883.33	9825.00	9837.50
$\hat{\bar{Y}}_{p1}^{2}$	**0.0153** 0.0156	**0.0107** 0.0108	**0.0074** 0.0075	1162.75	1101.87	1063.51
$\hat{\bar{Y}}_{p1}^{3}$	**0.0252** 0.0256	**0.0175** 0.0177	**0.0119** 0.0119	705.95	673.71	661.34
$\hat{\bar{Y}}_{p1}^{4}$	**0.0042** 0.0043	**0.0028** 0.0029	**0.0019** 0.0019	4235.71	4210.72	4142.11
$\hat{\bar{Y}}_{p1}^{5}$	**0.0111** 0.0113	**0.0073** 0.0075	**0.0049** 0.0050	1602.70	1615.06	1606.12
$\hat{\bar{Y}}_{p1}^{6}$	**0.0066** 0.0068	**0.0044** 0.0045	**0.0030** 0.0031	2695.45	2679.54	2623.33
$\hat{\bar{Y}}_{p1}^{7}$	**0.0034** 0.0035	**0.0021** 0.0022	**0.0015** 0.0015	5232.35	5614.29	5246.67
$\hat{\bar{Y}}_{p1}^{8}$	**0.0154** 0.0157	**0.0107** 0.0109	**0.0075** 0.0075	1155.19	1101.86	1049.33
$\hat{\bar{Y}}_{p1}^{9}$	**0.0026** 0.0027	**0.0016** 0.0017	**0.0011** 0.0011	6842.31	7368.75	7154.55
$\hat{\bar{Y}}_{p1}^{10}$	**0.0253** 0.0258	**0.0176** 0.0178	**0.0119** 0.0119	703.16	669.89	661.34
$\hat{\bar{Y}}_{p1}^{11}$	**0.0018** 0.0018	**0.0012** 0.0013	**0.0008** 0.0008	9883.33	9825.00	9837.50
$\hat{\bar{Y}}_{p1}^{12}$	**0.1979** 0.1985	**0.0943** 0.0945	**0.0838** 0.0830	89.89	125.03	93.91
$\hat{\bar{Y}}_{p1}^{13}$	**0.0474** 0.0480	**0.0291** 0.0294	**0.0206** 0.0205	375.32	405.15	382.04
$\hat{\bar{Y}}_{p1}^{14}$	**0.0233** 0.0237	**0.0134** 0.0136	**0.0095** 0.0096	763.52	879.85	828.42

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f = n / N$			$f = n / N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\hat{\bar{Y}}_{p1}^{15}$	**0.1886** 0.1890	**0.1023** 0.1025	**0.0817** 0.0809	94.33	115.25	96.32
$\hat{\bar{Y}}_{p1}^{16}$	**0.0141** 0.0144	**0.0080** 0.0082	**0.0058** 0.0059	1261.70	1473.75	1356.89
$\hat{\bar{Y}}_{p1}^{17}$	**0.1862** 0.1867	**0.1067** 0.1069	**0.0809** 0.0802	95.54	110.49	97.28
$\hat{\bar{Y}}_{p1}^{18}$	**0.0192** 0.0196	**0.0127** 0.0130	**0.0085** 0.0085	926.56	928.34	925.88
$\hat{\bar{Y}}_{p1}^{19}$	**0.0469** 0.0475	**0.0296** 0.0299	**0.0205** 0.0204	379.32	398.31	383.90
$\hat{\bar{Y}}_{p1}^{20}$	**0.0230** 0.0235	**0.0137** 0.0139	**0.0095** 0.0095	773.48	860.58	828.42
$\hat{\bar{Y}}_{p1}^{21}$	**0.0472** 0.0478	**0.0297** 0.0301	**0.0206** 0.0205	376.91	396.96	382.04
$\hat{\bar{Y}}_{p1}^{22}$	**0.0499** 0.0505	**0.0342** 0.0345	**0.0109** 0.0109	356.51	344.94	722.02
$\hat{\bar{Y}}_{p1}^{23}$	**0.0249** 0.0254	**0.0165** 0.0167	**0.0045** 0.0045	714.46	714.55	1748.88
$\hat{\bar{Y}}_{p1}^{24}$	**0.0502** 0.0507	**0.0344** 0.0347	**0.0110** 0.0111	354.38	342.73	715.45
$\hat{\bar{Y}}_{p1}^{25}$	**0.0464** 0.0469	**0.0301** 0.0304	**0.0204** 0.0203	383.41	391.69	385.78
$\hat{\bar{Y}}_{p1}^{26}$	**0.0226** 0.0231	**0.0139** 0.0140	**0.0094** 0.0095	787.17	848.20	837.23
$\hat{\bar{Y}}_{p1}^{27}$	**0.0466** 0.0471	**0.0303** 0.0305	**0.0205** 0.0204	381.76	389.11	383.90

Table 4.3
Theoretical (boldface) and Empirical MSE's, PRE's (with ME) of the Estimators Relative to Mean Per Unit Estimator in Simple Random Sampling for Population-I

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f = n / N$			$f = n / N$		
	500/5000	750/5000	1000/5000	500/5000	750/5000	1000/5000
$\bar{y}$	**0.1941** 0.1946	**0.1283** 0.1281	**0.0859** 0.0860	100.00	100.00	100.00
$\hat{\bar{Y}}_{pi}$	**0.0308** 0.0282	**0.0200** 0.0147	**0.0144** 0.0100	630.19	641.50	596.53
$\hat{\bar{Y}}_{p1}^{1}$	**0.0341** 0.0243	**0.0217** 0.0151	**0.0153** 0.0105	569.21	591.24	561.44
$\hat{\bar{Y}}_{p1}^{2}$	**0.0398** 0.0297	**0.0262** 0.0189	**0.0183** 0.0128	487.69	489.69	469.40
$\hat{\bar{Y}}_{p1}^{3}$	**0.0478** 0.0352	**0.0317** 0.0230	**0.0219** 0.0155	406.07	404.73	392.24
$\hat{\bar{Y}}_{p1}^{4}$	**0.0329** 0.0253	**0.0211** 0.0152	**0.0148** 0.0104	589.97	608.06	580.41
$\hat{\bar{Y}}_{p1}^{5}$	**0.0368** 0.0277	**0.0237** 0.0171	**0.0165** 0.0116	527.45	514.35	520.61
$\hat{\bar{Y}}_{p1}^{6}$	**0.0340** 0.0251	**0.0218** 0.0157	**0.0153** 0.0107	570.88	588.53	561.44
$\hat{\bar{Y}}_{p1}^{7}$	**0.0327** 0.0248	**0.0209** 0.0150	**0.0147** 0.0103	593.58	613.88	584.35
$\hat{\bar{Y}}_{p1}^{8}$	**0.0319** 0.0293	**0.0262** 0.0189	**0.0183** 0.0128	608.46	489.69	469.40
$\hat{\bar{Y}}_{p1}^{9}$	**0.0328** 0.0259	**0.0209** 0.0149	**0.0147** 0.0103	591.77	613.88	584.35
$\hat{\bar{Y}}_{p1}^{10}$	**0.0479** 0.0352	**0.0318** 0.0231	**0.0220** 0.0155	405.22	403.46	390.45
$\hat{\bar{Y}}_{p1}^{11}$	**0.0343** 0.0244	**0.0214** 0.0150	**0.0153** 0.0105	565.89	599.53	561.44
$\hat{\bar{Y}}_{p1}^{12}$	**0.2140** 0.2239	**0.1047** 0.0968	**0.0910** 0.0935	90.70	122.54	94.40
$\hat{\bar{Y}}_{p1}^{13}$	**0.0674** 0.0509	**0.0421** 0.0313	**0.0296** 0.0215	287.98	304.75	290.20
$\hat{\bar{Y}}_{p1}^{14}$	**0.0462** 0.0397	**0.0284** 0.0205	**0.0200** 0.0141	420.13	451.76	429.50

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f = n / N$			$f = n / N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\hat{\bar{Y}}_{p1}^{15}$	**0.2046** 0.2099	**0.1127** 0.1071	**0.0889** 0.0904	94.87	113.84	96.63
$\hat{\bar{Y}}_{p1}^{16}$	**0.0389** 0.0286	**0.0242** 0.0175	**0.0171** 0.0120	498.97	530.17	502.34
$\hat{\bar{Y}}_{p1}^{17}$	**0.2022** 0.2064	**0.1171** 0.1128	**0.0881** 0.0892	95.99	109.56	97.50
$\hat{\bar{Y}}_{p1}^{18}$	**0.0429** 0.0351	**0.0278** 0.0202	**0.0191** 0.0134	452.45	461.51	449.74
$\hat{\bar{Y}}_{p1}^{19}$	**0.0670** 0.0506	**0.0425** 0.0317	**0.0295** 0.0214	289.70	301.88	291.19
$\hat{\bar{Y}}_{p1}^{20}$	**0.0461** 0.0338	**0.0286** 0.0207	**0.0199** 0.0140	421.04	448.60	431.66
$\hat{\bar{Y}}_{p1}^{21}$	**0.0672** 0.0508	**0.0426** 0.0381	**0.0296** 0.0215	288.84	301.17	290.20
$\hat{\bar{Y}}_{p1}^{22}$	**0.0696** 0.0589	**0.0467** 0.0353	**0.0211** 0.0148	278.88	274.73	407.11
$\hat{\bar{Y}}_{p1}^{23}$	**0.0475** 0.0349	**0.0309** 0.0224	**0.0162** 0.0114	408.63	415.21	530.25
$\hat{\bar{Y}}_{p1}^{24}$	**0.0598** 0.0531	**0.0469** 0.0354	**0.0212** 0.0150	324.58	273.56	405.19
$\hat{\bar{Y}}_{p1}^{25}$	**0.0664** 0.0501	**0.0430** 0.0321	**0.0294** 0.0213	292.32	298.37	292.18
$\hat{\bar{Y}}_{p1}^{26}$	**0.0457** 0.0353	**0.0288** 0.0218	**0.0190** 0.0139	424.73	445.49	452.11
$\hat{\bar{Y}}_{p1}^{27}$	**0.0666** 0.0563	**0.0431** 0.0321	**0.0291** 0.0214	291.44	297.68	295.19

Table 4.4
Theoretical Biases (with/without ME) of the Estimators in Simple Random Sampling for Population-I

Estimators	Bias (without ME)			Bias (with ME)		
	$f = n / N$			$f = n / N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\bar{y}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{pi}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{p1}^{1}$	-0.00008	-0.00004	-0.000001	0.0031	0.0019	0.0015
$\hat{\bar{Y}}_{p1}^{2}$	-0.0069	-0.0047	-0.0033	-0.0052	-0.0036	-0.0025
$\hat{\bar{Y}}_{p1}^{3}$	-0.0079	-0.0054	-0.0037	-0.0067	-0.0046	-0.0032
$\hat{\bar{Y}}_{p1}^{4}$	-0.0036	-0.0024	-0.0017	-0.0011	-0.0008	-0.0005
$\hat{\bar{Y}}_{p1}^{5}$	-0.0061	-0.0041	-0.0028	-0.0042	-0.0028	-0.0019
$\hat{\bar{Y}}_{p1}^{6}$	-0.0047	-0.0032	-0.0022	-0.0025	-0.0018	-0.0012
$\hat{\bar{Y}}_{p1}^{7}$	-0.0029	-0.0019	-0.0013	-0.0003	-0.0002	-0.0001
$\hat{\bar{Y}}_{p1}^{8}$	-0.0069	-0.0047	-0.0033	-0.0052	-0.0036	-0.0025
$\hat{\bar{Y}}_{p1}^{9}$	-0.0021	-0.0014	-0.0009	0.0007	0.0004	0.0004
$\hat{\bar{Y}}_{p1}^{10}$	-0.0080	-0.0054	-0.0037	-0.0067	-0.0046	-0.0032
$\hat{\bar{Y}}_{p1}^{11}$	0.0001	-0.0003	-0.00006	0.0033	0.0016	0.0016
$\hat{\bar{Y}}_{p1}^{12}$	0.0020	-0.0022	0.0005	0.0020	-0.0021	0.0005
$\hat{\bar{Y}}_{p1}^{13}$	-0.0086	-0.0058	-0.0039	-0.0078	-0.0052	-0.0036
$\hat{\bar{Y}}_{p1}^{14}$	-0.0078	-0.0051	-0.0035	-0.0065	-0.0041	-0.0029
$\hat{\bar{Y}}_{p1}^{15}$	0.0011	-0.0015	0.0003	0.0011	-0.0015	0.0003

Estimators	Bias (without ME)			Bias (with ME)		
	$f = n / N$			$f = n / N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\hat{\bar{Y}}_{p1}^{16}$	-0.0067	-0.0042	-0.0030	-0.0049	-0.0031	-0.0022
$\hat{\bar{Y}}_{p1}^{17}$	0.0008	-0.0011	0.0002	0.0008	-0.0011	0.0002
$\hat{\bar{Y}}_{p1}^{18}$	-0.0074	-0.0049	-0.0034	-0.0059	-0.0040	-0.0027
$\hat{\bar{Y}}_{p1}^{19}$	-0.0086	-0.0057	-0.0039	-0.0078	-0.0053	-0.0036
$\hat{\bar{Y}}_{p1}^{20}$	-0.0078	-0.0051	-0.0035	-0.0064	-0.0042	-0.0029
$\hat{\bar{Y}}_{p1}^{21}$	-0.0086	-0.0057	-0.0039	-0.0078	-0.0053	-0.0036
$\hat{\bar{Y}}_{p1}^{22}$	-0.0086	-0.0058	-0.0037	-0.0078	-0.0053	-0.0031
$\hat{\bar{Y}}_{p1}^{23}$	-0.0079	-0.0053	-0.0027	-0.0066	-0.0045	-0.0018
$\hat{\bar{Y}}_{p1}^{24}$	-0.0086	-0.0057	-0.0037	-0.0079	-0.0053	-0.0031
$\hat{\bar{Y}}_{p1}^{25}$	-0.0086	-0.0058	-0.0039	-0.0078	-0.0053	-0.0036
$\hat{\bar{Y}}_{p1}^{26}$	-0.0078	-0.0051	-0.0035	-0.0064	-0.0042	-0.0029
$\hat{\bar{Y}}_{p1}^{27}$	-0.0086	-0.0058	-0.0039	-0.0078	-0.0053	-0.0036

Table 4.5
Theoretical (boldface) and Empirical MSE's, PRE's (without ME) of the Estimators Relative to Mean Per Unit Estimator in Simple Random Sampling for Population-II

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f = n / N$			$f = n / N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\bar{y}$	**0.1862** 0.1862	**0.1146** 0.1148	**0.0830** 0.0829	100.00	100.00	100.00
$\hat{\bar{Y}}_{pi}$	**0.0016** 0.0015	**0.0009** 0.0010	**0.0007** 0.0007	11637.50	12733.33	11857.14
$\hat{\bar{Y}}_{p1}^{1}$	**0.0018** 0.0019	**0.0011** 0.0011	**0.0008** 0.0008	10344.44	10418.18	10375.00
$\hat{\bar{Y}}_{p1}^{2}$	**0.0165** 0.0169	**0.0098** 0.0099	**0.0074** 0.0074	1128.48	1169.39	1121.62
$\hat{\bar{Y}}_{p1}^{3}$	**0.0264** 0.0271	**0.0159** 0.0161	**0.0119** 0.0121	705.30	720.75	697.48
$\hat{\bar{Y}}_{p1}^{4}$	**0.0045** 0.0047	**0.0026** 0.0026	**0.0019** 0.0019	4137.78	4407.69	4368.42
$\hat{\bar{Y}}_{p1}^{5}$	**0.0113** 0.0117	**0.0067** 0.0067	**0.0051** 0.0051	1647.79	1710.45	1627.45
$\hat{\bar{Y}}_{p1}^{6}$	**0.0069** 0.0072	**0.0040** 0.0041	**0.0030** 0.0031	2698.55	2865.00	2766.67
$\hat{\bar{Y}}_{p1}^{7}$	**0.0035** 0.0035	**0.0019** 0.0019	**0.0015** 0.0015	5320.00	6031.58	5533.33
$\hat{\bar{Y}}_{p1}^{8}$	**0.0165** 0.0170	**0.0098** 0.0100	**0.0074** 0.0075	1128.48	1169.39	1121.62
$\hat{\bar{Y}}_{p1}^{9}$	**0.0027** 0.0028	**0.0015** 0.0015	**0.0011** 0.0012	6896.30	7640.00	7545.45
$\hat{\bar{Y}}_{p1}^{10}$	**0.0266** 0.0272	**0.0160** 0.0162	**0.0120** 0.0121	700.00	716.25	691.67
$\hat{\bar{Y}}_{p1}^{11}$	**0.0018** 0.0019	**0.0011** 0.0011	**0.0008** 0.0008	10344.44	10418.18	10375.00
$\hat{\bar{Y}}_{p1}^{12}$	**0.1927** 0.1972	**0.1221** 0.1224	**0.0938** 0.0937	96.63	93.86	88.49
$\hat{\bar{Y}}_{p1}^{13}$	**0.0482** 0.0488	**0.0290** 0.0293	**0.0206** 0.0207	386.31	395.17	402.91
$\hat{\bar{Y}}_{p1}^{14}$	**0.0229** 0.0235	**0.0136** 0.0137	**0.0096** 0.0097	813.10	842.65	864.58

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f=n/N$			$f=n/N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\hat{\bar{Y}}_{p1}^{15}$	**0.1898** 0.1898	**0.1188** 0.1191	**0.0893** 0.0892	98.10	96.46	92.95
$\hat{\bar{Y}}_{p1}^{16}$	**0.0142** 0.0145	**0.0083** 0.0084	**0.0058** 0.0059	1311.27	1380.72	1431.03
$\hat{\bar{Y}}_{p1}^{17}$	**0.1889** 0.1889	**0.1177** 0.1178	**0.0874** 0.0873	98.57	97.37	94.97
$\hat{\bar{Y}}_{p1}^{18}$	**0.0206** 0.0211	**0.0126** 0.0127	**0.0089** 0.0089	903.88	909.52	932.58
$\hat{\bar{Y}}_{p1}^{19}$	**0.0482** 0.0488	**0.0293** 0.0295	**0.0212** 0.0213	386.31	391.13	391.51
$\hat{\bar{Y}}_{p1}^{20}$	**0.0229** 0.0235	**0.0138** 0.0139	**0.0099** 0.0100	813.10	830.43	838.38
$\hat{\bar{Y}}_{p1}^{21}$	**0.0484** 0.0491	**0.0294** 0.0296	**0.0213** 0.0214	384.71	389.80	389.67
$\hat{\bar{Y}}_{p1}^{22}$	**0.0201** 0.0206	**0.0271** 0.0274	**0.0335** 0.0335	926.37	422.88	247.76
$\hat{\bar{Y}}_{p1}^{23}$	**0.0085** 0.0087	**0.0125** 0.0126	**0.0184** 0.0184	2190.59	916.80	451.09
$\hat{\bar{Y}}_{p1}^{24}$	**0.0202** 0.0207	**0.0273** 0.0275	**0.0336** 0.0336	921.78	419.78	247.02
$\hat{\bar{Y}}_{p1}^{25}$	**0.0483** 0.0490	**0.0293** 0.0295	**0.0215** 0.0216	385.51	391.13	386.05
$\hat{\bar{Y}}_{p1}^{26}$	**0.0229** 0.0236	**0.0137** 0.0139	**0.0102** 0.0102	813.10	836.50	813.73
$\hat{\bar{Y}}_{p1}^{27}$	**0.0486** 0.0493	**0.0295** 0.0296	**0.0217** 0.0217	383.13	388.47	382.49

Table 4.6
Theoretical (boldface) and Empirical MSE's, PRE's (with ME) of the Estimators Relative to Mean Per Unit Estimator in Simple Random Sampling for Population-II

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f = n / N$			$f = n / N$		
	500/5000	750/5000	1000/5000	500/5000	750/5000	1000/5000
$\bar{y}$	**0.2319** 0.2279	**0.1428** 0.1438	**0.1029** 0.1012	100.00	100.00	100.00
$\hat{\bar{Y}}_{pi}$	**0.0826** 0.0624	**0.0520** 0.0401	**0.0366** 0.0270	280.75	274.62	281.15
$\hat{\bar{Y}}_{p1}^{1}$	**0.0922** 0.0667	**0.0581** 0.0431	**0.0404** 0.0288	251.52	245.78	254.70
$\hat{\bar{Y}}_{p1}^{2}$	**0.0855** 0.0636	**0.0529** 0.0417	**0.0374** 0.0283	271.23	269.94	275.13
$\hat{\bar{Y}}_{p1}^{3}$	**0.0903** 0.0674	**0.0557** 0.0441	**0.0398** 0.0301	256.81	256.37	258.54
$\hat{\bar{Y}}_{p1}^{4}$	**0.0851** 0.0627	**0.0532** 0.0409	**0.0372** 0.0276	272.50	268.42	276.61
$\hat{\bar{Y}}_{p1}^{5}$	**0.0839** 0.0624	**0.0522** 0.0408	**0.0367** 0.0276	276.40	273.56	280.38
$\hat{\bar{Y}}_{p1}^{6}$	**0.0839** 0.0622	**0.0524** 0.0407	**0.0367** 0.0274	276.40	272.52	280.38
$\hat{\bar{Y}}_{p1}^{7}$	**0.0862** 0.0634	**0.0539** 0.0413	**0.0377** 0.0277	269.03	264.94	272.94
$\hat{\bar{Y}}_{p1}^{8}$	**0.0855** 0.0636	**0.0529** 0.0416	**0.0375** 0.0283	271.23	269.94	274.40
$\hat{\bar{Y}}_{p1}^{9}$	**0.0876** 0.0641	**0.0549** 0.0416	**0.0384** 0.0280	264.73	260.11	267.97
$\hat{\bar{Y}}_{p1}^{10}$	**0.0904** 0.0675	**0.0558** 0.0442	**0.0397** 0.0302	256.53	255.91	259.19
$\hat{\bar{Y}}_{p1}^{11}$	**0.0924** 0.0667	**0.0583** 0.0431	**0.0407** 0.0289	250.97	244.94	252.83
$\hat{\bar{Y}}_{p1}^{12}$	**0.2385** 0.2374	**0.1504** 0.1548	**0.1138** 0.1174	97.23	94.95	90.42
$\hat{\bar{Y}}_{p1}^{13}$	**0.1051** 0.0799	**0.0645** 0.0581	**0.0456** 0.0351	220.65	221.40	225.66
$\hat{\bar{Y}}_{p1}^{14}$	**0.0884** 0.0658	**0.0545** 0.0431	**0.0384** 0.0291	262.33	262.02	267.62

Estimators	MSE(Without ME)			PRE (Without ME)		
	$f = n / N$			$f = n / N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\hat{\bar{Y}}_{p1}^{15}$	**0.2356** 0.2332	**0.1471** 0.1499	**0.1093** 0.1105	98.43	97.08	94.14
$\hat{\bar{Y}}_{p1}^{16}$	**0.0846** 0.0629	**0.0525** 0.0412	**0.0369** 0.0278	274.11	272.00	278.86
$\hat{\bar{Y}}_{p1}^{17}$	**0.2347** 0.2318	**0.1459** 0.1482	**0.1074** 0.1076	98.81	97.88	95.81
$\hat{\bar{Y}}_{p1}^{18}$	**0.0872** 0.0649	**0.0541** 0.0427	**0.0381** 0.0287	265.94	263.96	270.08
$\hat{\bar{Y}}_{p1}^{19}$	**0.1051** 0.0799	**0.0646** 0.0520	**0.0461** 0.0354	220.65	221.05	223.21
$\hat{\bar{Y}}_{p1}^{20}$	**0.0884** 0.0658	**0.0546** 0.0431	**0.0386** 0.0292	262.33	261.54	266.58
$\hat{\bar{Y}}_{p1}^{21}$	**0.1053** 0.0801	**0.0648** 0.0521	**0.0462** 0.0355	220.23	220.37	222.73
$\hat{\bar{Y}}_{p1}^{22}$	**0.0869** 0.0648	**0.0631** 0.0506	**0.0561** 0.0447	266.86	226.31	183.42
$\hat{\bar{Y}}_{p1}^{23}$	**0.0838** 0.0621	**0.0540** 0.0426	**0.0439** 0.0336	276.73	264.44	234.40
$\hat{\bar{Y}}_{p1}^{24}$	**0.0871** 0.0648	**0.0632** 0.0506	**0.0562** 0.0448	266.25	225.95	183.10
$\hat{\bar{Y}}_{p1}^{25}$	**0.1053** 0.0800	**0.0646** 0.0520	**0.0464** 0.0357	220.23	221.05	221.77
$\hat{\bar{Y}}_{p1}^{26}$	**0.0884** 0.0659	**0.0546** 0.0431	**0.0387** 0.0293	262.33	261.54	265.89
$\hat{\bar{Y}}_{p1}^{27}$	**0.1054** 0.0801	**0.0647** 0.0521	**0.0464** 0.0357	220.02	220.71	221.77

Table 4.7
Theoretical Biases (with/without ME) of the Estimators in Simple Random Sampling for Population-II

Estimators	**bias (without ME)**			**bias (with ME)**		
	$f = n / N$			$f = n / N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\bar{y}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{pi}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{p1}^{1}$	-0.0002	0.0001	-0.0000	0.0086	0.0058	0.0038
$\hat{\bar{Y}}_{p1}^{2}$	-0.0073	-0.0045	-0.0033	-0.0028	-0.0015	-0.0013
$\hat{\bar{Y}}_{p1}^{3}$	-0.0084	-0.0052	-0.0038	-0.0048	-0.0028	-0.0022
$\hat{\bar{Y}}_{p1}^{4}$	-0.0039	-0.0023	-0.0017	0.0029	0.0021	-0.0013
$\hat{\bar{Y}}_{p1}^{5}$	-0.0063	-0.0039	-0.0028	-0.0011	-0.0004	-0.0005
$\hat{\bar{Y}}_{p1}^{6}$	-0.0050	-0.0030	-0.0022	0.0011	0.0009	-0.0004
$\hat{\bar{Y}}_{p1}^{7}$	-0.0031	-0.0018	-0.0013	0.0042	0.0029	0.0018
$\hat{\bar{Y}}_{p1}^{8}$	-0.0073	-0.0045	-0.0033	-0.0028	-0.0016	-0.0013
$\hat{\bar{Y}}_{p1}^{9}$	-0.0023	-0.0013	-0.0009	0.0054	0.0037	0.0024
$\hat{\bar{Y}}_{p1}^{10}$	-0.0084	-0.0052	-0.0038	-0.0049	-0.0029	-0.0022
$\hat{\bar{Y}}_{p1}^{11}$	-0.0001	0.0002	0.0000	0.0087	0.0059	0.0039
$\hat{\bar{Y}}_{p1}^{12}$	0.0006	0.0007	0.0011	0.0006	0.0007	0.0011
$\hat{\bar{Y}}_{p1}^{13}$	-0.0090	-0.0056	-0.0040	-0.0069	-0.0042	-0.0030
$\hat{\bar{Y}}_{p1}^{14}$	-0.0081	-0.0049	-0.0036	-0.0042	-0.0024	-0.0018

Estimators	bias (without ME)			bias (with ME)		
	$f = n/N$			$f = n/N$		
	500/5000	**750/5000**	**1000/5000**	**500/5000**	**750/5000**	**1000/5000**
$\hat{\bar{Y}}_{p1}^{15}$	0.0004	0.0004	0.0006	0.0004	0.0004	0.0006
$\hat{\bar{Y}}_{p1}^{16}$	-0.0069	-0.0042	-0.0030	-0.0021	-0.0011	-0.0008
$\hat{\bar{Y}}_{p1}^{17}$	0.0003	0.0003	0.0004	0.0003	0.0003	0.0004
$\hat{\bar{Y}}_{p1}^{18}$	-0.0079	-0.0048	-0.0035	-0.0038	-0.0022	-0.0016
$\hat{\bar{Y}}_{p1}^{19}$	-0.0090	-0.0056	-0.0040	-0.0069	-0.0042	-0.0031
$\hat{\bar{Y}}_{p1}^{20}$	-0.0081	-0.0049	-0.0036	-0.0042	-0.0025	-0.0019
$\hat{\bar{Y}}_{p1}^{21}$	-0.0090	-0.0056	-0.0040	-0.0069	-0.0042	-0.0031
$\hat{\bar{Y}}_{p1}^{22}$	-0.0078	-0.0056	-0.0037	-0.0037	-0.0040	-0.0032
$\hat{\bar{Y}}_{p1}^{23}$	-0.0056	-0.0048	-0.0040	0.0002	-0.0022	-0.0029
$\hat{\bar{Y}}_{p1}^{24}$	-0.0078	-0.0056	-0.0037	-0.0037	-0.0041	-0.0032
$\hat{\bar{Y}}_{p1}^{25}$	-0.0090	-0.0056	-0.0040	-0.0069	-0.0042	-0.0031
$\hat{\bar{Y}}_{p1}^{26}$	-0.0081	-0.0049	-0.0036	-0.0042	-0.0025	-0.0019
$\hat{\bar{Y}}_{p1}^{27}$	-0.0090	-0.0056	-0.0040	-0.0069	-0.0042	-0.0031

Results in Tables (4.2-4.7) show that our proposed generalized estimator ($\hat{\bar{Y}}_{A4}$) and its special cases have zero or near zero bias and is performing well both in presence and absence of measurement error. The performance of all the estimators in the proposed series of ratio estimators ($\hat{\bar{Y}}_{p1}^{j}$) shows that the efficiency of these estimators is effected quite a bit under measurement error. With the contribution of measurement error, MSE of these estimators has increased and this causes decline in percent relative efficiency. Note that the classical ratio estimator ($\hat{\bar{Y}}_{p1}^{1}$) is doing the best without measurement errors but not so with measurement error. We also note, as expected, special cases that utilize those population parameters that reflect on mean estimation do better. For example, utilization of the Tri Mean (TM) produces better results as opposed to utilization of various quartiles such as QD.

4.7 NUMERICAL EXAMPLE

To check the performance of proposed estimator, we used the data taken from Gujarati and Sangeetha (2007, page 539). The population characteristics are described as:

Y_i = true consumption expenditure,
y_i = measured consumption expenditure,
X_i = true income,
x_i = measured income

Table 4.8
Summary Statistics for the Real Data

N	n	μ_Y	μ_X	S_Y	S_X	ρ_{YX}	σ_U^2	σ_V^2
70	10	981.29	1755.53	613.66	1406.13	0.778	36.00	36.00

Table 4.9
Theoretical MSE's and PRE's (with/without ME) of the Estimators Relative to Mean Per Unit Estimator in Simple Random Sampling for Real Data

Estimators	Mean Square Error			Percent Relative Efficiency	
	Without ME	Change due to ME	With ME	Without ME	With ME
$\bar{y}$ (Mean per unit estimator)	32278.17	3.08	32281.25	100.0000	100.0000
$\hat{\bar{Y}}_{p1}^{1}$ (Classical ratio estimator)	20901.40	4.04	20905.45	154.4307	154.4155
$\hat{\bar{Y}}_{pi}$ (Proposed generalized estimator)	12740.71	3.44	12744.15	253.3467	253.3025

When dealing with the real data, Table 4.9 shows that our proposed generalized estimator ($\hat{\bar{Y}}_{Pi}$) is better than the existing estimators $\bar{y}$ and $\hat{\bar{Y}}_{p1}^{1}$ under both scenarios when measurement error is present and when it is absent. The contribution of measurement error in generalized estimator is lower as compared to classical ratio estimator. However it is not so as compared to the mean per unit estimator because the mean per unit estimator has only one source of measurement errors (*Y*) but our proposed estimator has two (*Y* and *X*).

4.8 CONCLUSION

It is well known that proper use of auxiliary information can help produce better parameter estimates. It is more so at the estimation stage. In this chapter, we have proposed a generalized estimator for population mean which leads several new estimators as special cases. Particularly, the special cases ($\hat{\bar{Y}}_{p1}^{j}$) utilize some conventional and non-conventional measures with single auxiliary variable which is highly correlated with study variable. Results in Tables (4.2-4.7) show the effect of measurement error on the estimators of the proposed series using different sampling fractions. As expected, the theoretical and empirical mean square errors are in very good match. The amounts of biases are larger for small sample size but become negligible as the size of the sample increases. It can be readily seen that the efficiency of the estimators reduces if we take measurement errors into account. Thus there is a need to take due care of measurement errors. A case in point is the classical ratio estimator whose performance is very good when there are no measurement errors but it performs poorly when measurement errors are present. Numerical results based on real data corroborate our simulation results.

