National College of Business Administration and Economics Lahore

PROPERTIES OF NEW COMPOUND FAMILY OF EXPONENTIATED LINDLEY EXPONENTIAL POWER SERIES DISTRIBUTION AND APPLICATION

BY

USMAN ALI

MASTER OF PHILOSOPHY
IN
STATISTICS

SEPTEMBER, 2018

NATIONAL COLLEGE OF BUSINESS ADMINISTRATION AND ECONOMICS

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A dissertation submitted to
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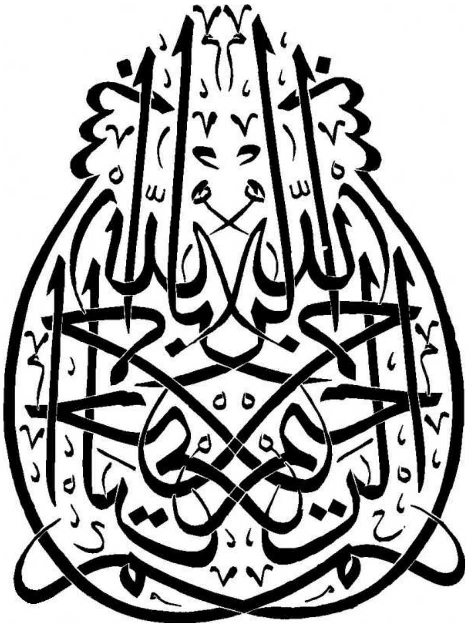

In the name of ALLAH,

The Most Beneficial,

The Most Merciful,

NATIONAL COLLEGE OF BUSINESS ADMINISTRATION AND ECONOMICS LAHORE

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Dissertation Committee:

Chairman

Member

Member

Director (Research)
National College of Business
Administration and Economics

DECLARATION

It is to declare that this research work has not been submitted for obtaining similar degree from any other university/college.

USMAN ALI
September, 2018

Dedicated
to

My Parents and my Wife
for their countless efforts
and lots of sacrifices
in developing me
what I am now.

ACKNOWLEDGEMENT

It is with profound feeling of gratitude that I express to Dr. Zafar Iqbal for his inordinate help, keen interest, skilled advice, constructive criticism and valuable guidance during the course of my research. Whenever I came across any difficulty, his sympathetic behavior and vast knowledge enabled me to get through.

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis entitled **“Properties of New Compound Family of Exponentiated Lindley Exponential Power Series Distribution and Application”** has been carried out and completed by **Usman Ali** under my supervision during his **M.Phil. Statistics** Programme.

(Dr. Zafar Iqbal)
Supervisor

SUMMARY

In 1997, some authors started to use the shape parameter(s) for the purpose of generalization of any probability distribution and such techniques are continuously in practice from the last two decades. Different distributions have been discussed in the literature to model lifetime data by compounding some useful continuous distributions with discrete distribution.

Adamidis and Loukas (1998) use the concept of compounding continues with discrete distribution.

In this thesis, a new family of lifetime distributions is introduced by compounding exponentiated Lindley distribution and power series distributions which is a quite flexible model in analyzing positive data. The ELEPS family of distributions allow for a high degree of flexibility in modeling a real data.

ABBREVIATIONS

ELEPS	Exponentiated Lindley Exponential Power Series
Pdf	Probability Density Function
Cdf	Cumulative Density Function
MLE	Method of Likelihood Estimation

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION OF COMPOUNDING DISTRIBUTIONS AND POWER SERIES DISTRIBUTION

For extracting a suitable model for the real life data has been studied massively in the literature, there are number of occasions where already developed models are not appropriate or less representative of real data, therefore, as a result to overcome this problem one needs to develop a general model. The mostly used and existing distributions are very limited in their characteristics, for example the distributions: Exponential, Rayleigh, Weibull, gamma and beta are unable to show flexibility in modeling many real life situations. In 1997, some authors prefer to use the shape parameter(s) for the purpose of generalization of any probability distribution and such techniques are continuously in practice from the last two decades. By compounding life time distributions with discrete distributions various distributions discussed in the literature for modeling lifetime data. Compounding lifetime distributions have been obtained by mixing up the distribution when the lifetime can be expressed as the minimum of a series of identically and independently distributed (iid) random variables with discrete random variable. This idea was first pioneered by Adamidis and Loukas (1998) by compounding the exponential random variable simultaneously with a geometric random variable. Several authors introduced new lifetime distributions (see for example; Kus (2007), Nadarajah et al. (2011), Barreto-Souza et al. (2011), and Lu and Shi (2012)).

In recent years, massive effort has been employed to find new compounding families of distributions by mixing power series and lifetime distributions. The new families extend some compound distributions and yield more flexibility in modeling several practical data. Some authors defined new families of lifetime distributions (see for example; exponential-power series family (Chahkandi and Ganjali (2009)), Weibull power series distributions (Morais and Barreto-Souza (2011)), generalized exponential power series family (Mahmoudi and Jafari (2012)), extended Weibull power series distributions (Silva et al. (2013)), Burr XII power series (Silva and Corderio (2015)).

In the literature, it is difficult to find an appropriate model for the real data sets and there are many position where existing models are not suitable or less representative of real data, therefore, to resolve such cases one needs to develop a general model. The existed distributions are very limited in their characteristics, for example, the distributions: exponential, Rayleigh, Weibull, gamma and beta are unable to show more flexibility in modeling many real cases. In 1997, some authors started to use the shape parameter(s) for the purpose of generalization of any probability distribution and such techniques are continuously in practice from the last two decades. Different distributions have been discussed in the literature to model lifetime data by compounding some useful continuous distributions with discrete distribution. Compounding lifetime distributions have been obtained by mixing up the distribution when the lifetime can be expressed as the minimum of a sequence of independent and identically distributed (iid) random variables with a discrete random variable. Adamidis and Loukas (1998) introduced new idea by compounding continues and power series distribution. For example; Kus (2007), Barreto-Souza et al. (2011), Lu and Shi (2012), Tahmasebi and Jafari (2015) and Silva et al. (2016) several authors introduced new lifetime distributions.

Newly, with great effort has been produced to new compounding families of distributions by mixing some beneficial lifetime and power series distributions. The new families expand some compound distributions and produce more tractability in modeling for various practical data. Some authors defined new families of lifetime distributions (see for example; exponential-power series family (Chahkandi and Ganjali (2009)), Weibull-power series distributions (Morais and Barreto-Souza (2011)), generalized exponential power series family (Mahmoudi and Jafari (2012)), extended Weibull power series distributions (Silva et al. (2013)), Asgharzadeh et al. (2014). Burr XII power series (Silva and Corderio (2015), Tahmasebi and Jafari (2015) and Silva et al. (2016).

1.2 LINDLEY DISTRIBUTION

In the context of Bayes' theorem a distribution proposed by Lindley (1958) as a counter example of fiducial statistics. Lindley distribution is mixture of exponential (β) and gamma (2, β)

Ghitany et al. (2011) proposed two-parameter weighted Lindley distribution and states its applications to survival data. Nadarajah et al. (2011) developed generalized Lindley distribution and obtained its various properties and applications. Bakouch et al. (2012) developed an extended Lindley distribution and obtained its various properties and applications. Merovci and Elbatal (2014) transmuted Lindley-geometric and its application. Oluyede and Yang (2014) developed a new class of generalized Lindley distribution called beta generalized Lindley distribution. Ashour and Eltehiwy (2014) proposed Exponentiated power Lindley distribution and discussed its application on real data set. Nedjar and Zeghdoudi (2017) give treatment of properties of pseudo Lindley distribution.

In this thesis, a new family of lifetime distributions is introduced by compounding exponentiated Lindley distribution and power series distributions that is a quite flexible model in analyzing positive data and we provide later a comprehensive description of some of its mathematical properties with the purpose that it will attract wider applications in reliability, engineering and in other areas of research.

1.3 OBJECTIVE

The objectives of this research are to

i) To find the density function as well as, cumulative, survival and hazard rate functions of ELEPS family distributions.

ii) To derive some mathematical properties such as quantile, moments, entropy and order statistics.

iii) To develop some special sub-models and also some of its mathematical properties for these special sub-models.

iv) To find maximum likelihood (ML) estimators for the unknown parameters on the basis of the family are obtained and a simulation study is carried out on the basis of ML estimates and of method of moments.

v) To rind applications to real data sets are given to show the flexibility and applicability of the proposed family of distributions.

The ELEPS family of distributions allow for a high degree of flexibility in modeling a real data. We shall see later that the ELEPS family distributions allow for major hazard shapes: increasing, decreasing and bathtub failure rates. We shall also see later that the ELE geometric distribution i.e. member of ELEPS family distributions will provide significantly better fits than Weibull, exponential and exponentiated exponential distributions for data sets.

CHAPTER 2

LITERATURE REVIEW

Adamidis and Loukas (1998) compounded the exponential random variable simultaneously with a geometric random variable and find out properties of this new distribution. Kus (2007) developed new two-parameter distribution with decreasing failure rate is introduced and discussed various properties of the introduced distribution are discussed. Barreto-Souza et al. (2011) introduced the Weibull-geometric (WG) distribution that contains the EEG, EG and Weibull distributions as special submodels and discussed some of its properties. Lu and Shi (2012) developed the Weibull–Poisson distribution and they proved the shape of failure rate function of the new compounding distribution is flexible, it can be decreasing, increasing, upside-down bathtub-shaped or unimodal. Chahkandi and Ganjali (2009) introduced a new two-parameter distribution family with decreasing failure rate arising by mixing power-series distribution and exponential distribution. This family includes some well-used mixing distributions. exponential-geometric (Adamidis et al., 2005; Adamidis and Loukas, 1998), exponential Poisson (Kus, 2007), and exponential-logarithmic (Tahmasbi and Rezaei, 2008) distributions; the Weibull-power series distributions is introduced by Morais and Barreto-Souza (2011) and is a generalization of the EPS distribution and is concluded the Weibull-geometric (Barreto-Souza et al., 2011) and Weibull-Poisson (Lu and Shi, 2011) distributions; the generalized exponential power series distribution is introduced by Mahmoudi and Jafari (2012) which is concluded the Poisson-exponential (Cancho et al., 2011; Louzada-Neto et al., 2011) complementary exponential-geometric (Louzada et al., 2011), and the complementary exponential power series (Flores et al., 2013) distributions; the extended Weibull-power series distributions (Silva et al., 2013), Birnbaum-Saunders power series distributions (Bourguignon et al., 2014), and exponentiated Weibull-Poisson distribution (Mahmoudi and Sepahdar, 2013). Similar procedures are used by Roman et al. (2012); Tojeiro et al. (2014); Louzada et al. (2014). Mahmoudi and Jafari (2012) introduced the generalized exponential power series distributions and some mathematical properties of the new class are studied, including the cumulative distribution function, density function, survival function, and hazard rate function. Sadaf (2014) introduced the moment exponential power series distributions and found its properties including the Shannon entropy, Reneyi entropy, survival function and hazard rate function. Silva and Corderio (2015) proposed a new compound family based on Burr XII distribution and power series distribution and found some mathematical properties including moments, quantile and generating functions, order statistics and their moments.

2.1 HAZARD RATE FUNCTION

Hazard rate function occurs in the situation of the study of the time to the incident and defines the recent casual of failure for the population that yet has not failed. Dealing with a life time data, this function is significantly useful in formulating and defining a model. Barlow (1963) used hazard rate function for the very first time. Leadbetter and Watson (1964) investigated and summarized properties of the hazard rate function. Dhillon (1978) provided awareness about the hazard rate function. Gross and Huber-Carol (1992) reflected the submissions of reverse hazard rate in medical education. Mirta and Basu (1996) considered some properties such as exponential bounds for survival function, moment, closure properties of bathtub hazard rate family of life distributions. Nandrajah and Kotz (2004) found a generalized Gumbel distribution. Navarro et al. (2009) found some general properties of hazard rate functions modeled by using generalization mixtures of two distributions and then applied the obtained results to determine the shape of an increasing hazard rate model and exponential model which are constructed by using generalized mixtures. Desai et al. (2011) pointed out that the reverse hazard rate is a reducing function for some standard distributions and possible for use in the field of maintenance engineering. Dara (2012) determined the reliability measures of the moment distributions such as survival function, hazard rate function, cumulative hazard rate function, mean residual function, reverse hazard rate and mean inactivity time.