CHAPTER 5

MEAN ESTIMATION FOR A NON-SENSITIVE STUDY VARIABLE IN THE PRESENCE OF MEASUREMENT ERRORS UNDER STRATIFIED RANDOM SAMPLING

5.1 INTRODUCTION

In this Chapter, we introduce a class of generalized mean estimators for a non-sensitive study variable where measurement errors can occur both in the study variable and the auxiliary variable. Also, unlike the previous chapter, it is done here using a stratified random sampling design. Once again, the focus is on studying the impact of measurement errors on mean estimation. We have also provided a comparison of proposed estimator with some existing mean estimators.

5.2 SAMPLING PROCEDURE AND NOTATIONS

We follow the usual notation for stratified sampling and consider a finite population $M = (M_1, M_2, ...M_N)$ of size N which is divided into L strata either to ensure greater homogeneity or because of some other consideration such as need for sub-domain level estimates. We assume that the h^{th} stratum contains N_h units such that $\sum_{h=1}^{L} N_h = N$ and the weight of the h^{th} stratum is $W_h = \frac{N_h}{N}$ where (h=1, 2,…, L). A simple random sample of size n_h is drawn without replacement from the h^{th} stratum such that $\sum_{h=1}^{L} n_h = n$. Let (y_{hi}, x_{hi}) be the observed pair of values (factoring in measurement errors) and $\left(Y_{hi}, X_{hi}\right)$ be the true pair of values of the study variable Y and the auxiliary variable X on the i^{th} unit of the h^{th} stratum. Let $\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$, $\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h$, where $\bar{Y}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} y_{hi}$, $\bar{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi}$ denote the population mean and sample mean of the study variable (Y). Also, let $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h, \bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h$, where $\bar{X}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} x_{hi}, \bar{x}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} x_{hi}$ denote the population mean and sample mean respectively of the auxiliary variable (X). The measurement errors associated with the non-sensitive study variable and the auxiliary variable in the h^{th} stratum, as defined in Equations 1.5.4 and 1.5.5 Chapter 1 are given by:

$$U_{hi} = y_{hi} - Y_{hi},$$
$$V_{hi} = x_{hi} - X_{hi},$$

We use the following notations:

Let $e_{ost} = \dfrac{\bar{y}_{st} - \bar{Y}}{\bar{Y}}$ and $e_{1st} = \dfrac{\bar{x}_{st} - \bar{X}}{\bar{X}}$ Such that, $E(e_{ost}) = E(e_{1st}) = 0,$ and

$$E\left(e_{ost}^2\right) = \frac{1}{\bar{Y}^2}\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Yh}^2}{\theta_{Yh}} = V_{20},\ E\left(e_{1st}^2\right) = \frac{1}{\bar{X}^2}\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}$$

$$= V_{02},\ E\left(e_{ost}\, e_{1st}\right) = \frac{1}{\overline{YX}}\sum_{h=1}^{L} W_h^2\gamma_h S_{YXh} = V_{11},$$

where

$$S_{Yh}^2 = \sum_{i=1}^{N_h}\frac{\left(y_{hi} - \bar{Y}_h\right)^2}{N_h - 1}, S_{Xh}^2 = \sum_{i=1}^{N_h}\frac{\left(x_{hi} - \bar{X}_h\right)^2}{N_h - 1}, S_{YXh} = \sum_{i=1}^{N_h}\frac{\left(y_{hi} - \bar{Y}_h\right)\left(x_{hi} - \bar{X}_h\right)}{N_h - 1},$$

$$\theta_{Yh} = \frac{S_{Yh}^2}{(S_{Uh}^2 + S_{Yh}^2)}, \theta_{Xh} = \frac{S_{Xh}^2}{(S_{Vh}^2 + S_{Xh}^2)}, \gamma_h = \left(\frac{1 - f_h}{n_h}\right) \text{ and } f_h = \frac{n_h}{N_h}. \tag{5.2.1}$$

5.3 PROPOSED GENERALIZED ESTIMATOR IN THE PRESENCE OF MEASUREMENT ERROR

In this section, we propose a generalized estimator for population mean in the presence of measurement errors on both the non-sensitive study variable (Y) and the auxiliary variable (X) in stratified random sampling.

The proposed estimator is:

$$\hat{\bar{Y}}_{pi,st} = \left[\bar{y}_{st} + k\left(\bar{X} - \bar{x}_{st}\right)\right]\left[\frac{a_{st}\bar{X} + b_{st}}{\lambda\left(a_{st}\bar{x}_{st} + b_{st}\right) + (1-\lambda)\left(a_{st}\bar{X} + b_{st}\right)}\right]^g, \tag{5.3.1}$$

Here the constants λ, k and g may be assumed known, or may be assumed unknown. In the latter case, their values will need to be estimated. As usual, the constants $a_{st}(\neq 0)$, and b_{st} are either real numbers or functions of the known parameters of the auxiliary variable *X*. Many estimators can be derived from the proposed class of estimators with specific choices of the constants. For example, $g=1$ gives us various combined ratio estimators and $g=-1$ gives us various combined product estimators.

Remark 1:

For $g=1$, $\hat{\hat{Y}}_{pi,st}$ can take the following form:

$$\hat{\hat{Y}}_{pi,st}=\left[\bar{y}_{st}+k\left(\bar{X}-\bar{x}_{st}\right)\right]\left[\frac{a_{st}\bar{X}+b_{st}}{\lambda\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda\right)\left(a_{st}\bar{X}+b_{st}\right)}\right], \tag{5.3.2}$$

By setting different values of unknown constants in Equation (5.3.2), various combined ratio estimators based on the single auxiliary variable may be obtained as a family of $\hat{\hat{Y}}_{pi,st}$. For example,

(i) By putting $k=0$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{p1,st}=\bar{y}_{st}\left(\frac{a_{st}\bar{X}+b_{st}}{a_{st}\bar{x}_{st}+b_{st}}\right). \tag{5.3.3}$$

(ii) By putting $k=1$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{p2,st}=\left(\bar{y}_{st}+(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{a_{st}\bar{x}_{st}+b_{st}}\right). \tag{5.3.4}$$

(iii) By putting $k=b_{YX}$ (the slope term in regression *Y* on *X*) and $\lambda=1$, we have

$$\hat{\hat{Y}}_{p3,st}=\left(\bar{y}_{st}+b_{YX}(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{a_{st}\bar{x}_{st}+b_{st}}\right). \tag{5.3.5}$$

(iv) By putting $k=0$ and $\lambda=\lambda_{opt}$ (optimized value of λ relative to the MSE of the proposed estimator), we have

$$\hat{\hat{Y}}_{p4,st}=\bar{y}_{st}\left(\frac{a_{st}\bar{X}+b_{st}}{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}\right). \tag{5.3.6}$$

(v) By putting $k=1$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{p5,st}=\left(\bar{y}_{st}+(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}\right). \tag{5.3.7}$$

(vi) By putting $k=b_{YX}$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{p6,st}=\left(\bar{y}_{st}+b_{YX}(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}\right). \tag{5.3.8}$$

Remark 2:

For $g=-1$, $\hat{\hat{Y}}_{pi,st}$ can take the following form:

$$\hat{\hat{Y}}_{pi,st}=\left[\bar{y}_{st}+k\left(\bar{X}-\bar{x}_{st}\right)\right]\left[\frac{\lambda\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda\right)\left(a_{st}\bar{X}+b_{st}\right)}{a_{st}\bar{X}+b_{st}}\right]. \tag{5.3.9}$$

By setting different values of unknown constants in Equation (5.3.9), various combined product estimators based on single auxiliary variable may be obtained as a family of $\hat{\hat{Y}}_{pi,st}$. For example,

(i) By putting $k=0$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{p7,st}=\bar{y}_{st}\left(\frac{a_{st}\bar{x}_{st}+b_{st}}{a_{st}\bar{X}+b_{st}}\right). \tag{5.3.10}$$

(ii) By putting $k=1$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{p8,st}=\left(\bar{y}_{st}+(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{x}_{st}+b_{st}}{a_{st}\bar{X}+b_{st}}\right). \tag{5.3.11}$$

(iii) By putting $k = b_{YX}$ and $\lambda = 1$, we have

$$\hat{\hat{Y}}_{p9,st} = \left(\bar{y}_{st} + b_{YX}(\bar{X} - \bar{x}_{st})\right)\left(\frac{a_{st}\bar{x}_{st} + b_{st}}{a_{st}\bar{X} + b_{st}}\right). \tag{5.3.12}$$

(iv) By putting $k = 0$ and $\lambda = \lambda_{opt}$, we have

$$\hat{\hat{Y}}_{p10,st} = \bar{y}_{st}\left(\frac{\lambda_{opt}\left(a_{st}\bar{x}_{st} + b_{st}\right) + \left(1-\lambda_{opt}\right)\left(a_{st}\bar{X} + b_{st}\right)}{a_{st}\bar{X} + b_{st}}\right). \tag{5.3.13}$$

(v) By putting $k = 1$ and $\lambda = \lambda_{opt}$, we have

$$\hat{\hat{Y}}_{p11,st} = \left(\bar{y}_{st} + (\bar{X} - \bar{x}_{st})\right)\left(\frac{\lambda_{opt}\left(a_{st}\bar{x}_{st} + b_{st}\right) + \left(1-\lambda_{opt}\right)\left(a_{st}\bar{X} + b_{st}\right)}{a_{st}\bar{X} + b_{st}}\right). \tag{5.3.14}$$

(vi) By putting $k = b_{YX}$ and $\lambda = \lambda_{opt}$, we have

$$\hat{\hat{Y}}_{p12,st} = \left(\bar{y}_{st} + b_{YX}(\bar{X} - \bar{x}_{st})\right)\left(\frac{\lambda_{opt}\left(a_{st}\bar{x}_{st} + b_{st}\right) + \left(1-\lambda_{opt}\right)\left(a_{st}\bar{X} + b_{st}\right)}{a_{st}\bar{X} + b_{st}}\right). \tag{5.3.15}$$

5.3.1 The Bias and Mean Square Error of the Proposed Generalized Estimator

Expressing Equation (5.3.1) in terms of e's, we have

$$\hat{\hat{Y}}_{pi,st} = \left(\bar{Y}\left(1+e_{ost}\right) + k\left(\bar{X} - \bar{X}\left(1+e_{1st}\right)\right)\right)$$
$$\left(\frac{a_{st}\bar{X} + b_{st}}{\lambda\left(a_{st}\bar{X}\left(1+e_{1st}\right) + b_{st}\right) + \left(1-\lambda\right)\left(a_{st}\bar{X} + b_{st}\right)}\right)^{g},$$

or

$$\hat{\hat{Y}}_{pi,st} = \left(\bar{Y} + \bar{Y}e_{ost} - k\bar{X}e_{1st}\right)\left(\frac{a_{st}\bar{X} + b_{st}}{a_{st}\bar{X} + b_{st} + \lambda a_{st}\bar{X}e_{1st}}\right)^{g},$$

or

$$\hat{\bar{Y}}_{pi,st} = \left(\bar{Y} + \bar{Y}e_{ost} - k\bar{X}e_{1st}\right)\left(1 + \frac{\lambda a_{st}\bar{X}e_{1st}}{a_{st}\bar{X} + b_{st}}\right)^{-g},$$

or

$$\hat{\bar{Y}}_{pi,st} = \left(\bar{Y} + \bar{Y}e_{ost} - k\bar{X}e_{1st}\right)\left(1 + \lambda\phi e_{1st}\right)^{-g},$$

where

$$\phi = \frac{a_{st}\bar{X}}{(a_{st}\bar{X} + b_{st})}.$$

By using Taylor series expansion

$$\left(1 + x\right)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \ldots,$$

we get

$$\hat{\bar{Y}}_{pi,st} \approx \left(\bar{Y} + \bar{Y}e_{ost} - k\bar{X}e_{1st}\right)\left(1 - g\lambda\phi e_{1st} + \frac{g(g+1)}{2}\left(\lambda\phi e_{1st}\right)^2\right),$$

or

$$\hat{\bar{Y}}_{pi,st} \approx \left(\begin{array}{l} \bar{Y} - g\lambda\phi\bar{Y}e_{1st} + \frac{g(g+1)}{2}\lambda^2\phi^2\bar{Y}e_{1st}^2 + \bar{Y}e_{ost} \\ \qquad -g\lambda\phi\bar{Y}e_{ost}e_{1st} + \frac{g(g+1)}{2}\lambda^2\phi^2\bar{Y}e_{ost}e_{1st}^2 \\ \qquad -k\bar{X}e_{1st} + g\lambda\phi k\bar{X}e_{1st}^2 - \frac{g(g+1)}{2}\lambda^2\phi^2 k\bar{X}e_{1st}^3 \end{array}\right) \quad (5.3.16)$$

In order to derive the expression of bias, using terms up to second order from Equation (5.3.16), we have

$$(\hat{\bar{Y}}_{pi,st} - \bar{Y}) \approx \left(\frac{g(g+1)}{2}\lambda^2\phi^2\bar{Y}e_{1st}^2 - g\lambda\phi\bar{Y}e_{0st}e_{1st} + g\lambda\phi k\bar{X}e_{1st}^2\right).$$

Taking expectation and using Equation (5.2.1), we get

$$E(\hat{\bar{Y}}_{pi,st}-\bar{Y})\approx\left(g\lambda\phi\right)\left(\begin{array}{c}\left(\left(\left(\frac{g+1}{2}\right)\lambda\phi\bar{Y}+k\bar{X}\right)\frac{1}{\bar{X}^2}\right.\\ \left.\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}-\frac{1}{\bar{X}}\sum_{h=1}^{L}W_h^2\gamma_h S_{YXh}\right)\end{array}\right),$$

or

$$Bias^*(\hat{\bar{Y}}_{pi,st})\approx\left(g\lambda\phi\right)\left(\left(\left(\frac{g+1}{2}\right)\phi\lambda\bar{Y}+k\bar{X}\right)V_{02}-\bar{Y}V_{11}\right). \qquad (5.3.17)$$

The expression of bias of the proposed generalized estimator without measurement error may be obtained by putting $S_{Vh}^2=0$ in Equation (5.3.17).

In order to derive the expression of MSE of proposed estimator, using terms up to first order from Equation (5.3.16), we have

$$(\hat{\bar{Y}}_{pi,st}-\bar{Y})\approx\left(\bar{Y}e_{0st}-g\lambda\phi\bar{Y}e_{1st}-k\bar{X}e_{1st}\right).$$

Squaring both sides and taking expectation, we have

$$E(\hat{\bar{Y}}_{pi,st}-\bar{Y})^2\approx\left(\begin{array}{l}\bar{Y}^2\left(\frac{1}{\bar{Y}^2}\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Yh}^2}{\theta_{Yh}}\right)\\ +\left((g\lambda\phi)^2\bar{Y}^2+k^2\bar{X}^2+2g\lambda\phi\bar{Y}k\bar{X}\right)\frac{1}{\bar{X}^2}\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\\ -\left(2g\lambda\phi\bar{Y}^2+2\bar{Y}k\bar{X}\right)\frac{1}{\overline{YX}}\sum_{h=1}^{L}W_h^2\gamma_h S_{YXh}\end{array}\right), \qquad (5.3.18)$$

After simplification of Equation (5.3.18), we get

$$\begin{aligned}MSE^*(\hat{\bar{Y}}_{pi,st})&\approx\bar{Y}^2V_{20}+\left((g\lambda\phi)^2\bar{Y}^2+k^2\bar{X}^2+2g\lambda\phi\overline{YX}k\right)V_{02}\\&-\left(2g\lambda\phi\bar{Y}^2+2\overline{YX}k\right)V_{11},\end{aligned} \qquad (5.3.19)$$

In order to obtain minimized $MSE^*(\hat{\bar{Y}}_{pi,st})$, we differentiate Equation (5.3.19) with respect to $(g\lambda\phi)$, and equate the results to zero, i.e.

$$\frac{\partial MSE^*(\hat{\bar{Y}}_{pi,st})}{\partial(g\lambda\phi)} = 0,$$

or

$$\frac{\partial MSE^*(\hat{\bar{Y}}_{pi,st})}{\partial(g\lambda\phi)} = 2\left(g\lambda\phi\bar{Y}^2 + \bar{Y}k\bar{X}\right)V_{02} - 2\bar{Y}^2V_{11}, \tag{5.3.20}$$

On solving (5.3.20), the optimum value of $(g\lambda\phi)$ is obtained as,

$$(g\lambda\phi)_{opt} = \left(\frac{V_{11}}{V_{02}} - \frac{\bar{X}}{\bar{Y}}k\right). \tag{5.3.21}$$

It may be noted that optimization with respect to $(g\lambda\phi)$ is mentioned only because of notational convenience, the key parameter being optimized is λ. The value of g will always be 1 or -1, and φ is a function of known parameters of the auxiliary variable. Also, k will always be 0, 1, or the regression coefficient between X and Y.

Substitution of (5.3.21) in (5.3.19) yields the minimized $MSE^*(\hat{\bar{Y}}_{pi,st})$ as:

$$MSE^*_{\min}(\hat{\bar{Y}}_{pi,st}) \approx \bar{Y}^2V_{20}\left(1-\rho^2_{st}\right), \tag{5.3.22}$$

where $\rho_{st} = V_{11}\Big/\left(\sqrt{V_{20}}\sqrt{V_{02}}\right)$.

The expression of minimized MSE of the proposed estimator without measurement error may be obtained by putting $S^2_{Uh} = S^2_{Vh} = 0$ in Equation (5.3.22).

5.4 ADDITIONAL SPECIAL CASES OF THE GENERALIZED RATIO ESTIMATOR

Many additional combined ratio estimators can be deduced from the generalized ratio estimator $\hat{\bar{Y}}_{p1,st}$ given in Equation (5.3.3). We denote the generalized ratio estimator by,

$$\hat{\bar{Y}}^j_{p1,st} = \bar{y}_{st}\left(\frac{a^j_{st}\bar{X} + b^j_{st}}{a^j_{st}\bar{x}_{st} + b^j_{st}}\right).$$

Various choices of a_{st}^{j}, b_{st}^{j} are given in the table below. The general expressions for the MSE and bias respectively with measurement error for this generalized ratio estimator $\hat{\bar{Y}}_{p1,st}^{j}$ are given by

$$\left.\begin{aligned}
&MSE^{*}(\hat{\bar{Y}}_{p1,st}^{j})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{j}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{j}-2\beta_{YXh}\theta_{Xh}\right)\right),\\
&Bias^{*}(\hat{\bar{Y}}_{p1,st}^{j})=\left(\frac{\phi_j}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{j}-\beta_{YXh}\theta_{Xh}\right),\\
&\phi_j=\frac{a_{st}^{j}\bar{X}}{(a_{st}^{j}\bar{X}+b_{st}^{j})},\quad R_{st}^{j}=\frac{\bar{Y}}{\bar{X}}\phi_j,\ \beta_{YXh}=\frac{S_{YXh}}{S_{Xh}^2}.
\end{aligned}\right]\qquad(5.4.1)$$

Table 5.1

Additional Special Cases of the Generalized Ratio Estimator ($\hat{\bar{Y}}^{j}_{P1,st}$)

Estimators	a^j_{st}	b^j_{st}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}^{1}_{p1,st} = \bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right)$	1	0	$MSE^*(\hat{\bar{Y}}^{1}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S^2_{Yh}}{\theta_{Yh}} + R^1_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^1_{st} - 2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{1}_{p1,st}) = \left(\frac{\phi_1}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^1_{st} - \beta_{YXh}\theta_{Xh}\right)$	$R^1_{st} = \frac{\bar{Y}}{\bar{X}}\phi_1$ $\phi_1 = 1$
$\hat{\bar{Y}}^{2}_{p1,st} = \bar{y}_{st}\left(\frac{\bar{X}+\Omega_1}{\bar{x}_{st}+\Omega_1}\right)$	1	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$MSE^*(\hat{\bar{Y}}^{2}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S^2_{Yh}}{\theta_{Yh}} + R^2_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^2_{st} - 2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{2}_{p1,st}) = \left(\frac{\phi_2}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^2_{st} - \beta_{YXh}\theta_{Xh}\right)$	$R^2_{st} = \frac{\bar{Y}}{\bar{X}}\phi_2$ $\phi_2 = \bar{X}/(\bar{X}+\Omega_1)$
$\hat{\bar{Y}}^{3}_{p1,st} = \bar{y}_{st}\left(\frac{\bar{X}+\Omega_2}{\bar{x}_{st}+\Omega_2}\right)$	1	$\Omega_2 = \sum_{h=1}^{L} W_h \beta_{2h}(x)$	$MSE^*(\hat{\bar{Y}}^{3}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S^2_{Yh}}{\theta_{Yh}} + R^3_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^3_{st} - 2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{3}_{p1,st}) = \left(\frac{\phi_3}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^3_{st} - \beta_{YXh}\theta_{Xh}\right)$	$R^3_{st} = \frac{\bar{Y}}{\bar{X}}\phi_3$ $\phi_3 = \bar{X}/(\bar{X}+\Omega_2)$
$\hat{\bar{Y}}^{4}_{p1,st} = \bar{y}_{st}\left(\frac{\Omega_2\bar{X}+\Omega_1}{\Omega_2\bar{x}_{st}+\Omega_1}\right)$	$\Omega_2 = \sum_{h=1}^{L} W_h \beta_{2h}(x)$	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$MSE^*(\hat{\bar{Y}}^{4}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S^2_{Yh}}{\theta_{Yh}} + R^4_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^4_{st} - 2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{4}_{p1,st}) = \left(\frac{\phi_4}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^4_{st} - \beta_{YXh}\theta_{Xh}\right)$	$R^4_{st} = \frac{\bar{Y}}{\bar{X}}\phi_4$ $\phi_4 = \Omega_2\bar{X}/(\Omega_2\bar{X}+\Omega_1)$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{p1,st}^{5} = \bar{y}_{st}\left(\dfrac{\Omega_1\bar{X}+\Omega_2}{\Omega_1\bar{x}_{st}+\Omega_2}\right)$	$\Omega_1=\sum_{h=1}^{L}W_h C_{Xh}$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{5})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{5}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{5}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{5})=\left(\dfrac{\phi_5}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{5}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{5}=\dfrac{\bar{Y}}{\bar{X}}\phi_5$ $\phi_5=\Omega_1\bar{X}\big/(\Omega_1\bar{X}+\Omega_2)$
$\hat{\bar{Y}}_{p1,st}^{6} = \bar{y}_{st}\left(\dfrac{\bar{X}+\Omega_3}{\bar{x}_{st}+\Omega_3}\right)$	1	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{YXh}$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{6})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{6}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{6}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{6})=\left(\dfrac{\phi_6}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{6}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{6}=\dfrac{\bar{Y}}{\bar{X}}\phi_6$ $\phi_6=\bar{X}\big/(\bar{X}+\Omega_3)$
$\hat{\bar{Y}}_{p1,st}^{7} = \bar{y}_{st}\left(\dfrac{\Omega_1\bar{X}+\Omega_3}{\Omega_1\bar{x}_{st}+\Omega_3}\right)$	$\Omega_1=\sum_{h=1}^{L}W_h C_{Xh}$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{YXh}$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{7})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{7}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{7}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{7})=\left(\dfrac{\phi_7}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{7}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{7}=\dfrac{\bar{Y}}{\bar{X}}\phi_7$ $\phi_7=\Omega_1\bar{X}\big/(\Omega_1\bar{X}+\Omega_3)$
$\hat{\bar{Y}}_{p1,st}^{8} = \bar{y}_{st}\left(\dfrac{\Omega_3\bar{X}+\Omega_1}{\Omega_3\bar{x}_{st}+\Omega_1}\right)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{YXh}$	$\Omega_1=\sum_{h=1}^{L}W_h C_{Xh}$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{8})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{8}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{8}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{8})=\left(\dfrac{\phi_8}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{8}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{8}=\dfrac{\bar{Y}}{\bar{X}}\phi_8$ $\phi_8=\Omega_3\bar{X}\big/(\Omega_3\bar{X}+\Omega_1)$
$\hat{\bar{Y}}_{p1,st}^{9} = \bar{y}_{st}\left(\dfrac{\Omega_2\bar{X}+\Omega_3}{\Omega_2\bar{x}_{st}+\Omega_3}\right)$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{YXh}$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{9})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{9}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{9}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{9})=\left(\dfrac{\phi_9}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{9}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{9}=\dfrac{\bar{Y}}{\bar{X}}\phi_9$ $\phi_9=\Omega_2\bar{X}\big/(\Omega_2\bar{X}+\Omega_3)$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{p1,st}^{10} = \bar{y}_{st}\left(\dfrac{\Omega_3\bar{X}+\Omega_2}{\Omega_3\bar{x}_{st}+\Omega_2}\right)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{YXh}$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{10})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{10}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{10}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{10})=\left(\dfrac{\phi_{10}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{10}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{10}=\dfrac{\bar{Y}}{\bar{X}}\phi_{10}$ $\phi_{10}=\Omega_3\bar{X}/(\Omega_3\bar{X}+\Omega_2)$
$\hat{\bar{Y}}_{p1,st}^{11} = \bar{y}_{st}\left(\dfrac{\bar{X}+\Omega_4}{\bar{x}_{st}+\Omega_4}\right)$	1	$\Omega_4=\sum_{h=1}^{L}W_h\beta_{1h}(x)$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{11})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{11}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{11}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{11})=\left(\dfrac{\phi_{11}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{11}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{11}=\dfrac{\bar{Y}}{\bar{X}}\phi_{11}$ $\phi_{11}=\bar{X}/(\bar{X}+\Omega_4)$
$\hat{\bar{Y}}_{p1,st}^{12} = \bar{y}_{st}\left(\dfrac{\Omega_4\bar{X}+\Omega_2}{\Omega_4\bar{x}_{st}+\Omega_2}\right)$	$\Omega_4=\sum_{h=1}^{L}W_h\beta_{1h}(x)$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{12})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{12}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{12}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{12})=\left(\dfrac{\phi_{12}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{12}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{12}=\dfrac{\bar{Y}}{\bar{X}}\phi_{12}$ $\phi_{12}=\Omega_4\bar{X}/(\Omega_4\bar{X}+\Omega_2)$
$\hat{\bar{Y}}_{p1,st}^{13} = \bar{y}_{st}\left(\dfrac{\bar{X}+\Omega_5}{\bar{x}_{st}+\Omega_5}\right)$	1	$\Omega_5=\sum_{h=1}^{L}W_hQ_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{13})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{13}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{13}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{13})=\left(\dfrac{\phi_{13}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{13}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{13}=\dfrac{\bar{Y}}{\bar{X}}\phi_{13}$ $\phi_{13}=\bar{X}/(\bar{X}+\Omega_5)$
$\hat{\bar{Y}}_{p1,st}^{14} = \bar{y}_{st}\left(\dfrac{\Omega_1\bar{X}+\Omega_5}{\Omega_1\bar{x}_{st}+\Omega_5}\right)$	$\Omega_1=\sum_{h=1}^{L}W_hC_{Xh}$	$\Omega_5=\sum_{h=1}^{L}W_hQ_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{p1,st}^{14})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\dfrac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{14}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{14}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{p1,st}^{14})=\left(\dfrac{\phi_{14}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{14}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{14}=\dfrac{\bar{Y}}{\bar{X}}\phi_{14}$ $\phi_{14}=\Omega_1\bar{X}/(\Omega_1\bar{X}+\Omega_5)$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}^{15}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\Omega_4\bar{X}+\Omega_5}{\Omega_4\bar{x}_{st}+\Omega_5}\right)$	$\Omega_4 = \sum_{h=1}^{L} W_h\beta_{1h}(x)$	$\Omega_5 = \sum_{h=1}^{L} W_h Q_{2h}(x)$	$MSE^*(\hat{\bar{Y}}^{15}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}} + R_{st}^{15}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{15}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{15}_{p1,st}) = \left(\frac{\phi_{15}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{15}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{15} = \frac{\bar{Y}}{\bar{X}}\phi_{15}$ $\phi_{15} = \Omega_4\bar{X}\big/(\Omega_4\bar{X}+\Omega_5)$
$\hat{\bar{Y}}^{16}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\Omega_2\bar{X}+\Omega_5}{\Omega_2\bar{x}_{st}+\Omega_5}\right)$	$\Omega_2 = \sum_{h=1}^{L} W_h\beta_{2h}(x)$	$\Omega_5 = \sum_{h=1}^{L} W_h Q_{2h}(x)$	$MSE^*(\hat{\bar{Y}}^{16}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}} + R_{st}^{16}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{16}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{16}_{p1,st}) = \left(\frac{\phi_{16}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{16}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{16} = \frac{\bar{Y}}{\bar{X}}\phi_{16}$ $\phi_{16} = \Omega_2\bar{X}\big/(\Omega_2\bar{X}+\Omega_5)$
$\hat{\bar{Y}}^{17}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\Omega_4\bar{X}+\Omega_6}{\Omega_4\bar{x}_{st}+\Omega_6}\right)$	$\Omega_4 = \sum_{h=1}^{L} W_h\beta_{1h}(x)$	$\Omega_6 = \sum_{h=1}^{L} W_h QD_h(x)$	$MSE^*(\hat{\bar{Y}}^{17}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}} + R_{st}^{17}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{17}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{17}_{p1,st}) = \left(\frac{\phi_{17}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{17}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{17} = \frac{\bar{Y}}{\bar{X}}\phi_{17}$ $\phi_{17} = \Omega_4\bar{X}\big/(\Omega_4\bar{X}+\Omega_6)$
$\hat{\bar{Y}}^{18}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\Omega_2\bar{X}+\Omega_6}{\Omega_2\bar{x}_{st}+\Omega_6}\right)$	$\Omega_2 = \sum_{h=1}^{L} W_h\beta_{2h}(x)$	$\Omega_6 = \sum_{h=1}^{L} W_h QD_h(x)$	$MSE^*(\hat{\bar{Y}}^{18}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}} + R_{st}^{18}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{18}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{18}_{p1,st}) = \left(\frac{\phi_{18}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{18}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{18} = \frac{\bar{Y}}{\bar{X}}\phi_{18}$ $\phi_{18} = \Omega_2\bar{X}\big/(\Omega_2\bar{X}+\Omega_6)$
$\hat{\bar{Y}}^{19}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\bar{X}+\Omega_7}{\bar{x}_{st}+\Omega_7}\right)$	1	$\Omega_7 = \sum_{h=1}^{L} W_h TM_h(x)$	$MSE^*(\hat{\bar{Y}}^{19}_{p1,st}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}} + R_{st}^{19}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{19}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{19}_{p1,st}) = \left(\frac{\phi_{19}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{19}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{19} = \frac{\bar{Y}}{\bar{X}}\phi_{19}$ $\phi_{19} = \bar{X}\big/(\bar{X}+\Omega_7)$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\hat{Y}}_{p1,st}^{20} = \bar{y}_{st}\left(\frac{\Omega_1\bar{X}+\Omega_7}{\Omega_1\bar{x}_{st}+\Omega_7}\right)$	$\Omega_1=\sum_{h=1}^{L}W_hC_{Xh}$	$\Omega_7=\sum_{h=1}^{L}W_hTM_h(x)$	$MSE^*(\hat{\hat{Y}}_{p1,st}^{20})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{20}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{20}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}_{p1,st}^{20})=\left(\frac{\phi_{20}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{20}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{20}=\frac{\bar{Y}}{\bar{X}}\phi_{20}$ $\phi_{20}=\Omega_1\bar{X}/(\Omega_1\bar{X}+\Omega_7)$
$\hat{\hat{Y}}_{p1,st}^{21} = \bar{y}_{st}\left(\frac{\Omega_3\bar{X}+\Omega_7}{\Omega_3\bar{x}_{st}+\Omega_7}\right)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{YXh}$	$\Omega_7=\sum_{h=1}^{L}W_hTM_h(x)$	$MSE^*(\hat{\hat{Y}}_{p1,st}^{21})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{21}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{21}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}_{p1,st}^{21})=\left(\frac{\phi_{21}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{21}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{21}=\frac{\bar{Y}}{\bar{X}}\phi_{21}$ $\phi_{21}=\Omega_3\bar{X}/(\Omega_3\bar{X}+\Omega_7)$
$\hat{\hat{Y}}_{p1,st}^{22} = \bar{y}_{st}\left(\frac{\bar{X}+\Omega_8}{\bar{x}_{st}+\Omega_8}\right)$	1	$\Omega_8=\sum_{h=1}^{L}W_hMR_h(x)$	$MSE^*(\hat{\hat{Y}}_{p1,st}^{22})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{22}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{22}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}_{p1,st}^{22})=\left(\frac{\phi_{22}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{22}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{22}=\frac{\bar{Y}}{\bar{X}}\phi_{22}$ $\phi_{22}=\bar{X}/(\bar{X}+\Omega_8)$
$\hat{\hat{Y}}_{p1,st}^{23} = \bar{y}_{st}\left(\frac{\Omega_1\bar{X}+\Omega_8}{\Omega_1\bar{x}_{st}+\Omega_8}\right)$	$\Omega_1=\sum_{h=1}^{L}W_hC_{Xh}$	$\Omega_8=\sum_{h=1}^{L}W_hMR_h(x)$	$MSE^*(\hat{\hat{Y}}_{p1,st}^{23})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{23}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{23}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}_{p1,st}^{23})=\left(\frac{\phi_{23}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{23}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{23}=\frac{\bar{Y}}{\bar{X}}\phi_{23}$ $\phi_{23}=\Omega_1\bar{X}/(\Omega_1\bar{X}+\Omega_8)$
$\hat{\hat{Y}}_{p1,st}^{24} = \bar{y}_{st}\left(\frac{\Omega_3\bar{X}+\Omega_8}{\Omega_3\bar{x}_{st}+\Omega_8}\right)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{YXh}$	$\Omega_8=\sum_{h=1}^{L}W_hMR_h(x)$	$MSE^*(\hat{\hat{Y}}_{p1,st}^{24})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Yh}^2}{\theta_{Yh}}+R_{st}^{24}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{24}-2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}_{p1,st}^{24})=\left(\frac{\phi_{24}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{24}-\beta_{YXh}\theta_{Xh}\right)$	$R_{st}^{24}=\frac{\bar{Y}}{\bar{X}}\phi_{24}$ $\phi_{24}=\Omega_3\bar{X}/(\Omega_3\bar{X}+\Omega_8)$

Estimators	a^j_{st}	b^j_{st}	**Mean Square Errors & Biases**	**Ratio's**
$\hat{\bar{Y}}^{25}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\bar{X}+\Omega_9}{\bar{x}_{st}+\Omega_9}\right)$	1	$\Omega_9 = \sum_{h=1}^{L} W_h HL_h(x)$	$MSE^*(\hat{\bar{Y}}^{25}_{p1,st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S^2_{Yh}}{\theta_{Yh}} + R^{25}_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^{25}_{st} - 2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{25}_{p1,st}) = \left(\frac{\phi_{25}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^{25}_{st} - \beta_{YXh}\theta_{Xh}\right)$	$R^{25}_{st} = \frac{\bar{Y}}{\bar{X}}\phi_{25}$ $\phi_{25} = \bar{X} \big/ (\bar{X}+\Omega_9)$
$\hat{\bar{Y}}^{26}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\Omega_1\bar{X}+\Omega_9}{\Omega_1\bar{x}_{st}+\Omega_9}\right)$	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$\Omega_9 = \sum_{h=1}^{L} W_h HL_h(x)$	$MSE^*(\hat{\bar{Y}}^{26}_{p1,st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S^2_{Yh}}{\theta_{Yh}} + R^{26}_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^{26}_{st} - 2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{26}_{p1,st}) = \left(\frac{\phi_{26}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^{26}_{st} - \beta_{YXh}\theta_{Xh}\right)$	$R^{26}_{st} = \frac{\bar{Y}}{\bar{X}}\phi_{26}$ $\phi_{26} = \Omega_1\bar{X} \big/ (\Omega_1\bar{X}+\Omega_9)$
$\hat{\bar{Y}}^{27}_{p1,st}$ $= \bar{y}_{st}\left(\frac{\Omega_3\bar{X}+\Omega_9}{\Omega_3\bar{x}_{st}+\Omega_9}\right)$	$\Omega_3 = \sum_{h=1}^{L} W_h \rho_{YXh}$	$\Omega_9 = \sum_{h=1}^{L} W_h HL_h(x)$	$MSE^*(\hat{\bar{Y}}^{27}_{p1,st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S^2_{Yh}}{\theta_{Yh}} + R^{27}_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^{27}_{st} - 2\beta_{YXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^{27}_{p1,st}) = \left(\frac{\phi_{27}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^{27}_{st} - \beta_{YXh}\theta_{Xh}\right)$	$R^{27}_{st} = \frac{\bar{Y}}{\bar{X}}\phi_{27}$ $\phi_{27} = \Omega_3\bar{X} \big/ (\Omega_3\bar{X}+\Omega_9)$

Various terms used in Table 5.1 are described in the Appendix-B.