2.2 REVERSE HAZARD RATE

The reverse hazard rate function is determined as the relation among the probability density and its conforming distribution function. The reverse Hazard rate function is beneficial in the expanses of medical studies as well as in the study of lifetimes with the reversed time scale:

$$r(x) = \frac{g(x)}{G(x)}.$$

2.3 ENTROPY

In general, entropy is reprocessed as a key device in information system and in almost every division of science and engineering, while statistical entropy is a measure of uncertainty in probability term or random experiment outcome’s ignorance. Information theoretic ideas and approaches have become essential parts of probability and statistics and have been applied in several branches of statistics and associated grounds. Jaynes (1980) gave short historical account of different notions of entropy and Ma (1981) estimated the entropies of continuous probability distributions. Zellner and Highfield (1988), Kapour (1989) and Soofi et al. (1995), developed maximum entropy distributions subject to various types of constraints. Paninski (2003) showed simple estimators of entropy, short variances but high prejudices that are difficult to analyze due to the divergence of the logarithm adjacent zero. Nemenman and Bialek (2004) presented Bayesian entropy estimator to records and organized theoretical analysis of neural replies and further biological data.

Kennel et al. (2005) solved this problem of entropy in computational biology applications with the moderately under sampled régime. They also developed the sympathetic that it is incredible to approximate entropy with zero partiality uniformly over totally distributions for a lesser N. Orlitsky et al. (2006) determined the entropy rate of shapes of certain random procedures containing all finite-entropy stationary processes. Lesne and Benecke (2008) changed the level of the account and investigated relations between distribution of probabilities in order to imprisonment invariants and expectable evidences. Zografos and Balakrishnan (2009) provided suitable constraints for the gamma-generated distributions to prove that the maximum entropy distribution is unique. Shams (2011) replaced entropy function by its second-order Taylor approximation. Nemenman (2011) performed an asymptotic analysis of entropy estimator and the analysis illuminated the requirement of the approximations on the amount of chances in the trial and displayed that the estimator has a definite limit for a huge cardinality of the studied variable.

CHAPTER 3

SOME RESULTS ON COMPOUND FAMILY OF ELEPS

3.1. INTRODUCTION

In this chapter some final results about compound power series are provided. The results are related to

i. A compound class of Weibull PS distributions
ii. Extended Weibull PS family of distributions
iii. The compound class of linear failure rate PS distributions

3.2 A COMPOUND CLASS OF WEIBULL PS DISTRIBUTIONS

Morais and Barreto-Souza (2011) introduced the Weibull PS distributions which is obtained by using the procedure of Adamidis and Loukas (1998). It has sub models as the two-parameter exponential PS distribution by Chahkandi and Ganjali, (2009), exponential geometric by Adamidis and Loukas (1998), exponential Poisson (Kus, 2007) and exponential logarithmic (Tahmasbi and Rezaei, 2008) distribution s Some major results of this research are described as under.

3.2.1 Some Major Results of Weibull PS Distributions

The Weibull PS distribution function is defined as

$$F\left(X\right)=1-\frac{C\left(\theta e^{-(\beta x)^{\alpha}}\right)}{C\left(\theta\right)},x>0 \qquad (3.1)$$

$C\left(\theta\right)=\sum_{n=1}^{\infty} a_n\theta^n$ and $\theta\in\left(0,s\right)$ (s can be ∞) is such that $C\left(\theta\right)$ is finite.

The PDF associated to (3.1) is given by

$$f\left(x\right)=\alpha\theta\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}\,\frac{C'\left(\theta e^{-(\beta x)^{\alpha}}\right)}{C\left(\theta\right)}, \qquad x>0.$$

The survival function of the WPS distributions is given by

$$S(x)=\frac{C\left(\theta e^{-(\beta x)^{\alpha}}\right)}{C(\theta)}, \qquad x>0,$$

and the hazard function is

$$h(x)=\alpha\theta\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}\,\frac{C'\left(\theta e^{-(\beta x)^{\alpha}}\right)}{C\left(\theta e^{-(\beta x)^{\alpha}}\right)}, \qquad x>0.$$

The quantile ξ of a WPS distribution is given by

$$X_{\xi}=\beta^{-1}\left\{-\log C\left((1-\xi)C(\theta)\right)^{-1}+\log\theta\right\}^{1/\alpha},$$

where $C(.)^{-1}$ is the inverse function of $C(.)$.

The rth moment of *a* WPS (β,α,θ) distribution is given by

$$E\left(X^{r}\right)=\frac{\Gamma(r/\alpha+1)}{\beta^{r}C(\theta)}\sum_{n=1}^{\infty}\frac{a_n\theta^{n}}{n^{r/\alpha}}.$$

The pdf of the ith order statistic $X_{i:n}$ is given by

$$f_{i:n}(x)=\frac{n!\,f(x)}{(n-i)!(i-1)!}\left[1-\frac{C\left(\theta e^{-(\beta x)^{\alpha}}\right)}{C(\theta)}\right]^{t-1}\left[\frac{C\left(\theta e^{-(\beta x)^{\alpha}}\right)}{C(\theta)}\right]^{n-1}, \quad x>0$$

$F_{i:n}$, becomes

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{(-1)^k\binom{n-i}{k}}{k+i}\left\{1-\frac{C\left(e^{-(\beta x)^{\alpha}}\right)}{C(\theta)}\right\}^{k+i}, \quad x>0.$$

$$E\left(X_{i:n}^{r}\right)=r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\begin{pmatrix}k-1\\n-i\end{pmatrix}\begin{pmatrix}n\\k\end{pmatrix}\int_0^{\infty}x^{r-1}S(x)^k\,dx$$

$$=r\sum_{k=n-i+1}^{n}\frac{(-1)^{k-n+i-1}}{C(\theta)^k}\begin{pmatrix}k-1\\n-i\end{pmatrix}\begin{pmatrix}n\\k\end{pmatrix}\int_0^{\infty}x^{r-1}C\left(\theta e^{-(\beta x)^{\alpha}}\right)^k dx,$$

The weibull binomial (WB) distribution is defined by the cdf (2) with $C(\theta)=(\theta+1)^m-1$, which is given by

$$F(x)=1-\frac{\left(\theta e^{-(\beta x)^{\alpha}}+1\right)}{(\theta+1)^m-1}, \qquad x>0,$$

The pdf and survival function of the Weibull Poisson (WP) distribution are given by

$$f(x)=\frac{\theta\alpha\beta^{\alpha}}{e^{\theta}-1}x^{\alpha-1}e^{-(\beta x)^{\alpha}}e^{\theta e^{-(\beta x)^{\alpha}}}$$

Let X b a random variable with WP (β,α,θ) distribution. Then, the rth moment of X is given by

$$E\left(X^r\right)=\frac{\Gamma(r/\alpha+1)}{\beta^r\left(e^{\theta}-1\right)}\sum_{n=1}^{\infty}\frac{\theta^n}{n!n^{r/\alpha}}.$$

3.3 GENERALIZED EXTENDED WEIBULL PS FAMILY OF DISTRIBUTIONS

Alkarni (2015 introduced a new family of models for lifetime data called generalized extended Weibull power series family of distributions by compounding generalized extended Weibull distributions and power series distributions. The compounding procedure follows the same setup carried out by Adamidis (1998).

The CDF of extended Weibull PS distributions

$$F_{X_{(n)}}(x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{c(\theta)} \left(1 - e^{-\alpha H(x)}\right)^{\delta n} = \frac{c\left(\theta\left(1 - e^{-\alpha H(x)}\right)^{\delta}\right)}{c(\theta)}, \quad x > 0,$$

The survival functions are given by

$$s_{X_{(n)}}(x) = 1 - \frac{c(\theta G(x))}{c(\theta)} = 1 - \frac{c\left(\theta\left(1 - e^{-\alpha H(x)}\right)^{\delta}\right)}{c(\theta)}$$

The quantile function of

$$Q_{X_{(n)}}(p) = H^{-1}\left\{-\frac{1}{\alpha}\log\left[\left(1 - \frac{c^{-1}(pc(\theta))}{\theta}\right)^{\frac{1}{\delta}}\right]\right\},$$

The mgf of

$$\begin{aligned} M_{X_{(1)}}(t) &= \int_0^{\infty} e^{tx} f_{X(1)}(x)\,dx = \sum_{n=1}^{\infty} P(N = n) \int_0^{\infty} e^{tx} g_{X(1)}(x)\,dx \\ &= \sum_{n=1}^{\infty} P(N = n) E\left(Y_{(1)}^{k}\right). \end{aligned}$$

3.3.1 Some Major Results of Extended Weibull PS Distributions

$$\begin{aligned} \ln = \mathrm{n}&\left[\log\theta + \log\alpha + \log\delta(c(\theta))\right] \\ &-\alpha\sum_{i=1}^{n} H(x_i) + (\delta - 1)\sum_{i=1}^{n} \log\left(1 - e^{-\alpha H(x_i)}\right) \\ &+\sum_{i=1}^{n} \log\left(c'\left(\theta\left(1 - e^{-\alpha H(x_i)}\right)^{\delta}\right)\right) \end{aligned}$$

Consider $p_i = 1 - e^{-\alpha H(x_i:\eta)}$. Then the score function is given by

$$U_n(\Theta) = (\partial \ln / \partial\theta, \partial \ln / \partial\alpha, \partial \ln / \partial\delta, \partial \ln / \partial\xi)^r,$$

$$\frac{\partial \ln}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{p_i^{\delta} c''\left(\theta p_i^{\delta}\right)}{c'\left(\theta p_i^{\delta}\right)} - n\frac{c'(\theta)}{c(\theta)},$$

$$\frac{\partial \ln}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} H(x_i) + (\delta - 1)\sum_{i=1}^{n} \alpha h(x_i)\frac{(1 - p_i)}{p_i}$$
$$+\sum_{i=1}^{n} \frac{\theta\delta H(x_i)(1 - p_i)\, p_i^{\delta-1} c^{n}\left(\theta p_i^{\delta}\right)}{c'\left(\theta p_i^{\delta}\right)},$$

$$\frac{\partial \ln}{\partial\theta} = \frac{n}{\delta} + \sum_{i=1}^{n} \log(p_i) + \sum_{i=1}^{n} \frac{\theta p_i^{\delta} \log(p_i) c''\left(\theta p_i^{\delta}\right)}{c'\left(\theta p_i^{\delta}\right)},$$

$$\frac{\partial \ln}{\partial\xi_k} = \sum_{i=1}^{n} \frac{\partial \log h(x_i)}{\partial\xi_k} - \alpha\sum_{i=1}^{n} \frac{\partial H(x_i)}{\partial\xi_k}$$
$$\left[1 + (\delta - 1)\frac{1 - p_i}{p_i} + \theta\delta(1 - p_i)\, p_i^{\delta-1}\frac{c''\left(\theta p_i^{\delta}\right)}{c'\left(\theta p_i^{\delta}\right)}\right].$$

3.4 THE COMPOUND CLASS OF LINEAR FAILURE RATE PS DISTRIBUTIONS

Mahmoudi and Jafari (2015) introduced a new class of distributions which generalizes the linear failure rate distribution and is obtained by compounding this distribution and power series class of distributions. This new class of distributions is called the linear failure rate-power series distributions and contains some new distributions such as linear failure rate-geometric, linear failure rate-Poisson, linear failure rate-logarithmic, linear failure rate-binomial distributions and Rayleigh-power series class of distributions.