5.5 EFFICIENCY COMPARISON

To check the efficiency of the proposed generalized ratio estimator ($\hat{\bar{Y}}_{p1,st}^{j}$) against combined sample mean estimator ($\bar{y}_{st}$), the mathematical conditions have been derived by using MSE's formulas with measurement errors. The algebraic expressions are:

$$MSE^*(\hat{\bar{Y}}_{p1,st}^{j}) \le Var^*(\bar{y}_{st})$$

if

$$\sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S_{Yh}^2}{\theta_{Yh}} + R_{st}^j \frac{S_{Xh}^2}{\theta_{Xh}} \left(R_{st}^j - 2\beta_{YXh}\theta_{Xh} \right) \right) \le \sum_{h=1}^{L} \frac{W_h^2 \gamma_h S_{Yh}^2}{\theta_{Yh}},$$

or if

$$\sum_{h=1}^{L} W_h^2 \gamma_h \left(R_{st}^j \frac{S_{Xh}^2}{\theta_{Xh}} \left(R_{st}^j - 2\frac{S_{YXh}}{S_{Xh}^2}\theta_{Xh} \right) \right) < 0 ,$$

or if

$$R_{st}^j < 2\frac{\bar{Y}}{\bar{X}}\frac{V_{11}}{V_{02}},$$

or if

$$R_{st}^j < 2\frac{A_1}{A_2}, \qquad (5.5.1)$$

where $A_1 = \sum_{h=1}^{L} W_h^2 \gamma_h S_{YXh}$ and $A_2 = \sum_{h=1}^{L} W_h^2 \gamma_h \frac{S_{Xh}^2}{\theta_{Xh}}$

The generalized combined ratio estimator $\hat{\bar{Y}}_{p1,st}^{j}$ will be more efficient than the combined sample mean estimator $\bar{y}_{st}$ under the condition (5.5.1). When the data are recorded without measurement error, the corresponding condition may be obtained by setting $S_{Vh}^2 = 0$.

5.6 SIMULATION RESULTS

In this section, we conduct a simulation study with particular focus on the following two issues:

a. How does the generalized estimator $(\hat{\bar{Y}}_{pi,st})$ and the generalized ratio estimator $(\hat{\bar{Y}}^{j}_{p1,st})$ compare with the combined sample mean estimator $(\bar{y}_{st})$ in the presence and absence of measurement errors?

b. How are the MSE, PRE and bias, influenced with the contribution of measurement errors?

We consider two bivariate normal populations with different covariance matrices to represent the distribution of the study variable (*Y*) and the auxiliary variable (*X*). All of the simulated populations have theoretical mean of (*Y*, X) as $\bar{Y} = [5 \quad 5]$ and covariance matrices as defined below:

Population-I	**Population-II**
N=1000	N=1000
$\sum_1 = \begin{bmatrix} 9 & 3.2 \\ 3.2 & 4 \end{bmatrix},\ \rho_{yx} = 0.5154$	$\sum_2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix},\ \rho_{yx} = 0.9511$

For both populations, the methodology used to get the observed values of (y_{hi}, x_{hi}) in h^{th} stratum is as given below:

$$y_{hi} = Y_{hi} + Q_{1i} \text{ and } x_{hi} = X_{hi} + Q_{2i},$$

where $Q_{1i}, Q_{2i} \sim uncorrelated\ N(0,1)$

We have considered three choices for sample sizes, namely n= 60, 150, and 300. For simulation study, stratification is based on the auxiliary variable (*X*). The sample size from each stratum is based on the Neyman allocation, so we give only the total sample size in each case. The following steps were used in a R-program:

Step 1: Twenty thousand samples of size *n* were selected from both populations, using (simple random sampling without replacement) in each stratum.

Step 2: Using the data from Step 1, twenty thousand values of an estimator (say $\hat{\bar{Y}}^{*}_{st}$) are obtained for each sample size.

Step 3: The empirical MSE of the estimators is computed by

$$EMSE(\hat{\bar{Y}}_{st}^{*}) = \frac{1}{20,000}\sum_{L=1}^{20,000}\left(\hat{\bar{Y}}_{st}^{*} - \bar{Y}\right)^{2},$$

where $\hat{\bar{Y}}_{st}^{*}$ represents an estimator deduced from (5.3.1) and $\bar{Y}$ is the population mean of the study variable. The following expression is used to calculate the percent relative efficiency (*PRE*) of proposed estimators as compared to the combined sample mean estimator.

$$PRE = \frac{VAR(\bar{y}_{st})}{MSE(\hat{\bar{Y}}_{st}^{*})} * 100$$

The MSE's, PRE's and biases of the estimators for both populations on different sample sizes are presented in Tables (5.2-5.5).

Table 5.2
Theoretical (boldface) and Empirical MSE's, PRE's (with/without ME) of the Estimators Relative to Combined Sample Mean Estimator in Stratified Random Sampling for Population-I
$(\rho_{xy1} = 0.5119, \rho_{xy2} = 0.5547)$

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		without ME	Change due to ME	with ME	Without ME	With ME
$\bar{y}_{st}$	60	**0.1450** 0.1465	**0.0138** 0.0132	**0.1588** 0.1597	100.0000	100.0000
	150	**0.0513** 0.051	**0.0058** 0.0058	**0.0571** 0.0568	100.0000	100.0000
	300	**0.0219** 0.0216	**0.0025** 0.0027	**0.0243** 0.0243	100.0000	100.0000
$\hat{\bar{Y}}_{pi,st}$	60	**0.0918** 0.0923	**0.0103** 0.0101	**0.1021** 0.1024	157.9521	155.5338
	150	**0.0319** 0.0305	**0.0083** 0.0090	**0.0402** 0.0395	160.8151	142.0398
	300	**0.0139** 0.0141	**0.0017** 0.0020	**0.0156** 0.0163	157.5539	155.7692
$\hat{\bar{Y}}^{1}_{p1,st}$	60	**0.1020** 0.1020	**0.0293** 0.0231	**0.1313** 0.1251	142.1569	120.9444
	150	**0.0375** 0.0389	**0.0113** 0.0105	**0.0488** 0.0494	136.8000	117.0082
	300	**0.0163** 0.0165	**0.0047** 0.0053	**0.0209** 0.0218	134.3558	116.2679
$\hat{\bar{Y}}^{2}_{p1,st}$	60	**0.1012** 0.1013	**0.0272** 0.0216	**0.1285** 0.1229	143.2806	123.5798
	150	**0.0369** 0.0382	**0.0105** 0.0096	**0.0474** 0.0478	139.0244	120.4641
	300	**0.0161** 0.0162	**0.0044** 0.0050	**0.0204** 0.0212	136.0248	119.1176
$\hat{\bar{Y}}^{3}_{p1,st}$	60	**0.1015** 0.1015	**0.0281** 0.0222	**0.1296** 0.1238	142.8571	122.5309
	150	**0.0373** 0.0387	**0.0111** 0.0102	**0.0483** 0.0489	137.5335	118.2195
	300	**0.0162** 0.0164	**0.0046** 0.0053	**0.0209** 0.0217	135.1852	116.2679

Estimators	*n*	**Mean Square Error**			**Percent Relative Efficiency**	
		without ME	**Change due to ME**	**with ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}^{4}_{p1,st}$	60	**0.1020** 0.1033	**0.0222** 0.0184	**0.1242** 0.1217	142.1569	127.8583
	150	**0.0381** 0.0376	**0.0075** 0.0086	**0.0456** 0.0462	134.6457	125.2193
	300	**0.0186** 0.0190	**0.0026** 0.0035	**0.0212** 0.0224	117.7419	114.6226
$\hat{\bar{Y}}^{5}_{p1,st}$	60	**0.1011** 0.1012	**0.0265** 0.0209	**0.1275** 0.1222	143.4224	124.5490
	150	**0.0370** 0.0384	**0.0107** 0.0099	**0.0477** 0.0483	138.6486	119.7065
	300	**0.0162** 0.0164	**0.0046** 0.0052	**0.0208** 0.0216	135.1852	116.8269
$\hat{\bar{Y}}^{6}_{p1,st}$	60	**0.1011** 0.1012	**0.0264** 0.0210	**0.1275** 0.1222	143.4224	124.549
	150	**0.0368** 0.0380	**0.0103** 0.0094	**0.0470** 0.0474	139.4022	121.4894
	300	**0.0161** 0.0161	**0.0043** 0.0049	**0.0203** 0.0210	136.0248	119.7044
$\hat{\bar{Y}}^{7}_{p1,st}$	60	**0.1014** 0.1018	**0.0232** 0.0186	**0.1247** 0.1204	142.9980	127.3456
	150	**0.0365** 0.0376	**0.0092** 0.0084	**0.0457** 0.0460	140.5479	124.9453
	300	**0.0160** 0.0160	**0.0038** 0.0043	**0.0198** 0.0203	136.8750	122.7273
$\hat{\bar{Y}}^{8}_{p1,st}$	60	**0.1010** 0.1012	**0.0258** 0.0204	**0.1269** 0.1217	143.5644	125.1379
	150	**0.0366** 0.0378	**0.0099** 0.0091	**0.0465** 0.0469	140.1639	122.7957
	300	**0.0160** 0.0161	**0.0041** 0.0046	**0.0201** 0.0207	136.8750	120.8955
$\hat{\bar{Y}}^{9}_{p1,st}$	60	**0.1035** 0.1049	**0.0206** 0.0171	**0.1240** 0.1220	140.0966	128.0645
	150	**0.0390** 0.0385	**0.0071** 0.0082	**0.0461** 0.0467	131.5385	123.8612
	300	**0.0191** 0.0193	**0.0026** 0.0033	**0.0217** 0.0226	114.6597	111.9816

Estimators	***n***	**Mean Square Error**			**Percent Relative Efficiency**	
		without ME	**Change due to ME**	**with ME**	**Without ME**	**With ME**
$\hat{\hat{Y}}^{10}_{p1,st}$	60	**0.1012** 0.1014	**0.0272** 0.0215	**0.1285** 0.1229	143.2806	123.5798
	150	**0.0371** 0.0385	**0.0109** 0.0100	**0.0479** 0.0485	138.2749	119.2067
	300	**0.0162** 0.0164	**0.0046** 0.0052	**0.0209** 0.0216	135.1852	116.2679
$\hat{\hat{Y}}^{11}_{p1,st}$	60	**0.1024** 0.1023	**0.0300** 0.0237	**0.1324** 0.1260	141.6016	119.9396
	150	**0.0372** 0.0386	**0.0110** 0.0101	**0.0481** 0.0487	137.9032	118.7110
	300	**0.0162** 0.0164	**0.0046** 0.0053	**0.0208** 0.0217	135.1852	116.8269
$\hat{\hat{Y}}^{12}_{p1,st}$	60	**0.1077** 0.1096	**0.0395** 0.0477	**0.1472** 0.1574	134.6332	107.8804
	150	**0.0404** 0.0413	**0.0072** 0.0085	**0.0476** 0.0498	126.9802	119.9580
	300	**0.0161** 0.0165	**0.0043** 0.0046	**0.0204** 0.0211	136.0248	119.1176
$\hat{\hat{Y}}^{13}_{p1,st}$	60	**0.1090** 0.1099	**0.0176** 0.0147	**0.1266** 0.1246	133.0275	125.4344
	150	**0.0385** 0.0391	**0.0072** 0.0065	**0.0457** 0.0456	133.2468	124.9453
	300	**0.0169** 0.0167	**0.0030** 0.0034	**0.0199** 0.0201	129.5858	122.1106
$\hat{\hat{Y}}^{14}_{p1,st}$	60	**0.1216** 0.1228	**0.0150** 0.0132	**0.1366** 0.1360	119.2434	116.2518
	150	**0.0423** 0.0426	**0.0063** 0.0058	**0.0486** 0.0484	121.2766	117.4897
	300	**0.0185** 0.0182	**0.0026** 0.0030	**0.0212** 0.0212	118.3784	114.6226
$\hat{\hat{Y}}^{15}_{p1,st}$	60	**0.1319** 0.1381	**0.0163** 0.0161	**0.1482** 0.1542	109.9318	107.1525
	150	**0.0462** 0.0465	**0.0059** 0.0061	**0.0522** 0.0526	111.0390	109.3870
	300	**0.0211** 0.0208	**0.0025** 0.0027	**0.0235** 0.0235	103.7915	103.4043

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		without ME	**Change due to ME**	**with ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}^{16}_{p1,st}$	60	**0.1294** 0.1309	**0.0143** 0.0130	**0.1437** 0.1439	112.0556	110.5080
	150	**0.0475** 0.0477	**0.0059** 0.0061	**0.0534** 0.0538	108.0000	106.9288
	300	**0.0215** 0.0212	**0.0025** 0.0028	**0.0239** 0.0240	101.8605	101.6736
$\hat{\bar{Y}}^{17}_{p1,st}$	60	**0.1160** 0.1217	**0.0178** 0.0188	**0.1338** 0.1405	125.0000	118.6846
	150	**0.0404** 0.0436	**0.0066** 0.0075	**0.0470** 0.0512	126.9802	121.4894
	300	**0.0195** 0.0191	**0.0025** 0.0028	**0.0220** 0.0219	112.3077	110.4545
$\hat{\bar{Y}}^{18}_{p1,st}$	60	**0.1115** 0.1130	**0.0168** 0.0144	**0.1284** 0.1274	130.0448	123.6760
	150	**0.0421** 0.0459	**0.0063** 0.0077	**0.0484** 0.0536	121.8527	117.9752
	300	**0.0205** 0.0204	**0.0025** 0.0029	**0.0230** 0.0233	106.8293	105.6522
$\hat{\bar{Y}}^{19}_{p1,st}$	60	**0.1089** 0.1098	**0.0177** 0.0147	**0.1266** 0.1245	133.1497	125.4344
	150	**0.0385** 0.0391	**0.0072** 0.0064	**0.0457** 0.0456	133.2468	124.9453
	300	**0.0169** 0.0167	**0.0030** 0.0034	**0.0199** 0.0201	129.5858	122.1106
$\hat{\bar{Y}}^{20}_{p1,st}$	60	**0.1215** 0.1227	**0.0150** 0.0132	**0.1365** 0.1359	119.3416	116.3370
	150	**0.0424** 0.0426	**0.0063** 0.0058	**0.0487** 0.0484	120.9906	117.2485
	300	**0.0185** 0.0182	**0.0026** 0.0030	**0.0212** 0.0212	118.3784	114.6226
$\hat{\bar{Y}}^{21}_{p1,st}$	60	**0.1164** 0.1174	**0.0158** 0.0136	**0.1321** 0.1310	124.5704	120.2120
	150	**0.0411** 0.0415	**0.0065** 0.0059	**0.0476** 0.0474	124.8175	119.9580
	300	**0.0180** 0.0177	**0.0027** 0.0030	**0.0207** 0.0207	121.6667	117.3913

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		without ME	Change due to ME	with ME	Without ME	With ME
$\hat{\bar{Y}}_{p1,st}^{22}$	60	**0.1084** 0.1092	**0.0179** 0.0149	**0.1262** 0.1241	133.7638	125.8320
	150	**0.0389** 0.0395	**0.0070** 0.0063	**0.0460** 0.0458	131.8766	124.1304
	300	**0.0170** 0.0167	**0.0030** 0.0034	**0.0200** 0.0201	128.8235	121.5000
$\hat{\bar{Y}}_{p1,st}^{23}$	60	**0.1208** 0.1220	**0.0151** 0.0132	**0.1359** 0.1352	120.0331	116.8506
	150	**0.0430** 0.0432	**0.0062** 0.0057	**0.0492** 0.0489	119.3023	116.0569
	300	**0.0186** 0.0183	**0.0026** 0.0029	**0.0213** 0.0213	117.7419	114.0845
$\hat{\bar{Y}}_{p1,st}^{24}$	60	**0.1157** 0.1167	**0.0159** 0.0137	**0.1316** 0.1304	125.3241	120.6687
	150	**0.0417** 0.0420	**0.0064** 0.0058	**0.0480** 0.0478	123.0216	118.9583
	300	**0.0181** 0.0178	**0.0027** 0.0030	**0.0208** 0.0208	120.9945	116.8269
$\hat{\bar{Y}}_{p1,st}^{25}$	60	**0.1089** 0.1098	**0.0177** 0.0147	**0.1266** 0.1245	133.1497	125.4344
	150	**0.0385** 0.0391	**0.0072** 0.0064	**0.0457** 0.0456	133.2468	124.9453
	300	**0.0169** 0.0167	**0.0030** 0.0034	**0.0199** 0.0201	129.5858	122.1106
$\hat{\bar{Y}}_{p1,st}^{26}$	60	**0.1215** 0.1227	**0.0150** 0.0132	**0.1365** 0.1359	119.3416	116.3370
	150	**0.0424** 0.0427	**0.0063** 0.0058	**0.0487** 0.0484	120.9906	117.2485
	300	**0.0185** 0.0182	**0.0026** 0.0030	**0.0212** 0.0212	118.3784	114.6226
$\hat{\bar{Y}}_{p1,st}^{27}$	60	**0.1164** 0.1174	**0.0158** 0.0136	**0.1321** 0.1310	124.5704	120.2120
	150	**0.0411** 0.0415	**0.0065** 0.0059	**0.0476** 0.0474	124.8175	119.9580
	300	**0.0180** 0.0177	**0.0027** 0.0030	**0.0207** 0.0207	121.6667	117.3913

Table 5.3
Theoretical Biases (with/without ME) of the Estimators in Stratified Random Sampling for Population-I ($\rho_{xy1} = 0.5119, \rho_{xy2} = 0.5547$)

Estimators	bias (without ME)			bias (with ME)		
	n			*n*		
	60	150	300	60	150	300
$\bar{y}_{st}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{pi,st}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}^{1}_{p1,st}$	0.0016	0.0009	0.0003	0.0047	0.002	0.0008
$\hat{\bar{Y}}^{2}_{p1,st}$	0.0007	0.0005	0.0002	0.0034	0.0015	0.0006
$\hat{\bar{Y}}^{3}_{p1,st}$	0.0011	0.0008	0.0003	0.004	0.0019	0.0007
$\hat{\bar{Y}}^{4}_{p1,st}$	-0.0011	-0.0006	-0.0003	0.0006	-0.0003	-0.0002
$\hat{\bar{Y}}^{5}_{p1,st}$	0.0004	0.0006	0.0003	0.003	0.0016	0.0007
$\hat{\bar{Y}}^{6}_{p1,st}$	0.0004	0.0004	0.0001	0.0029	0.0013	0.0005
$\hat{\bar{Y}}^{7}_{p1,st}$	-0.0008	0.0000	0.0000	0.0011	0.0007	0.0002
$\hat{\bar{Y}}^{8}_{p1,st}$	0.0002	0.0003	0.0001	0.0026	0.0011	0.0004
$\hat{\bar{Y}}^{9}_{p1,st}$	-0.0016	-0.0007	-0.0002	-0.0003	-0.0005	-0.0002
$\hat{\bar{Y}}^{10}_{p1,st}$	0.0007	0.0005	0.0003	0.0034	0.0017	0.0007
$\hat{\bar{Y}}^{11}_{p1,st}$	0.0019	0.0008	0.0003	0.0051	0.0018	0.0007
$\hat{\bar{Y}}^{12}_{p1,st}$	0.0132	-0.0007	0.0001	0.0209	-0.0005	0.0005
$\hat{\bar{Y}}^{13}_{p1,st}$	-0.0022	-0.0007	-0.0003	-0.0014	-0.0004	-0.0002
$\hat{\bar{Y}}^{14}_{p1,st}$	-0.0019	-0.0007	-0.0003	-0.0017	-0.0006	-0.0002
$\hat{\bar{Y}}^{15}_{p1,st}$	-0.0013	-0.0004	-0.0001	-0.0011	-0.0004	-0.0001

Estimators	bias (without ME)			bias (with ME)		
	n			n		
	60	150	300	60	150	300
$\hat{\bar{Y}}_{p1,st}^{16}$	-0.0014	-0.0003	0.0000	-0.0013	-0.0003	0.0000
$\hat{\bar{Y}}_{p1,st}^{17}$	-0.0021	-0.0007	-0.0002	-0.0018	-0.0006	-0.0002
$\hat{\bar{Y}}_{p1,st}^{18}$	-0.0022	-0.0007	-0.0001	-0.0016	-0.0006	-0.0001
$\hat{\bar{Y}}_{p1,st}^{19}$	-0.0022	-0.0007	-0.0003	-0.0014	-0.0004	-0.0002
$\hat{\bar{Y}}_{p1,st}^{20}$	-0.0019	-0.0007	-0.0003	-0.0017	-0.0006	-0.0002
$\hat{\bar{Y}}_{p1,st}^{21}$	-0.0021	-0.0007	-0.0003	-0.0018	-0.0006	-0.0002
$\hat{\bar{Y}}_{p1,st}^{22}$	-0.0021	-0.0007	-0.0003	-0.0013	-0.0005	-0.0002
$\hat{\bar{Y}}_{p1,st}^{23}$	-0.002	-0.0007	-0.0003	-0.0017	-0.0006	-0.0002
$\hat{\bar{Y}}_{p1,st}^{24}$	-0.0022	-0.0007	-0.0003	-0.0017	-0.0006	-0.0002
$\hat{\bar{Y}}_{p1,st}^{25}$	-0.0022	-0.0007	-0.0003	-0.0014	-0.0004	-0.0002
$\hat{\bar{Y}}_{p1,st}^{26}$	-0.0019	-0.0007	-0.0003	-0.0017	-0.0006	-0.0002
$\hat{\bar{Y}}_{p1,st}^{27}$	-0.0021	-0.0007	-0.0003	-0.0018	-0.0006	-0.0002

Table 5.4
Theoretical (boldface) and Empirical MSE's, PRE's (with/without ME) of the Estimators Relative to Combined Sample Mean Estimator in Stratified Random Sampling for Population-II
$(\rho_{xy1} = 0.9525, \rho_{xy2} = 0.9454)$

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\bar{y}_{st}$	60	**0.0777** 0.0783	**0.0161** 0.0168	**0.0938** 0.0951	100.0000	100.0000
	150	**0.0278** 0.0280	**0.0061** 0.0048	**0.0339** 0.0328	100.0000	100.0000
	300	**0.0118** 0.0117	**0.0021** 0.0028	**0.0139** 0.0144	100.0000	100.0000
$\hat{\bar{Y}}_{pi,st}$	60	**0.0075** 0.0069	**0.0023** 0.0018	**0.0098** 0.0087	1036.0000	957.1429
	150	**0.0026** 0.0032	**0.0008** 0.0007	**0.0034** 0.0039	1069.2308	997.0588
	300	**0.0011** 0.0014	**0.0006** 0.0005	**0.0017** 0.0019	1072.7273	817.6471
$\hat{\bar{Y}}^{1}_{p1,st}$	60	**0.0152** 0.0157	**0.0319** 0.0316	**0.0472** 0.0473	511.1842	198.7288
	150	**0.0052** 0.0053	**0.0114** 0.0104	**0.0166** 0.0157	534.6154	204.2169
	300	**0.0023** 0.0023	**0.0047** 0.0051	**0.0070** 0.0075	513.0435	198.5714
$\hat{\bar{Y}}^{2}_{p1,st}$	60	**0.0169** 0.0175	**0.0303** 0.0299	**0.0473** 0.0474	459.7633	198.3087
	150	**0.0058** 0.0059	**0.0109** 0.0099	**0.0167** 0.0158	479.3103	202.9940
	300	**0.0026** 0.0026	**0.0044** 0.0049	**0.0070** 0.0075	453.8462	198.5714
$\hat{\bar{Y}}^{3}_{p1,st}$	60	**0.0175** 0.0181	**0.0298** 0.0294	**0.0474** 0.0475	444.0000	197.8903
	150	**0.0051** 0.0052	**0.0115** 0.0106	**0.0166** 0.0158	545.0980	204.2169
	300	**0.0024** 0.0024	**0.0046** 0.0051	**0.0070** 0.0075	491.6667	198.5714

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\hat{\hat{Y}}_{p1,st}^{4}$	60	**0.0197** 0.0211	**0.0281** 0.0289	**0.0478** 0.0500	394.4162	196.2343
	150	**0.0095** 0.0101	**0.0084** 0.0078	**0.0179** 0.0179	292.6316	189.3855
	300	**0.0065** 0.0111	**0.0031** 0.0043	**0.0096** 0.0154	181.5385	144.7917
$\hat{\hat{Y}}_{p1,st}^{5}$	60	**0.0232** 0.0238	**0.0259** 0.0255	**0.0492** 0.0493	334.9138	190.6504
	150	**0.0047** 0.0048	**0.0119** 0.0110	**0.0166** 0.0158	591.4894	204.2169
	300	**0.0024** 0.0025	**0.0045** 0.0050	**0.0070** 0.0075	491.6667	198.5714
$\hat{\hat{Y}}_{p1,st}^{6}$	60	**0.0209** 0.0216	**0.0273** 0.0268	**0.0483** 0.0484	371.7703	194.2029
	150	**0.0072** 0.0074	**0.0099** 0.0087	**0.0171** 0.0161	386.1111	198.2456
	300	**0.0032** 0.0032	**0.0039** 0.0044	**0.0071** 0.0076	368.7500	195.7746
$\hat{\hat{Y}}_{p1,st}^{7}$	60	**0.0328** 0.0335	**0.0217** 0.0214	**0.0546** 0.0549	236.8902	171.7949
	150	**0.0115** 0.0116	**0.0080** 0.0067	**0.0195** 0.0183	241.7391	173.8462
	300	**0.0050** 0.0050	**0.0031** 0.0035	**0.0080** 0.0085	236.0000	173.7500
$\hat{\hat{Y}}_{p1,st}^{8}$	60	**0.0171** 0.0175	**0.0302** 0.0299	**0.0472** 0.0474	454.3860	198.7288
	150	**0.0059** 0.0060	**0.0108** 0.0098	**0.0167** 0.0158	471.1864	202.9940
	300	**0.0026** 0.0026	**0.0044** 0.0048	**0.0070** 0.0075	453.8462	198.5714
$\hat{\hat{Y}}_{p1,st}^{9}$	60	**0.0291** 0.0302	**0.0231** 0.0235	**0.0523** 0.0537	267.0103	179.3499
	150	**0.0153** 0.0157	**0.0064** 0.0060	**0.0217** 0.0217	181.6993	156.2212
	300	**0.0096** 0.0110	**0.0026** 0.0029	**0.0122** 0.0139	122.9167	113.9344

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\hat{\bar{Y}}^{10}_{p1,st}$	60	**0.0176** 0.0182	**0.0297** 0.0293	**0.0474** 0.0475	441.4773	197.8903
	150	**0.0051** 0.0052	**0.0115** 0.0106	**0.0166** 0.0158	545.0980	204.2169
	300	**0.0024** 0.0024	**0.0046** 0.0051	**0.0070** 0.0075	491.6667	198.5714
$\hat{\bar{Y}}^{11}_{p1,st}$	60	**0.0152** 0.0157	**0.0319** 0.0316	**0.0472** 0.0473	511.1842	198.7288
	150	**0.0053** 0.0054	**0.0113** 0.0103	**0.0166** 0.0157	524.5283	204.2169
	300	**0.0022** 0.0022	**0.0048** 0.0053	**0.0070** 0.0075	536.3636	198.5714
$\hat{\bar{Y}}^{12}_{p1,st}$	60	**0.0759** 0.0789	**0.0161** 0.0201	**0.0921** 0.0990	102.3715	101.8458
	150	**0.0208** 0.0211	**0.0059** 0.0072	**0.0267** 0.0283	133.6538	126.9663
	300	**0.0030** 0.0031	**0.0045** 0.0045	**0.0075** 0.0077	393.3333	185.3333
$\hat{\bar{Y}}^{13}_{p1,st}$	60	**0.0386** 0.0393	**0.0200** 0.0198	**0.0587** 0.0591	201.2953	159.7956
	150	**0.0136** 0.0138	**0.0074** 0.0060	**0.0210** 0.0198	204.4118	161.4286
	300	**0.0059** 0.0059	**0.0028** 0.0033	**0.0086** 0.0091	200.0000	161.6279
$\hat{\bar{Y}}^{14}_{p1,st}$	60	**0.0586** 0.0593	**0.0168** 0.0172	**0.0755** 0.0765	132.5939	124.2384
	150	**0.0208** 0.0210	**0.0063** 0.0049	**0.0271** 0.0259	133.6538	125.0923
	300	**0.0088** 0.0088	**0.0023** 0.0028	**0.0111** 0.0116	134.0909	125.2252
$\hat{\bar{Y}}^{15}_{p1,st}$	60	**0.0776** 0.0781	**0.0161** 0.0168	**0.0937** 0.0949	100.1289	100.1067
	150	**0.0272** 0.0274	**0.0061** 0.0048	**0.0333** 0.0322	102.2059	101.8018
	300	**0.0113** 0.0114	**0.0025** 0.0025	**0.0138** 0.0138	104.4248	100.7246