3.4.1 Some Major Results of Compound Class of Linear Failure Rate PS Distributions

If $X_{(1)} = \min\left(X_1,....,X_N\right)$ then the conditional cdf of $X_{(1)}$ given $N = n$ is

$$G_{X_{(1)}|N=n}\left(x\right) = 1 - \exp\left(-anx - \frac{bn}{2}x^2\right),$$

The CDF of compound class of linear failure rate PS distributions

$$F\left(x\right) = 1 - \frac{C\left(\theta\exp\left(-ax - \frac{b}{2}x^2\right)\right)}{C\left(\theta\right)}, \qquad x > 0,$$

The PDF of compound class of linear failure rate PS distributions

$$f\left(x\right) = \theta\left(a + bx\right)\exp\left(-ax - \frac{b}{2}x^2\right)\frac{C'\left(\theta\exp\left(-ax - \frac{b}{2}x^2\right)\right)}{C\left(\theta\right)}.$$

The survival function and the hrf of the proposed class are given, respectively, by

$$S\left(x\right) \ = \frac{C\left(\theta\exp\left(ax - \frac{b}{2}x^2\right)\right)}{C\left(\theta\right)},$$

$$h\left(x\right) \ = \frac{\theta\left(a + bx\right)\exp\left(-ax - \frac{b}{2}x^2\right)C'\left(\theta\exp\left(-ax - \frac{b}{2}x^2\right)\right)}{C\left(\theta\exp\left(-ax - \frac{b}{2}x^2\right)\right)}.$$

The kth moment is

If $a \geq 0, b > 0,$ from Nadarajah and Mitov (2005), we have

$$E\left(X_{(1)}^k\right) - \frac{k}{\sqrt{2bn}}e^{\frac{na^2}{2b}}\sum_{l=0}^{k-1}\binom{k-1}{l}\left(-\frac{a}{b}\right)^l\left(\frac{2}{bn}\right)^{\frac{k-1-l}{2}}\Gamma\left(\frac{k-l}{2}, \frac{na^2}{2b}\right),$$

If $a=0$, then the LFRPS distributions gives a new class of Rayleigh distribution with the following pdf

$$f(x)=\theta bx\exp\left(-\frac{b}{2}x^2\right)\frac{C'\left(\theta\exp\left(-\tfrac{b}{2}x^2\right)\right)}{C(\theta)},$$

which Rayleigh power series distribution.

The geometric distribution (truncated at zero) is a special case of power series distributions with $a_n=1$ and $C(\theta)=\frac{\theta}{1-\theta}(0<\theta<1)$. the pdf of LFRG distribution is given by

$$f(x)=\frac{(1-\theta)(a+bx)\exp\left(-ax-\tfrac{b}{2}x^2\right)}{\left(1-\theta\exp\left(-ax-\tfrac{b}{2}x^2\right)\right)^2}=\frac{a(a+bx)\exp\left(-ax-\tfrac{b}{2}x^2\right)}{\left(1-(1-\alpha)\exp\left(-ax-\tfrac{b}{2}x^2\right)\right)^2},$$

where $\theta=1-\alpha$. The hrf is given by

$$h(x)=\frac{(a+bx)}{1-\theta\exp\left(-ax-\tfrac{b}{2}x^2\right)}.$$

The Poisson distribution (truncated at zero is a special case of power series distribution with $a_n=\frac{1}{n!}$ and $C(\theta)=c^\theta-1(\theta>0)$. The pdf of LFRP distribution is given as

$$h(x)=\theta(a+bx)\exp\left(-ax-\frac{b}{2}x^2\right)\frac{c^\theta\exp\left(-ax-\tfrac{b}{2}x^2\right)}{c^\theta\exp\left(-ax-\tfrac{b}{2}x^2\right)-1}.$$

The pdf of LFR-binomial (LFRB) distribution is given by

$$f(x)=\theta m(a+bx)\exp\left(-ax-\frac{b}{2}x^2\right)\frac{\left(\theta\exp\left(-ax-\tfrac{b}{2}x^2\right)+1\right)^{m-1}}{(\theta+1)^m-1},$$

and its hrf is given as

$$h(x)=\frac{m\theta(a+bx)\exp\left(-ax-\frac{b}{2}x^2\right)\left(\theta\exp\left(-ax-\frac{b}{2}x^2\right)+1\right)^{m-1}}{\left(\theta\exp\left(-ax-\frac{b}{2}x^2\right)+1\right)^{m}-1}.$$

The logarithmic distribution (truncated at zero) is a special case of power series distributions with $a_n=\frac{1}{n}$ and $C(\theta)=-\log(1-\theta)(0<\theta<1)$. The pdf of LFR-logarithmic (LFRL) distribution is given by

$$f(x)=\frac{\theta(a+bx)\exp\left(-ax-\frac{b}{2}x^2\right)}{-\log(1-\theta)\left(1-\theta\exp\left(-ax-\frac{b}{2}x^2\right)\right)},$$

and its hrf is given by

$$h(x)=\frac{\theta(a+bx)\exp\left(-ax-\frac{b}{2}x^2\right)}{\left(\theta\exp\left(-ax-\frac{b}{2}x^2\right)-1\right)\log\left(1-\theta\exp\left(-ax-\frac{b}{2}x^2\right)\right)}.$$

CHAPTER 4

PROPERTIES OF NEW COMPOUND FAMILY OF EXPONENTIATED LINDLEY EXPONENTIAL POWER SERIES DISTRIBUTION AND APPLICATION

4.1 INTRODUCTION

This chapter introduces a new family of lifetime distributions called the exponentiated Lindley exponential power series (ELEPS). This new family is obtained by compounding the exponentiated Lindley and truncated power series distributions, where the compounding procedure follows same way that was previously carried out by Adamidis and Loukas (1998). The new family contains some new distributions such as exponentiated Lindley geometric distribution, exponentiated Lindley Poisson distribution, exponentiated Lindley logarithmic distribution and exponentiated Lindley binomial distribution. We obtain several properties of ELEPS family, among them; quantile function, order statistics, moments and entropy. Some special models in the exponentiated moment exponential power series family of distributions are provided. Maximum likelihood (ML) method is applied to obtain parameter estimates of the ELEPS family. A simulation study is carried out to check the consistency of the ML estimators of the parameters. Two real data sets are used to validate the distributions and the results demonstrate that the sub-models from the family can be considered as suitable models under several real situations.

4.2 EXPONENTIATED LINDLEY EXPONENTIAL AND POWER SERIES DISTRIBUTION

In the context of Bayes' theorem a distribution proposed by Lindley (1958) as a counter example of fiducial statistics with probability density function (pdf):

$$g(x;\beta)=\frac{\beta^2}{\beta+1}(1+x)e^{-\beta x} \quad x,\beta>0. \tag{4.1}$$

The *pdf* of the exponentiated Lindley distribution is as under

$$g(x;\gamma,\beta)=\frac{\gamma\beta^2}{\beta+1}(1+x)e^{-\beta x}\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma-1},\ x,\beta,\gamma>0. \qquad (4.2)$$

The probability mass function of a random variable, Z which is a discrete random variable also a member of power series distribution (truncated at zero).

$$P(Z=z;\theta)=\frac{a_z\theta^z}{K(\theta)},\ z=1,2,3,.... \qquad (4.3)$$

where, $\theta>0$ is the scale parameter. The coefficients depend only on z, $K(\theta)=\sum_{z=1}^{\infty}a_z\theta^z$ is finite, $K'(.)$ and $K''(.)$ denote its first and second derivatives, respectively. The “power series distribution” is credited to Noack (1950). This family of distributions includes many popular distributions, which are the binomial, Poisson, geometric, negative binomial, and logarithmic distributions.

Let $X_1,X_2,...,X_z$ be iid random variables having *ELE* distribution with pdf (1) and the following cumulative distribution function (cdf):

$$G(x;\beta,\gamma)=\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma} \qquad (4.4)$$

Suppose that Z has a zero truncated power series distribution with the pmf (3). Let $X_{(1)}=\min\{X_1,X_2,...,X_z\}$ independent of $X's$, then the conditional probability density function of $X_{(1)}\,|\,Z$ is obtained as follows

$$f_{X_{(1)}|Z}(x|\,z;\beta,\gamma)=z\frac{\gamma\beta^2}{\beta+1}(1+x)e^{-\beta x}\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma-1}$$
$$\left(1-\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma}\right)^{z-1}.$$

The joint probability density function of $X_{(1)}$ and Z is obtained as follows

$$f_{X_{(1)},Z}(x,z;\beta)=\frac{z\gamma\beta^2 a_z\theta^z(1+x)e^{-\beta x}\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma-1}}{\left(\beta+1\right)K(\theta)}$$
$$\left(1-\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma}\right)^{z-1}.$$

The probability density function of a ELE power series family of distributions can be defined by the marginal density of X, that is,

$$f(x;\psi)=\frac{\gamma\theta\beta^2}{\beta+1}(1+x)e^{-\beta x}\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma-1}$$
$$\frac{K'\left(\theta\left(1-\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)}, x,\beta,\theta,>0. \qquad (4.5)$$

where $\psi\equiv(\beta,\gamma,\theta)$ is a set of parameters. A random variable X with density function (4.5) is denoted by X~*ELEPS* (β,γ,θ).

Furthermore, the cumulative distribution function of *ELEPS* family of distributions corresponding to (4.5) is obtained as follows

$$F(x;\psi)=1-\frac{K\left(\theta\left(1-\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)}. \qquad (4.6)$$

In addition, the reliability and hazard rate functions for *ELEPS* family of distributions, respectively, take the following forms

$$R(x;\psi)=\frac{K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)},$$

and

$$h(x;\psi)=\frac{\frac{\beta^2}{\beta+1}\gamma\theta(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1} K'\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}. \qquad (4.7)$$

4.3 SOME MATHEMATICAL PROPERTIES

In this section, some mathematical properties of the *ELEPS* family including, expansion for pdf (5), quantile function, rth moment, Re'nyi entropy and distribution of order statistics are obtained.

4.3.1 Useful Expansion

In this subsection, two important propositions are provided. The first proposition indicates that the new family has the *ELE* distribution as a limiting case while the second proposition provides useful expansion for the pdf of *ELEPS* distribution.

Proposition (1)

The *EL* distribution with parameters β *and* γ is a limiting special case of *ELEPS* family of distributions when $\theta \to 0^+$.

Proof:

By applying $f(\theta)=\sum_{z=1}^{\infty} a_z\theta^z,$ for $x>0$ in cdf (4.6), then we obtain

$$\lim_{\theta\to 0^+} F(x;\psi)=1-\lim_{\theta\to 0^+}\frac{\sum_{z=1}^{\infty} a_z\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^z}{\sum_{z=1}^{\infty} a_z\theta^z}.$$

By using L’Hˆospital’s rule, we have

$$\underset{\theta\to 0^+}{Lim}\, F(x;\psi) = 1 - \frac{\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\left[1+a_1^{-1}\underset{\theta\to 0^+}{Lim}\sum_{z=2}^{\infty} z a_z\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}\right]}{1+a_1^{-1}\underset{\theta\to 0^+}{Lim}\sum_{z=2}^{\infty} z a_z \theta^{z-1}}.$$

Hence,

$$\underset{\theta\to 0^+}{Lim}\, F(x;\psi) = \left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma},$$

which is the distribution function of the *EL* distribution.

Proposition (2)

The density function of *ELEPS* family can be expressed as a linear combination of the density of $X_{(1)} = \min\{X_1, X_2, ...X_z\}$

Proof:

Since $f'(\theta) = \sum_{z=1}^{\infty} z a_z \theta^{z-1}$, then the pdf (4.4) can be expressed as follows

$$f(x;\psi) = \sum_{z=1}^{\infty} P(Z=z;\theta)\, g_{X_{(1)}}(x;z), \tag{4.8}$$

where $g_{X_{(1)}}(x;z)$ is the pdf of $X_{(1)} = \min\{X_1, X_2, ..., X_z\}$ given by:

$$g_{X_{(1)}|Z}(x|\,z;\beta,\gamma) = z\frac{\gamma\beta^2}{\beta+1}(1+x)e^{-\beta x}\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma-1}$$
$$\left(1-\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma}\right)^{z-1}.$$

4.3.2 The Lambert W Function

The Lambert W function has attracted a great deal of attention beginning with Lambert in 1758 and Euler in 1779. The name “Lambert W function” has become a standard after its implementation in the computer algebra system Maple in the 1980s and subsequent publication by Corless et al. (1996) of a comprehensive survey of the history, theory and applications of this function. The Lambert W function is a multivalued complex function defined as the solution of the equation

$$W(z)\exp\left(W(z)\right)=z \tag{4. 9}$$

where z is a complex number. If z is a real number such that $z \geq -1/e$ then W(z) becomes a real function and there are two possible real branches. The real branch taking on values in $(-\infty,-1]$ is called the negative branch and denoted by W_{-1}. The real branch taking on values in $[-1,\infty)$ is called the principal branch and denoted by W_0

Lemma 1

Let $a, b \, and \, c$ complex numbers, the solution of the equation $z+ab^z=c$ with respect to $z \in C$ is

$$z=c-\frac{1}{\log(b)}W\left(ab^c\log(b)\right) \tag{4.10}$$

where W denotes the lambert W function.