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\hat{\bar{Y}}^{16}_{p1,st}$	60	**0.0544** 0.0551	**0.0173** 0.0177	**0.0717** 0.0728	142.8309	130.8229
	150	**0.0237** 0.0241	**0.0057** 0.0046	**0.0294** 0.0287	117.2996	115.3061
	300	**0.0112** 0.0112	**0.0022** 0.0030	**0.0134** 0.0142	105.3571	103.7313
$\hat{\bar{Y}}^{17}_{p1,st}$	60	**0.0770** 0.0777	**0.0161** 0.0177	**0.0931** 0.0954	100.9091	100.7519
	150	**0.0248** 0.0253	**0.0061** 0.0049	**0.0309** 0.0302	112.0968	109.7087
	300	**0.0075** 0.0075	**0.0029** 0.0030	**0.0103** 0.0105	157.3333	134.9515
$\hat{\bar{Y}}^{18}_{p1,st}$	60	**0.0286** 0.0297	**0.0233** 0.0237	**0.0520** 0.0534	271.6783	180.3846
	150	**0.0149** 0.0154	**0.0068** 0.0059	**0.0217** 0.0213	186.5772	156.2212
	300	**0.0097** 0.0110	**0.0026** 0.0028	**0.0123** 0.0139	121.6495	113.0081
$\hat{\bar{Y}}^{19}_{p1,st}$	60	**0.0386** 0.0392	**0.0200** 0.0199	**0.0586** 0.0591	201.2953	160.0683
	150	**0.0136** 0.0138	**0.0074** 0.0060	**0.0210** 0.0198	204.4118	161.4286
	300	**0.0059** 0.0059	**0.0028** 0.0033	**0.0086** 0.0091	200.0000	161.6279
$\hat{\bar{Y}}^{20}_{p1,st}$	60	**0.0587** 0.0593	**0.0168** 0.0172	**0.0756** 0.0765	132.3680	124.0741
	150	**0.0208** 0.0210	**0.0064** 0.0049	**0.0272** 0.0259	133.6538	124.6324
	300	**0.0088** 0.0088	**0.0023** 0.0028	**0.0111** 0.0116	134.0909	125.2252
$\hat{\bar{Y}}^{21}_{p1,st}$	60	**0.0393** 0.0400	**0.0198** 0.0197	**0.0593** 0.0597	197.7099	158.1788
	150	**0.0139** 0.0141	**0.0073** 0.0060	**0.0212** 0.0201	200.0000	159.9057
	300	**0.0060** 0.0060	**0.0027** 0.0032	**0.0087** 0.0092	196.6667	159.7701

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}^{22}_{p1,st}$	60	**0.0364** 0.0371	**0.0206** 0.0204	**0.0571** 0.0575	213.4615	164.2732
	150	**0.0139** 0.0141	**0.0074** 0.0060	**0.0213** 0.0201	200.0000	159.1549
	300	**0.0056** 0.0056	**0.0028** 0.0033	**0.0085** 0.0090	210.7143	163.5294
$\hat{\bar{Y}}^{23}_{p1,st}$	60	**0.0567** 0.0573	**0.0171** 0.0173	**0.0737** 0.0746	137.0370	127.2727
	150	**0.0211** 0.0213	**0.0063** 0.0049	**0.0274** 0.0262	131.7536	123.7226
	300	**0.0086** 0.0086	**0.0023** 0.0028	**0.0109** 0.0115	137.2093	127.5229
$\hat{\bar{Y}}^{24}_{p1,st}$	60	**0.0372** 0.0379	**0.0204** 0.0202	**0.0576** 0.0581	208.8710	162.8472
	150	**0.0142** 0.0144	**0.0073** 0.0059	**0.0215** 0.0203	195.7746	157.6744
	300	**0.0058** 0.0058	**0.0028** 0.0033	**0.0086** 0.0091	203.4483	161.6279
$\hat{\bar{Y}}^{25}_{p1,st}$	60	**0.0386** 0.0393	**0.0200** 0.0198	**0.0586** 0.0591	201.2953	160.0683
	150	**0.0136** 0.0138	**0.0074** 0.0060	**0.0210** 0.0198	204.4118	161.4286
	300	**0.0059** 0.0059	**0.0028** 0.0033	**0.0086** 0.0091	200.0000	161.6279
$\hat{\bar{Y}}^{26}_{p1,st}$	60	**0.0586** 0.0593	**0.0168** 0.0172	**0.0756** 0.0765	132.5939	124.0741
	150	**0.0208** 0.0210	**0.0063** 0.0049	**0.0271** 0.0259	133.6538	125.0923
	300	**0.0088** 0.0088	**0.0023** 0.0028	**0.0111** 0.0116	134.0909	125.2252
$\hat{\bar{Y}}^{27}_{p1,st}$	60	**0.0394** 0.0401	**0.0198** 0.0196	**0.0593** 0.0597	197.2081	158.1788
	150	**0.0139** 0.0141	**0.0073** 0.0060	**0.0212** 0.0201	200.0000	159.9057
	300	**0.0060** 0.0060	**0.0027** 0.0032	**0.0087** 0.0092	196.6667	159.7701

Table 5.5
Theoretical Biases (with/without ME) of the Estimators in Stratified Random Sampling for Population-II $(\rho_{xy1} = 0.9525, \rho_{xy2} = 0.9454)$

Estimators	**bias (without ME)**			**bias (with ME)**		
	N			***n***		
	60	**150**	**300**	**60**	**150**	**300**
$\bar{y}_{st}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{pi,st}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}^{1}_{p1,st}$	-0.0031	-0.0011	-0.0005	0.0000	0.0000	0.0000
$\hat{\bar{Y}}^{2}_{p1,st}$	-0.0033	-0.0011	-0.0005	-0.0005	-0.0001	0.0000
$\hat{\bar{Y}}^{3}_{p1,st}$	-0.0033	-0.0011	-0.0005	-0.0005	0.0000	0.0000
$\hat{\bar{Y}}^{4}_{p1,st}$	-0.0034	0.0718	-0.0005	-0.0012	0.1145	-0.0002
$\hat{\bar{Y}}^{5}_{p1,st}$	-0.0035	-0.0010	-0.0005	-0.0014	0.0002	0.0000
$\hat{\bar{Y}}^{6}_{p1,st}$	-0.0034	-0.0012	-0.0005	-0.0013	-0.0005	-0.0002
$\hat{\bar{Y}}^{7}_{p1,st}$	-0.0033	-0.0012	-0.0005	-0.0022	-0.0008	-0.0003
$\hat{\bar{Y}}^{8}_{p1,st}$	-0.0033	-0.0011	-0.0005	-0.0005	-0.0002	0.0000
$\hat{\bar{Y}}^{9}_{p1,st}$	-0.0033	0.0026	-0.0002	-0.0021	0.0028	-0.0002
$\hat{\bar{Y}}^{10}_{p1,st}$	-0.0032	-0.0011	-0.0005	-0.0006	0.0000	0.0000
$\hat{\bar{Y}}^{11}_{p1,st}$	-0.0031	-0.0011	-0.0005	0.0000	0.0000	0.0001
$\hat{\bar{Y}}^{12}_{p1,st}$	-0.0017	0.0124	-0.0004	-0.0015	0.0230	0.0001
$\hat{\bar{Y}}^{13}_{p1,st}$	-0.0031	-0.0011	-0.0005	-0.0023	-0.0008	-0.0003
$\hat{\bar{Y}}^{14}_{p1,st}$	-0.0017	-0.0006	-0.0003	-0.0015	-0.0005	-0.0002
$\hat{\bar{Y}}^{15}_{p1,st}$	-0.0001	0.0000	0.0002	-0.0001	0.0000	0.0002

$\hat{\hat{Y}}_{p1,st}^{16}$	-0.0018	0.0002	-0.0001	-0.0016	0.0003	-0.0001
$\hat{\hat{Y}}_{p1,st}^{17}$	-0.0007	-0.0002	0.0048	-0.0007	-0.0002	0.0061
$\hat{\hat{Y}}_{p1,st}^{18}$	-0.0034	0.0024	-0.0002	-0.0022	0.0027	-0.0002
$\hat{\hat{Y}}_{p1,st}^{19}$	-0.0031	-0.0011	-0.0005	-0.0023	-0.0008	-0.0003
$\hat{\hat{Y}}_{p1,st}^{20}$	-0.0017	-0.0006	-0.0003	-0.0015	-0.0006	-0.0002
$\hat{\hat{Y}}_{p1,st}^{21}$	-0.0030	-0.0011	-0.0005	-0.0023	-0.0008	-0.0003
$\hat{\hat{Y}}_{p1,st}^{22}$	-0.0030	-0.0011	-0.0005	-0.0023	-0.0008	-0.0003
$\hat{\hat{Y}}_{p1,st}^{23}$	-0.0017	-0.0006	-0.0003	-0.0015	-0.0005	-0.0003
$\hat{\hat{Y}}_{p1,st}^{24}$	-0.0029	-0.0011	-0.0005	-0.0023	-0.0008	-0.0003
$\hat{\hat{Y}}_{p1,st}^{25}$	-0.0031	-0.0011	-0.0005	-0.0023	-0.0008	-0.0003
$\hat{\hat{Y}}_{p1,st}^{26}$	-0.0017	-0.0006	-0.0003	-0.0016	-0.0005	-0.0002
$\hat{\hat{Y}}_{p1,st}^{27}$	-0.0030	-0.0011	-0.0005	-0.0023	-0.0008	-0.0003

Tables 5.2-5.5 give the contribution of measurement error, the amount of MSE's, PRE's and biases for the estimators ($\bar{y}_{st}$), ($\hat{\bar{Y}}_{pi,st}$) and ($\hat{\bar{Y}}^{j}_{p1,st}$) based on different sample sizes. It is clear from the simulation results that measurement errors play a significant role in increasing the MSE of an estimator. The reduction in PRE for ($\hat{\bar{Y}}^{j}_{p1,st}$), when measurement errors occur, is much higher as compared to the proposed generalized estimator ($\hat{\bar{Y}}_{pi,st}$). Also note that the performance of the proposed generalized estimator is better than the existing estimators both in the presence and absence of measurement errors, more so in the high correlation case as compared to the low correlation case.

5.7 NUMERICAL EXAMPLE

Data Statistics

The data choice used for numerical study is motivated by Sousa et al. (2014). While the actual data used there is subject to confidentiality protocol, we have used here simulated data which follows the summary information in Sousa et al. (2014). The following notations are used below:

Y = True Purchase orders, X = True Turnover of enterprises
y = Measured Purchase orders, x = Measured Turnover of enterprises

The methodology used to get the observed values of y and x in h^th^ stratum is $y_{hi} = Y_{hi} + Q_{1i}$ and $x_{hi} = X_{hi} + Q_{2i}$ where $Q_{1i}, Q_{2i} \sim \text{uncorrelated}\, N(0,1)$. We generated a bivariate normal population of size 1698 with these parameters.

Table 5.6
Summary Statistics for the Numerical Example

Stratum (h)	N_h	ρ_{yxh}	$\bar{Y}_h$	S_{Yh}	$\bar{X}_h$	S_{Xh}	Population
1	979	0.7802	2.15	2.46	3.12	2.68	N=1698, $\rho_{yx} = 0.9368$
2	362	0.7952	16.67	6.86	20.31	6.02	$\bar{Y} = 14.44$, $\bar{X} = 17.97$
3	357	0.8408	45.88	30.21	56.33	30.18	$S_y = 22.39$, $S_x = 25.31$

Table 5.7
Theoretical MSE's, PRE's (with/without ME) of the Estimators Relative to Combined Sample Mean Estimator in Stratified Random Sampling for the Numerical Example

Estimators	***n***	**Mean Square Error**			**Percent Relative Efficiency**	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\bar{y}_{st}$ **(Combined Sample Mean Estimator)**	250	0.7204	0.0034	0.7238	100.0000	100.0000
	500	0.3087	0.0026	0.3113	100.0000	100.0000
$\hat{\bar{Y}}_{pi,st}$ **(Proposed Generalized Estimator)**	250	0.2060	0.0013	0.2073	349.7087	349.1558
	500	0.0857	0.0009	0.0866	360.2100	359.4688
$\hat{\bar{Y}}^{1}_{p1,st}$ **(Combined Ratio Estimator)**	250	0.2126	0.0053	0.2179	338.8523	332.1707
	500	0.0865	0.0028	0.0893	356.8786	348.6002

Table 5.7 shows that our proposed generalized estimator ($\hat{\bar{Y}}_{pi,st}$) performs better than existing estimators both in the presence and absence of measurement errors. We saw the same pattern in Tables (5.2-5.5). Also presence of measurement errors impacts the proposed generalized estimator ($\hat{\bar{Y}}_{pi,st}$) less than the combined ratio estimator ($\hat{\bar{Y}}^{1}_{p1,st}$).

5.8 CONCLUSION

The main contribution of this chapter is the introduction of a generalized mean estimator in the presence of measurement errors in stratified random sampling. The generalized estimator leads to several combined ratio and combined product estimators as special cases. A specific generalized combined ratio estimator has been studied by using various functions of the parameters of the auxiliary variable. The asymptotic bias and MSE formulae have been derived. The results are validated through a simulation and numerical study. Tables 5.2-5.5 show the effect of measurement errors on the proposed estimators using different sampling fractions. As expected, the theoretical and empirical mean square errors are quite similar. All of the estimators are more efficient as compared to the combined sample mean estimator. Improvement in efficiency is more significant in the high correlation case as compared to the low correlation case. It is obvious from the simulation and numerical results that measurement errors hurt the efficiency of all estimators Thus there is a need to make a considerable effort to eliminate measurement errors in the survey.

CHAPTER 6

MEAN ESTIMATION FOR A SENSITIVE STUDY VARIABLE IN THE PRESENCE OF MEASUREMENT ERRORS UNDER SIMPLE RANDOM SAMPLING

6.1 INTRODUCTION

In this Chapter, a class of generalized estimators of population mean for a sensitive study variable has been proposed when both the study variable and the auxiliary variable are contaminated by measurement errors, and the sampling design is simple random sampling without replacement. The focus is on studying the impact of measurement errors on the efficiency of RRT mean estimators. We have also compared our proposed estimator with some existing estimators.

6.2 SAMPLING PROCEDURE AND NOTATIONS

Consider a finite population of size N having identifiable units $M=(M_1, M_2, ...M_N)$. Let Y be the sensitive study variable, which cannot be observed directly due to potential respondent bias, and X be a non-sensitive auxiliary variable which has a positive correlation with Y. Let S be a zero-mean scrambling random variable with known distribution. The respondent is asked to report a scrambled response for study variable (Y) given by $Z=Y+S$, but is asked to provide a true response for the auxiliary variable (X). Let a simple random sample of size n be drawn without replacement from the population. Let (y_i, x_i, z_i) be the observed values (factoring in measurement errors) and (Y_i, X_i, Z_i) be the true values for the study variable Y, auxiliary variable X and scrambled response variable Z respectively associated with the i^{th} $(i=1,2,...,n)$ sample unit. To estimate $\bar{Y}$, it is assumed that $\bar{X}$ is known. Note that $\bar{Z}=\bar{Y}$ is the population mean for the scrambled variable Z since S has zero mean.

We recall that the measurement errors associated with the scrambled response variable Z and the auxiliary variable X, as defined in Equations 1.5.2 and 1.5.3 in Chapter-1 are given by:

$$T_i = (z_i - Z_i),$$

$$V_i = (x_i - X_i),$$

We will use the following notations:

$$\psi_Z = \sum_{i=1}^{n}\left(Z_i - \bar{Z}\right), \tag{6.2.1}$$

$$\psi_T = \sum_{i=1}^{n} T_i, \tag{6.2.2}$$

$$\psi_X = \sum_{i=1}^{n}\left(X_i - \bar{X}\right), \tag{6.2.3}$$

and

$$\psi_V = \sum_{i=1}^{n} V_i. \tag{6.2.4}$$

In (6.2.1), $\left(Z_i - \bar{Z}\right)$ is the deviation of the true Z-values from the population mean of the sensitive study variable for the ith unit and these deviations are summed over the entire sample.

Adding Equations (6.2.1) and (6.2.2), we have

$$\psi_Z + \psi_T = \sum_{i=1}^{n}\left(Z_i - \bar{Z}\right) + \sum_{i=1}^{n} T_i. \tag{6.2.5}$$

Using Equation (1.5.3) in Equation (6.2.5), we have

$$\psi_Z + \psi_T = \sum_{i=1}^{n}\left(Z_i - \bar{Z}\right) + \sum_{i=1}^{n}\left(z_i - Z_i\right).$$

Dividing both sides by n, we have

$$\frac{1}{n}(\psi_Z + \psi_T) = \frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \bar{Z}\right) + \frac{1}{n}\sum_{i=1}^{n}\left(z_i - Z_i\right),$$

or

$$\frac{1}{n}(\psi_Z + \psi_T) = \frac{1}{n}\sum_{i=1}^{n}\left(z_i - \bar{Z}\right).$$

After simplification, we get

$$\bar{z} = \bar{Z} + \bar{Z}', \text{ where } \bar{Z}' = \frac{1}{n}\left(\psi_Z + \psi_T\right). \tag{6.2.6}$$

Similarly, from Equations (6.2.3) and (6.2.4), we have

$$\bar{x} = \bar{X} + \bar{X}', \text{ where } \bar{X}' = \frac{1}{n}\left(\psi_X + \psi_V\right). \tag{6.2.7}$$

Squaring both sides of (6.2.5), we have

$$(\psi_Z + \psi_T)^2 = \left[\sum_{i=1}^{n}\left(Z_i - \bar{Z}\right) + \sum_{i=1}^{n} T_i\right]^2,$$

or

$$(\psi_Z + \psi_T)^2 = \left[\left\{\sum_{i=1}^{n}\left(Z_i - \bar{Z}\right)\right\}^2 + \left\{\sum_{i=1}^{n} T_i\right\}^2 + 2\left\{\sum_{i=1}^{n}\left(Z_i - \bar{Z}\right)\right\}\left\{\sum_{i=1}^{n} T_i\right\}\right].$$

Taking expectation on both sides, we have

$$E(\psi_Z + \psi_T)^2 = E\left[\sum_{i=1}^{n}\left(Z_i - \bar{Z}\right)^2 + \sum_{i=1}^{n}\left(T_i - \bar{T}\right)^2 + 2\left\{\sum_{i=1}^{n}\left(Z_i - \bar{Z}\right)\right\}\left\{\sum_{i=1}^{n} T_i\right\}\right].$$

Applying expectation, the cross-product term becomes zero since the true values of non-sensitive study variable Y are independent of measurement error, thus

$$E(\psi_Z + \psi_T)^2 = \sum_{i=1}^{n} E\left(Z_i - \bar{Z}\right)^2 + \sum_{i=1}^{n} E(T_i - \bar{T})^2,$$

or

$$E(\psi_Z + \psi_T)^2 = \sum_{i=1}^{n} S_Z^2 + \sum_{i=1}^{n} S_T^2 = n(S_Z^2 + S_T^2).$$

If the finite population correction factor is used,

$$E(\psi_Z + \psi_T)^2 = n(1 - f)(S_Z^2 + S_T^2),$$

where $f = \dfrac{n}{N}$. In this situation,

$$E(\psi_Z + \psi_T)^2 = n^2 \frac{(1-f)}{n}(S_Z^2 + S_T^2),$$

or

$$E(\psi_Z + \psi_T)^2 = n^2\theta(S_Z^2 + S_T^2),$$

Dividing both sides by n^2, we get

$$E\left(\frac{\psi_Z + \psi_T}{n}\right)^2 = E(\bar{Z}')^2 = \theta\left(S_Z^2 + S_T^2\right),$$

Also

$$\left.\begin{aligned} &E\left(\frac{\psi_X + \psi_V}{n}\right)^2 = E(\bar{X}')^2 = \theta\left(S_X^2 + S_V^2\right), \\ &E\left\{\left(\frac{\psi_Z + \psi_T}{n}\right)\left(\frac{\psi_X + \psi_V}{n}\right)\right\} = E(\bar{Z}'\bar{X}') = \theta\rho_{ZX} S_Z S_X, \\ &\theta = \frac{(1-f)}{n} = \frac{(N-n)}{Nn},\ S_Z^2 = S_Y^2 + S_S^2,\ \rho_{ZX} = \frac{\rho_{YX}}{\sqrt{1+\frac{S_S^2}{S_Y^2}}}. \end{aligned}\right| \qquad (6.2.8)$$

6.3 PROPOSED GENERALIZED RRT ESTIMATOR IN THE PRESENCE OF MEASUREMENT ERROR

In this section, we propose a generalized RRT estimator for population mean in the presence of measurement errors on both the sensitive study variable (*Y*) and the non-sensitive auxiliary variable (*X*) in Simple random sampling without replacement. As pointed out earlier, a scrambled version of *Y* is observed in the form of $Z = Y + S,$ where *S* is a scrambling variable.

The proposed estimator is:

$$\hat{\bar{Y}}_{Gi} = \left[\bar{z} + k\left(\bar{X} - \bar{x}\right)\right]\left[\frac{\bar{D}}{\bar{d}}\right]^g, \qquad (6.3.1)$$

where

$$\bar{d} = \lambda\left(a_i\bar{x} + b_i\right) + \left(1-\lambda\right)\left(a_i\bar{X} + b_i\right) \text{ and } \bar{D} = a_i\bar{X} + b_i.$$

Here k and g are suitable constants, and λ is assumed to be an unknown constant whose value is to be determined from optimality considerations. Also a_i and b_i are the parameters or functions of parameters of the auxiliary variable (X). Many RRT mean estimators can be deduced from the proposed class of estimators. For example, with $g=1$, we get various RRT ratio estimators and with $g=-1$, we get various RRT product estimators.

Remark 1:

For $g=1$, $\hat{\hat{Y}}_{Gi}$ can take the following form:

$$\hat{\hat{Y}}_{Gi}=\left[\bar{z}+k\left(\bar{X}-\bar{x}\right)\right]\left[\frac{(a_i\bar{X}+b_i)}{\lambda\left(a_i\bar{x}+b_i\right)+\left(1-\lambda\right)\left(a_i\bar{X}+b_i\right)}\right]. \qquad (6.3.2)$$

By setting different values of unknown constants in Equation (6.3.2), various RRT ratio estimators based on single auxiliary variable may be obtained as a family of $\hat{\hat{Y}}_{Gi}$. For example,

(i) By putting $k=0$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G1}=\bar{z}\left[\frac{(a_i\bar{X}+b_i)}{(a_i\bar{x}+b_i)}\right]. \qquad (6.3.3)$$

(ii) By putting $k=1$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G2}=\left[\bar{z}+(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{X}+b_i)}{(a_i\bar{x}+b_i)}\right]. \qquad (6.3.4)$$

(iii) By putting $k=b_{ZX}$ (the slope term in regression Z on X) and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G3}=\left[\bar{z}+b_{ZX}(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{X}+b_i)}{(a_i\bar{x}+b_i)}\right]. \qquad (6.3.5)$$

(iv) By putting $k=0$ and $\lambda=\lambda_{opt}$ (optimized value of λ relative to the MSE of the proposed estimator), we have

$$\hat{\hat{Y}}_{G4}=\bar{z}\left[\frac{(a_i\bar{X}+b_i)}{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}\right]. \qquad (6.3.6)$$

(v) By putting $k=1$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G5}=\left[\bar{z}+(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{X}+b_i)}{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}\right]. \quad (6.3.7)$$

(vi) By putting $k=b_{ZX}$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G6}=\left[\bar{z}+b_{ZX}(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{X}+b_i)}{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}\right]. \quad (6.3.8)$$

Remark 2:

For $g=-1$, $\hat{\hat{Y}}_{Gi}$ can take the following form:

$$\hat{\hat{Y}}_{Gi}=\left[\bar{z}+k\left(\bar{X}-\bar{x}\right)\right]\left[\frac{\lambda\left(a_i\bar{x}+b_i\right)+(1-\lambda)\left(a_i\bar{X}+b_i\right)}{(a_i\bar{X}+b_i)}\right]. \quad (6.3.9)$$

By setting different values of unknown constants in Equation (6.3.9), various RRT product estimators based on single auxiliary variable may be obtained as a family of $\hat{\hat{Y}}_{Gi}$. For example,

(i) By putting $k=0$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G7}=\bar{z}\left[\frac{(a_i\bar{x}+b_i)}{(a_i\bar{X}+b_i)}\right]. \quad (6.3.10)$$

(ii) By putting $k=1$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G8}=\left[\bar{z}+(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{x}+b_i)}{(a_i\bar{X}+b_i)}\right]. \quad (6.3.11)$$

(iii) By putting $k=b_{ZX}$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G9}=\left[\bar{z}+b_{ZX}(\bar{X}-\bar{x})\right]\left[\frac{(a_i\bar{x}+b_i)}{(a_i\bar{X}+b_i)}\right]. \quad (6.3.12)$$

(iv) By putting $k=0$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G10}=\bar{z}\left[\frac{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}{(a_i\bar{X}+b_i)}\right]. \tag{6.3.13}$$

(v) By putting $k=1$and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G11}=\left[\bar{z}+(\bar{X}-\bar{x})\right]\left[\frac{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}{(a_i\bar{X}+b_i)}\right]. \tag{6.3.14}$$

(vi) By putting $k=b_{ZX}$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G12}=\left[\bar{z}+b_{ZX}(\bar{X}-\bar{x})\right]\left[\frac{\lambda_{opt}\left(a_i\bar{x}+b_i\right)+\left(1-\lambda_{opt}\right)\left(a_i\bar{X}+b_i\right)}{(a_i\bar{X}+b_i)}\right]. \tag{6.3.15}$$

6.3.1 The Bias and Mean Square Error of the Proposed Generalized RRT Estimator

Using Equations (6.2.6) and (6.2.7) in Equation (6.3.1), we have

$$\hat{\hat{Y}}_{Gi}=\left[\left(\bar{Z}+\bar{Z}'\right)+k\left(\bar{X}-\left(\bar{X}+\bar{X}'\right)\right)\right]$$
$$\left[\frac{\left(a_i\bar{X}+b_i\right)}{\lambda\left(a_i(\bar{X}+\bar{X}')+b_i\right)+\left(1-\lambda\right)\left(a_i\bar{X}+b_i\right)}\right]^g,$$

or

$$\hat{\hat{Y}}_{Gi}=\left[\bar{Z}+\bar{Z}'-k\bar{X}'\right]\left[\frac{\left(a_i\bar{X}+b_i\right)}{\left(a_i\bar{X}+b_i+\lambda a_i\bar{X}'\right)}\right]^g,$$

or

$$\hat{\hat{Y}}_{Gi}=\left[\bar{Z}+\bar{Z}'-k\bar{X}'\right]\left[1+\frac{\lambda a_i\bar{X}'}{\left(a_i\bar{X}+b_i\right)}\right]^{-g}.$$

By using Taylor series expansion

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \dots,$$

we get

$$\hat{\hat{Y}}_{Gi} \approx \left[\bar{Z} + \bar{Z}' - k\bar{X}'\right]\left[1 - g\frac{\lambda a_i \bar{X}'}{(a_i\bar{X} + b_i)} + \frac{g(g+1)}{2}\left(\frac{\lambda a_i \bar{X}'}{(a_i\bar{X} + b_i)}\right)^2\right].$$

(6.3.16)

In order to derive the expression of bias, using second order approximation, we have

$$\hat{\hat{Y}}_{Gi} - \bar{Z} \approx \frac{1}{\bar{\bar{Z}}}\left[\frac{g(g+1)}{2}\left(\frac{\lambda a_i \bar{Z}\bar{X}'}{(a_i\bar{X} + b_i)}\right)^2 - g\frac{\lambda a_i \bar{Z}\bar{Z}'\bar{X}'}{(a_i\bar{X} + b_i)} + g\frac{\lambda k a_i \bar{Z}\bar{X}'^2}{(a_i\bar{X} + b_i)}\right].$$

Taking expectation and using Equation (6.2.8), we get

$$E(\hat{\hat{Y}}_{Gi} - \bar{Z}) \approx \frac{\theta}{\bar{\bar{Z}}}\left[\begin{array}{l}\frac{g(g+1)}{2}\lambda^2 R_i^2\left(S_X^2 + S_V^2\right) \\ \qquad + g\lambda k R_i\left(S_X^2 + S_V^2\right) - g\lambda R_i \rho_{ZX} S_Z S_X\end{array}\right],$$

where $R_i = \dfrac{a_i \bar{Z}}{(a_i\bar{X} + b_i)}$.

On simplification, we get

$$Bias^*(\hat{\hat{Y}}_{Gi}) \approx \frac{\theta}{\bar{\bar{Z}}}\left[\frac{g(g+1)}{2}\lambda^2 R_i^2 S_X^2 + g\lambda k R_i S_X^2 - g\lambda R_i \rho_{ZX} S_Z S_X\right]$$
$$+ \frac{\theta}{\bar{\bar{Z}}}\left[\frac{g(g+1)}{2}\lambda^2 R_i^2 S_V^2 + g\lambda k R_i S_V^2\right], \qquad (6.3.17)$$

or

$$Bias^*(\hat{\hat{Y}}_{Gi}) \approx Bias(\hat{\hat{Y}}_{Gi}) + ME'(\hat{\hat{Y}}_{Gi}), \qquad (6.3.18)$$

where

$$Bias(\hat{\hat{Y}}_{Gi}) \approx \frac{\theta}{\bar{\bar{Z}}}\left[\frac{g(g+1)}{2}\lambda^2 R_i^2 S_X^2 + g\lambda k R_i S_X^2 - g\lambda R_i \rho_{ZX} S_Z S_X\right], (6.3.19)$$

is the bias of proposed estimator ($\hat{\bar{Y}}_{Gi}$) without measurement error, and

$$ME'(\hat{\bar{Y}}_{Gi}) \approx \frac{\theta}{\bar{Z}}\left[\frac{g(g+1)}{2}\lambda^2 R_i^2 S_V^2 + g\lambda k R_i S_V^2\right], \qquad (6.3.20)$$

is the term expressing the contribution of measurement error.

In order to obtain the mean square error of the proposed estimator, up to the first order approximation, we note that Equation (6.3.16) reduces to

$$\left(\hat{\bar{Y}}_{Gi} - \bar{Z}\right) \approx \left[\bar{Z}' - g\lambda \frac{a_i \bar{Z}\bar{X}'}{(a_i \bar{X} + b_i)} - k\bar{X}'\right].$$

Squaring both sides and taking expectation, we have

$$E\left(\hat{\bar{Y}}_{Gi} - \bar{Z}\right)^2 \approx \left[\begin{array}{c} E(\bar{Z}'^2) + g^2\lambda^2 R_i^2 E(\bar{X}'^2) + k^2 E(\bar{X}'^2) \\ -2g\lambda R_i E(\bar{Z}'\bar{X}') - 2kE(\bar{Z}'\bar{X}') + 2g\lambda k R_i E(\bar{X}'^2) \end{array}\right].$$

After simplification, the MSE of the proposed estimator is:

$$MSE^*(\hat{\bar{Y}}_{Gi}) \approx \theta\left[\begin{array}{c} S_Z^2 + g^2\lambda^2 R_i^2 S_X^2 + k^2 S_X^2 - 2g\lambda R_i \rho_{ZX} S_Z S_X \\ -2k\rho_{ZX} S_Z S_X + 2g\lambda k R_i S_X^2 \end{array}\right] + \theta\left[S_T^2 + g^2\lambda^2 R_i^2 S_V^2 + k^2 S_V^2 + 2g\lambda k R_i S_V^2\right], \qquad (6.3.21)$$

or

$$MSE^*(\hat{\bar{Y}}_{Gi}) \approx MSE(\hat{\bar{Y}}_{Gi}) + ME(\hat{\bar{Y}}_{Gi}), \qquad (6.3.22)$$

where

$$MSE(\hat{\bar{Y}}_{Gi}) \approx \theta\left[\begin{array}{c} S_Z^2 + g^2\lambda^2 R_i^2 S_X^2 + k^2 S_X^2 - 2g\lambda R_i \rho_{ZX} S_Z S_X \\ -2k\rho_{ZX} S_Z S_X + 2g\lambda k R_i S_X^2 \end{array}\right], \qquad (6.3.23)$$

is the MSE of the proposed estimator without measurement error, and

$$ME(\hat{\bar{Y}}_{Gi}) \approx \theta\left[S_T^2 + g^2\lambda^2 R_i^2 S_V^2 + k^2 S_V^2 + 2g\lambda k R_i S_V^2\right], \qquad (6.3.24)$$

is the increase due to measurement error.