4.3.3 Quantile Function

In this subsection, the quantile function of the ELEPS distribution is derived. The quantile function, denoted by, $Q(p)$, defined by , is the root of the following equation

$$1-\frac{K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}(Q(p)))e^{-\beta(Q(p))}\right)^{\gamma}\right)\right)}{K(\theta)}=p,\ 0<p<1.$$

Let

$$D(p) = -(1+\frac{\beta}{\beta+1}(Q(p)).$$

Then,

$$1-\left(1+D(p)e^{D(p)+1}\right)^{\gamma} = \frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}.$$

$$D(p)e^{D(p)(\beta+1)} = \frac{1}{e^{\beta+1}}\left(-1-\left(-1+\left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right).$$

$$\left(\beta+1\right)D(p) = C(p)$$

$$D(p) = \frac{C(p)}{\left(\beta+1\right)}$$

$$\frac{C(p)}{\beta+1}e^{C(p)} = \left(\frac{1}{e^{\beta+1}}\left(-1-\left(-1+\left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right)\right)$$

$$C(p)e^{C(p)} = \left(\beta+1\right)\left(\frac{1}{e^{\beta+1}}\left(-1-\left(-1+\left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right)\right)$$

$$C(p) = W_{-1}\left[\left(\beta+1\right)\left(\frac{1}{e^{\beta+1}}\left(-1-\left(-1+\left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]$$

Then solution for $D(p)$ is

$$D(p)=\frac{1}{\beta+1}W_{-1}\left[(\beta+1)\left(\frac{1}{e^{\beta+1}}\left(-1-\left(-1+\left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]$$

where $W(.)$ is the negative branch of the Lambert W function (see Corless et al. (1996)). Consequently, the quantile function of the *ELEPS* family is given by solving the following equation for $Q(p)$.

$$Q(p)=\left(\frac{1}{\beta+1}W_{-1}\left[(\beta+1)\left(e^{-\beta-1}\left(-1-\left(-1+\left(\frac{K^{-1}\left((1-p)K(\theta)\right)}{\theta}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\right) \tag{4.11}$$

4.3.4 Moments and Moment Generating Function

The rth moment of a random variable X from the *ELEPS* distribution, is given by using pdf (4.5) as the following

$$\mu_r'=\int_0^{\infty}x^r f(x;\psi)dx$$

$$\mu_r'=\int_0^{\infty}x^r\frac{\gamma\theta\beta^2}{\beta+1}(1+x)e^{-\beta x}\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma-1}\frac{K'\left(\theta\left(1-\left(1-\left(1+\frac{\beta}{\beta+1}x\right)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)}dx.$$

Based on the first four moments of the *ELEPS* family, the measures of skewness (SK) and kurtosis (K) can be obtained from following relations respectively

$$SK=\frac{\mu'_3-3\mu'_2\,\mu'_1+2\mu'^3_1}{\left(\mu'_2-\mu'^2_1\right)^{\frac{3}{2}}},\qquad K=\frac{\mu'_4-4\mu'_3\,\mu'_1+6\mu'_2\,\mu'^2_1-3\mu'^4_1}{\left(\mu'_2-\mu'^2_1\right)^{2}},$$

where, μ_1', μ_2', μ_3' and μ_4' can be obtained from (9), by substituting $r = 1, 2, 3, 4.$

Also, it is easy to show that, the moment generating $M_X(t)$ function can be written as follows

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r',$$

4.3.5 Order Statistics

In this subsection, an expression for the pdf of the ith order statistics from the *ELEPS* distribution is derived. In addition, the distributions of the smallest and largest order statistics are obtained.

Let $X_1, X_2, ... X_n$ be a simple random sample from a *ELEPS* family with pdf (4.2) and cdf (4.3). Let $X_{1:n} < X_{2:n} < ... < X_{n:n}$ denote the corresponding order statistics from the sample. The pdf of $X_{i:n}$, where $i = 1, ... n$ is given by

$$f_{i:n}(x;\psi) = \frac{1}{\mathrm{B}(i, n-i+1)} f(x;\psi)[F(x;\psi)]^{i-1}[1 - F(x;\psi)]^{n-i}, \qquad (4.12)$$

where, $\mathrm{B}(.,.)$ is the beta function. By using cdf (4) and applying the binomial expansion in (4.12), then we get

$$f_{i:n}(x;\psi) = \frac{f(x;\psi)}{\mathrm{B}(i,n-i+1)} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \left(\frac{K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)} \right)^{n+j-i}.$$

Now, since an expansion for $\left(K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}$ can be written as follows

$$\left(K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}$$

$$=\left(\sum_{z=1}^{\infty}a_z\theta^{z}\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z}\right)^{n+j-i},$$

$$\left(K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}$$

$$=\left(a_1\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}$$

$$\times\left[\begin{array}{l}1+\dfrac{a_2}{a_1}\left(\theta\left(1-\left(1-(1+\dfrac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\\ \qquad+\dfrac{a_3}{a_2}\left(\theta\left(1-\left(1-(1+\dfrac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{2}+\dots\end{array}\right]^{n+j-i}.$$

Hence,

$$\left(K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}=a_1^{\,n+j-i}$$

$$\times\left(\sum_{m=0}^{\infty}\ell_m\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}\right)^{n+j-i},\ell_m=\frac{a_{m+1}}{a_1},\ m=1,2,\dots \tag{4.13}$$

According to Gradshteyn and Ryzhik (2000) for a positive integer, we have the following relation

$$\left(\sum_{m=0}^{\infty}\ell_m Y^m\right)^{n+j-i}=\sum_{m=0}^{\infty}d_{n+j-i,m}Y^m.$$

Then (11) can be written as follows

$$\left(K\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\right)^{n+j-i}$$
$$=(a_1)^{n+j-i}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m},\qquad(4.14)$$

where, $d_{n+j-i,0}=1$ and the coefficients $d_{n+j-i,m}$ are easily determined from the following recurrence equation

$$d_{n+j-i,t}=t^{-1}\sum_{m=1}^{t}[m(n+j-i+1)-t]\ell_m d_{n+j-i,t-m},t\geq 1.$$

In addition,

$$K'\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)$$
$$=\sum_{z=1}^{\infty}z\,a_z\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}.$$

Let $k=z-1$, then the previous equation can be expressed as

$$K'\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)$$
$$=\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{k},\ \ell_k=\frac{a_{k+1}}{a_1}\qquad(4.15)$$

Then, the pdf of the ith order statistic from *ELEPS* family of distributions is obtained by substituting expansions (4.11) and (4.12) in pdf (4.9) as follows

$$f_{i:n}(x;\psi)=\frac{\frac{\beta^2}{\beta+1}\gamma\theta(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{k}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}$$

$$\times\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j a_1^{n+j-i+1}\sum_{m=0}^{\infty}d_{n+j-i,m}$$

$$\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m}$$

Thus, the pdf of the ith order statistics can be formed as follows

$$f_{i:n}(x;\psi)=\frac{\frac{\beta^2}{\beta+1}\gamma(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}}{\mathrm{B}(i,n-i+j)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}(-1)^j\binom{i-1}{j}\ell_k(k+1)$$

$$\times\frac{d_{n+j-i,m}a_1^{n+j-i+1}\theta^{n+j-i+m+k+1}}{(K(\theta))^{n+j-i+1}}$$

$$\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m+k},\quad x>0.$$

as follows or it can be written

$$f_{i:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\tau_{j,k,m}\gamma\frac{\beta}{\beta+1}(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}$$
$$\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m+k}$$

where,

$$\tau_{j,k,m}=\left(-1\right)^{j}\binom{i-1}{j}\frac{\beta\ell_k(k+1)\theta^{n+j-i+m+k+1}a_1^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}.$$

Another form can be written by using binomial expansion as follows:

$$f_{i:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\tau_{j,k,m}\gamma\frac{\beta}{\beta+1}(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}$$
$$\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{n+j-i+m+k},$$

$$f_{i:n}(x;\psi)=\frac{\beta}{\beta+1}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\sum_{l=0}^{\infty}\left(-1\right)^{h+l}\theta^{n+j-i+m+k}$$
$$\binom{n+j-i+m+k}{h}\xi_{j,k,m,h}\ \gamma(1+x)e^{-\beta x}\left((1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{l},$$

$$f_{i:n}(x;\psi)=\frac{\beta}{\beta+1}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\sum_{l=0}^{\infty}\sum_{s=0}^{l}\left(-1\right)^{h+l}\theta^{n+j-i+m+k}\binom{l}{s}$$
$$\binom{n+j-i+m+k}{h}\xi_{j,k,m,h}\left(\frac{\beta}{\beta+1}\right)^{s}(1+x)x^{s}e^{-(l+\beta)x}\qquad(4.16)$$

where,

$$\xi_{j,k,m,h}=\left(-1\right)^{j}\binom{i-1}{j}\binom{m+n+j-i+k}{h}$$
$$\frac{\beta^{h+1}\left(\frac{1}{\beta+1}\right)^{h}\theta^{n+j-i+m+k+1}\ell_k(k+1)a_1^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(K(\theta))^{n+j-i+1}}.$$

In particular, the pdf of the smallest and the largest order statistics of the *ELEPS* distribution is obtained by substituting $i = 1, n$, in (4.16), respectively, as follows

$$f_{1:n}(x;\psi) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{h=0}^{n+j-i+m+k} \eta_{k,m,h} \frac{\beta}{\beta+1}(1+x)\, x^{(h+1)} e^{-(n+m+k)\beta x},$$

where

$$\eta_{k,m,h} = \binom{m+n-1+k}{h} \frac{n\beta^{h+1}\left(\frac{1}{\beta+1}\right)^{h} \ell_k (k+1)\theta^{n+m+k} a_1^{n} d_{n-1,m}}{(K(\theta))^{n}}.$$

and,

$$f_{n:n}(x;\psi) = \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \sum_{m=0}^{\infty} \sum_{h=0}^{j+m+k} \varsigma_{j,k,m,h} \frac{\beta}{\beta+1}(1+x)\, x^{(h+1)} e^{-(j+m+k+1)\beta x},$$

where,

$$\varsigma_{k,m,h} = \binom{m+j+k}{h}\binom{n-1}{j}(-1)^{j} \frac{n\beta^{h+1}\left(\frac{1}{\beta+1}\right)^{h} \ell_k (k+1)\theta^{j+m+k+1} a_1^{j+1} d_{j,m}}{(K(\theta))^{j+1}}.$$

4.3.6 Re'nyi Entropy

Entropy has been used in various situations in science and engineering. The entropy of a random variable X is a measure of variation of the uncertainty. If X is a random variable which distributed as *ELEPS*, then the Re'nyi entropy, for $\rho > 0$, and $\rho \neq 1$, is defined as

$$I_R(x) = (1-\rho)^{-1} \log_b \left(\int_0^{\infty} \left(f(x;\psi) \right)^{\rho} dx \right).$$

Let, $IP = \int_0^{\infty} \left(f(x;\psi) \right)^{\rho} dx,$ then IP can be written as follows:

$$IP=\int_0^{\infty}\left(\frac{\beta^2}{\beta+1}\gamma\theta(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}\right)^{\rho}$$

$$\left\{\frac{\sum_{z=1}^{\infty} z a_z\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}}{K(\theta)}\right\}^{\rho} dx.$$

But

$$\left(\sum_{z=1}^{\infty} z a_z\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}\right)^{\rho}$$

$$=a_1^{\rho}\left(\sum_{m=0}^{\infty}\delta_m\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}\right)^{\rho},$$

$$\delta_m=\frac{a_{m+1}}{a_1},\ m=1,2,...$$

Using the same rule as provided by Gradshteyn and Ryzhik (2000), then we obtain

$$\left(\sum_{z=1}^{\infty}\delta_m\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}\right)^{\rho}$$

$$=\sum_{m=0}^{\infty} d_{\rho,m}\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}.$$

Therefore,

$$\left(\sum_{z=1}^{\infty} z a_z \left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}\right)^{\rho}$$

$$= a_1^{\rho} \sum_{z=1}^{\infty} d_{\rho,m} \left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{m}. \qquad (4.17)$$

The coefficients for $t>1$ are computed from the following recurrence equation:

$$d_{\rho,t} = t^{-1} \sum_{m=1}^{t} [m(\rho+1)-t]\delta_m d_{\rho,t-m}, d_{\rho,0} = 1$$

Using binomial expansion for $(1+\alpha x)^m$, then (4.17) can be written as follows:

$$\left(\sum_{z=1}^{\infty} z a_z \left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)^{z-1}\right)^{\rho}$$

$$= a_1^{\rho} \sum_{z=1}^{\infty}\sum_{k=0}^{m} \binom{m}{k} d_{\rho,m}\theta^{m} e^{-m\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma k}$$

$$= a_1^{\rho} \sum_{z=1}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\sum_{i=0}^{j} (-1)^{j+k}\left(\frac{\beta}{\beta+1}\right)^{j}\binom{m}{k}\binom{\gamma k}{j}\binom{j}{i} d_{\rho,m}\theta^{m} e^{-m\beta x} x^{j} e^{-\beta j x}$$

Then the IP can be rewritten as follows

$$IP = \left(\frac{\beta^2}{\beta+1}\gamma\theta a_1(1+x)\right)^{\rho} \sum_{z=1}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\sum_{i=0}^{j} (-1)^{j+k}\left(\frac{\beta}{\beta+1}\right)^{j}\binom{m}{k}$$

$$\binom{\gamma(k+\rho)-\rho}{j}\binom{j}{i} d_{\rho,m}\theta^{m}\int_{0}^{\infty} e^{-(m+j+\rho)\beta x} x^{j} dx$$

After some simplification, then the Re'nyi entropy takes the following form

$$I_R(x) = (1-\rho)^{-1}\log_b \left[\left(\frac{\beta^2}{\beta+1}\gamma\theta a_1(1+x)\right)^{\rho} \sum_{z=1}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{(-1)^{j+k}\left(\frac{\beta}{\beta+1}\right)^{j}\Gamma(j+1)\binom{m}{k}\binom{\alpha(k+\rho)-\rho}{j}\binom{j}{i}d_{\rho,m}\theta^{m}}{\left((m+j+\rho)\left(\frac{\beta}{\beta+1}\right)\right)^{j+1}} \right]. \tag{4.18}$$

4.4 SPECIAL MODELS OF THE FAMILY

Some sub-models from *ELEPS* family of distributions for selected values of the parameters are presented in this section. Also, some sub-models; which are the generalized exponentiated Lindley exponential Poisson and exponentiated Lindley exponential geometric distributions are discussed in more details.

The sub models are considered as follows:

1. For $K(\theta)=e^{\theta}-1$, then the *ELEPS* distribution reduces to exponentiated Lindley exponential Poisson *(ELEP)* distribution with the following cdf:

$$F(x;\psi) = \frac{e^{\theta}-\exp\left[\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right]}{e^{\theta}-1}, \quad x,\theta,\beta,\gamma>0.$$

2. For $K(\theta)=e^{\theta}-1, \gamma=1$, then the *ELEPS* distribution reduces to Lindley exponential Poisson *(LEP)* distribution with the following cdf:

$$F(x;\psi)=\frac{e^{\theta}-\exp\left[\theta\left((1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)\right]}{e^{\theta}-1},\qquad x,\theta,\beta>0.$$

3. For $K(\theta)=-\ln(1-\theta)$ then the *ELEPS* distribution reduces to exponentiated Lindley exponential logarithmic (*ELEL*) distribution with the following cdf:

$$F(x;\psi)=1-\frac{\ln\left[1-\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right]}{\ln(1-\theta)},$$
$$x,\beta,\gamma>0,\quad 0<\theta<1.$$

4. For $K(\theta)=-\ln(1-\theta),\gamma=1$ then the *ELEPS* distribution reduces to Lindley exponential logarithmic (*LEL*) distribution with the following cdf:

$$F(x;\psi)=1-\frac{\ln\left[1-\theta\left((1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)\right]}{\ln(1-\theta)},\ x,\beta>0,\ 0<\theta<1.$$

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5. For $K(\theta)=\theta(1-\theta)^{-1}$, then the *ELEPS* distribution reduces to exponentiated Lindley exponential geometric (*ELEG*) distribution with the following cdf:

$$F(x;\psi)=\frac{\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}}{1-\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)},\ x,\beta,\gamma>0,\ 0<\theta<1.$$

6. For $K(\theta)=\theta(1-\theta)^{-1},\gamma=1$ then the *ELEPS* distribution reduces to Lindley exponential geometric (*LEG*) distribution with the following cdf:

$$F(x;\psi)=\frac{\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)}{1-\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)\right)},\ x,\beta>0,\ 0<\theta<1.$$

7. For $K(\theta)=\theta(1-\theta)^{-1}$, then the *ELEPS* distribution reduces to exponentiated Lindley exponential geometric (*ELEG*) distribution with the following cdf:

$$F(x;\psi)=\frac{\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}}{1-\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)},\quad x,\beta,\gamma>0,\ 0<\theta<1.$$

8. For $K(\theta)=\theta(1-\theta)^{-1}$ & $\gamma=1$ then the *ELEPS* distribution reduces to Lindley exponential geometric (*LEG*) distribution with the following cdf:

$$F(x;\psi)=\frac{\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)}{1-\theta\left((1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)},\quad x,\beta>0,\ 0<\theta<1.$$

9. For $K(\theta)=(1-\theta)^{m}-1$, then the *ELEPS* distribution reduces to exponentiated Lindley exponential binomial (*ELEB*) distribution with the following cdf:

$$F(x;\psi)=\frac{(1-\theta)^{m}-\left[1-\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right]^{m}}{(1-\theta)^{m}-1},$$

$$x,\beta,\gamma>0,\quad 0<\theta<1.$$

10. For $K(\theta)=(1-\theta)^{m}-1$ & $\gamma=1$ then the *ELEPS* distribution reduces to Lindley exponential binomial (*LEB*) distribution with the following cdf:

$$F(x;\psi)=\frac{(1-\theta)^{m}-\left[1-\theta(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right]^{m}}{(1-\theta)^{m}-1},\ x,\beta>0,\ 0<\theta<1.$$

4.5 EXPONENTIATED LINDLEY EXPONENTIAL POISSON DISTRIBUTION

As mentioned above the *ELEP* distribution is obtained from *ELEPS* family distribution as a special case. The *pdf* of the *ELEP* distribution corresponding to (17) takes the following form

$$f(x;\psi)=\frac{\gamma\frac{\beta^2}{\beta+1}(1+x)e^{-\beta x}\theta\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}\exp\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}{(e^{\theta}-1)}\quad x,\beta,\gamma,\theta>0.$$

In addition, the reliability and hazard rate function take the following form respectively:

$$R(x;\psi)=\frac{\exp\left[\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right]}{e^{\theta}-1}-1,$$

and,

$$h(x;\psi)=\frac{\frac{\beta^2}{\beta+1}\gamma\theta(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}\exp\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}{\left[\exp\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)-1\right]}.$$

Figure 4.1, gives plots of the pdf of the *ELEP* distribution for some selected values of parameters exhibiting the behavior of density.

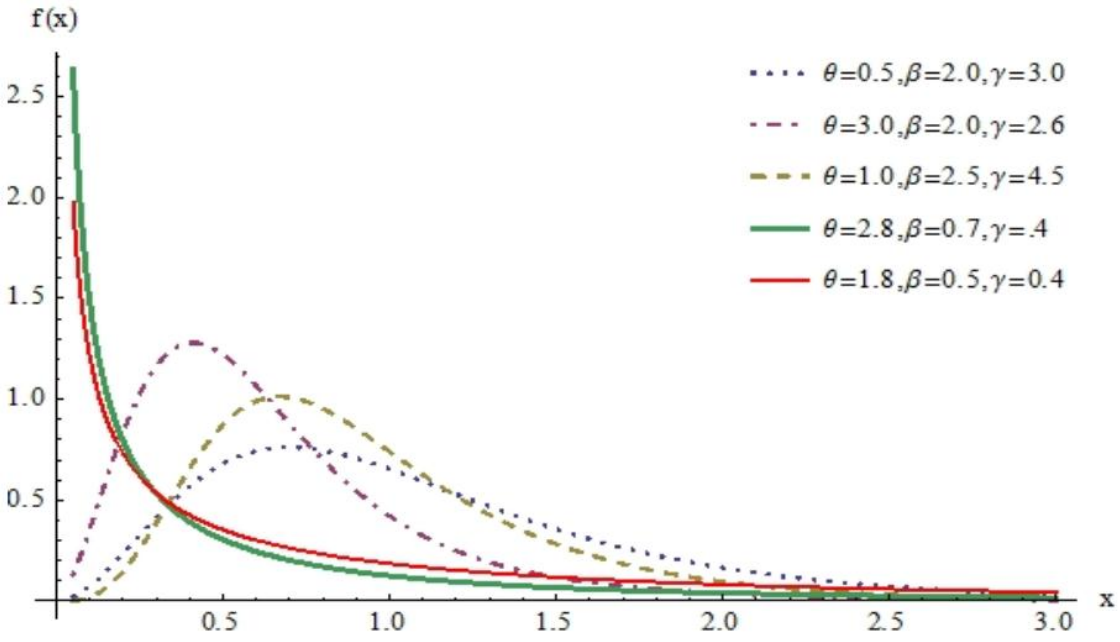

Figure 4.1: The pdf Plots of the *ELEP* Distribution

The following figure gives the hazard rate function plots for *ELEP* distribution for some selected values of parameters.

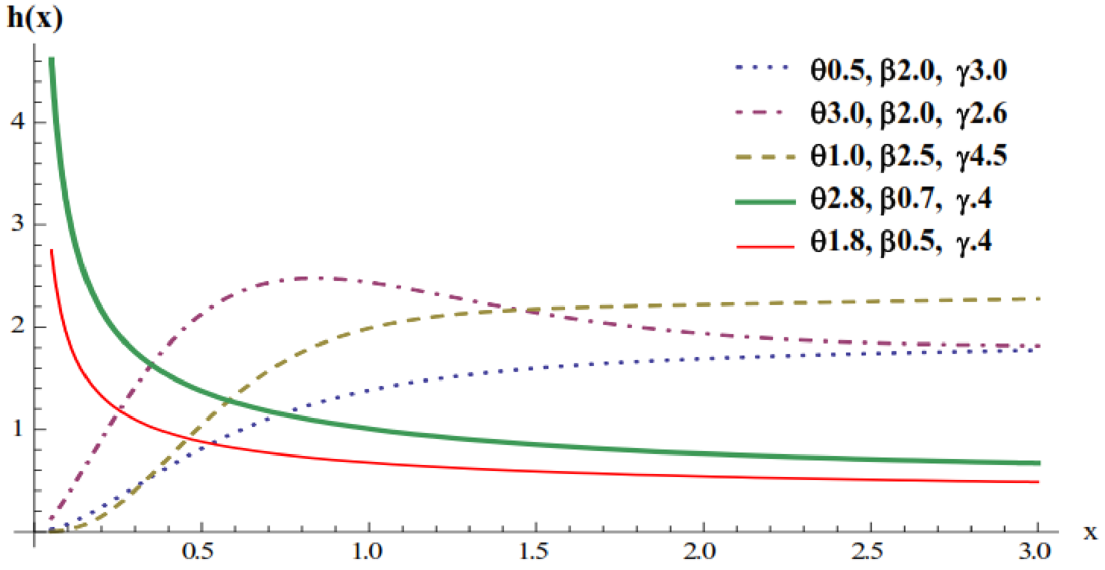

Figure 4.2: The Hazard Rate Plots for the *ELEP* Distribution

It is clear from Figure 4.2 that the *ELEP* distribution has increasing, decreasing and constant failure rates.

The quantile function for the *ELEP* distribution is obtained directly from expression (8) with $K(\theta) = e^{\theta} - 1,$ and $K^{-1}(\theta) = \ln(1+\theta)$ as follows:

$$(Q(p)) = \left(\frac{1}{\beta+1} W_{-1}[\beta+1(-\frac{\ln(p+(1-p)e^{\theta})}{\theta e^{\beta+1}})] \right)^{\frac{1}{\gamma}}.$$

Solving this equation for $Q(p),$ the quantile function of *ELEP* is obtained.