To find the optimal value of λ, we differentiate the expression in Equation (6.3.21) with respect to λ, and then equate to zero, to get

$$\lambda = \frac{\rho_{ZX} S_Z S_X - k(S_X^2 + S_V^2)}{gR_i(S_X^2 + S_V^2)} = \lambda_{opt} \tag{6.3.25}$$

Substitution of (6.3.25) in (6.3.21) yields the minimum MSE of $(\hat{\hat{Y}}_{Gi})$ as:

$$MSE^*_{\min}(\hat{\hat{Y}}_{Gi}) \approx \theta\left(S_Z^2 + S_T^2 - \frac{\rho_{ZX}^2 S_Z^2 S_X^2}{\left(S_X^2 + S_V^2\right)} \right). \tag{6.3.26}$$

The expression of minimized MSE of proposed estimator without measurement error can be obtained by putting $S_T^2 = S_V^2 = 0$ in Equation (6.3.26). It is given by,

$$MSE_{\min}(\hat{\hat{Y}}_{Gi}) = \theta S_Z^2 \left(1 - \rho_{ZX}^2\right). \tag{6.3.27}$$

Note that this expression of $MSE_{\min}(\hat{\hat{Y}}_{Gi})$ is same as that of the approximate variance of the usual linear regression estimator.

6.4 ADDITIONAL SPECIAL CASES OF THE GENERALIZED RRT RATIO ESTIMATOR

Many additional estimators can be deduced from the generalized RRT ratio estimator ($\hat{\hat{Y}}_{G1}$**)** given in Equation (6.3.3). We denote the generalized RRT ratio estimator by,

$$\hat{\hat{Y}}_{G1}^{j} = \bar{z}\left[\frac{(a_j \bar{X} + b_j)}{(a_j \bar{x} + b_j)}\right],$$

Various choices of a_j, b_j are given in the table below. The general expressions for the mean square error and bias respectively with measurement error for this generalized RRT ratio estimator $\hat{\hat{Y}}_{G1}^{j}$ are given by

$$\left.\begin{aligned} MSE^*(\hat{\bar{Y}}_{G1}^{j}) &\approx \theta\left(S_Z^2 + R_j^2 S_X^2 - 2R_j \rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_j^2 S_V^2\right), \\ Bias^*(\hat{\bar{Y}}_{G1}^{j}) &\approx \frac{\theta}{\bar{Z}}\left(R_j^2 S_X^2 - R_j \rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_j^2 S_V^2\right), \\ R_j &= \frac{a_j \bar{Z}}{(a_j \bar{X} + b_j)}, \end{aligned}\right] \quad (6.4.1)$$

Table 6.1

Additional Special Cases of the Generalized RRT Ratio Estimator ($\hat{\bar{Y}}_{G1}^{j}$)

Proposed Estimators	a_j	b_j	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{G1}^{1} = \bar{z}\left(\frac{\bar{X}}{\bar{x}}\right)$	1	0	$MSE^*(\hat{\bar{Y}}_{G1}^{1}) = \theta\left(S_Z^2 + R_1^2 S_X^2 - 2R_1\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_1^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{1}) \approx \frac{\theta}{\bar{Z}}\left(R_1^2 S_X^2 - R_1\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_1^2 S_V^2\right)$	$R_1 = \frac{\bar{Z}}{\bar{X}}$
$\hat{\bar{Y}}_{G1}^{2} = \bar{z}\left(\frac{\bar{X} + C_x}{\bar{x} + C_x}\right)$	1	C_x	$MSE^*(\hat{\bar{Y}}_{G1}^{2}) = \theta\left(S_Z^2 + R_2^2 S_X^2 - 2R_2\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_2^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{2}) \approx \frac{\theta}{\bar{Z}}\left(R_2^2 S_X^2 - R_2\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_2^2 S_V^2\right)$	$R_2 = \frac{\bar{Z}}{\bar{X} + C_x}$
$\hat{\bar{Y}}_{G1}^{3} = \bar{z}\left(\frac{\bar{X} + \beta_2(x)}{\bar{x} + \beta_2(x)}\right)$	1	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{G1}^{3}) = \theta\left(S_Z^2 + R_3^2 S_X^2 - 2R_3\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_3^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{3}) \approx \frac{\theta}{\bar{Z}}\left(R_3^2 S_X^2 - R_3\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_3^2 S_V^2\right)$	$R_3 = \frac{\bar{Z}}{\bar{X} + \beta_2(x)}$
$\hat{\bar{Y}}_{G1}^{4} = \bar{z}\left(\frac{\bar{X}\beta_2(x) + C_x}{\bar{x}\beta_2(x) + C_x}\right)$	$\beta_2(x)$	C_x	$MSE^*(\hat{\bar{Y}}_{G1}^{4}) = \theta\left(S_Z^2 + R_4^2 S_X^2 - 2R_4\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_4^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{4}) \approx \frac{\theta}{\bar{Z}}\left(R_4^2 S_X^2 - R_4\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_4^2 S_V^2\right)$	$R_4 = \frac{\bar{Z}\beta_2(x)}{\bar{X}\beta_2(x) + C_x}$
$\hat{\bar{Y}}_{G1}^{5} = \bar{z}\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}C_x + \beta_2(x)}\right)$	C_x	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{G1}^{5}) = \theta\left(S_Z^2 + R_5^2 S_X^2 - 2R_5\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_5^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{5}) \approx \frac{\theta}{\bar{Z}}\left(R_5^2 S_X^2 - R_5\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_5^2 S_V^2\right)$	$R_5 = \frac{\bar{Z}C_x}{\bar{X}C_x + \beta_2(x)}$
$\hat{\bar{Y}}_{G1}^{6} = \bar{z}\left(\frac{\bar{X} + \rho_{ZX}}{\bar{x} + \rho_{ZX}}\right)$	1	ρ_{ZX}	$MSE^*(\hat{\bar{Y}}_{G1}^{6}) = \theta\left(S_Z^2 + R_6^2 S_X^2 - 2R_6\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_6^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{6}) \approx \frac{\theta}{\bar{Z}}\left(R_6^2 S_X^2 - R_6\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_6^2 S_V^2\right)$	$R_6 = \frac{\bar{Z}}{\bar{X} + \rho_{ZX}}$

Proposed Estimators	a_j	b_j	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{G1}^{7}=\bar{z}\left(\frac{\bar{X}C_x+\rho_{ZX}}{\bar{x}C_x+\rho_{ZX}}\right)$	C_x	ρ_{ZX}	$MSE^*(\hat{\bar{Y}}_{G1}^{7})=\theta\left(S_Z^2+R_7^2S_X^2-2R_7\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_7^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{7})\approx\frac{\theta}{\bar{Z}}\left(R_7^2S_X^2-R_7\rho_{ZX}S_ZS_X\right)+\frac{\theta}{\bar{Z}}\left(R_7^2S_V^2\right)$	$R_7=\frac{\bar{Z}C_x}{\bar{X}C_x+\rho_{ZX}}$
$\hat{\bar{Y}}_{G1}^{8}=\bar{z}\left(\frac{\bar{X}\rho_{ZX}+C_x}{\bar{x}\rho_{ZX}+C_x}\right)$	ρ_{ZX}	C_x	$MSE^*(\hat{\bar{Y}}_{G1}^{8})=\theta\left(S_Z^2+R_8^2S_X^2-2R_8\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_8^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{8})\approx\frac{\theta}{\bar{Z}}\left(R_8^2S_X^2-R_8\rho_{ZX}S_ZS_X\right)+\frac{\theta}{\bar{Z}}\left(R_8^2S_V^2\right)$	$R_8=\frac{\bar{Z}\rho_{ZX}}{\bar{X}\rho_{ZX}+C_x}$
$\hat{\bar{Y}}_{G1}^{9}=\bar{z}\left(\frac{\bar{X}\beta_2(x)+\rho_{ZX}}{\bar{x}\beta_2(x)+\rho_{ZX}}\right)$	$\beta_2(x)$	ρ_{ZX}	$MSE^*(\hat{\bar{Y}}_{G1}^{9})=\theta\left(S_Z^2+R_9^2S_X^2-2R_9\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_9^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{9})\approx\frac{\theta}{\bar{Z}}\left(R_9^2S_X^2-R_9\rho_{ZX}S_ZS_X\right)+\frac{\theta}{\bar{Z}}\left(R_9^2S_V^2\right)$	$R_9=\frac{\bar{Z}\beta_2(x)}{\bar{X}\beta_2(x)+\rho_{ZX}}$
$\hat{\bar{Y}}_{G1}^{10}=\bar{z}\left(\frac{\bar{X}\rho_{ZX}+\beta_2(x)}{\bar{x}\rho_{ZX}+\beta_2(x)}\right)$	ρ_{ZX}	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{G1}^{10})=\theta\left(S_Z^2+R_{10}^2S_X^2-2R_{10}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{10}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{10})\approx\frac{\theta}{\bar{Z}}\left(R_{10}^2S_X^2-R_{10}\rho_{ZX}S_ZS_X\right)+\frac{\theta}{\bar{Z}}\left(R_{10}^2S_V^2\right)$	$R_{10}=\frac{\bar{Z}\rho_{ZX}}{\bar{X}\rho_{ZX}+\beta_2(x)}$
$\hat{\bar{Y}}_{G1}^{11}=\bar{z}\left(\frac{\bar{X}+\beta_1(x)}{\bar{x}+\beta 1(x)}\right)$	1	$\beta_1(x)$	$MSE^*(\hat{\bar{Y}}_{G1}^{11})=\theta\left(S_Z^2+R_{11}^2S_X^2-2R_{11}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{11}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{11})\approx\frac{\theta}{\bar{Z}}\left(R_{11}^2S_X^2-R_{11}\rho_{ZX}S_ZS_X\right)+\frac{\theta}{\bar{Z}}\left(R_{11}^2S_V^2\right)$	$R_{11}=\frac{\bar{Z}}{\bar{X}+\beta_1(x)}$
$\hat{\bar{Y}}_{G1}^{12}=\bar{z}\left(\frac{\bar{X}\beta_1(x)+\beta_2(x)}{\bar{x}\beta 1(x)+\beta_2(x)}\right)$	$\beta_1(x)$	$\beta_2(x)$	$MSE^*(\hat{\bar{Y}}_{G1}^{12})=\theta\left(S_Z^2+R_{12}^2S_X^2-2R_{12}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{12}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{12})\approx\frac{\theta}{\bar{Z}}\left(R_{12}^2S_X^2-R_{12}\rho_{ZX}S_ZS_X\right)+\frac{\theta}{\bar{Z}}\left(R_{12}^2S_V^2\right)$	$R_{12}=\frac{\bar{Z}\beta_1(x)}{\bar{X}\beta_1(x)+\beta_2(x)}$
$\hat{\bar{Y}}_{G1}^{13}=\bar{z}\left(\frac{\bar{X}+Q_2}{\bar{x}+Q_2}\right)$	1	Q_2	$MSE^*(\hat{\bar{Y}}_{G1}^{13})=\theta\left(S_Z^2+R_{13}^2S_X^2-2R_{13}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{13}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{13})\approx\frac{\theta}{\bar{Z}}\left(R_{13}^2S_X^2-R_{13}\rho_{ZX}S_ZS_X\right)+\frac{\theta}{\bar{Z}}\left(R_{13}^2S_V^2\right)$	$R_{13}=\frac{\bar{Z}}{\bar{X}+Q_2}$

Proposed Estimators	a_j	b_j	Mean Square Errors & Biases	Ratio’s
$\hat{\bar{Y}}_{G1}^{14} = \bar{z}\left(\frac{\bar{X}C_x + Q_2}{\bar{x}C_x + Q_2}\right)$	C_x	Q_2	$MSE^*(\hat{\bar{Y}}_{G1}^{14}) = \theta\left(S_Z^2 + R_{14}^2 S_X^2 - 2R_{14}\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_{14}^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{14}) \approx \frac{\theta}{\bar{Z}}\left(R_{14}^2 S_X^2 - R_{14}\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_{14}^2 S_V^2\right)$	$R_{14} = \frac{\bar{Z}C_x}{\bar{X}C_x + Q_2}$
$\hat{\bar{Y}}_{G1}^{15} = \bar{z}\left(\frac{\bar{X}\beta_1(x) + Q_2}{\bar{x}\beta_1(x) + Q_2}\right)$	$\beta_1(x)$	Q_2	$MSE^*(\hat{\bar{Y}}_{G1}^{15}) = \theta\left(S_Z^2 + R_{15}^2 S_X^2 - 2R_{15}\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_{15}^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{15}) \approx \frac{\theta}{\bar{Z}}\left(R_{15}^2 S_X^2 - R_{15}\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_{15}^2 S_V^2\right)$	$R_{15} = \frac{\bar{Z}\beta_1(x)}{\bar{X}\beta_1(x) + Q_2}$
$\hat{\bar{Y}}_{G1}^{16} = \bar{z}\left(\frac{\bar{X}\beta_2(x) + Q_2}{\bar{x}\beta_2(x) + Q_2}\right)$	$\beta_2(x)$	Q_2	$MSE^*(\hat{\bar{Y}}_{G1}^{16}) = \theta\left(S_Z^2 + R_{16}^2 S_X^2 - 2R_{16}\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_{16}^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{16}) \approx \frac{\theta}{\bar{Z}}\left(R_{16}^2 S_X^2 - R_{16}\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_{16}^2 S_V^2\right)$	$R_{16} = \frac{\bar{Z}\beta_2(x)}{\bar{X}\beta_2(x) + Q_2}$
$\hat{\bar{Y}}_{G1}^{17} = \bar{z}\left(\frac{\bar{X}\beta_1(x) + QD}{\bar{x}\beta_1(x) + QD}\right)$	$\beta_1(x)$	QD	$MSE^*(\hat{\bar{Y}}_{G1}^{17}) = \theta\left(S_Z^2 + R_{17}^2 S_X^2 - 2R_{17}\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_{17}^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{17}) \approx \frac{\theta}{\bar{Z}}\left(R_{17}^2 S_X^2 - R_{17}\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_{17}^2 S_V^2\right)$	$R_{17} = \frac{\bar{Z}\beta_1(x)}{\bar{X}\beta_1(x) + QD}$
$\hat{\bar{Y}}_{G1}^{18} = \bar{z}\left(\frac{\bar{X}\beta_2(x) + QD}{\bar{x}\beta_2(x) + QD}\right)$	$\beta_2(x)$	QD	$MSE^*(\hat{\bar{Y}}_{G1}^{18}) = \theta\left(S_Z^2 + R_{18}^2 S_X^2 - 2R_{18}\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_{18}^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{18}) \approx \frac{\theta}{\bar{Z}}\left(R_{18}^2 S_X^2 - R_{18}\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_{18}^2 S_V^2\right)$	$R_{18} = \frac{\bar{Z}\beta_2(x)}{\bar{X}\beta_2(x) + QD}$
$\hat{\bar{Y}}_{G1}^{19} = \bar{z}\left(\frac{\bar{X} + TM}{\bar{x} + TM}\right)$	1	TM	$MSE^*(\hat{\bar{Y}}_{G1}^{19}) = \theta\left(S_Z^2 + R_{19}^2 S_X^2 - 2R_{19}\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_{19}^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{19}) \approx \frac{\theta}{\bar{Z}}\left(R_{19}^2 S_X^2 - R_{19}\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_{19}^2 S_V^2\right)$	$R_{19} = \frac{\bar{Z}}{\bar{X} + TM}$
$\hat{\bar{Y}}_{G1}^{20} = \bar{z}\left(\frac{\bar{X}C_x + TM}{\bar{x}C_x + TM}\right)$	C_x	TM	$MSE^*(\hat{\bar{Y}}_{G1}^{20}) = \theta\left(S_Z^2 + R_{20}^2 S_X^2 - 2R_{20}\rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_{20}^2 S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{20}) \approx \frac{\theta}{\bar{Z}}\left(R_{20}^2 S_X^2 - R_{20}\rho_{ZX} S_Z S_X\right) + \frac{\theta}{\bar{Z}}\left(R_{20}^2 S_V^2\right)$	$R_{20} = \frac{\bar{Z}C_x}{\bar{X}C_x + TM}$

Proposed Estimators	a_j	b_j	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{G1}^{21}=\bar{z}\left(\dfrac{\bar{X}\rho_{ZX}+TM}{\bar{x}\rho_{ZX}+TM}\right)$	ρ_{ZX}	TM	$MSE^*(\hat{\bar{Y}}_{G1}^{21})=\theta\left(S_Z^2+R_{21}^2S_X^2-2R_{21}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{21}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{21})\approx\dfrac{\theta}{\bar{Z}}\left(R_{21}^2S_X^2-R_{21}\rho_{ZX}S_ZS_X\right)+\dfrac{\theta}{\bar{Z}}\left(R_{21}^2S_V^2\right)$	$R_{21}=\dfrac{\bar{Z}\rho_{ZX}}{\bar{X}\rho_{ZX}+TM}$
$\hat{\bar{Y}}_{G1}^{22}=\bar{z}\left(\dfrac{\bar{X}+MR}{\bar{x}+MR}\right)$	1	MR	$MSE^*(\hat{\bar{Y}}_{G1}^{22})=\theta\left(S_Z^2+R_{22}^2S_X^2-2R_{22}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{22}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{22})\approx\dfrac{\theta}{\bar{Z}}\left(R_{22}^2S_X^2-R_{22}\rho_{ZX}S_ZS_X\right)+\dfrac{\theta}{\bar{Z}}\left(R_{22}^2S_V^2\right)$	$R_{22}=\dfrac{\bar{Z}}{\bar{X}+MR}$
$\hat{\bar{Y}}_{G1}^{23}=\bar{z}\left(\dfrac{\bar{X}C_x+MR}{\bar{x}C_x+MR}\right)$	C_x	MR	$MSE^*(\hat{\bar{Y}}_{G1}^{23})=\theta\left(S_Z^2+R_{23}^2S_X^2-2R_{23}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{23}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{23})\approx\dfrac{\theta}{\bar{Z}}\left(R_{23}^2S_X^2-R_{23}\rho_{ZX}S_ZS_X\right)+\dfrac{\theta}{\bar{Z}}\left(R_{23}^2S_V^2\right)$	$R_{23}=\dfrac{\bar{Z}C_x}{\bar{X}C_x+MR}$
$\hat{\bar{Y}}_{G1}^{24}=\bar{z}\left(\dfrac{\bar{X}\rho_{ZX}+MR}{\bar{x}\rho_{ZX}+MR}\right)$	ρ_{ZX}	MR	$MSE^*(\hat{\bar{Y}}_{G1}^{24})=\theta\left(S_Z^2+R_{24}^2S_X^2-2R_{24}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{24}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{24})\approx\dfrac{\theta}{\bar{Z}}\left(R_{24}^2S_X^2-R_{24}\rho_{ZX}S_ZS_X\right)+\dfrac{\theta}{\bar{Z}}\left(R_{24}^2S_V^2\right)$	$R_{24}=\dfrac{\bar{Z}\rho_{ZX}}{\bar{X}\rho_{ZX}+MR}$
$\hat{\bar{Y}}_{G1}^{25}=\bar{z}\left(\dfrac{\bar{X}+HL}{\bar{x}+HL}\right)$	1	HL	$MSE^*(\hat{\bar{Y}}_{G1}^{25})=\theta\left(S_Z^2+R_{25}^2S_X^2-2R_{25}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{25}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{25})\approx\dfrac{\theta}{\bar{Z}}\left(R_{25}^2S_X^2-R_{25}\rho_{ZX}S_ZS_X\right)+\dfrac{\theta}{\bar{Z}}\left(R_{25}^2S_V^2\right)$	$R_{25}=\dfrac{\bar{Z}}{\bar{X}+HL}$
$\hat{\bar{Y}}_{G1}^{26}=\bar{z}\left(\dfrac{\bar{X}C_x+HL}{\bar{x}C_x+HL}\right)$	C_x	HL	$MSE^*(\hat{\bar{Y}}_{G1}^{26})=\theta\left(S_Z^2+R_{26}^2S_X^2-2R_{26}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{26}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{26})\approx\dfrac{\theta}{\bar{Z}}\left(R_{26}^2S_X^2-R_{26}\rho_{ZX}S_ZS_X\right)+\dfrac{\theta}{\bar{Z}}\left(R_{26}^2S_V^2\right)$	$R_{26}=\dfrac{\bar{Z}C_x}{\bar{X}C_x+HL}$
$\hat{\bar{Y}}_{G1}^{27}=\bar{z}\left(\dfrac{\bar{X}\rho_{ZX}+HL}{\bar{x}\rho_{ZX}+HL}\right)$	ρ_{ZX}	HL	$MSE^*(\hat{\bar{Y}}_{G1}^{27})=\theta\left(S_Z^2+R_{27}^2S_X^2-2R_{27}\rho_{ZX}S_ZS_X\right)+\theta\left(S_T^2+R_{27}^2S_V^2\right)$ $Bias^*(\hat{\bar{Y}}_{G1}^{27})\approx\dfrac{\theta}{\bar{Z}}\left(R_{27}^2S_X^2-R_{27}\rho_{ZX}S_ZS_X\right)+\dfrac{\theta}{\bar{Z}}\left(R_{27}^2S_V^2\right)$	$R_{27}=\dfrac{\bar{Z}\rho_{ZX}}{\bar{X}\rho_{ZX}+HL}$

Various terms used in Table 6.1 are described in the Appendix-A.

6.5 EFFICIENCY COMPARISON

To check the efficiency of the proposed generalized RRT ratio estimator ($\hat{\hat{Y}}_{G1}^{j}$) against ordinary RRT mean estimator ($\bar{z}$), the mathematical conditions have been derived by using the MSE expressions in Equations (6.4.1) and (3.2.13). These conditions are given by:

$$MSE^{*}(\hat{\hat{Y}}_{G1}^{j}) \le Var^{*}(\bar{z})$$

if

$$\theta\left(S_Z^2 + R_j^2 S_X^2 - 2R_j \rho_{ZX} S_Z S_X\right) + \theta\left(S_T^2 + R_j^2 S_V^2\right) \le \theta\left(S_Z^2 + S_T^2\right),$$

or if

$$R_j S_X^2 + R_j S_V^2 \le 2\rho_{ZX} S_Z S_X,$$

or if

$$\rho_{ZX} \ge \frac{R_j}{2}\left(\frac{S_X^2 + S_V^2}{S_X S_Z}\right) \tag{6.5.1}$$

When observations are recorded without measurement errors (ME), Condition (6.5.1) reduces to

$$\rho_{ZX} \ge \frac{R_j}{2}\frac{S_X}{S_Z} \tag{6.5.2}$$

6.6 SIMULATION RESULTS

In this section, we conduct a simulation study with particular focus on the following two issues:

a. How does the generalized RRT estimator ($\hat{\hat{Y}}_{Gi}$) and the generalized RRT ratio estimator ($\hat{\hat{Y}}_{G1}^{j}$) compare with the ordinary RRT mean estimator ($\bar{z}$) in the presence and absence of measurement errors?

b. How are the MSE, PRE and bias, influenced with the contribution of measurement errors?

We consider two finite sub-populations of size 5000 each from bivariate normal populations having different means and covariance matrices to represent the distribution of study variable (Y) and auxiliary variable (X). The scrambling variable S is taken to be a normal variate with mean equal to zero and standard deviation ($\sigma_s = 0.01\sigma_x$). The true response is given by $Z=Y+S$. Both populations have theoretical mean μ and covariance matrices $\sum$ as given below.

Population-1 **Population-2**

$$\mu = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\ \Sigma = \begin{bmatrix} 9 & 3.2 \\ 3.2 & 4 \end{bmatrix},\ \rho_{yx} = 0.5154 \qquad \mu = \begin{bmatrix} 9 \\ 5 \end{bmatrix},\ \Sigma = \begin{bmatrix} 16 & 6.26 \\ 6.26 & 5 \end{bmatrix},\ \rho_{yx} = 0.7203$$

For both populations, the methodology used to get the observed values of y, x and s is as follows:

$y = Y + \tau_1$, $x = X + \tau_2$ and $s = S + \tau_3$ where τ_1, τ_2, τ_3

each having a normal distribution with zero mean and unit variance. The observed response is given by $z = y + s$.

We consider sample sizes: n = 500, 1000. The following steps which were coded in R-program, summarize the simulation procedures used to find the empirical *MSE's* of any specific estimator.

Step-1: Fifty thousand samples of size n were selected from both populations, using simple random sampling without replacement.

Step-2: Using the data from Step-1, 50,000 estimates ($\hat{\bar{Y}}^*$) are obtained for each sample size.

Step-3: The empirical MSE of $\hat{\bar{Y}}^*$ is computed by

$$EMSE(\hat{\bar{Y}}^*) = \frac{1}{50000} \sum_{i=1}^{50000} (\hat{\bar{Y}}^* - \bar{Y})^2,$$

where $\hat{\bar{Y}}^*$ is the estimator, deduced from Equation (6.3.1) and $\bar{Y}$ is the population mean of the sensitive study variable. The percent relative efficiency (PRE) of the estimators under study is calculated by using following equation:

$$PRE = \frac{VAR(\bar{z})}{MSE(\hat{\bar{Y}}^*)} \times 100,$$

The MSE's, PRE's and biases of the estimators for both populations on different sampling fractions are presented in Tables (6.2-6.5).

Table 6.2
Theoretical (boldface) and Empirical MSE's, PRE's (with/without ME) of the RRT Estimators Relative to Ordinary RRT Mean Estimator in Simple Random Sampling for Population-I

Estimators	***n***	**Mean Square Error**			**Percent Relative Efficiency**	
		Without ME	**Change due to ME**	**with ME**	**Without ME**	**With ME**
$\bar{z}$	500	**0.0161** 0.0158	**0.0035** 0.0038	**0.0197** 0.0196	100.0000	100.0000
	1000	**0.0075** 0.0076	**0.0016** 0.0013	**0.0091** 0.0089	100.0000	100.0000
$\hat{\bar{Y}}_{Gi}$	500	**0.0114** 0.0111	**0.0045** 0.0044	**0.0158** 0.0154	142.0048	124.2741
	1000	**0.0052** 0.0053	**0.0020** 0.0018	**0.0073** 0.0071	142.1627	124.3370
$\hat{\bar{Y}}_{G1}^{1}$	500	**0.0117** 0.0114	**0.0053** 0.0052	**0.0170** 0.0166	138.3921	115.7383
	1000	**0.0054** 0.0054	**0.0024** 0.0022	**0.0078** 0.0076	139.0761	116.2487
$\hat{\bar{Y}}_{G1}^{2}$	500	**0.0115** 0.0112	**0.0044** 0.0042	**0.0159** 0.0154	140.4218	124.2299
	1000	**0.0053** 0.0054	**0.0020** 0.0017	**0.0073** 0.0071	140.1069	124.3298
$\hat{\bar{Y}}_{G1}^{3}$	500	**0.0116** 0.0113	**0.0053** 0.0051	**0.0169** 0.0164	139.2437	116.8545
	1000	**0.0053** 0.0054	**0.0023** 0.0021	**0.0076** 0.0075	140.7188	118.6686
$\hat{\bar{Y}}_{G1}^{4}$	500	**0.0151** 0.0147	**0.0036** 0.0037	**0.0186** 0.0184	107.2310	105.7560
	1000	**0.0065** 0.0066	**0.0016** 0.0013	**0.0081** 0.0079	114.4981	111.2018
$\hat{\bar{Y}}_{G1}^{5}$	500	**0.0116** 0.0113	**0.0053** 0.0051	**0.0169** 0.0164	139.2451	116.8564
	1000	**0.0053** 0.0054	**0.0023** 0.0021	**0.0076** 0.0075	140.7011	118.6392
$\hat{\bar{Y}}_{G1}^{6}$	500	**0.0114** 0.0111	**0.0047** 0.0045	**0.0160** 0.0156	141.9902	122.8023
	1000	**0.0052** 0.0053	**0.0021** 0.0018	**0.0074** 0.0072	142.0875	123.0898

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	**Change due to ME**	**with ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}_{G1}^{7}$	500	**0.0114** 0.0111	**0.0047** 0.0045	**0.0160** 0.0156	141.9894	122.8085
	1000	**0.0052** 0.0053	**0.0021** 0.0018	**0.0074** 0.0072	142.0993	123.0456
$\hat{\bar{Y}}_{G1}^{8}$	500	**0.0119** 0.0116	**0.0040** 0.0040	**0.0160** 0.0156	135.2293	123.2262
	1000	**0.0055** 0.0056	**0.0018** 0.0015	**0.0074** 0.0072	134.5646	123.0350
$\hat{\bar{Y}}_{G1}^{9}$	500	**0.0144** 0.0140	**0.0036** 0.0037	**0.0180** 0.0177	112.2735	109.5463
	1000	**0.0061** 0.0062	**0.0017** 0.0014	**0.0078** 0.0075	122.6989	116.7322
$\hat{\bar{Y}}_{G1}^{10}$	500	**0.0115** 0.0113	**0.0052** 0.0050	**0.0167** 0.0163	139.8472	117.7014
	1000	**0.0053** 0.0053	**0.0023** 0.0020	**0.0075** 0.0073	141.5538	120.2357
$\hat{\bar{Y}}_{G1}^{11}$	500	**0.0117** 0.0115	**0.0054** 0.0052	**0.0171** 0.0167	137.7247	114.9102
	1000	**0.0053** 0.0054	**0.0024** 0.0022	**0.0077** 0.0076	139.6170	116.9927
$\hat{\bar{Y}}_{G1}^{12}$	500	**0.0122** 0.0122	**0.0040** 0.0037	**0.0162** 0.0159	131.9672	121.6049
	1000	**0.0059** **0.0060**	**0.0017** **0.0014**	**0.0076** **0.0074**	125.4992	118.4422
$\hat{\bar{Y}}_{G1}^{13}$	500	**0.0121** 0.0117	**0.0040** 0.0039	**0.0161** 0.0157	133.8789	122.6941
	1000	**0.0056** 0.0057	**0.0018** 0.0015	**0.0074** 0.0072	133.7668	122.7111
$\hat{\bar{Y}}_{G1}^{14}$	500	**0.0121** 0.0117	**0.0040** 0.0039	**0.0161** 0.0157	133.8561	122.6845
	1000	**0.0056** 0.0056	**0.0018** 0.0015	**0.0074** 0.0072	133.9428	122.7843
$\hat{\bar{Y}}_{G1}^{15}$	500	**0.0162** 0.0161	**0.0036** 0.0033	**0.0198** 0.0194	101.5081	101.2268
	1000	**0.0073** 0.0074	**0.0016** 0.0013	**0.0089** 0.0087	102.4879	102.0297

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	**Change due to ME**	**with ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}_{G1}^{16}$	500	**0.0156** 0.0152	**0.0036** 0.0037	**0.0191** 0.0190	103.6937	102.9845
	1000	**0.0069** 0.0070	**0.0016** 0.0013	**0.0085** 0.0083	108.2094	106.5367
$\hat{\bar{Y}}_{G1}^{17}$	500	**0.0161** 0.0160	**0.0036** 0.0033	**0.0197** 0.0193	102.2784	101.8476
	1000	**0.0072** 0.0073	**0.0016** 0.0013	**0.0088** 0.0086	103.6844	102.9912
$\hat{\bar{Y}}_{G1}^{18}$	500	**0.0153** 0.0150	**0.0036** 0.0037	**0.0189** 0.0187	105.4683	104.3859
	1000	**0.0067** 0.0068	**0.0016** 0.0013	**0.0083** 0.0081	111.6895	109.1587
$\hat{\bar{Y}}_{G1}^{19}$	500	**0.0121** 0.0117	**0.0040** 0.0039	**0.0160** 0.0157	133.9829	122.7373
	1000	**0.0056** 0.0057	**0.0018** 0.0015	**0.0074** 0.0072	133.8510	122.7462
$\hat{\bar{Y}}_{G1}^{20}$	500	**0.0121** **0.0117**	**0.0040** 0.0039	**0.0160** 0.0157	133.9602	122.7279
	1000	**0.0056** 0.0056	**0.0018** 0.0015	**0.0074** 0.0072	134.0263	122.8187
$\hat{\bar{Y}}_{G1}^{21}$	500	**0.0129** 0.0125	**0.0038** 0.0038	**0.0166** 0.0163	125.3772	118.3023
	1000	**0.0060** 0.0060	**0.0017** 0.0014	**0.0076** 0.0074	125.2908	118.3186
$\hat{\bar{Y}}_{G1}^{22}$	500	**0.0118** 0.0115	**0.0041** 0.0040	**0.0159** 0.0155	136.4626	123.6480
	1000	**0.0054** 0.0055	**0.0019** 0.0016	**0.0073** 0.0071	138.0793	124.1324
$\hat{\bar{Y}}_{G1}^{23}$	500	**0.0118** 0.0115	**0.0041** 0.0040	**0.0159** 0.0155	136.4434	123.6420
	1000	**0.0054** 0.0055	**0.0019** 0.0016	**0.0073** 0.0071	138.2026	124.1573
$\hat{\bar{Y}}_{G1}^{24}$	500	**0.0126** 0.0122	**0.0038** 0.0038	**0.0164** 0.0161	128.5455	120.1027
	1000	**0.0057** 0.0058	**0.0018** 0.0015	**0.0075** 0.0072	130.9884	121.4378