Furthermore, the r$^{\text{th}}$ moment of the *ELEP* distribution about the origin is given by substituting the following pmf of truncated Poisson distribution

$$P(Z = z;\theta) = \frac{e^{-\theta}\theta^{z}}{z!(1-e^{-\theta})},\ z = 1,2,...$$

in (9) as follows

$$\mu_r' = \sum_{z=1}^{\infty} \sum_{j=0}^{z-1} \sum_{i=0}^{j+1} \binom{z-1}{j} \binom{j+1}{i} \frac{\theta^{z}\Gamma\left(r+i+1\right)}{z!(e^{\theta}-1)z^{r+i}\lambda^{r}}, \qquad r = 1,2....$$

Additionally the Re'nyi entropy is obtained by substituting in (16) as follows

$$I_R(x) = (1-\rho)^{-1}\log_b \left[\sum_{m=0}^{\infty} \sum_{k=0}^{m} \sum_{h=0}^{\rho} \binom{m}{k} \binom{\rho}{h} \frac{d_{\rho,m}\theta^{m+\rho}a_1^{\rho}\Gamma(1+k+h)}{(e^{\theta}-1)^{\rho}(m+\rho)^{1+k+h}} \right].$$

4.6 EXPONENTIATED LINDLEY EXPONENTIAL GEOMETRIC DISTRIBUTION

The moment exponential geometric distribution is discussed as the second special model from *ELEPS* family. The pdf of the *ELEG* distribution corresponding to (18) takes the following form

$$f(x;\psi)=\frac{\frac{\beta^{2}}{\beta+1}\gamma\left(1+x\right)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}(1-\theta)}{\left[1-\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)\right]^{2}},$$

$$x>0,0<\theta<1,\beta,\gamma>0.$$

In addition, the reliability and hazard rate function take the following form:

$$R(x;\psi)=\frac{(1-\theta)\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)}{1-\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)},$$

and,

$$h(x;\psi)=\frac{\frac{\beta^{2}}{\beta+1}\gamma(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}}{\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\left[1-\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right]}.$$

Figures 4.3 and 4.4 represent probability density and hazard rate functions plots for *ELEG* distribution for some selected values of parameters.

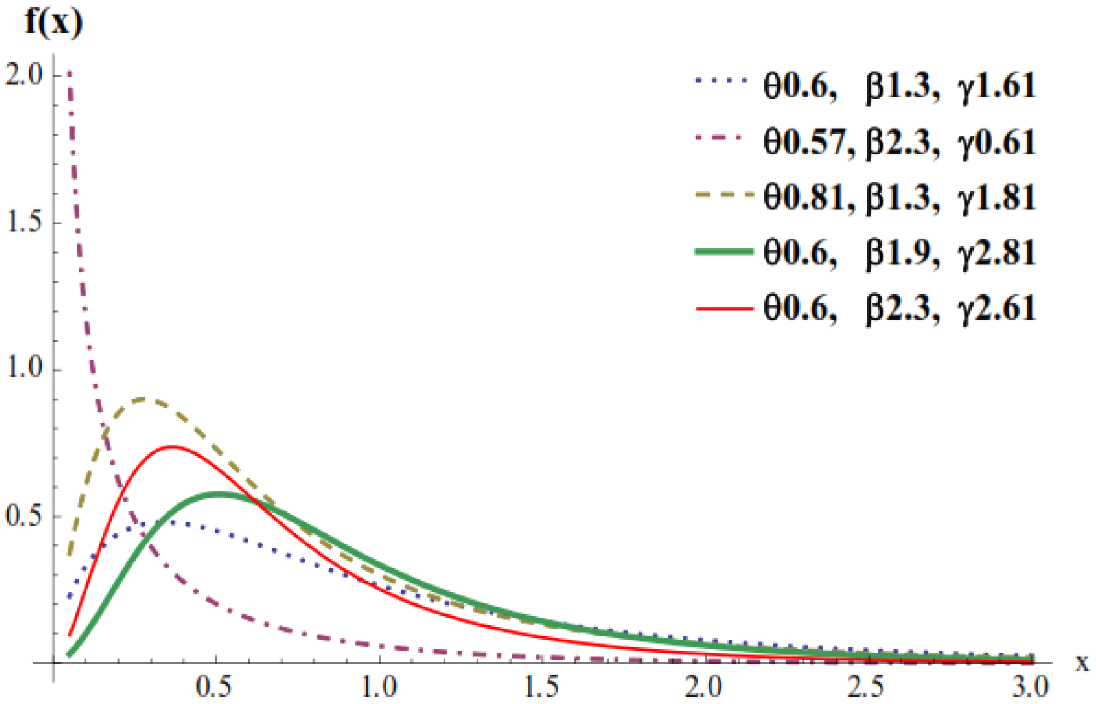

Figure 4.3: The pdf Plots of the *ELEG* Distribution

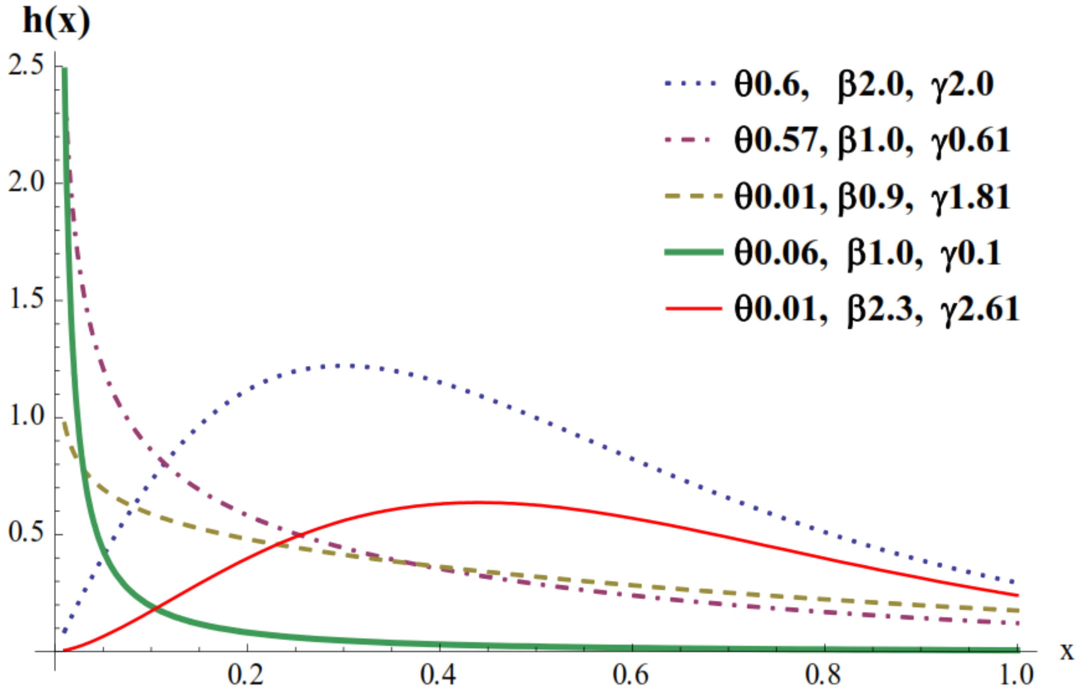

Figure 4.4: The Hazard Rate Plots of the *ELEG* Distribution

From this figure, it is observed that the shapes of the hazard rate are increasing at some selected values. For some choices of parameters; the distribution has increasing, decreasing and constant patterns of cumulative instantaneous failure.

The quantile function for the *ELEG* distribution is obtained directly from expression (8) with $K(\theta)=\theta(1-\theta)^{-1}$, and $K^{-1}(\theta)=\theta(1+\theta)^{-1}$ as follows

$$Q(p)=\left(\frac{1}{\beta+1}W_{-1}[(\beta+1)(-\frac{(1-p)\theta}{(1-\theta p)e^{\beta+1}})]\right)^{\frac{1}{\gamma}}.$$

Solving this equation for $Q(p)$, the quantile function *ELEG* is obtained.

Additionally, the rth moment of the *ELEG* distribution about the origin is given by substituting the following pmf of truncated geometric

$P(Z=z;\theta)=(1-\theta)\theta^{z-1},\ z=1,2,...$ in (9) as follows

$$\mu_r{}'=\sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z-1}(1-\theta)\Gamma\left(r+i+1\right)}{z^{r+i}\left(\frac{\beta}{\beta+1}\right)^{r}},\ \ r=1,2....$$

Further, the Re'nyi entropy is obtained by substituting in (16) as follows

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m}\lambda^{\rho+h+k}a_1^{\rho}\Gamma(1+k+h)}{(1-\theta)^{-\rho}(m+\rho)^{1+k+h}}\right].$$

4.7 PARAMETER ESTIMATION OF THE FAMILY

In this section estimation of the model parameters of *ELEPS* family of distributions is obtained by using the maximum likelihood method.

Let $X_1,X_2,...X_n$ be a simple random sample from the *ELEPS* family with set of parameters $\psi=(\beta,\gamma,\theta)$. The log-likelihood function based on the observed random sample of size n is given by:

$$f(x;\psi)=\frac{\beta^{2}}{\beta+1}\gamma\theta(1+x)e^{-\beta x}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}$$

$$\frac{K'\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}{K(\theta)},x,\beta,\theta,\gamma>0.$$

$$L(x;\psi)=\frac{\beta^{2n}}{\left(\beta+1\right)^{n}}\gamma^{n}\theta^{n}\left(\prod_{i=1}^{n}(1+x)\right)e^{-\beta\sum\limits_{i=1}^{n}x}\prod_{i=1}^{n}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}$$

$$\frac{\prod\limits_{i=1}^{n}K'\left(\theta\left(1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\right)\right)}{\left(K(\theta)\right)^{n}}$$

$$\ln L(x;\psi)=2n\ln\beta-n\ln(\beta+1)+n\ln\gamma+n\ln\theta$$

$$+\sum_{i=1}^{n}\ln(1+x_{i})-\beta\sum_{i=1}^{n}x+\sum_{i=1}^{n}\ln\left(K'(\theta S_{i})\right)-n\ln(K(\theta)).$$

where, $\ln L=\ln L(x;\psi)$ and $S_{i}=1-\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}$.

The partial derivatives of the log-likelihood function with respect to the unknown parameters are given by:

$$\frac{\partial\ln L}{\partial\beta}=\frac{2n}{\beta}-\frac{n}{\beta+1}-\sum_{i=1}^{n}x_{i}+\theta\sum_{i=1}^{n}\frac{K''(\theta S_{i})}{K'(\theta S_{i})}\frac{\partial S_{i}}{\partial\beta},$$

$$\frac{\partial\ln L}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\left[\frac{K''(\theta S_{i})}{K'(\theta S_{i})}\right]S_{i}-\frac{nK'(\theta)}{K(\theta)},$$

where,

$$\frac{\partial S_{i}}{\partial\beta}=-\frac{\beta\gamma}{\left(\beta+1\right)^{2}}\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma-1}e^{-\beta x}\left(2x\left(\beta+1\right)+1\right)$$

$$\frac{\partial S_{i}}{\partial\gamma}=\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right)^{\gamma}\ln\left(1-(1+\frac{\beta}{\beta+1}x)e^{-\beta x}\right).$$

The maximum likelihood estimators of the model parameters are determined by solving the non-linear equations $\frac{\ln L}{\partial\beta}=0, \frac{\ln L}{\partial\gamma}=0 \; and \; \frac{\ln L}{\partial\theta}=0$. These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative technique.

4.8 A SIMULATION STUDIES

We adopt the Monte Carlo simulation study to access the performance of the MLE's of $\Theta=(\beta,\gamma,\theta)$ through Mathematica 9 version. We generate different n sample observation from the quantile function in equation (20) above of the model ELEG distribution. The parameters are estimated by maximum likelihood method. We considered different sample size =20, 30, 50, 100, 300 and 500 and the number of repetition is 10000. The true parameters value as γ,β and θ with three different sets of values, in Tables 4.1 and 4.2, of below shows the bias and mean squared error (MSE) of the estimate parameters at different parameter values. We observed that, when we increase sample sizes "n" the bias and Mean square error for the ELEP and ELEG models given below as: (β,γ,θ) decreases with respect to the best estimation.

Table 4.1
The Bias and MSE on Monte Carlo Simulation for Parameters Values of ELEP Distribution

Parameter	**True Value**	**Sample Size *n***	**Mean**	**Bias**	**MSE**
β	3	$n = 20$	3.2851	0.2851	1.0815
		$n = 30$	3.2426	0.2426	0.8638
		$n = 50$	3.2271	0.2271	0.7814
		$n = 100$	3.2115	0.2115	0.7465
		$n = 300$	3.1636	0.1634	0.4319
		$n = 500$	3.0119	0.0119	0.2126
γ	3	$n = 20$	3.3615	0.3615	0.8614
		$n = 30$	3.2425	0.2425	0.8017
		$n = 50$	3.2049	0.2049	0.6019
		$n = 100$	3.1519	0.1519	0.5442
		$n = 300$	3.1414	0.1414	0.3610
		$n = 500$	3.0123	0.0123	0.1012
θ	0.5	$n = 20$	0.6523	0.1513	0.4536
		$n = 30$	0.6421	0.122	0.4198
		$n = 50$	0.6021	0.1021	0.3357
		$n = 100$	0.5227	0.0527	0.2029
		$n = 300$	0.5076	0.0126	0.0772
		$n = 500$	0.3069	0.0067	0.0129

Table 4.2
The Bias and MSE on Monte Carlo Simulation for Parameters Values for ELEG Distribution

Parameter	**True Value**	**Sample Size *n***	**Mean**	**Bias**	**MSE**
β	3	$n=20$	3.3284	0.3284	1.0503
		$n=30$	2.3045	0.2601	0.9211
		$n=50$	4.1344	0.2328	0.8434
		$n=100$	3.1243	0.1243	0.6462
		$n=300$	3.1204	0.1204	0.4219
		$n=500$	2.9736	-0.0164	0.1235
γ	3	$n=20$	3.2574	0.2574	0.9128
		$n=30$	2.3613	0.2320	0.8412
		$n=50$	3.1173	0.1173	0.7442
		$n=100$	3.1210	0.1210	0.6015
		$n=300$	3.1024	0.1024	0.4512
		$n=500$	3.0134	0.0134	0.1336
θ	0.5	$n=20$	0.6921	0.1921	0.4764
		$n=30$	0.6574	0.1274	0.3226
		$n=50$	0.6221	0.1221	0.3015
		$n=100$	0.5023	0.0423	0.1369
		$n=300$	0.4976	0.0126	0.1125
		$n=500$	0.4069	0.069	0.0275

Given first three sample moments, the corresponding $\Theta=(\beta,\gamma,\theta)$ values are estimated from the actual theoretical first three population moments derived from (The sampling distributions of estimated $\Theta=(\beta,\gamma,\theta)$ are given in Table 4.3 based on various sample sizes. For small samples, the percentage of estimates falling in the indicated interval increases with larger sample size. Using this range, we estimate Θ by the method of moments. If we include omitted data, we expect larger Mean Square Error (MSE). This MSE, however, decreases with increasing sample size

Table 4.3

Percentage of Sample Estimates of $\Theta = (\beta, \gamma, \theta)$ through Method of Moments (MM) for the ELEP Model

n	% Estimated Values of Parameter in Indicated Interval with $\beta = 3$	% Estimated Values of Parameter in Indicated Interval with $\gamma = 3$	% Estimated Values of Parameter in Indicated Interval with $\theta = 0.5$
	$2.5 < \hat{\beta} < 3.5$	$2.5 < \hat{\gamma} < 3.5$	$0.3 < \hat{\theta} < 0.7$
30	85.28%	84.30%	80.52%
50	91.26%	88.32%	86.52%
100	94.94%	92.13%	89.71%
250	97.62%	96.44%	94.76%
500	99.23%	98.64%	96.89%

Table 4.4

Percentage of Sample Estimates of $\Theta = (\beta, \gamma, \theta)$ through Method of Moments (MM) for the ELEG Model

n	% Estimated Values of Parameter in Indicated Interval with $\beta = 3$	% Estimated Values of Parameter in Indicated Interval with $\gamma = 3$	% Estimated Values of Parameter in Indicated Interval with $\theta = 0.5$
	$2.5 < \hat{\beta} < 3.5$	$2.5 < \hat{\gamma} < 3.5$	$0.3 < \hat{\theta} < 0.7$
30	88.38%	84.31%	83.12%
50	93.45%	87.62%	86.34%
100	95.14%	93.43%	89.67%
250	98.62%	98.24%	97.25%
500	99.61%	99.12%	98.37%

4.9 APPLICATIONS

In this section, the flexibility of some special models of *ELEPS* family is examined using three real data sets. We illustrate the superiority of new selected distribution as compared with some sub-models.

Based on the maximum-likelihood method, the unknown parameters of each distribution are estimated. Some selected measures as; Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*), the correct Akaike information criterion (*CAIC*), and the Kolmogorov-Smirnov (*k*-*s*) are obtained to compare the fitted models (as seen in Table 4.1). The mathematical form of these measures is as follows:

$$AIC = 2k - 2\ln L, \qquad CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = k\ln(n) - 2\ln L,$$

where k is the number of models parameter, n is the sample size and $\ln L$ is the maximized value of the log- likelihood function under the fitted models.

Also, $k - s = \sup_y \left[F_n(y) - F(y)\right]$, where $F_n(y) = \frac{1}{n}$ (number of observation $\leq y$), and $F(y)$ denotes the cdf. The best distribution is the distribution corresponding to the lower values of, *AIC*, *AICC*, *BIC*, and *k*-*s* statistics. The results for mentioned measures for all models are reported in Tables 4.4 and 4.6.

4.9.1 Aircraft Windshield Data Set

The first data set correspond the failure times of 84 for a particular model aircraft windshield. This data are reported in the book "Weibull Models" by Murthy et al. (2004, p.297)[12]. This data consist of 84 failed windshield, the unit for measurement is 1000 h. The data are:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

We estimated unknown parameters of the distribution by maximum likelihood method as describe in section 5 by using the R code to find the best fit of the data. We use some measures of goodness of fit, including Kolmogorov Smirnov (K-S). For this real data set, we have fitted exponentiated Lindley exponential binomial (ELEB) distribution, exponentiated Lindley exponential geometric (ELEG) distribution, exponentiated Lindley exponential logarithmic (ELEL) distribution, exponentiated Lindley exponential Poisson (ELEP) distribution.

Table 4.5
Criteria for Comparison for First Data Set

Model	$k-s$	*AIC*	*CAIC*	*BIC*
ELEB	0.7411	268.50	269.010	278.227
ELEG	0.0820	262.19	262.492	269.485
ELEL	0.3329	262.68	263.126	275.403
ELEP	0.0530	265.59	265.892	272.884

Values of K- S, AIC, CAIC, and BIC are listed in Tables 4.4 and 4.5. According to the criterion K- S, AIC, AICC and BIC, we found that the *ELEB, ELEG, ELEL and ELEP* distributions are all fit for the aircraft windshield data set. The histogram of two data sets and the estimated PDFs, CDFs and P-P plots for the fitted data model are displayed in Figures (4.5, 4.6, 4.7, 4.8, 4.9 and 4.10). It is clear from Tables 4.4 and 4.5 and Figures (4.5, 4.6, 4.7, 4.8, 4.9 and 4.10) that the EME family of distributions fit to the histogram and therefore could be chosen as the best model for both data set.

Also the plots of the estimated densities and estimated cumulative of the fitted models are achieved in Figures 4.5 and 4.6.