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	with ME	Without ME	With ME
$\hat{\bar{Y}}_{G1}^{25}$	500	**0.0120** 0.0117	**0.0040** 0.0039	**0.0160** 0.0157	134.0181	122.7518
	1000	**0.0056** 0.0057	**0.0018** 0.0015	**0.0074** 0.0072	133.8405	122.7419
$\hat{\bar{Y}}_{G1}^{26}$	500	**0.0120** 0.0117	**0.0040** 0.0039	**0.0160** 0.0157	133.9954	122.7425
	1000	**0.0056** 0.0056	**0.0018** 0.0015	**0.0074** 0.0072	134.0159	122.8145
$\hat{\bar{Y}}_{G1}^{27}$	500	**0.0129** 0.0125	**0.0038** 0.0038	**0.0166** 0.0163	125.4190	118.3270
	1000	**0.0060** 0.0060	**0.0017** 0.0014	**0.0077** 0.0074	125.2784	118.3113

Table 6.3
Theoretical Biases (with/without ME) of the RRT Estimators in Simple Random Sampling for Population-I

Estimators	Bias (Without ME)		Bias (With ME)	
	$f = n / N$		$f = n / N$	
	500/5000	**1000/5000**	**500/5000**	**1000/5000**
$\bar{z}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{Gi}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{G1}^{1}$	0.0007	0.0003	0.0016	0.0007
$\hat{\bar{Y}}_{G1}^{2}$	-0.0003	-0.0002	0.0001	0.0000
$\hat{\bar{Y}}_{G1}^{3}$	0.0006	0.0002	0.0015	0.0006
$\hat{\bar{Y}}_{G1}^{4}$	-0.0002	-0.0002	-0.0002	-0.0002
$\hat{\bar{Y}}_{G1}^{5}$	0.0006	0.0002	0.0015	0.0006
$\hat{\bar{Y}}_{G1}^{6}$	0.0000	0.0000	0.0005	0.0002
$\hat{\bar{Y}}_{G1}^{7}$	0.0000	0.0000	0.0005	0.0002

Estimators	Bias (Without ME)		Bias (With ME)	
	$f=n/N$		$f=n/N$	
	500/5000	**1000/5000**	**500/5000**	**1000/5000**
$\hat{\bar{Y}}_{G1}^{8}$	-0.0005	-0.0003	-0.0003	-0.0001
$\hat{\bar{Y}}_{G1}^{9}$	-0.0004	-0.0003	-0.0004	-0.0002
$\hat{\bar{Y}}_{G1}^{10}$	0.0005	0.0001	0.0013	0.0005
$\hat{\bar{Y}}_{G1}^{11}$	0.0008	0.0003	0.0017	0.0007
$\hat{\bar{Y}}_{G1}^{12}$	-0.0006	-0.0003	-0.0004	-0.0002
$\hat{\bar{Y}}_{G1}^{13}$	-0.0006	-0.0003	-0.0003	-0.0002
$\hat{\bar{Y}}_{G1}^{14}$	-0.0006	-0.0003	-0.0003	-0.0002
$\hat{\bar{Y}}_{G1}^{15}$	-0.0001	0.0000	-0.0001	0.0000
$\hat{\bar{Y}}_{G1}^{16}$	-0.0001	-0.0001	-0.0001	-0.0001
$\hat{\bar{Y}}_{G1}^{17}$	-0.0001	-0.0001	-0.0001	-0.0001
$\hat{\bar{Y}}_{G1}^{18}$	0.0002	-0.0002	0.0002	-0.0002
$\hat{\bar{Y}}_{G1}^{19}$	-0.0006	-0.0003	-0.0003	-0.0002
$\hat{\bar{Y}}_{G1}^{20}$	-0.0006	-0.0003	-0.0003	-0.0002
$\hat{\bar{Y}}_{G1}^{21}$	-0.0006	-0.0003	-0.0005	-0.0002
$\hat{\bar{Y}}_{G1}^{22}$	-0.0005	-0.0002	-0.0002	-0.0001
$\hat{\bar{Y}}_{G1}^{23}$	-0.0005	-0.0002	-0.0002	-0.0001
$\hat{\bar{Y}}_{G1}^{24}$	-0.0006	-0.0003	-0.0004	-0.0002
$\hat{\bar{Y}}_{G1}^{25}$	-0.0006	-0.0003	-0.0003	-0.0002
$\hat{\bar{Y}}_{G1}^{26}$	-0.0006	-0.0003	-0.0003	-0.0002
$\hat{\bar{Y}}_{G1}^{27}$	-0.0006	-0.0003	-0.0005	-0.0002

Table 6.4
Theoretical (boldface) and Empirical MSE's, PRE's (with/without ME) of the RRT Estimators Relative to Ordinary RRT Mean Estimator in Simple Random Sampling for Population-II

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\bar{z}$	500	**0.0299** 0.0298	**0.0037** 0.0033	**0.0336** 0.0330	100.0000	100.0000
	1000	**0.0126** 0.0126	**0.0015** 0.0015	**0.0142** 0.0142	100.0000	100.0000
$\hat{\bar{Y}}_{Gi}$	500	**0.0148** 0.0147	**0.0062** 0.0060	**0.0210** 0.0207	201.5565	159.5839
	1000	**0.0066** 0.0066	**0.0025** 0.0024	**0.0091** 0.0090	191.4149	155.0619
$\hat{\bar{Y}}_{G1}^{1}$	500	**0.0173** 0.0173	**0.0097** 0.0098	**0.0270** 0.0271	172.6218	124.2879
	1000	**0.0079** 0.0079	**0.0041** 0.0039	**0.0121** 0.0118	159.1616	117.4852
$\hat{\bar{Y}}_{G1}^{2}$	500	**0.0161** 0.0160	**0.0087** 0.0088	**0.0248** 0.0248	185.7514	135.2156
	1000	**0.0073** 0.0073	**0.0037** 0.0035	**0.0110** 0.0109	172.1465	128.1698
$\hat{\bar{Y}}_{G1}^{3}$	500	**0.0170** 0.0169	**0.0094** 0.0096	**0.0264** 0.0265	176.2042	127.1363
	1000	**0.0078** 0.0077	**0.0040** 0.0038	**0.0118** 0.0116	162.5111	120.1433
$\hat{\bar{Y}}_{G1}^{4}$	500	**0.0157** 0.0155	**0.0055** 0.0053	**0.0211** 0.0208	190.9059	158.9821
	1000	**0.0069** 0.0068	**0.0023** 0.0022	**0.0091** 0.0090	183.6993	154.7679
$\hat{\bar{Y}}_{G1}^{5}$	500	**0.0166** 0.0165	**0.0092** 0.0093	**0.0258** 0.0258	180.1742	130.3947
	1000	**0.0076** 0.0076	**0.0039** 0.0037	**0.0115** 0.0113	166.3740	123.2837
$\hat{\bar{Y}}_{G1}^{6}$	500	**0.0156** 0.0156	**0.0083** 0.0083	**0.0239** 0.0239	191.1859	140.3322
	1000	**0.0071** 0.0071	**0.0035** 0.0034	**0.0106** 0.0104	177.6169	133.1065

Estimat ors	*n*	**Mean Square Error**			**Percent Relative Efficiency**	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}_{G1}^{7}$	500	**0.0149** 0.0148	**0.0072** 0.0071	**0.0221** 0.0219	200.5704	152.1840
	1000	**0.0067** 0.0066	**0.0030** 0.0029	**0.0097** 0.0096	188.8642	145.5922
$\hat{\bar{Y}}_{G1}^{8}$	500	**0.0158** 0.0157	**0.0084** 0.0085	**0.0242** 0.0241	189.7873	138.9617
	1000	**0.0071** 0.0071	**0.0036** 0.0034	**0.0107** 0.0105	176.6779	132.2313
$\hat{\bar{Y}}_{G1}^{9}$	500	**0.0172** 0.0170	**0.0048** 0.0046	**0.0220** 0.0215	174.3095	152.9617
	1000	**0.0074** 0.0074	**0.0020** 0.0019	**0.0094** 0.0093	169.6508	149.8735
$\hat{\bar{Y}}_{G1}^{10}$	500	**0.0168** 0.0168	**0.0093** 0.0094	**0.0262** 0.0262	177.5881	128.2588
	1000	**0.0077** 0.0077	**0.0040** 0.0038	**0.0117** 0.0115	163.9386	121.2936
$\hat{\bar{Y}}_{G1}^{11}$	500	**0.0173** 0.0173	**0.0097** 0.0098	**0.0270** 0.0271	172.7435	124.3834
	1000	**0.0079** 0.0079	**0.0041** 0.0039	**0.0120** 0.0118	159.2410	117.5476
$\hat{\bar{Y}}_{G1}^{12}$	500	**0.0245** 0.0244	**0.0038** 0.0034	**0.0283** 0.0278	121.8517	118.5114
	1000	**0.0109** 0.0109	**0.0016** 0.0015	**0.0125** 0.0125	115.3731	113.2220
$\hat{\bar{Y}}_{G1}^{13}$	500	**0.0162** 0.0160	**0.0052** 0.0050	**0.0213** 0.0209	185.2084	157.4071
	1000	**0.0070** 0.0070	**0.0022** 0.0021	**0.0092** 0.0091	179.8491	153.8482
$\hat{\bar{Y}}_{G1}^{14}$	500	**0.0196** 0.0194	**0.0043** 0.0040	**0.0238** 0.0233	152.8050	140.9036
	1000	**0.0084** 0.0084	**0.0018** 0.0017	**0.0102** 0.0101	150.3012	139.0507
$\hat{\bar{Y}}_{G1}^{15}$	500	**0.0298** 0.0296	**0.0037** 0.0033	**0.0334** 0.0329	100.5047	100.4491
	1000	**0.0126** 0.0126	**0.0015** 0.0015	**0.0141** 0.0141	100.3305	100.2942

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}_{G1}^{16}$	500	**0.0260** 0.0259	**0.0037** 0.0033	**0.0298** 0.0292	114.8836	112.8245
	1000	**0.0111** 0.0111	**0.0016** 0.0015	**0.0126** 0.0126	113.8565	111.9617
$\hat{\bar{Y}}_{G1}^{17}$	500	**0.0294** 0.0293	**0.0037** 0.0033	**0.0331** 0.0326	101.6381	101.4541
	1000	**0.0125** 0.0125	**0.0015** 0.0015	**0.0140** 0.0140	101.1200	100.9953
$\hat{\bar{Y}}_{G1}^{18}$	500	**0.0209** 0.0207	**0.0041** 0.0038	**0.0249** 0.0244	143.3757	134.6261
	1000	**0.0089** 0.0089	**0.0017** 0.0017	**0.0106** 0.0106	141.6145	133.2657
$\hat{\bar{Y}}_{G1}^{19}$	500	**0.0162** 0.0160	**0.0052** 0.0050	**0.0213** 0.0209	185.1414	157.3848
	1000	**0.0070** 0.0070	**0.0022** 0.0022	**0.0092** 0.0091	179.7828	153.8287
$\hat{\bar{Y}}_{G1}^{20}$	500	**0.0196** 0.0194	**0.0043** 0.0040	**0.0238** 0.0233	152.7306	140.8562
	1000	**0.0084** 0.0084	**0.0018** 0.0017	**0.0102** 0.0101	150.2200	138.9989
$\hat{\bar{Y}}_{G1}^{21}$	500	**0.0174** 0.0172	**0.0047** 0.0045	**0.0221** 0.0217	171.5713	151.6349
	1000	**0.0076** 0.0075	**0.0020** 0.0019	**0.0095** 0.0094	166.9542	148.5763
$\hat{\bar{Y}}_{G1}^{22}$	500	**0.0161** 0.0159	**0.0052** 0.0050	**0.0213** 0.0209	186.0606	157.6842
	1000	**0.0075** 0.0074	**0.0020** 0.0019	**0.0095** 0.0094	168.8147	149.4802
$\hat{\bar{Y}}_{G1}^{23}$	500	**0.0195** 0.0193	**0.0043** 0.0040	**0.0237** 0.0232	153.7663	141.5134
	1000	**0.0091** 0.0091	**0.0017** 0.0016	**0.0108** 0.0107	139.1048	131.5073
$\hat{\bar{Y}}_{G1}^{24}$	500	**0.0173** 0.0171	**0.0047** 0.0045	**0.0221** 0.0216	172.6574	152.1697
	1000	**0.0082** 0.0081	**0.0018** 0.0018	**0.0100** 0.0099	154.6636	141.7636

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\hat{\bar{Y}}_{G1}^{25}$	500	**0.0162** 0.0160	**0.0052** 0.0050	**0.0213** 0.0210	185.0210	157.3445
	1000	**0.0070** 0.0070	**0.0022** 0.0021	**0.0092** 0.0091	179.8893	153.8600
$\hat{\bar{Y}}_{G1}^{26}$	500	**0.0196** 0.0194	**0.0043** 0.0040	**0.0239** 0.0234	152.5972	140.7710
	1000	**0.0084** 0.0084	**0.0018** 0.0017	**0.0102** 0.0101	150.3506	139.0822
$\hat{\bar{Y}}_{G1}^{27}$	500	**0.0174** 0.0172	**0.0047** 0.0045	**0.0222** 0.0217	171.4303	151.5646
	1000	**0.0075** 0.0075	**0.0020** 0.0019	**0.0095** 0.0094	167.0878	148.6425

Table 6.5
Theoretical Biases (with/without ME) of the RRT Estimators in Simple Random Sampling for Population-II

Estimators	bias (Without ME)		bias (With ME)	
	$f = n / N$		$f = n / N$	
	500/5000	**1000/5000**	**500/5000**	**1000/5000**
$\bar{z}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{Gi}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{G1}^{1}$	0.0010	0.0005	0.0016	0.0007
$\hat{\bar{Y}}_{G1}^{2}$	0.0006	0.0003	0.0012	0.0006
$\hat{\bar{Y}}_{G1}^{3}$	0.0009	0.0004	0.0015	0.0007
$\hat{\bar{Y}}_{G1}^{4}$	-0.0003	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1}^{5}$	0.0008	0.0004	0.0014	0.0006
$\hat{\bar{Y}}_{G1}^{6}$	0.0005	0.0003	0.0010	0.0005
$\hat{\bar{Y}}_{G1}^{7}$	0.0001	0.0001	0.0005	0.0003

$\hat{\hat{Y}}_{G1}^{8}$	0.0005	0.0003	0.0010	0.0005
$\hat{\hat{Y}}_{G1}^{9}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{10}$	0.0008	0.0004	0.0015	0.0007
$\hat{\hat{Y}}_{G1}^{11}$	0.0010	0.0005	0.0016	0.0007
$\hat{\hat{Y}}_{G1}^{12}$	-0.0003	-0.0001	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{13}$	-0.0003	-0.0001	-0.0002	-0.0001
$\hat{\hat{Y}}_{G1}^{14}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{15}$	0.0000	0.0000	0.0000	0.0000
$\hat{\hat{Y}}_{G1}^{16}$	-0.0002	-0.0001	-0.0002	-0.0001
$\hat{\hat{Y}}_{G1}^{17}$	0.0000	0.0000	0.0000	0.0000
$\hat{\hat{Y}}_{G1}^{18}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{19}$	-0.0003	-0.0001	-0.0002	-0.0001
$\hat{\hat{Y}}_{G1}^{20}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{21}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{22}$	-0.0003	-0.0002	-0.0002	-0.0001
$\hat{\hat{Y}}_{G1}^{23}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{24}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{25}$	-0.0003	-0.0001	-0.0002	-0.0001
$\hat{\hat{Y}}_{G1}^{26}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1}^{27}$	-0.0004	-0.0002	-0.0003	-0.0001

Results in Tables (6.2-6.5) show that our proposed generalized RRT estimator ($\hat{\hat{Y}}_{Gi}$) and its special cases have zero or near zero bias and perform well both in the presence and absence of measurement errors. The performance of all the estimators in the proposed series of RRT ratio estimators ($\hat{\hat{Y}}_{G1}^{j}$) shows that the efficiency of these estimators is negatively affected by measurement errors. Because of measurement error, mean square error of these RRT estimators increases and this causes a decline in percent relative efficiency. It is clear from Tables 6.2 and 6.4 that the performance of the proposed generalized RRT estimator ($\hat{\hat{Y}}_{Gi}$) is always better than that of the ordinary RRT mean estimator ($\bar{z}$) and RRT ratio estimators ($\hat{\hat{Y}}_{G1}^{j}$) in the presence of measurement errors. The same is true for almost all cases even when measurement errors are not present.

6.7 NUMERICAL EXAMPLE

Data Statistics

Characteristics of the real data set used by Sousa et al. (2014) are considered here for numerical illustration. In 2010, a survey was conducted on Information and Communication Technologies (ICT) usage in enterprises with seat in Portugal (Smilhily and Storm, 2010). The following notations are used below:

Y = True Purchase orders, X = True Turnover of enterprises
y = Measured Purchase orders, x = Measured Turnover of enterprises

Let S be a scrambling random variable with distribution $S \sim N(0, 0.01\sigma_X)$, just as in Sousa et al. (2014). The scrambled response on purchase orders Y is given by $Z = Y + S$. We consider sample sizes: n = 250, 500. The methodology used to get the observed values of z, x and s is same as described in the simulation section above.

Table 6.6
Summary Statistics for the Numerical Example

N	$\bar{Y}$	$\bar{X}$	S_Y^2	S_X^2	ρ_{yx}	S_T^2	S_V^2
1	14.44	17.97	501.31	640.59	0.9368	1.93	0.99

Table 6.7
Theoretical MSE's, PRE's (with/without ME) of the RRT Estimators Relative to Ordinary RRT Mean Estimator in Simple Random Sampling for the Numerical Example

Estimators	***n***	**Mean Square Error**			**Percent Relative Efficiency**	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\bar{z}$ (Ordinary RRT Estimator)	250	0.7157	0.0029	0.7186	100.0000	100.0000
	500	1.7260	0.0066	1.7326	100.0000	100.0000
$\hat{\bar{Y}}_{G1}^{1}$ (RRT Ratio Estimator)	250	0.0911	0.0038	0.0949	785.6202	757.2181
	500	0.2120	0.0099	0.2219	814.1509	780.8022
$\hat{\bar{Y}}_{Gi}$ (Proposed Generalized RRT Estimator)	250	0.0902	0.0023	0.0925	793.4589	776.8649
	500	0.2118	0.0089	0.2207	814.9197	785.0476

When dealing with the real data, Table 6.7 shows that our proposed estimator ($\hat{\bar{Y}}_{Gi}$) is better than the commonly used estimators ($\bar{z}$) and ($\hat{\bar{Y}}_{G1}^{1}$) both when measurement error is present and when it is absent. The contribution of measurement error in generalized estimator is lower as compared to RRT ratio estimator ($\hat{\bar{Y}}_{G1}^{1}$). However it is not so as compared to the mean per unit estimator for the smaller sample size. We believe the reason for this is that the mean per unit estimator has only one source of measurement errors (Y) but our proposed estimator has two (Y and X). This extra source of measurement error is offset when the sample size is larger.

6.8 CONCLUSION

In this chapter, we have proposed a RRT version of the generalized mean estimator introduced in Chapter 4 to deal with sensitive study variables. The proposed estimator leads to several new RRT estimators as special cases. Particularly, the special cases ($\hat{\bar{Y}}_{G1}^{j}$) utilize some conventional and non-conventional measures with a non-sensitive auxiliary variable which is highly correlated with sensitive study variable. Results in Tables (6.2-6.5) show the effect of measurement error on the estimators of the proposed series using two sampling fractions. As expected, the theoretical and empirical mean square errors are in very good match. The amounts of biases are larger for small sample size but become negligible as the size of the sample increases. The efficiency of the estimators reduces if we take measurement errors into account. It is concluded that the generalized randomized response estimator performs better than the ordinary RRT mean estimator and RRT ratio estimator, particularly if the correlation between the study and the auxiliary variable is high. Numerical results in Table 6.7 corroborate our simulation results.

CHAPTER 7

MEAN ESTIMATION FOR A SENSITIVE STUDY VARIABLE IN THE PRESENCE OF MEASUREMENT ERRORS UNDER STRATIFIED RANDOM SAMPLING

7.1 INTRODUCTION

In this Chapter, we revisit the class of generalized mean estimators introduced in Chapter 6 for sensitive study variable where measurement errors can occur both in the study variable and the auxiliary variable. But unlike the previous chapter, it is done here using a stratified random sampling design. Once again, the focus is on studying the impact of measurement errors on mean estimation. We have also provided a comparison of the proposed estimator with some existing mean estimators.

7.2 SAMPLING PROCEDURE AND NOTATIONS

We follow the usual notation for stratified sampling and consider a finite population $M = (M_1, M_2, ...M_N)$ of size N divided in L homogenous strata with N_h units (h=1, 2,…, L) in the h^{th} stratum such that $\sum_{h=1}^{L} N_h = N$ and the weight of the h^{th} stratum is $W_h = \frac{N_h}{N}$. Let Y be the sensitive study variable which is not directly observable. Let a non-sensitive auxiliary variable X be available, which is strongly correlated with Y. Let S be a zero-mean scrambling random variable with known distribution. The respondent is asked to report an additively scrambled response for the study variable Y given by $Z = Y + S$ and is also asked to provide a true response for the auxiliary variable X. A simple random sample of size n_h is drawn without replacement from the h^{th} stratum such that $\sum_{h=1}^{L} n_h = n$. Let (y_{hi}, x_{hi}, z_{hi}) be the observed pair of values (factoring in measurement errors) and $\left(Y_{hi}, X_{hi}, Z_{hi}\right)$ be the true pair of values of the study variable Y, the auxiliary variable X and the scrambled response variable Z respectively, associated with the i^{th} $(i = 1, 2, ..., N_h)$ unit of the h^{th} stratum. Let $\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h$, $\bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h$ and $\bar{z}_{st} = \sum_{h=1}^{L} W_h \bar{z}_h$ be the stratified sample means where $\bar{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi}$, $\bar{x}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} x_{hi}$ and $\bar{z}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} z_{hi}$ are the stratum sample means and $\bar{Y}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} y_{hi}$, $\bar{X}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} x_{hi}$ and $\bar{Z}_h = \frac{1}{N_h}\sum_{i=1}^{N_h} z_{hi}$ are the corresponding population stratum means. To estimate $\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$, it is assumed that $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$ is known. Note that $\bar{Z} = \bar{Y}$ is the population mean for the scrambled variable Z since S has zero mean. The measurement errors associated with the scrambled response variable Z and the auxiliary variable X in the h^{th} stratum, as defined in Equations 1.5.5 and 1.5.6 Chapter 1 are given by:

$$T_{hi} = z_{hi} - Z_{hi},$$

$$V_{hi} = x_{hi} - X_{hi},$$

The following notations will be needed in this chapter:

Let $e'_{ost} = \frac{\bar{z}_{st} - \bar{Z}}{\bar{Z}}$ and $e'_{1st} = \frac{\bar{x}_{st} - \bar{X}}{\bar{X}}$ such that, $E\left(e'_{ost}\right) = E\left(e'_{1st}\right) = 0,$ and

$$E\left(e'^2_{ost}\right) = \frac{1}{\bar{Z}^2}\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Zh}^2}{\theta_{Zh}} = V'_{20},\ E\left(e'^2_{1st}\right) = \frac{1}{\bar{X}^2}\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}} = V'_{02},$$

$E\left(e'_{ost}\, e'_{1st}\right) = \frac{1}{\bar{Z}\bar{X}}\sum_{h=1}^{L} W_h^2 \gamma_h S_{ZXh} = V'_{11},$ where $S_{Zh}^2 = S_{Yh}^2 + S_{Sh}^2$ and

$$S_{Yh}^2 = \sum_{i=1}^{N_h}\frac{\left(y_{hi} - \bar{Y}_h\right)^2}{N_h - 1}, S_{Sh}^2 = \sum_{i=1}^{N_h}\frac{\left(s_{hi} - \bar{S}_h\right)^2}{N_h - 1}, S_{Xh}^2 = \sum_{i=1}^{N_h}\frac{\left(x_{hi} - \bar{X}_h\right)^2}{N_h - 1},$$

$$S_{ZXh} = \rho_{ZXh} S_{Zh} S_{Xh},$$

$$\rho_{ZXh} = \frac{\rho_{YXh}}{\sqrt{1 + \left(S_{Sh}^2 / S_{Yh}^2\right)}}, \theta_{zh} = \frac{S_{Zh}^2}{S_{Th}^2 + S_{Zh}^2}, \theta_{Xh} = \frac{S_{Xh}^2}{(S_{Vh}^2 + S_{Xh}^2)}, \gamma_h = \left(\frac{1 - f_h}{n_h}\right)$$

and $f_h = \frac{n_h}{N_h}$.

(7.2.1)

7.3 PROPOSED GENERALIZED RRT ESTIMATOR IN THE PRESENCE OF MEASUREMENT ERROR

In this section, we propose a generalized randomized response estimator for population mean in the presence of measurement errors on both the sensitive study variable (*Y*) and the non-sensitive auxiliary variable (*X*) in stratified random sampling. As pointed out earlier, a scrambled version of *Y* is observed in the form of *Z*=*Y*+*S*, where *S* is a zero-mean scrambling variable.

The proposed estimator is:

$$\hat{\bar{Y}}_{Gi,st} = \left[\bar{z}_{st} + k\left(\bar{X} - \bar{x}_{st}\right)\right]\left[\frac{a_{st}\bar{X} + b_{st}}{\lambda\left(a_{st}\bar{x}_{st} + b_{st}\right) + (1-\lambda)\left(a_{st}\bar{X} + b_{st}\right)}\right]^{g}, \tag{7.3.1}$$

Here k and g are suitable constants, and λ is assumed to be an unknown constant whose value is to be determined from optimality considerations. As usual, the constants $a_{st}(\neq 0)$, and b_{st} are either real numbers or functions of the known parameters of the auxiliary variable *X*. Many RRT mean estimators can be deduced from the proposed class of estimators with specific choices of the constants. For example, $g = 1$ gives us various combined RRT ratio estimators and $g = -1$ gives us various combined RRT product estimators.

Remark 1:

For $g = 1$, $\hat{\bar{Y}}_{Gi,st}$ takes the following form:

$$\hat{\bar{Y}}_{Gi,st} = \left[\bar{z}_{st} + k\left(\bar{X} - \bar{x}_{st}\right)\right]\left[\frac{a_{st}\bar{X} + b_{st}}{\lambda\left(a_{st}\bar{x}_{st} + b_{st}\right) + (1-\lambda)\left(a_{st}\bar{X} + b_{st}\right)}\right], \tag{7.3.2}$$

By setting different values of the unknown constants in Equation (7.3.2), various combined RRT ratio estimators based on the single auxiliary variable may be obtained. For example,

(i) By putting $k = 0$ and $\lambda = 1$, we have

$$\hat{\bar{Y}}_{G1,st} = \bar{z}_{st}\left(\frac{a_{st}\bar{X} + b_{st}}{a_{st}\bar{x}_{st} + b_{st}}\right). \tag{7.3.3}$$

(ii) By putting $k=1$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G2,st}=\left(\bar{z}_{st}+(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{a_{st}\bar{x}_{st}+b_{st}}\right). \tag{7.3.4}$$

(iii) By putting $k=b_{ZX}$ (the slope term in regression *Z* on *X*) and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G3,st}=\left(\bar{z}_{st}+b_{ZX}(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{a_{st}\bar{x}_{st}+b_{st}}\right). \tag{7.3.5}$$

(iv) By putting $k=0$ and $\lambda=\lambda_{opt}$ (optimized value of λ relative to the MSE of the proposed estimator), we have

$$\hat{\hat{Y}}_{G4,st}=\bar{z}_{st}\left(\frac{a_{st}\bar{X}+b_{st}}{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}\right). \tag{7.3.6}$$

(v) By putting $k=1$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G5,st}=\left(\bar{z}_{st}+(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}\right). \tag{7.3.7}$$

(vi) By putting $k=b_{ZX}$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G6,st}=\left(\bar{z}_{st}+b_{ZX}(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{X}+b_{st}}{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}\right). \tag{7.3.8}$$

Remark 2:

For $g=-1$, $\hat{\hat{Y}}_{Gi,st}$ takes the following form:

$$\hat{\hat{Y}}_{Gi,st}=\left[\bar{z}_{st}+k\left(\bar{X}-\bar{x}_{st}\right)\right]\left[\frac{\lambda\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda\right)\left(a_{st}\bar{X}+b_{st}\right)}{a_{st}\bar{X}+b_{st}}\right]. \tag{7.3.9}$$

By setting different values of the unknown constants in Equation (7.3.9), various combined RRT product estimators based on single auxiliary variable may be obtained. For example,

(i) By putting $k=0$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G7,st}=\bar{z}_{st}\left(\frac{a_{st}\bar{x}_{st}+b_{st}}{a_{st}\bar{X}+b_{st}}\right). \tag{7.3.10}$$

(ii) By putting $k=1$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G8,st}=\left(\bar{z}_{st}+(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{x}_{st}+b_{st}}{a_{st}\bar{X}+b_{st}}\right). \tag{7.3.11}$$

(iii) By putting $k=b_{ZX}$ and $\lambda=1$, we have

$$\hat{\hat{Y}}_{G9,st}=\left(\bar{z}_{st}+b_{ZX}(\bar{X}-\bar{x}_{st})\right)\left(\frac{a_{st}\bar{x}_{st}+b_{st}}{a_{st}\bar{X}+b_{st}}\right). \tag{7.3.12}$$

(iv) By putting $k=0$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G10,st}=\bar{z}_{st}\left(\frac{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}{a_{st}\bar{X}+b_{st}}\right). \tag{7.3.13}$$

(v) By putting $k=1$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G11,st}=\left(\bar{z}_{st}+(\bar{X}-\bar{x}_{st})\right)\left(\frac{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}{a_{st}\bar{X}+b_{st}}\right). \tag{7.3.14}$$

(vi) By putting $k=b_{ZX}$ and $\lambda=\lambda_{opt}$, we have

$$\hat{\hat{Y}}_{G12,st}=\left(\bar{z}_{st}+b_{ZX}(\bar{X}-\bar{x}_{st})\right)\left(\frac{\lambda_{opt}\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\lambda_{opt}\right)\left(a_{st}\bar{X}+b_{st}\right)}{a_{st}\bar{X}+b_{st}}\right). \tag{7.3.15}$$

7.3.1 The Bias and Mean Square Error of the Proposed Generalized RRT Estimator

Expressing Equation (7.3.1) in terms of e's, we have

$$\hat{\hat{Y}}_{Gi,st} = \left(\bar{Z}\left(1+e'_{ost}\right) + k\left(\bar{X} - \bar{X}\left(1+e'_{1st}\right)\right)\right)$$
$$\left(\frac{a_{st}\bar{X} + b_{st}}{\lambda\left(a_{st}\bar{X}\left(1+e'_{1st}\right) + b_{st}\right) + (1-\lambda)\left(a_{st}\bar{X} + b_{st}\right)}\right)^{g},$$

or

$$\hat{\hat{Y}}_{Gi,st} = \left(\bar{Z} + \bar{Z}e'_{ost} - k\bar{X}e'_{1st}\right)\left(\frac{a_{st}\bar{X} + b_{st}}{a_{st}\bar{X} + b_{st} + \lambda a_{st}\bar{X}e'_{1st}}\right)^{g},$$

or

$$\hat{\hat{Y}}_{Gi,st} = \left(\bar{Z} + \bar{Z}e'_{ost} - k\bar{X}e'_{1st}\right)\left(1 + \frac{\lambda a_{st}\bar{X}e'_{1st}}{a_{st}\bar{X} + b_{st}}\right)^{-g},$$

or

$$\hat{\hat{Y}}_{Gi,st} = \left(\bar{Z} + \bar{Z}e'_{ost} - k\bar{X}e'_{1st}\right)\left(1 + \lambda\phi e'_{1st}\right)^{-g},$$

where $\phi = \dfrac{a_{st}\bar{X}}{(a_{st}\bar{X} + b_{st})}$

By using Taylor series expansion

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^{2} - \ldots,$$

we get

$$\hat{\hat{Y}}_{Gi,st} \approx \left(\bar{Z} + \bar{Z}e'_{ost} - k\bar{X}e'_{1st}\right)\left(1 - g\lambda\phi e'_{1st} + \frac{g(g+1)}{2}\left(\lambda\phi e'_{1st}\right)^{2}\right),$$

or

$$\hat{\hat{Y}}_{Gi,st} \approx \left(\begin{array}{c} \bar{Z} - g\lambda\phi\bar{Z}e'_{1st} + \frac{g(g+1)}{2}\lambda^{2}\phi^{2}\bar{Z}e'^{2}_{1st} + \bar{Z}e'_{ost} \\ -g\lambda\phi\bar{Z}e'_{ost}e'_{1st} + \frac{g(g+1)}{2}\lambda^{2}\phi^{2}\bar{Z}e'_{ost}e'^{2}_{1st} \\ -k\bar{X}e'_{1st} + g\lambda\phi k\bar{X}e'^{2}_{1st} - \frac{g(g+1)}{2}\lambda^{2}\phi^{2}k\bar{X}e'^{3}_{1st} \end{array}\right) \qquad (7.3.16)$$

In order to derive the expression of bias, using error terms up to second order from Equation (7.3.16), we have

$$(\hat{\hat{Y}}_{Gi,st} - \bar{Z}) \approx \left(\frac{g(g+1)}{2} \lambda^2 \phi^2 \bar{Z} e_{1st}'^2 - g\lambda\phi \bar{Z} e_{0st}' e_{1st}' + g\lambda\phi k \bar{X} e_{1st}'^2 \right).$$

Taking expectation and using Equation (7.2.1), we get

$$E(\hat{\hat{Y}}_{Gi,st} - \bar{Z}) \approx \left(g\lambda\phi \right) \left(\left(\left(\frac{g+1}{2} \right) \lambda\phi \bar{Z} + k\bar{X} \right) \frac{1}{\bar{X}^2} \sum_{h=1}^{L} \frac{W_h^2 \gamma_h S_{Xh}^2}{\theta_{Xh}} - \frac{1}{\bar{X}} \sum_{h=1}^{L} W_h^2 \gamma_h S_{ZXh} \right),$$

or

$$Bias^*(\hat{\hat{Y}}_{Gi,st}) \approx \left(g\lambda\phi \right) \left(\left(\left(\frac{g+1}{2} \right) \phi\lambda \bar{Z} + k\bar{X} \right) V_{02}' - \bar{Z} V_{11}' \right). \tag{7.3.17}$$

The expression of bias for the proposed generalized RRT estimator without measurement error may be obtained by putting $S_{Vh}^2 = 0$ in Equation (7.3.17).

In order to derive the expression of MSE of proposed estimator, using error terms up to first order from Equation (7.3.16), we have

$$(\hat{\hat{Y}}_{Gi,st} - \bar{Z}) \approx \left(\bar{Z} e_{0st}' - g\lambda\phi \bar{Z} e_{1st}' - k\bar{X} e_{1st}' \right).$$

Squaring both sides and taking expectation, we have

$$E(\hat{\hat{Y}}_{Gi,st} - \bar{Z})^2 \approx \left(\begin{array}{l} \bar{Z}^2 \left(\frac{1}{\bar{Z}^2} \sum_{h=1}^{L} \frac{W_h^2 \gamma_h S_{Zh}^2}{\theta_{Zh}} \right) \\ + \left((g\lambda\phi)^2 \bar{Z}^2 + k^2 \bar{X}^2 + 2g\lambda\phi \bar{Z} k \bar{X} \right) \frac{1}{\bar{X}^2} \sum_{h=1}^{L} \frac{W_h^2 \gamma_h S_{Xh}^2}{\theta_{Xh}} \\ - \left(2g\lambda\phi \bar{Z}^2 + 2\bar{Z} k \bar{X} \right) \frac{1}{\bar{Z}\bar{X}} \sum_{h=1}^{L} W_h^2 \gamma_h S_{ZXh} \end{array} \right), \tag{7.3.18}$$

After simplification of Equation (7.3.18), we get

$$MSE^*(\hat{\bar{Y}}_{Gi,st}) \approx \bar{Z}^2 V'_{20} + \left((g\lambda\phi)^2 \bar{Z}^2 + k^2 \bar{X}^2 + 2g\lambda\phi\bar{Z}\bar{X}k\right)V'_{02}$$
$$-\left(2g\lambda\phi\bar{Z}^2 + 2\bar{Z}\bar{X}k\right)V'_{11}, \qquad (7.3.19)$$

In order to obtain minimized $MSE^*(\hat{\bar{Y}}_{Gi,st})$, we differentiate Equation (7.3.19) with respect to $(g\lambda\phi)$, and equate the results to zero, i.e.

$$\frac{\partial MSE^*(\hat{\bar{Y}}_{Gi,st})}{\partial(g\lambda\phi)} = 0,$$

or

$$\frac{\partial MSE^*(\hat{\bar{Y}}_{Gi,st})}{\partial(g\lambda\phi)} = 2\left(g\lambda\phi\bar{Z}^2 + \bar{Z}k\bar{X}\right)V'_{02} - 2\bar{Z}^2 V'_{11}, \qquad (7.3.20)$$

On solving (7.3.20), the optimum value of $(g\lambda\phi)$ is obtained as,

$$(g\lambda\phi)_{opt} = \left(\frac{V'_{11}}{V'_{02}} - \frac{\bar{X}}{\bar{Z}}k\right). \qquad (7.3.21)$$

It may be noted, as in Chapter 5, that optimization with respect to $(g\lambda\phi)$ is mentioned only because of notational convenience, the key parameter being optimized is λ.

Substitution of (7.3.21) in (7.3.19) yields the minimized $MSE^*(\hat{\bar{Y}}_{Gi,st})$ as:

$$MSE^*_{\min}(\hat{\bar{Y}}_{Gi,st}) \approx \bar{Z}^2 V'_{20}\left(1 - \rho'^2_{st}\right), \qquad (7.3.22)$$

where $\rho'_{st} = V'_{11}\Big/\left(\sqrt{V'_{20}}\sqrt{V'_{02}}\right)$.

The expression of minimized MSE of the proposed generalized RRT estimator without measurement error may be obtained by putting $S^2_{Th} = S^2_{Vh} = 0$ in Equation (7.3.22).

7.4 ADDITIONAL SPECIAL CASES OF THE GENERALIZED RRT RATIO ESTIMATOR

Many additional combined RRT ratio estimators can be deduced from the generalized RRT ratio estimator ($\hat{\hat{Y}}_{G1,st}$) given in Equation (7.3.3). We denote the generalized RRT ratio estimator by,

$$\hat{\hat{Y}}^{j}_{G1,st} = \bar{z}_{st}\left(\frac{a^{j}_{st}\bar{X} + b^{j}_{st}}{a^{j}_{st}\bar{x}_{st} + b^{j}_{st}}\right).$$

Various choices of a^{j}_{st}, b^{j}_{st} are given in the table below. The general expressions for the mean square error and bias respectively with measurement error for this generalized RRT ratio estimator $\hat{\hat{Y}}^{j}_{G1,st}$ are given by

$$\left.\begin{aligned}
MSE^{*}(\hat{\hat{Y}}^{j}_{G1,st}) &= \sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S^2_{Zh}}{\theta_{Zh}} + R^{j}_{st}\frac{S^2_{Xh}}{\theta_{Xh}}\left(R^{j}_{st} - 2\beta_{ZXh}\theta_{Xh}\right)\right),\\
Bias^{*}(\hat{\hat{Y}}^{j}_{G1,st}) &= \left(\frac{\phi_j}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^{j}_{st} - \beta_{ZXh}\theta_{Xh}\right),\\
\phi_j &= \frac{a^{j}_{st}\bar{X}}{(a^{j}_{st}\bar{X} + b^{j}_{st})}, \quad R^{j}_{st} = \frac{\bar{Z}}{\bar{X}}\phi_j,\ \beta_{ZXh} = \frac{S_{ZXh}}{S^2_{Xh}}.
\end{aligned}\right] \qquad (7.4.1)$$

Table 7.1

Additional Special Cases of the Generalized RRT Ratio Estimator ($\hat{\hat{Y}}^{j}_{G1,st}$)

Estimators	a^j_{st}	b^j_{st}	Mean Square Errors & Biases	Ratio's
$\hat{\hat{Y}}^{1}_{G1,st} = \bar{z}_{st}\left(\dfrac{\bar{X}}{\bar{x}_{st}}\right)$	1	0	$MSE^*(\hat{\hat{Y}}^{1}_{G1,st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\dfrac{S^2_{Zh}}{\theta_{Zh}} + R^1_{st}\dfrac{S^2_{Xh}}{\theta_{Xh}}\left(R^1_{st} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}^{1}_{G1,st}) = \left(\dfrac{\phi_1}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^1_{st} - \beta_{ZXh}\theta_{Xh}\right)$	$R^1_{st} = \dfrac{\bar{Z}}{\bar{X}}\phi_1$ $\phi_1 = 1$
$\hat{\hat{Y}}^{2}_{G1,st} = \bar{z}_{st}\left(\dfrac{\bar{X}+\Omega_1}{\bar{x}_{st}+\Omega_1}\right)$	1	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$MSE^*(\hat{\hat{Y}}^{2}_{G1,st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\dfrac{S^2_{Zh}}{\theta_{Zh}} + R^2_{st}\dfrac{S^2_{Xh}}{\theta_{Xh}}\left(R^2_{st} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}^{2}_{G1,st}) = \left(\dfrac{\phi_2}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^2_{st} - \beta_{ZXh}\theta_{Xh}\right)$	$R^2_{st} = \dfrac{\bar{Z}}{\bar{X}}\phi_2$ $\phi_2 = \bar{X}/(\bar{X}+\Omega_1)$
$\hat{\hat{Y}}^{3}_{G1,st} = \bar{z}_{st}\left(\dfrac{\bar{X}+\Omega_2}{\bar{x}_{st}+\Omega_2}\right)$	1	$\Omega_2 = \sum_{h=1}^{L} W_h \beta_{2h}(x)$	$MSE^*(\hat{\hat{Y}}^{3}_{G1,st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\dfrac{S^2_{Zh}}{\theta_{Zh}} + R^3_{st}\dfrac{S^2_{Xh}}{\theta_{Xh}}\left(R^3_{st} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}^{3}_{G1,st}) = \left(\dfrac{\phi_3}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^3_{st} - \beta_{ZXh}\theta_{Xh}\right)$	$R^3_{st} = \dfrac{\bar{Z}}{\bar{X}}\phi_3$ $\phi_3 = \bar{X}/(\bar{X}+\Omega_2)$
$\hat{\hat{Y}}^{4}_{G1,st} = \bar{z}_{st}\left(\dfrac{\Omega_2\bar{X}+\Omega_1}{\Omega_2\bar{x}_{st}+\Omega_1}\right)$	$\Omega_2 = \sum_{h=1}^{L} W_h \beta_{2h}(x)$	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$MSE^*(\hat{\hat{Y}}^{4}_{G1,st}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\dfrac{S^2_{Zh}}{\theta_{Zh}} + R^4_{st}\dfrac{S^2_{Xh}}{\theta_{Xh}}\left(R^4_{st} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\hat{Y}}^{4}_{G1,st}) = \left(\dfrac{\phi_4}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S^2_{Xh}}{\theta_{Xh}}\left(R^4_{st} - \beta_{ZXh}\theta_{Xh}\right)$	$R^4_{st} = \dfrac{\bar{Z}}{\bar{X}}\phi_4$ $\phi_4 = \Omega_2\bar{X}/(\Omega_2\bar{X}+\Omega_1)$

Estimators	a_{st}^j	b_{st}^j	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}^5_{G1,st} = \bar{z}_{st}\left(\frac{\Omega_1\bar{X}+\Omega_2}{\Omega_1\bar{x}_{st}+\Omega_2}\right)$	$\Omega_1=\sum_{h=1}^{L}W_hC_{Xh}$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$MSE^*(\hat{\bar{Y}}^5_{G1,st})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^5\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^5-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^5_{G1,st})=\left(\frac{\phi_5}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^5-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^5=\frac{\bar{Z}}{\bar{X}}\phi_5$ $\phi_5=\Omega_1\bar{X}\big/(\Omega_1\bar{X}+\Omega_2)$
$\hat{\bar{Y}}^6_{G1,st} = \bar{z}_{st}\left(\frac{\bar{X}+\Omega_3}{\bar{x}_{st}+\Omega_3}\right)$	1	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{ZXh}$	$MSE^*(\hat{\bar{Y}}^6_{G1,st})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^6\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^6-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^6_{G1,st})=\left(\frac{\phi_6}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^6-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^6=\frac{\bar{Z}}{\bar{X}}\phi_6$ $\phi_6=\bar{X}\big/(\bar{X}+\Omega_3)$
$\hat{\bar{Y}}^7_{G1,st} = \bar{z}_{st}\left(\frac{\Omega_1\bar{X}+\Omega_3}{\Omega_1\bar{x}_{st}+\Omega_3}\right)$	$\Omega_1=\sum_{h=1}^{L}W_hC_{Xh}$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{ZXh}$	$MSE^*(\hat{\bar{Y}}^7_{G1,st})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^7\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^7-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^7_{G1,st})=\left(\frac{\phi_7}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^7-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^7=\frac{\bar{Z}}{\bar{X}}\phi_7$ $\phi_7=\Omega_1\bar{X}\big/(\Omega_1\bar{X}+\Omega_3)$
$\hat{\bar{Y}}^8_{G1,st} = \bar{z}_{st}\left(\frac{\Omega_3\bar{X}+\Omega_1}{\Omega_3\bar{x}_{st}+\Omega_1}\right)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{ZXh}$	$\Omega_1=\sum_{h=1}^{L}W_hC_{Xh}$	$MSE^*(\hat{\bar{Y}}^8_{G1,st})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^8\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^8-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^8_{G1,st})=\left(\frac{\phi_8}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^8-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^8=\frac{\bar{Z}}{\bar{X}}\phi_8$ $\phi_8=\Omega_3\bar{X}\big/(\Omega_3\bar{X}+\Omega_1)$
$\hat{\bar{Y}}^9_{G1,st} = \bar{z}_{st}\left(\frac{\Omega_2\bar{X}+\Omega_3}{\Omega_2\bar{x}_{st}+\Omega_3}\right)$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{ZXh}$	$MSE^*(\hat{\bar{Y}}^9_{G1,st})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^9\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^9-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}^9_{G1,st})=\left(\frac{\phi_9}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_hS_{Xh}^2}{\theta_{Xh}}\left(R_{st}^9-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^9=\frac{\bar{Z}}{\bar{X}}\phi_9$ $\phi_9=\Omega_2\bar{X}\big/(\Omega_2\bar{X}+\Omega_3)$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{G1,st}^{10} = \bar{z}_{st}\left(\frac{\Omega_3\bar{X}+\Omega_2}{\Omega_3\bar{x}_{st}+\Omega_2}\right)$	$\Omega_3=\sum_{h=1}^{L}W_h\rho_{ZXh}$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{10})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^{10}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{10}-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{10})=\left(\frac{\phi_{10}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{10}-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{10}=\frac{\bar{Z}}{\bar{X}}\phi_{10}$ $\phi_{10}=\Omega_3\bar{X}\Big/(\Omega_3\bar{X}+\Omega_2$
$\hat{\bar{Y}}_{G1,st}^{11} = \bar{z}_{st}\left(\frac{\bar{X}+\Omega_4}{\bar{x}_{st}+\Omega_4}\right)$	1	$\Omega_4=\sum_{h=1}^{L}W_h\beta_{1h}(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{11})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^{11}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{11}-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{11})=\left(\frac{\phi_{11}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{11}-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{11}=\frac{\bar{Z}}{\bar{X}}\phi_{11}$ $\phi_{11}=\bar{X}\Big/(\bar{X}+\Omega_4)$
$\hat{\bar{Y}}_{G1,st}^{12} = \bar{z}_{st}\left(\frac{\Omega_4\bar{X}+\Omega_2}{\Omega_4\bar{x}_{st}+\Omega_2}\right)$	$\Omega_4=\sum_{h=1}^{L}W_h\beta_{1h}(x)$	$\Omega_2=\sum_{h=1}^{L}W_h\beta_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{12})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^{12}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{12}-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{12})=\left(\frac{\phi_{12}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{12}-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{12}=\frac{\bar{Z}}{\bar{X}}\phi_{12}$ $\phi_{12}=\Omega_4\bar{X}\Big/(\Omega_4\bar{X}+\Omega_2$
$\hat{\bar{Y}}_{G1,st}^{13} = \bar{z}_{st}\left(\frac{\bar{X}+\Omega_5}{\bar{x}_{st}+\Omega_5}\right)$	1	$\Omega_5=\sum_{h=1}^{L}W_hQ_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{13})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^{13}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{13}-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{13})=\left(\frac{\phi_{13}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{13}-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{13}=\frac{\bar{Z}}{\bar{X}}\phi_{13}$ $\phi_{13}=\bar{X}\Big/(\bar{X}+\Omega_5)$
$\hat{\bar{Y}}_{G1,st}^{14} = \bar{z}_{st}\left(\frac{\Omega_1\bar{X}+\Omega_5}{\Omega_1\bar{x}_{st}+\Omega_5}\right)$	$\Omega_1=\sum_{h=1}^{L}W_hC_{Xh}$	$\Omega_5=\sum_{h=1}^{L}W_hQ_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{14})=\sum_{h=1}^{L}W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}}+R_{st}^{14}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{14}-2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{14})=\left(\frac{\phi_{14}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{14}-\beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{14}=\frac{\bar{Z}}{\bar{X}}\phi_{14}$ $\phi_{14}=\Omega_1\bar{X}\Big/(\Omega_1\bar{X}+\Omega_5$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{G1,st}^{15}$ $= \bar{z}_{st}\left(\frac{\Omega_4\bar{X}+\Omega_5}{\Omega_4\bar{x}_{st}+\Omega_5}\right)$	$\Omega_4 = \sum_{h=1}^{L} W_h\beta_{1h}(x)$	$\Omega_5 = \sum_{h=1}^{L} W_h Q_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{15}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{15}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{15} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{15}) = \left(\frac{\phi_{15}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{15} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{15} = \frac{\bar{Z}}{\bar{X}}\phi_{15}$ $\phi_{15} = \Omega_4\bar{X} \big/ (\Omega_4\bar{X} + \Omega_5$
$\hat{\bar{Y}}_{G1,st}^{16}$ $= \bar{z}_{st}\left(\frac{\Omega_2\bar{X}+\Omega_5}{\Omega_2\bar{x}_{st}+\Omega_5}\right)$	$\Omega_2 = \sum_{h=1}^{L} W_h\beta_{2h}(x)$	$\Omega_5 = \sum_{h=1}^{L} W_h Q_{2h}(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{16}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{16}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{16} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{16}) = \left(\frac{\phi_{16}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{16} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{16} = \frac{\bar{Z}}{\bar{X}}\phi_{16}$ $\phi_{16} = \Omega_2\bar{X} \big/ (\Omega_2\bar{X} + \Omega_5$
$\hat{\bar{Y}}_{G1,st}^{17}$ $= \bar{z}_{st}\left(\frac{\Omega_4\bar{X}+\Omega_6}{\Omega_4\bar{x}_{st}+\Omega_6}\right)$	$\Omega_4 = \sum_{h=1}^{L} W_h\beta_{1h}(x)$	$\Omega_6 = \sum_{h=1}^{L} W_h QD_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{17}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{17}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{17} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{17}) = \left(\frac{\phi_{17}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{17} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{17} = \frac{\bar{Z}}{\bar{X}}\phi_{17}$ $\phi_{17} = \Omega_4\bar{X} \big/ (\Omega_4\bar{X} + \Omega_6$
$\hat{\bar{Y}}_{G1,st}^{18}$ $= \bar{z}_{st}\left(\frac{\Omega_2\bar{X}+\Omega_6}{\Omega_2\bar{x}_{st}+\Omega_6}\right)$	$\Omega_2 = \sum_{h=1}^{L} W_h\beta_{2h}(x)$	$\Omega_6 = \sum_{h=1}^{L} W_h QD_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{18}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{18}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{18} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{18}) = \left(\frac{\phi_{18}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{18} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{18} = \frac{\bar{Z}}{\bar{X}}\phi_{18}$ $\phi_{18} = \Omega_2\bar{X} \big/ (\Omega_2\bar{X} + \Omega_6$
$\hat{\bar{Y}}_{G1,st}^{19}$ $= \bar{z}_{st}\left(\frac{\bar{X}+\Omega_7}{\bar{x}_{st}+\Omega_7}\right)$	1	$\Omega_7 = \sum_{h=1}^{L} W_h TM_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{19}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\frac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{19}\frac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{19} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{19}) = \left(\frac{\phi_{19}}{\bar{X}}\right)\sum_{h=1}^{L}\frac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{19} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{19} = \frac{\bar{Z}}{\bar{X}}\phi_{19}$ $\phi_{19} = \bar{X} \big/ (\bar{X} + \Omega_7)$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{G1,st}^{20} = \bar{z}_{st}\left(\dfrac{\Omega_1\bar{X}+\Omega_7}{\Omega_1\bar{x}_{st}+\Omega_7}\right)$	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$\Omega_7 = \sum_{h=1}^{L} W_h TM_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{20}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{20}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{20} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{20}) = \left(\dfrac{\phi_{20}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{20} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{20} = \dfrac{\bar{Z}}{\bar{X}}\phi_{20}$ $\phi_{20} = \Omega_1\bar{X} \big/ (\Omega_1\bar{X}+\Omega_7$
$\hat{\bar{Y}}_{G1,st}^{21} = \bar{z}_{st}\left(\dfrac{\Omega_3\bar{X}+\Omega_7}{\Omega_3\bar{x}_{st}+\Omega_7}\right)$	$\Omega_3 = \sum_{h=1}^{L} W_h \rho_{ZXh}$	$\Omega_7 = \sum_{h=1}^{L} W_h TM_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{21}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{21}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{21} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{21}) = \left(\dfrac{\phi_{21}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{21} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{21} = \dfrac{\bar{Z}}{\bar{X}}\phi_{21}$ $\phi_{21} = \Omega_3\bar{X} \big/ (\Omega_3\bar{X}+\Omega_7$
$\hat{\bar{Y}}_{G1,st}^{22} = \bar{z}_{st}\left(\dfrac{\bar{X}+\Omega_8}{\bar{x}_{st}+\Omega_8}\right)$	1	$\Omega_8 = \sum_{h=1}^{L} W_h MR_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{22}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{22}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{22} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{22}) = \left(\dfrac{\phi_{22}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{22} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{22} = \dfrac{\bar{Z}}{\bar{X}}\phi_{22}$ $\phi_{22} = \bar{X} \big/ (\bar{X}+\Omega_8)$
$\hat{\bar{Y}}_{G1,st}^{23} = \bar{z}_{st}\left(\dfrac{\Omega_1\bar{X}+\Omega_8}{\Omega_1\bar{x}_{st}+\Omega_8}\right)$	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$\Omega_8 = \sum_{h=1}^{L} W_h MR_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{23}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{23}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{23} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{23}) = \left(\dfrac{\phi_{23}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{23} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{23} = \dfrac{\bar{Z}}{\bar{X}}\phi_{23}$ $\phi_{23} = \Omega_1\bar{X} \big/ (\Omega_1\bar{X}+\Omega_8$
$\hat{\bar{Y}}_{G1,st}^{24} = \bar{z}_{st}\left(\dfrac{\Omega_3\bar{X}+\Omega_8}{\Omega_3\bar{x}_{st}+\Omega_8}\right)$	$\Omega_3 = \sum_{h=1}^{L} W_h \rho_{ZXh}$	$\Omega_8 = \sum_{h=1}^{L} W_h MR_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{24}) = \sum_{h=1}^{L} W_h^2\gamma_h\left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{24}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{24} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{24}) = \left(\dfrac{\phi_{24}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{24} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{24} = \dfrac{\bar{Z}}{\bar{X}}\phi_{24}$ $\phi_{24} = \Omega_3\bar{X} \big/ (\Omega_3\bar{X}+\Omega_8$

Estimators	a_{st}^{j}	b_{st}^{j}	Mean Square Errors & Biases	Ratio's
$\hat{\bar{Y}}_{G1,st}^{25} = \bar{z}_{st}\left(\dfrac{\bar{X}+\Omega_9}{\bar{x}_{st}+\Omega_9}\right)$	1	$\Omega_9 = \sum_{h=1}^{L} W_h HL_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{25}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{25}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{25} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{25}) = \left(\dfrac{\phi_{25}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{25} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{25} = \dfrac{\bar{Z}}{\bar{X}}\phi_{25}$ $\phi_{25} = \bar{X} \big/ (\bar{X}+\Omega_9)$
$\hat{\bar{Y}}_{G1,st}^{26} = \bar{z}_{st}\left(\dfrac{\Omega_1\bar{X}+\Omega_9}{\Omega_1\bar{x}_{st}+\Omega_9}\right)$	$\Omega_1 = \sum_{h=1}^{L} W_h C_{Xh}$	$\Omega_9 = \sum_{h=1}^{L} W_h HL_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{26}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{26}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{26} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{26}) = \left(\dfrac{\phi_{26}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{26} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{26} = \dfrac{\bar{Z}}{\bar{X}}\phi_{26}$ $\phi_{26} = \Omega_1\bar{X} \big/ (\Omega_1\bar{X}+\Omega_9$
$\hat{\bar{Y}}_{G1,st}^{27} = \bar{z}_{st}\left(\dfrac{\Omega_3\bar{X}+\Omega_9}{\Omega_3\bar{x}_{st}+\Omega_9}\right)$	$\Omega_3 = \sum_{h=1}^{L} W_h \rho_{ZXh}$	$\Omega_9 = \sum_{h=1}^{L} W_h HL_h(x)$	$MSE^*(\hat{\bar{Y}}_{G1,st}^{27}) = \sum_{h=1}^{L} W_h^2 \gamma_h \left(\dfrac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^{27}\dfrac{S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{27} - 2\beta_{ZXh}\theta_{Xh}\right)\right)$ $Bias^*(\hat{\bar{Y}}_{G1,st}^{27}) = \left(\dfrac{\phi_{27}}{\bar{X}}\right)\sum_{h=1}^{L}\dfrac{W_h^2\gamma_h S_{Xh}^2}{\theta_{Xh}}\left(R_{st}^{27} - \beta_{ZXh}\theta_{Xh}\right)$	$R_{st}^{27} = \dfrac{\bar{Z}}{\bar{X}}\phi_{27}$ $\phi_{27} = \Omega_3\bar{X} \big/ (\Omega_3\bar{X}+\Omega_9$

Various auxiliary variable attributes used in Table 7.1 are described in the Appendix-B.

7.5 EFFICIENCY COMPARISON

To check the efficiency of the proposed generalized RRT ratio estimator ($\hat{\hat{Y}}^{j}_{G1,st}$) against combined RRT mean estimator ($\bar{z}_{st}$), the mathematical conditions have been derived by using the MSE expressions in Equations (7.4.1) and (3.3.10). These conditions are given by:

$$MSE^{*}(\hat{\hat{Y}}^{j}_{G1,st}) \le Var^{*}(\bar{z}_{st})$$

if

$$\sum_{h=1}^{L} W_h^2 \gamma_h \left(\frac{S_{Zh}^2}{\theta_{Zh}} + R_{st}^j \frac{S_{Xh}^2}{\theta_{Xh}} \left(R_{st}^j - 2\beta_{ZXh}\theta_{Xh} \right) \right) \le \sum_{h=1}^{L} \frac{W_h^2 \gamma_h S_{Zh}^2}{\theta_{Zh}}$$

or if

$$\sum_{h=1}^{L} W_h^2 \gamma_h \left(R_{st}^j \frac{S_{Xh}^2}{\theta_{Xh}} \left(R_{st}^j - 2\frac{S_{ZXh}}{S_{Xh}^2}\theta_{Xh} \right) \right) < 0$$

or if

$$R_{st}^j < 2\frac{\bar{Z}}{\bar{X}}\frac{V'_{11}}{V'_{02}}$$

or if

$$R_{st}^j < 2\frac{A_1}{A_2}. \tag{7.5.1}$$

where $A_1 = \sum_{h=1}^{L} W_h^2 \gamma_h S_{ZXh}$ and $A_2 = \sum_{h=1}^{L} W_h^2 \gamma_h \frac{S_{Xh}^2}{\theta_{Xh}}$.

The generalized combined RRT ratio estimator $\hat{\hat{Y}}^{j}_{G1,st}$ will be more efficient than the combined RRT mean estimator $\bar{z}_{st}$ under the condition (7.5.1). When the data are recorded without measurement error, the corresponding condition may be obtained by setting $S_{Vh}^2 = 0$.

7.6 SIMULATION RESULTS

In this section, we conduct a simulation study with particular focus on the following two issues:

a. How does the generalized RRT estimator ($\hat{\hat{Y}}_{Gi,st}$) and the generalized RRT ratio estimator ($\hat{\hat{Y}}^{j}_{G1,st}$) compare with the combined RRT mean estimator ($\bar{z}_{st}$) in the presence and absence of measurement errors?

b. How are the MSE, PRE and bias, influenced with the contribution of measurement errors?

We consider two finite sub-populations of size 1000 each from bivariate normal populations having different means and covariance matrices to represent the distribution of sensitive study variable (Y) and auxiliary variable (X). The scrambling variable S is taken to be a normal variate with mean equal to zero and standard deviation ($\sigma_s = 0.1\sigma_x$). The true response is given by $Z=Y+S$. Both populations have theoretical mean μ and covariance matrices $\sum$ as given below."

Population-I **Population-II**

$$\mu = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \ \sum = \begin{bmatrix} 9 & 3.2 \\ 3.2 & 4 \end{bmatrix}, \ \rho_{yx} = 0.5139 \qquad \mu = \begin{bmatrix} 9 \\ 5 \end{bmatrix}, \ \sum = \begin{bmatrix} 16 & 6.26 \\ 6.26 & 5 \end{bmatrix}, \ \rho_{yx} = 0.7203$$

For both populations, the methodology used to get the observed values of y, x and s in h^{th} stratum is $y_{hi} = Y_{hi} + Q_{1i}$, $x_{hi} = X_{hi} + Q_{2i}$ and $s_{hi} = S_{hi} + Q_{3i}$, where Q_{1i}, Q_{2i}, Q_{3i} are independent $N(0,1)$ random variables . The observed response is given by $z_{hi} = y_{hi} + s_{hi}$.

Each population is divided in two strata of equal sizes using ordered values of the auxiliary variable. We have considered two choices for sample size, namely n= 150, and 300. The following steps were used in a R-program:

Step 1: Twenty thousand samples of size n were selected from both populations, using (simple random sampling without replacement) in each stratum.

Step 2: Using the data from Step 1, twenty thousand values of an estimator (say $\hat{\bar{Y}}_{st}^*$) are obtained for each sample size.

Step 3: The empirical MSE of the estimators is computed by

$$EMSE(\hat{\bar{Y}}_{st}^*) = \frac{1}{20,000} \sum_{L=1}^{20,000} \left(\hat{\bar{Y}}_{st}^* - \bar{Y} \right)^2 ,$$

where $\hat{\bar{Y}}_{st}^{*}$ represents an estimator deduced from (7.3.1) and $\bar{Y}$ is the population mean of the sensitive study variable. The following expression is used to calculate the percent relative efficiency (PRE) of proposed RRT mean estimators as compared to the combined RRT mean estimator $(\bar{z}_{st})$.

$$PRE = \frac{VAR^{*}(\bar{z}_{st})}{MSE^{*}(\hat{\bar{Y}}_{st}^{*})} * 100$$

The MSE's, PRE's and biases of the estimators for both populations based on different sample sizes are presented in Tables (7.2-7.5).