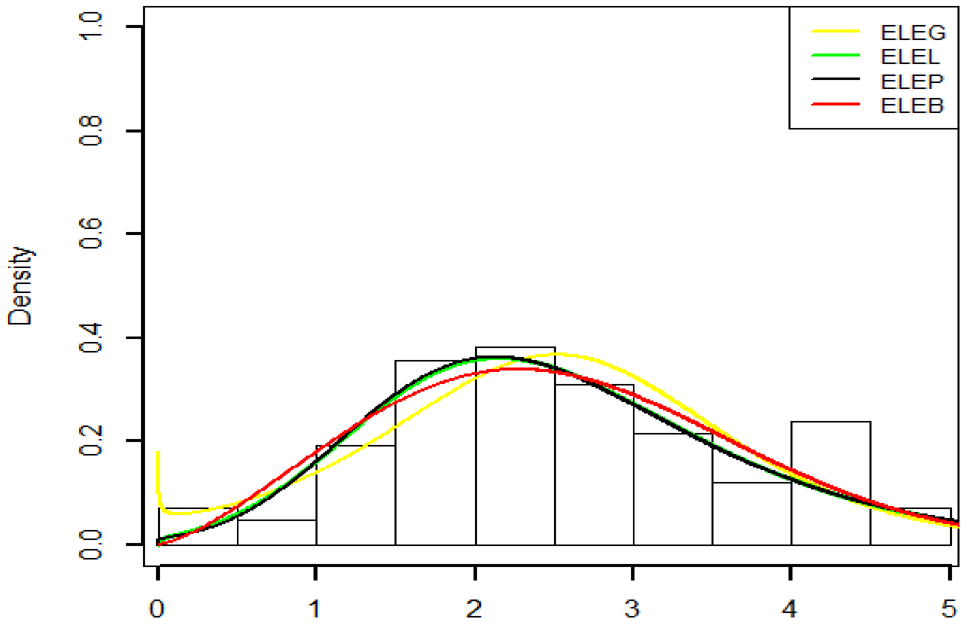

Figure 4.5: Estimated Densities of Models for the First Data Set

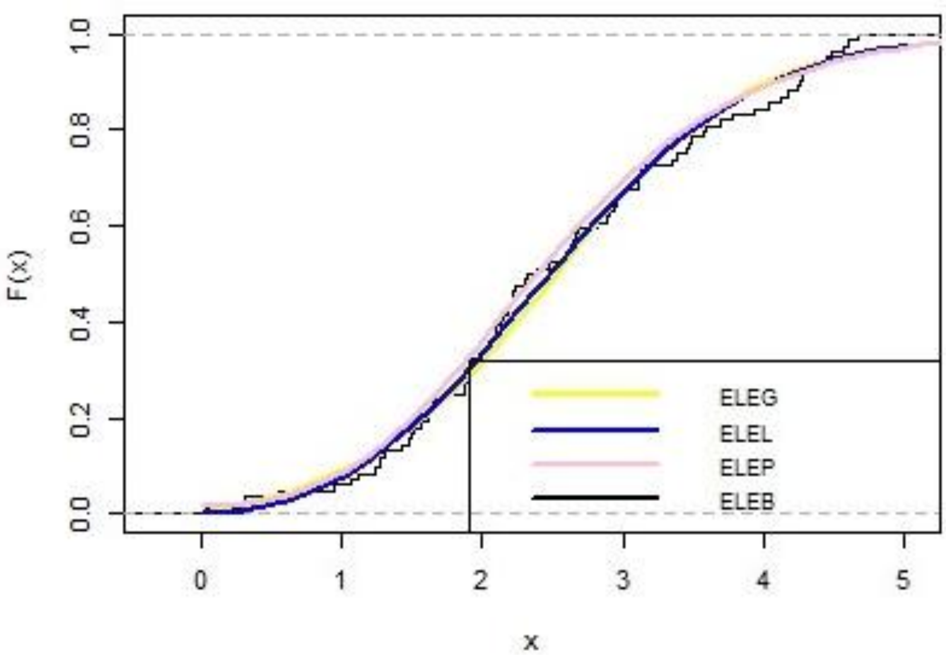

Figure 4.6: Estimated Cumulative Densities of Models for the First Data Set

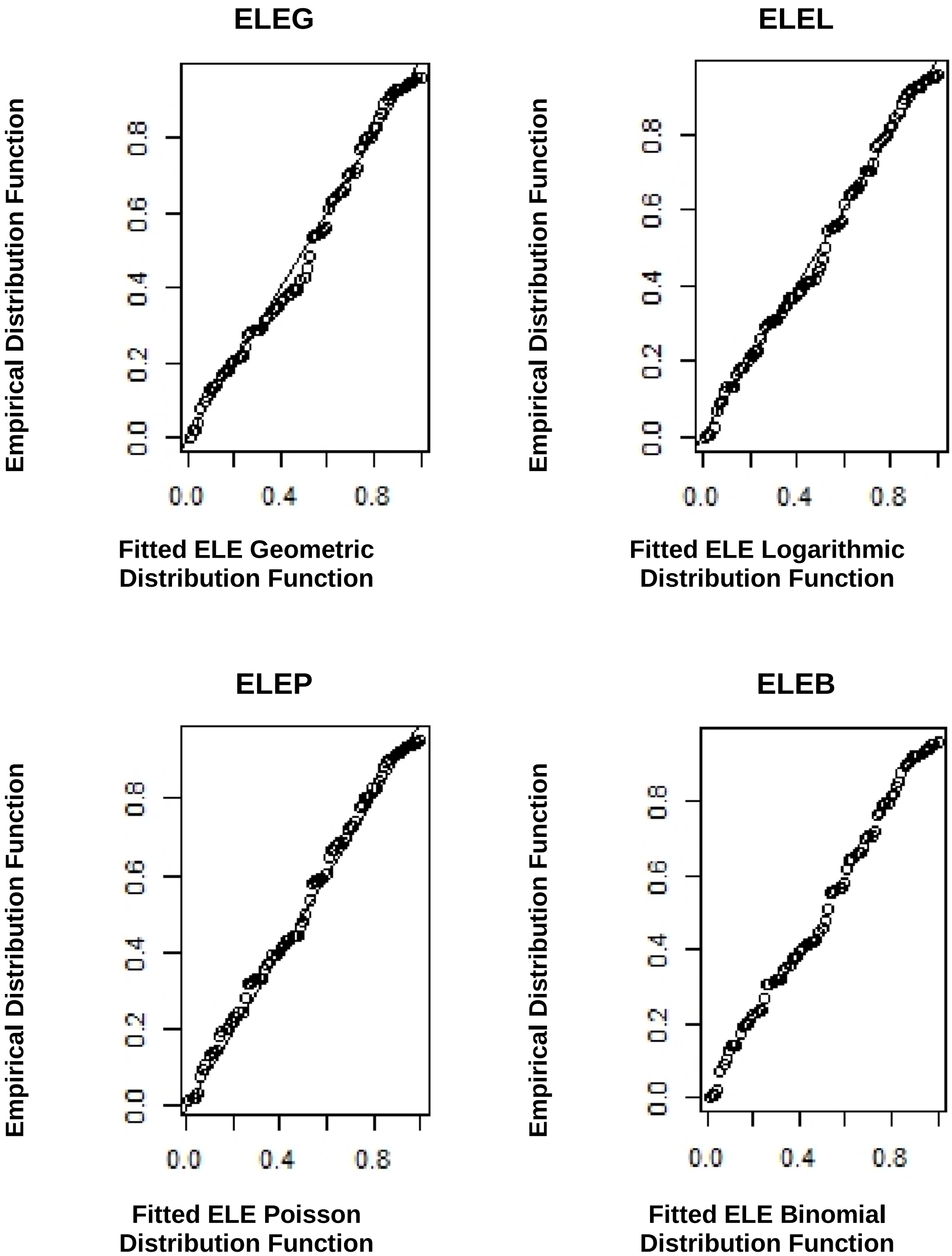

Figure 4.7: The Probability–Probability Plots for the Aircraft Windshield Data Set

4.9.2 2nd Data Set

The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:

> 0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

Table 4.6
Criteria for Comparison for 2nd Data Set

Model	$k-s$	*AIC*	*CAIC*	*BIC*
ELEB	0.359	157.23	158.773	162.97
ELEG	0.131	154.51	155.401	158.81
ELEL	0.191	168.95	159.422	163.63
ELEP	0.354	155.43	156.320	159.73

For the second data set, the values of *k-s*, *AIC*, *BIC* and *CAIC* are record in Table 4.6.

The plots of the estimated cumulative and estimated densities of the fitted models are achieved in Figures 4.8 and 4.9 respectively.

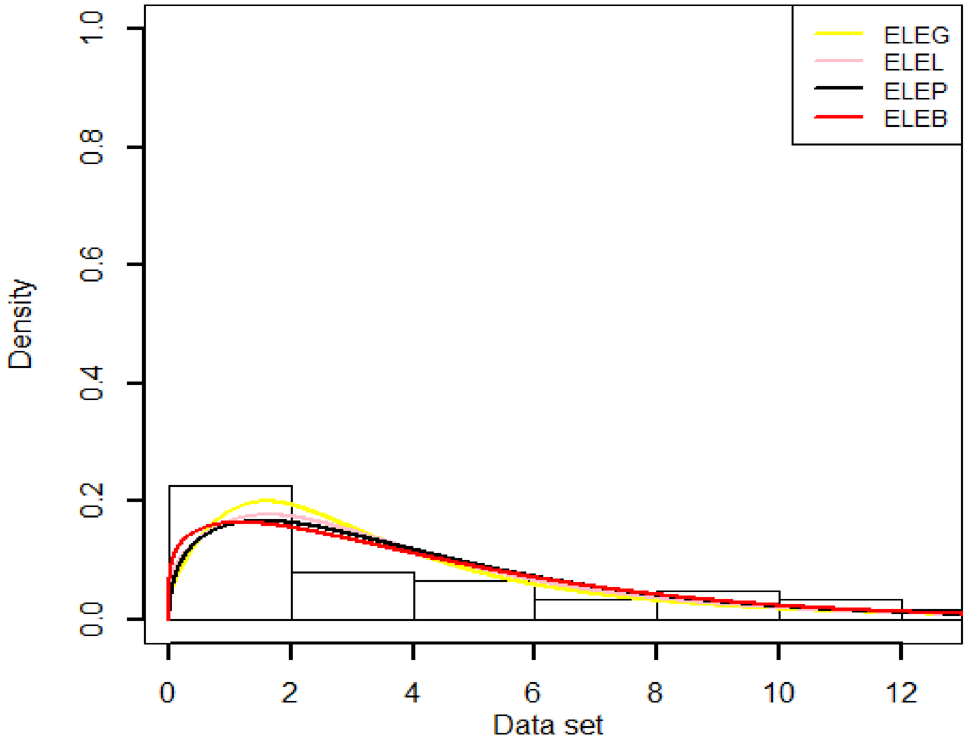

Figure 4.8: Estimated Densities of Models for the Bjerkedal (1960) Data Set

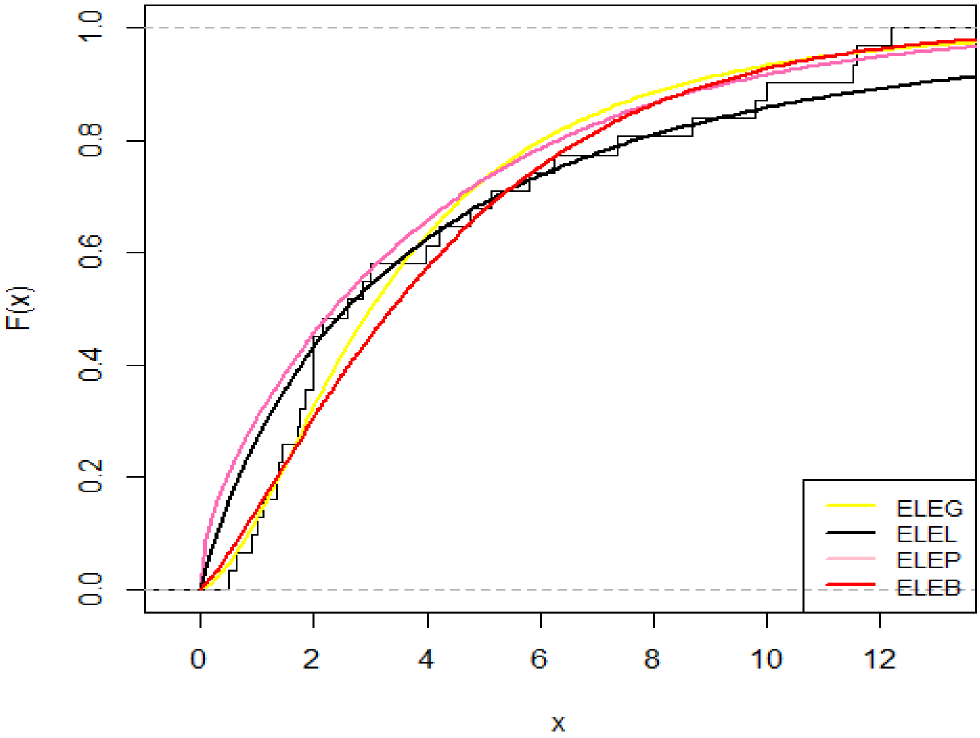

Figure 4.9: Estimated Cumulative Densities of Models for the Second Data Set

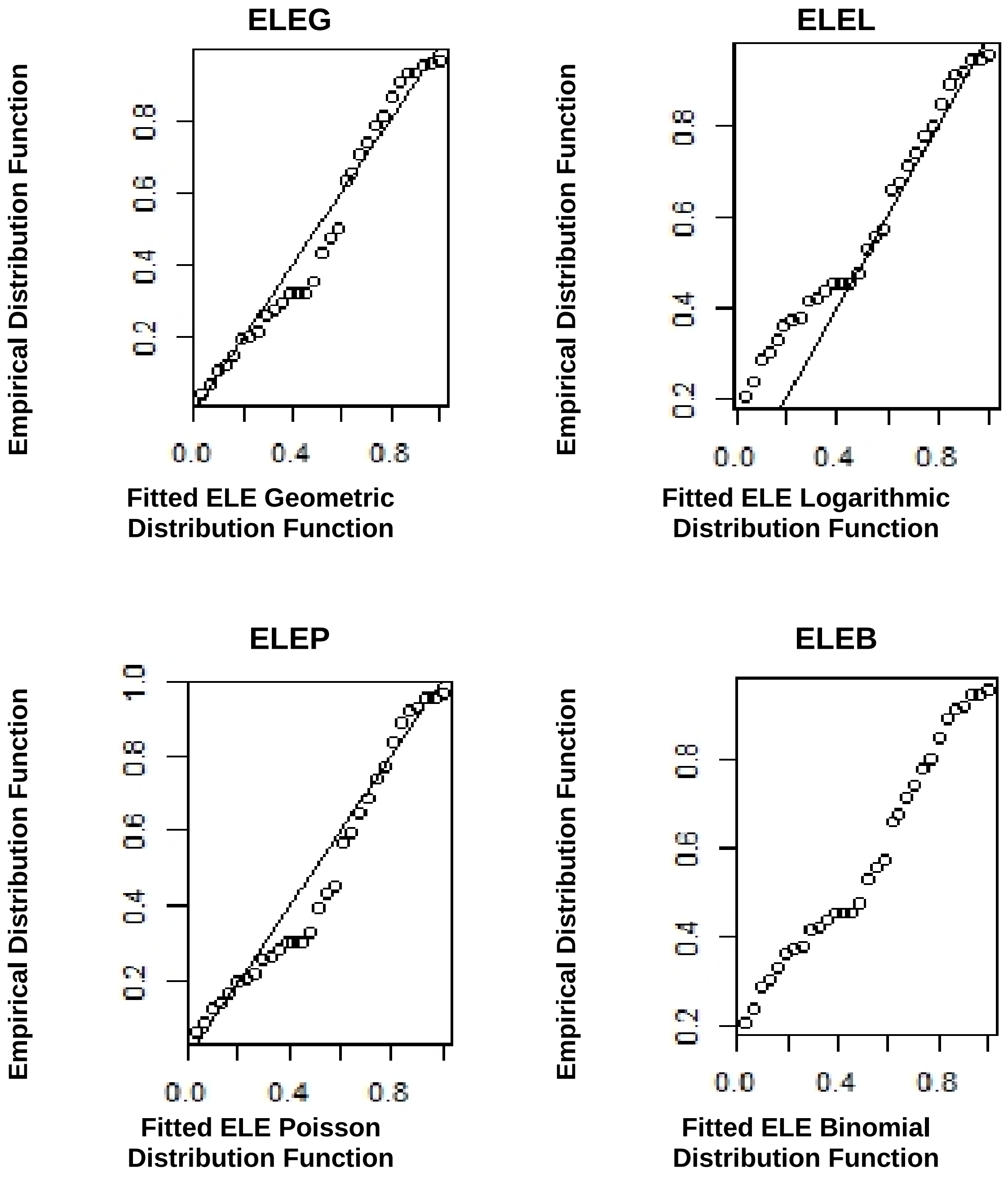

Figure 4.10: The Probability–Probability Plots for the Bjerkedal (1960) Data Set

It is clear from the above two figures that the family of *ELEPS distribution* has the fit on real data set.

CHAPTER 5

CONCLUDING REMARKS

We introduce a new class of lifetime models called the exponentiated Lindley exponential power series**.** This new family is obtained by compounding the exponentiated Lindley exponential distribution and truncated power series distributions. More specifically, the exponentiated Lindley exponential power series covers several new distributions. Also, mathematical properties of the new family, including expressions for density function, moments, moment generating function, quantile function, order statistics and entropy are provided. The hazard rate function has various shapes such as constant, increasing, decreasing, and bathtub. By simulation procedures it is discovered that the ML estimators are consistent since the bias and MSE approach to zero when the sample size increases. The usefulness of the model associated with this family is illustrated by two real data sets and the new model provides a better fit than the models provided in literature.

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