Table 7.2
Theoretical (boldface) and Empirical MSE's, PRE's (with/without ME) of the RRT Estimators Relative to Combined RRT Mean Estimator in Stratified Random Sampling for Population-I
$(\rho_{xy1} = 0.5201, \rho_{xy2} = 0.4930)$

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\bar{z}_{st}$	150	**0.0408** 0.0410	**0.0111** 0.0132	**0.0519** 0.0542	100.0000	100.0000
	300	**0.0168** 0.0167	**0.0043** 0.0049	**0.0211** 0.0216	100.0000	100.0000
$\hat{\bar{Y}}_{Gi,st}$	150	**0.0364** 0.0364	**0.0100** 0.0119	**0.0464** 0.0483	112.0900	111.8500
	300	**0.0149** 0.0156	**0.0040** 0.0045	**0.0189** 0.0201	112.7500	111.6400
$\hat{\bar{Y}}_{G1,st}^{1}$	150	**0.0369** 0.0367	**0.0168** 0.0204	**0.0537** 0.0571	110.5700	96.6500
	300	**0.0151** 0.0152	**0.0070** 0.0068	**0.0221** 0.0220	111.2600	95.4800
$\hat{\bar{Y}}_{G1,st}^{2}$	150	**0.0367** 0.0362	**0.0159** 0.0161	**0.0526** 0.0523	110.9747	97.9672
	300	**0.0151** 0.0150	**0.0071** 0.0064	**0.0223** 0.0214	110.9757	97.1211
$\hat{\bar{Y}}_{G1,st}^{3}$	150	**0.0365** 0.0361	**0.0137** 0.0142	**0.0502** 0.0503	111.7645	102.7292
	300	**0.0150** 0.0149	**0.0062** 0.0056	**0.0212** 0.0205	111.7655	102.1215

Estimators	***n***	**Mean Square Error**			**Percent Relative Efficiency**	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}_{G1,st}^{4}$	150	**0.0368** 0.0364	**0.0162** 0.0164	**0.0530** 0.0527	110.7573	97.3544
	300	**0.0152** 0.0151	**0.0073** 0.0064	**0.0224** 0.0215	110.7583	96.4859
$\hat{\bar{Y}}_{G1,st}^{5}$	150	**0.0376** 0.0374	**0.0116** 0.0124	**0.0492** 0.0497	108.4103	104.8733
	300	**0.0155** 0.0153	**0.0052** 0.0051	**0.0207** 0.0204	108.4110	104.6275
$\hat{\bar{Y}}_{G1,st}^{6}$	150	**0.0367** 0.0361	**0.0157** 0.0160	**0.0524** 0.0521	111.0903	98.3159
	300	**0.0151** 0.0150	**0.0071** 0.0063	**0.0222** 0.0213	111.0913	97.4833
$\hat{\bar{Y}}_{G1,st}^{7}$	150	**0.0365** 0.0365	**0.0142** 0.0150	**0.0506** 0.0515	111.7976	101.8332
	300	**0.0150** 0.0151	**0.0064** 0.0061	**0.0214** 0.0212	111.7986	101.1680
$\hat{\bar{Y}}_{G1,st}^{8}$	150	**0.0365** 0.0361	**0.0150** 0.0154	**0.0515** 0.0515	111.5688	100.0869
	300	**0.0150** 0.0150	**0.0067** 0.0062	**0.0218** 0.0211	111.5698	99.3297
$\hat{\bar{Y}}_{G1,st}^{9}$	150	**0.0368** 0.0364	**0.0161** 0.0163	**0.0529** 0.0526	110.8262	97.5433
	300	**0.0151** 0.0151	**0.0072** 0.0064	**0.0224** 0.0214	110.8272	96.6816
$\hat{\bar{Y}}_{G1,st}^{10}$	150	**0.0372** 0.0370	**0.0119** 0.0126	**0.0491** 0.0496	109.6253	104.9801
	300	**0.0153** 0.0151	**0.0053** 0.0051	**0.0207** 0.0203	109.6263	104.6588
$\hat{\bar{Y}}_{G1,st}^{11}$	150	**0.0369** 0.0363	**0.0164** 0.0166	**0.0533** 0.0528	110.5076	96.7028
	300	**0.0152** 0.0151	**0.0074** 0.0066	**0.0226** 0.0216	110.5085	95.8117
$\hat{\bar{Y}}_{G1,st}^{12}$	150	**0.0371** 0.0376	**0.0130** 0.0118	**0.0501** 0.0494	109.9433	104.6737
	300	**0.0153** 0.0157	**0.0054** 0.0054	**0.0207** 0.0211	109.9432	104.5977
$\hat{\bar{Y}}_{G1,st}^{13}$	150	**0.0370** 0.0368	**0.0122** 0.0129	**0.0492** 0.0496	110.2625	104.8649

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
	300	**0.0152** 0.0151	**0.0055** 0.0052	**0.0207** 0.0202	110.2634	104.4931
$\hat{\hat{Y}}^{14}_{G1,st}$	150	**0.0389** 0.0388	**0.0110** 0.0118	**0.0499** 0.0506	104.7124	103.2915
	300	**0.0160** 0.0158	**0.0049** 0.0049	**0.0210** 0.0207	104.7128	103.1925
$\hat{\hat{Y}}^{15}_{G1,st}$	150	**0.0383** 0.0364	**0.0121** 0.0136	**0.0504** 0.0500	106.3131	104.0079
	300	**0.0158** 0.0161	**0.0050** 0.0052	**0.0208** 0.0213	106.3130	103.9830
$\hat{\hat{Y}}^{16}_{G1,st}$	150	**0.0365** 0.0364	**0.0133** 0.0139	**0.0498** 0.0503	111.6104	103.4523
	300	**0.0150** 0.0150	**0.0060** 0.0055	**0.0210** 0.0204	111.6114	102.9000
$\hat{\hat{Y}}^{17}_{G1,st}$	150	**0.0365** 0.0359	**0.0146** 0.0149	**0.0511** 0.0508	111.7136	102.6275
	300	**0.0150** **0.0148**	**0.0061** **0.0062**	**0.0211** **0.0210**	111.7134	102.4751
$\hat{\hat{Y}}^{18}_{G1,st}$	150	**0.0367** 0.0363	**0.0157** 0.0159	**0.0524** 0.0522	111.1444	98.4861
	300	**0.0151** 0.0150	**0.0070** 0.0062	**0.0221** 0.0212	111.1454	97.6601
$\hat{\hat{Y}}^{19}_{G1,st}$	150	**0.0370** 0.0368	**0.0122** 0.0129	**0.0492** 0.0496	110.2685	104.8630
	300	**0.0152** 0.0151	**0.0055** 0.0052	**0.0207** 0.0202	110.2694	104.4906
$\hat{\hat{Y}}^{20}_{G1,st}$	150	**0.0389** 0.0388	**0.0110** 0.0118	**0.0499** 0.0506	104.7196	103.2957
	300	**0.0160** 0.0158	**0.0049** 0.0049	**0.0210** 0.0207	104.7200	103.1965
$\hat{\hat{Y}}^{21}_{G1,st}$	150	**0.0385** 0.0384	**0.0111** 0.0119	**0.0496** 0.0503	105.9405	103.9561
	300	**0.0158** 0.0156	**0.0050** 0.0049	**0.0208** 0.0205	105.9412	103.8178
$\hat{\hat{Y}}^{22}_{G1,st}$	150	**0.0371** 0.0369	**0.0121** 0.0128	**0.0491** 0.0496	110.0091	104.9302
	300	**0.0153** 0.0151	**0.0054** 0.0051	**0.0207** 0.0202	110.0100	104.5797

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\hat{\bar{Y}}_{G1,st}^{23}$	150	**0.0390** 0.0389	**0.0110** 0.0118	**0.0500** 0.0507	104.4259	103.1220
	300	**0.0161** 0.0158	**0.0049** 0.0049	**0.0210** 0.0207	104.4262	103.0311
$\hat{\bar{Y}}_{G1,st}^{24}$	150	**0.0386** 0.0385	**0.0111** 0.0119	**0.0497** 0.0504	105.6088	103.7873
	300	**0.0159** 0.0157	**0.0050** 0.0049	**0.0209** 0.0206	105.6094	103.6604
$\hat{\bar{Y}}_{G1,st}^{25}$	150	**0.0370** 0.0368	**0.0122** 0.0129	**0.0492** 0.0496	110.2683	104.8631
	300	**0.0152** 0.0151	**0.0055** 0.0052	**0.0207** 0.0202	110.2692	104.4907
$\hat{\bar{Y}}_{G1,st}^{26}$	150	**0.0389** 0.0388	**0.0110** 0.0118	**0.0499** 0.0506	104.7194	103.2956
	300	**0.0160** 0.0158	**0.0049** 0.0049	**0.0210** 0.0207	104.7198	103.1964
$\hat{\bar{Y}}_{G1,st}^{27}$	150	**0.0385** 0.0384	**0.0111** 0.0119	**0.0496** 0.0503	105.9403	103.9560
	300	**0.0158** 0.0156	**0.0050** 0.0049	**0.0208** 0.0205	105.9409	103.8177

Table 7.3
Theoretical Biases (with/without ME) of the RRT Estimators in Stratified Random Sampling for Population-I ($\rho_{xy1} = 0.5201, \rho_{xy2} = 0.4930$)

Estimators	bias (without ME)		bias (with ME)	
	n		n	
	150	**300**	**150**	**300**
$\bar{z}_{st}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{Gi,st}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{G1,st}^{1}$	0.0004	0.0001	0.0015	0.0006
$\hat{\bar{Y}}_{G1,st}^{2}$	0.0003	0.0001	0.0013	0.0006
$\hat{\bar{Y}}_{G1,st}^{3}$	0.0000	0.0000	0.0005	0.0002
$\hat{\bar{Y}}_{G1,st}^{4}$	0.0003	0.0001	0.0014	0.0006

Estimators	bias (without ME)		bias (with ME)	
	n		n	
	150	300	150	300
$\hat{\bar{Y}}_{G1,st}^{5}$	-0.0002	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{6}$	0.0002	0.0001	0.0012	0.0005
$\hat{\bar{Y}}_{G1,st}^{7}$	0.0000	0.0000	0.0007	0.0003
$\hat{\bar{Y}}_{G1,st}^{8}$	0.0001	0.0001	0.0010	0.0004
$\hat{\bar{Y}}_{G1,st}^{9}$	0.0003	0.0001	0.0013	0.0006
$\hat{\bar{Y}}_{G1,st}^{10}$	-0.0002	-0.0001	0.0000	0.0000
$\hat{\bar{Y}}_{G1,st}^{11}$	0.0004	0.0001	0.0015	0.0006
$\hat{\bar{Y}}_{G1,st}^{12}$	-0.0002	-0.0001	0.0000	0.0000
$\hat{\bar{Y}}_{G1,st}^{13}$	-0.0002	-0.0001	0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{14}$	-0.0002	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{15}$	-0.0002	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{16}$	-0.0001	0.0000	0.0004	0.0002
$\hat{\bar{Y}}_{G1,st}^{17}$	-0.0001	0.0000	0.0005	0.0002
$\hat{\bar{Y}}_{G1,st}^{18}$	0.0002	0.0001	0.0012	0.0005
$\hat{\bar{Y}}_{G1,st}^{19}$	-0.0002	-0.0001	0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{20}$	-0.0002	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{21}$	-0.0002	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{22}$	-0.0002	-0.0001	0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{23}$	-0.0001	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{24}$	-0.0002	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{25}$	-0.0002	-0.0001	0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{26}$	-0.0002	-0.0001	-0.0001	0.0000
$\hat{\bar{Y}}_{G1,st}^{27}$	-0.0002	-0.0001	-0.0001	0.0000

Table 7.4

Theoretical (boldface) and Empirical MSE's, PRE's (with/without ME) of the RRT Estimators Relative to Combined RRT Mean Estimator in Stratified Random Sampling for Population-II

($\rho_{xy1} = 0.6321, \rho_{xy2} = 0.7023$)

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\bar{z}_{st}$	150	**0.0633** 0.0643	**0.0109** 0.0108	**0.0742** 0.0751	100.0000	100.0000
	300	**0.0261** 0.0260	**0.0047** 0.0039	**0.0308** 0.0299	100.0000	100.0000
$\hat{\bar{Y}}_{Gi,st}$	150	**0.0450** 0.0456	**0.0081** 0.0027	**0.0531** 0.0483	140.6670	139.7360
	300	**0.0173** 0.0192	**0.0036** 0.0023	**0.0209** 0.0215	150.8700	147.3700
$\hat{\bar{Y}}^{1}_{G1,st}$	150	**0.0476** 0.0478	**0.0304** 0.0291	**0.0780** 0.0769	132.9830	95.1280
	300	**0.0196** 0.0204	**0.0128** 0.0118	**0.0324** 0.0322	133.1600	95.0600
$\hat{\bar{Y}}^{2}_{G1,st}$	150	**0.0466** 0.0468	**0.0281** 0.0329	**0.0747** 0.0797	135.9429	99.2914
	300	**0.0192** 0.0190	**0.0116** 0.0122	**0.0308** 0.0312	135.9383	100.3743
$\hat{\bar{Y}}^{3}_{G1,st}$	150	**0.0460** 0.0462	**0.0265** 0.0310	**0.0725** 0.0772	137.5703	102.2555
	300	**0.0190** 0.0188	**0.0110** 0.0116	**0.0299** 0.0304	137.5654	103.2599
$\hat{\bar{Y}}^{4}_{G1,st}$	150	**0.0459** 0.0571	**0.0260** 0.0402	**0.0719** 0.0973	138.0417	103.2390
	300	**0.0189** 0.0235	**0.0108** 0.0158	**0.0297** 0.0393	138.0367	104.2147
$\hat{\bar{Y}}^{5}_{G1,st}$	150	**0.0452** 0.0475	**0.0204** 0.0272	**0.0656** 0.0747	140.0442	113.0388
	300	**0.0186** 0.0195	**0.0086** 0.0106	**0.0272** 0.0301	140.0389	113.6290
$\hat{\bar{Y}}^{6}_{G1,st}$	150	**0.0461** 0.0463	**0.0267** 0.0313	**0.0728** 0.0776	137.3633	101.8456
	300	**0.0190** 0.0188	**0.0111** 0.0117	**0.0301** 0.0305	137.3583	102.8612
$\hat{\bar{Y}}^{7}_{G1,st}$	150	**0.0452** 0.0477	**0.0208** 0.0278	**0.0660** 0.0755	140.1306	112.4111

Estimators	*n*	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
	300	**0.0186** 0.0196	**0.0087** 0.0108	**0.0273** 0.0304	140.1254	113.0332
$\hat{\bar{Y}}^{8}_{G1,st}$	150	**0.0460** 0.0462	**0.0265** 0.0308	**0.0725** 0.0770	137.6089	102.3334
	300	**0.0190** 0.0188	**0.0110** 0.0115	**0.0299** 0.0302	137.6041	103.3357
$\hat{\bar{Y}}^{9}_{G1,st}$	150	**0.0454** 0.0542	**0.0237** 0.0360	**0.0691** 0.0902	139.5935	107.4167
	300	**0.0187** 0.0223	**0.0099** 0.0142	**0.0286** 0.0365	139.5882	108.2531
$\hat{\bar{Y}}^{10}_{G1,st}$	150	**0.0454** 0.0456	**0.0240** 0.0281	**0.0695** 0.0736	139.4141	106.7945
	300	**0.0187** 0.0186	**0.0100** 0.0105	**0.0287** 0.0291	139.4090	107.6542
$\hat{\bar{Y}}^{11}_{G1,st}$	150	**0.0476** 0.0478	**0.0306** 0.0357	**0.0782** 0.0835	133.0400	94.9195
	300	**0.0196** 0.0194	**0.0126** 0.0132	**0.0322** 0.0326	133.0398	96.0988
$\hat{\bar{Y}}^{12}_{G1,st}$	150	**0.0452** 0.0450	**0.0215** 0.0192	**0.0666** 0.0642	140.1807	111.3412
	300	**0.0186** 0.0183	**0.0090** 0.0106	**0.0276** 0.0289	140.1754	112.0155
$\hat{\bar{Y}}^{13}_{G1,st}$	150	**0.0470** 0.0470	**0.0158** 0.0187	**0.0628** 0.0657	134.8189	118.1418
	300	**0.0193** 0.0193	**0.0068** 0.0073	**0.0261** 0.0266	134.8145	118.3206
$\hat{\bar{Y}}^{14}_{G1,st}$	150	**0.0542** 0.0534	**0.0118** 0.0139	**0.0659** 0.0673	116.9124	112.4851
	300	**0.0223** 0.0221	**0.0052** 0.0058	**0.0275** 0.0279	116.9105	112.3920
$\hat{\bar{Y}}^{15}_{G1,st}$	150	**0.0531** 0.0518	**0.0121** 0.0105	**0.0651** 0.0623	119.3491	113.8750
	300	**0.0219** 0.0208	**0.0053** 0.0056	**0.0272** 0.0264	119.3469	113.7873
$\hat{\bar{Y}}^{16}_{G1,st}$	150	**0.0509** 0.0503	**0.0129** 0.0163	**0.0638** 0.0666	124.4780	116.3241
	300	**0.0210** 0.0208	**0.0056** 0.0066	**0.0266** 0.0274	124.4751	116.2718

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	**Change due to ME**	**With ME**	**Without ME**	**With ME**
$\hat{\bar{Y}}_{G1,st}^{17}$	150	**0.0455** **0.0428**	**0.0187** **0.0193**	**0.0642** **0.0621**	139.1269	115.4825
	300	**0.0187** 0.0179	**0.0079** 0.0086	**0.0267** 0.0265	139.1218	115.9270
$\hat{\bar{Y}}_{G1,st}^{18}$	150	**0.0452** 0.0511	**0.0208** 0.0306	**0.0660** 0.0817	140.1271	112.4455
	300	**0.0186** 0.0210	**0.0087** 0.0121	**0.0273** 0.0332	140.1218	113.0663
$\hat{\bar{Y}}_{G1,st}^{19}$	150	**0.0470** 0.0470	**0.0158** 0.0187	**0.0628** 0.0657	134.7232	118.1581
	300	**0.0194** 0.0193	**0.0068** 0.0073	**0.0261** 0.0266	134.7188	118.3331
$\hat{\bar{Y}}_{G1,st}^{20}$	150	**0.0542** 0.0534	**0.0118** 0.0139	**0.0660** 0.0673	116.8034	112.4199
	300	**0.0223** 0.0221	**0.0052** 0.0058	**0.0275** 0.0279	116.8016	112.3268
$\hat{\bar{Y}}_{G1,st}^{21}$	150	**0.0501** 0.0502	**0.0133** 0.0156	**0.0634** 0.0658	126.4298	117.0517
	300	**0.0206** 0.0207	**0.0058** 0.0063	**0.0264** 0.0270	126.4262	117.0231
$\hat{\bar{Y}}_{G1,st}^{22}$	150	**0.0469** 0.0469	**0.0160** 0.0189	**0.0628** 0.0658	135.1582	118.0754
	300	**0.0193** 0.0192	**0.0068** 0.0074	**0.0261** 0.0266	135.1537	118.2682
$\hat{\bar{Y}}_{G1,st}^{23}$	150	**0.0540** 0.0532	**0.0118** 0.0140	**0.0658** 0.0672	117.3095	112.7203
	300	**0.0222** 0.0220	**0.0052** 0.0059	**0.0274** 0.0279	117.3076	112.6277
$\hat{\bar{Y}}_{G1,st}^{24}$	150	**0.0499** 0.0500	**0.0134** 0.0158	**0.0633** 0.0658	127.0124	117.2429
	300	**0.0205** 0.0206	**0.0058** 0.0064	**0.0264** 0.0270	127.0087	117.2228
$\hat{\bar{Y}}_{G1,st}^{25}$	150	**0.0470** 0.0471	**0.0158** 0.0187	**0.0628** 0.0657	134.7020	118.1616
	300	**0.0194** 0.0193	**0.0068** 0.0073	**0.0261** 0.0266	134.6976	118.3358
$\hat{\bar{Y}}_{G1,st}^{26}$	150	**0.0542** 0.0534	**0.0118** 0.0139	**0.0660** 0.0673	116.7794	112.4055

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
	300	**0.0223** 0.0221	**0.0052** 0.0058	**0.0275** 0.0279	116.7776	112.3124
$\hat{\bar{Y}}^{27}_{G1,st}$	150	**0.0501** 0.0502	**0.0133** 0.0156	**0.0634** 0.0658	126.4018	117.0422
	300	**0.0206** 0.0207	**0.0058** 0.0063	**0.0264** 0.0270	126.3983	117.0132

Table 7.5
Theoretical Biases (with/without ME) of the RRT Estimators in Stratified Random Sampling for Population-II
($\rho_{xy1} = 0.6321, \rho_{xy2} = 0.7023$)

Estimators	bias (without ME)		bias (with ME)	
	n		n	
	150	300	150	300
$\bar{z}_{st}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}_{Gi,st}$	0.0000	0.0000	0.0000	0.0000
$\hat{\bar{Y}}^{1}_{G1,st}$	0.0010	0.0004	0.0032	0.0013
$\hat{\bar{Y}}^{2}_{G1,st}$	0.0007	0.0003	0.0027	0.0011
$\hat{\bar{Y}}^{3}_{G1,st}$	0.0005	0.0002	0.0023	0.0009
$\hat{\bar{Y}}^{4}_{G1,st}$	0.0005	0.0002	0.0022	0.0009
$\hat{\bar{Y}}^{5}_{G1,st}$	-0.0001	0.0000	0.0010	0.0004
$\hat{\bar{Y}}^{6}_{G1,st}$	0.0006	0.0002	0.0024	0.0009
$\hat{\bar{Y}}^{7}_{G1,st}$	-0.0001	0.0000	0.0011	0.0004
$\hat{\bar{Y}}^{8}_{G1,st}$	0.0005	0.0002	0.0023	0.0009
$\hat{\bar{Y}}^{9}_{G1,st}$	0.0002	0.0001	0.0017	0.0007
$\hat{\bar{Y}}^{10}_{G1,st}$	0.0003	0.0001	0.0018	0.0007
$\hat{\bar{Y}}^{11}_{G1,st}$	0.0010	0.0004	0.0032	0.0013
$\hat{\bar{Y}}^{12}_{G1,st}$	0.0000	0.0000	0.0012	0.0005

Estimators	bias (without ME)		bias (with ME)	
	n		n	
	150	**300**	**150**	**300**
$\hat{\hat{Y}}_{G1,st}^{13}$	-0.0004	-0.0002	0.0001	0.0000
$\hat{\hat{Y}}_{G1,st}^{14}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1,st}^{15}$	-0.0005	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1,st}^{16}$	-0.0005	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1,st}^{17}$	-0.0002	-0.0001	0.0006	0.0002
$\hat{\hat{Y}}_{G1,st}^{18}$	-0.0001	0.0000	0.0011	0.0004
$\hat{\hat{Y}}_{G1,st}^{19}$	-0.0004	-0.0002	0.0001	0.0000
$\hat{\hat{Y}}_{G1,st}^{20}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1,st}^{21}$	-0.0005	-0.0002	-0.0002	-0.0001
$\hat{\hat{Y}}_{G1,st}^{22}$	-0.0004	-0.0002	0.0001	0.0000
$\hat{\hat{Y}}_{G1,st}^{23}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1,st}^{24}$	-0.0005	-0.0002	-0.0002	-0.0001
$\hat{\hat{Y}}_{G1,st}^{25}$	-0.0004	-0.0002	0.0001	0.0000
$\hat{\hat{Y}}_{G1,st}^{26}$	-0.0004	-0.0002	-0.0003	-0.0001
$\hat{\hat{Y}}_{G1,st}^{27}$	-0.0005	-0.0002	-0.0002	-0.0001

Tables 7.2-7.5 give the contribution of measurement error, the amount of MSE's, PRE's and biases for the RRT estimators $(\bar{z}_{st}),(\hat{\hat{Y}}_{Gi,st})$ and $(\hat{\hat{Y}}_{G1,st}^{j})$ based on different sample sizes. It is clear from the simulation results that measurement errors play a significant role in increasing the MSE of an estimator. An important observation is that the ratio estimator is impacted a lot more because of the measurement error in the auxiliary variable but the proposed estimator is not, even though the auxiliary variable is used here as well. Also note that in general the ratio estimator performs better than the ordinary mean estimator when there are no measurement errors and there is high correlation between the study variable and the auxiliary variable, but this is not so when we factor in measurement errors. The reduction in PRE for the series of RRT ratio estimators $(\hat{\hat{Y}}_{G1,st}^{j})$, when measurement errors occur, is much higher as compared to proposed generalized RRT estimator $(\hat{\hat{Y}}_{Gi,st})$. The

performance of the proposed generalized RRT estimator is better than the existing RRT estimators both in the presence and absence of measurement errors, more so in the high correlation case as compared to the low correlation case.

7.7 NUMERICAL EXAMPLE

Data Statistics

For the numerical example, we generate a data set which has the same characteristics of the real data set used by Sousa et al. (2014) which was based on 2010 survey of Information and Communication Technologies (ICT) usage in enterprises with seat in Portugal (Smilhily and Storm, 2010). The following notations are used below:

Y = True Purchase orders, X = True Turnover of enterprises
y = Measured Purchase orders, x = Measured Turnover of enterprises

Let S be a scrambling random variable with distribution $S \sim N(0, 0.1\sigma_X)$, just as in Sousa et al. (2014). The scrambled response on purchase orders Y is given by $Z = Y + S$. We consider sample sizes: n = 250, 500. The methodology used to get the observed values of z, x and s in hth stratum is same as described in the in the simulation section above. We generated a bivariate normal population of size 1698 with these parameters.

Table 7.6
Summary Statistics for the Numerical Example

Stratum (h)	N_h	ρ_{yxh}	$\bar{Y}_h$	S_{Yh}	$\bar{X}_h$	S_{Xh}	**Population**
1	979	0.7802	2.15	2.46	3.12	2.68	N=1698, $\rho_{yx} = 0.9368$
2	362	0.7952	16.67	6.86	20.31	6.02	$\bar{Y} = 14.44$, $\bar{X} = 17.97$
3	357	0.8408	45.88	30.21	56.33	30.18	$S_y = 22.39$, $S_x = 25.31$

Table 7.7
Theoretical MSE's, PRE's (with/without ME) of the RRT Estimators Relative to Combined RRT Mean Estimator in Stratified Random Sampling for the Numerical Example

Estimators	n	Mean Square Error			Percent Relative Efficiency	
		Without ME	Change due to ME	With ME	Without ME	With ME
$\bar{z}_{st}$ (Combined RRT Mean Estimator)	250	0.6938	0.0068	0.7006	100.0000	100.0000
	500	0.3001	0.0027	0.3028	100.0000	100.0000
$\hat{\bar{Y}}_{Gi,st}$ (Proposed Generalized RRT Estimator)	250	0.2119	0.0025	0.2144	327.4200	326.7700
	500	0.0916	0.0010	0.0926	327.6200	327.0000
$\hat{\bar{Y}}^{1}_{G1,st}$ (Combined RRT ratio Estimator)	250	0.2155	0.0090	0.2245	321.9500	312.0700
	500	0.0920	0.0037	0.0957	326.2000	316.4100

Table 7.7 shows that our proposed generalized RRT estimator ($\hat{\bar{Y}}_{Gi,st}$) performs better than existing RRT estimators both in the presence and absence of measurement errors. We saw the same pattern in Tables (7.2-7.5). Also presence of measurement errors impacts the proposed generalized RRT estimator ($\hat{\bar{Y}}_{Gi,st}$) less than the combined RRT ratio estimator ($\hat{\bar{Y}}^{1}_{G1,st}$) .

7.8 CONCLUSION

The main contribution of this chapter is the introduction of a generalized RRT mean estimator in the presence of measurement errors in stratified random sampling. The generalized RRT estimator leads to several combined RRT ratio and combined RRT product estimators as special cases. The usual transformed RRT ratio estimator $\hat{\bar{Y}}_{G1,st}$ has been studied by using various transformations of the parameters of the auxiliary variable. The asymptotic bias and MSE formulae have been derived. The results are validated through a simulation and numerical study. Tables (7.2-7.5) show the effect of measurement errors on the proposed RRT estimators using different sampling fractions. As expected, the theoretical and empirical mean square errors are quite close. Simulation results show that all of the estimators are more efficient as compared to the usual combined RRT mean estimator. Improvement in efficiency is more significant in the high correlation case as compared to the low correlation case. Unlike the combined ratio estimator, the proposed estimator is not adversely affected by the extra source of measurement errors through the auxiliary variable. It may also be observed that the optimal MSE of the proposed generalized RRT estimator matches the MSE of the combined RRT regression estimator but it has the added attraction of being able to produce many existing RRT estimators depending on how the auxiliary information is exploited. Also it is obvious from the simulation and numerical results that measurement errors hurt the efficiency of all estimators Thus there is a need to make a considerable effort to eliminate measurement errors in the survey.

CHAPTER 8

CONCLUDING REMARKS AND FUTURE DIRECTIONS

8.1 CONCLUDING REMARKS

It is obvious from this study that the MSE of mean estimators increases when measurement errors are present as compared to when there are no measurement errors. Also the impact of measurement errors is more when study variable is sensitive as compared to when it is not, largely because there is an extra source of measurement error when dealing with sensitive variables. A very significant observation is that ratio estimators which do better than ordinary mean estimator in the absence of measurement errors, perform less efficiently when measurement errors are factored in.

The most critical finding of this thesis is that both in Simple Random Sampling and Stratified Random Sampling, our proposed generalized estimator performs better than both the simple mean estimator and the ratio estimator.

Another crucial point is that the proposed generalized estimator encompasses a wide variety of existing ratio and product, mean estimators depending on how the auxiliary information is exploited.

Yet another important observation is that superiority of our proposed generalized estimator is not conditional; instead it is always superior to the simple mean estimator and the ratio estimator. Finally we may point out that the simulation and numerical results clearly validate the theoretical findings.

8.2 FUTURE DIRECTIONS

Although research is a lifelong pursuit, two problems that come to mind for near term effort are:

(i) Examining the impact of measurement errors in the context of Optional RRT Models. In particular, we will examine if optionality reduces the impact of measurement errors.

(ii) Examining the impact of measurement errors on variance estimation.

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APPENDIX-A

VARIOUS NOTATIONS USED IN GENERALIZED RATIO ESTIMATORS IN SIMPLE RANDOM SAMPLING

$C_X = \frac{S_X}{\bar{X}}$	Coefficient of variation of auxiliary variable
$\beta_1(x) = \frac{N\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^3}{\left(N-1\right)\left(N-2\right)S_x^3}$	Coefficient of skewness of auxiliary variable
$\beta_2(x) = \frac{N\left(N+1\right)\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^3}{\left(N-1\right)\left(N-2\right)\left(N-3\right)S_x^4} - \frac{3\left(N-1\right)^2}{\left(N-2\right)\left(N-3\right)}$	Coefficient of kurtosis of auxiliary variable
$\rho_{XY} = \frac{S_{XY}}{S_X S_Y}$	Population Correlation Coefficient between *Y* and *X*
$\rho_{ZX} = \frac{\rho_{XY}}{\sqrt{1+\frac{S_S^2}{S_Y^2}}}$	Population Correlation Coefficient between *Z* and *X*
$Q_2 = \frac{(n+1)}{2}^{th}$ value	Median of auxiliary variable
$QD = \frac{(Q_3 - Q_1)}{2}$	Quartile Deviation of auxiliary variable
$TM = \frac{(Q_1 + 2Q_2 + Q_3)}{4}$	Tri-mean of auxiliary variable
$MR = \frac{(X_{(1)} + X_{(N)})}{2}$	Mid-range of auxiliary variable
$1 \le \mathrm{j} \le \mathrm{k} \le \mathrm{N}$ $HL = median\frac{(X_{(j)} + X_{(K)})}{2}$	Hodges-Lehmann estimator of auxiliary variable

APPENDIX-B

VARIOUS NOTATIONS USED IN GENERALIZED COMBINED RATIO ESTIMATOR IN STRATIFIED RANDOM SAMPLING

$C_{Xh} = S_{Xh} / \bar{X}_h$	Co-efficient of variation of auxiliary variable in h[th] stratum
$\beta_1(X_h) = \dfrac{N_h \sum_{i=1}^{N_h} (X_{ih} - \bar{X}_h)^3}{(N_h - 1)(N_h - 2) S_{xh}^3}$	Coefficient of skewness of auxiliary variable in h[th] stratum
$\beta_2(X_h) = \dfrac{N_h (N_h + 1) \sum_{i-1}^{N_h} (X_{ih} - \bar{X}_h)^3}{(N_h - 1)(N_h - 2)(N_h - 3) S_{xh}^4} - \dfrac{3(N_h - 1)^2}{(N_h - 2)(N_h - 3)}$	Co-efficient of kurtosis of auxiliary variable in h[th] stratum
$\rho_{XYh} = \dfrac{S_{XYh}}{S_{Xh} S_{Yh}}$	Population Correlation Coefficient between *Y* and *X* in h[th] stratum
$\rho_{ZXh} = \dfrac{\rho_{XYh}}{\sqrt{1 + \dfrac{S_{Sh}^2}{S_{Yh}^2}}}$	Population Correlation Coefficient between *Z* and *X* in h[th] stratum
$Q_{2h} = \dfrac{(n+1)}{2}^{th}$ value	Median of auxiliary variable in h[th] stratum
$QD_h = \dfrac{(Q_3(X_h) - Q_1(X_h))}{2}$	Quartile Deviation of auxiliary variable in h[th] stratum
$TM_h = \dfrac{(Q_1(X_h) + 2Q_2(X_h) + Q_3(X_h))}{4}$	Tri-mean of auxiliary variable in h[th] stratum
$MR_h = \dfrac{(X_{h(1)} + X_{h(N_h)})}{2}$	Mid-range of auxiliary variable in h[th] stratum
$HL_h = median \dfrac{(X_{h(j)} + X_{h(K)})}{2}$ $1 \le j \le k \le N$	Hodges-Lehmann estimator of auxiliary Variable in h[th] stratum