National College of Business Administration & Economics Lahore

ESTIMATORS IN SAMPLING FOR LINEAR AND NON-LINEAR RELATIONSHIP

BY

UZMA YASMEEN

**MASTER OF PHILOSOPHY
IN
STATISTICS**

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UZMA YASMEEN

A dissertation submitted to Faculty of Social Sciences

In Partial Fulfillment of the Requirements for the Degree of

MASTER OF PHILOSOPHY IN STATISTICS

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بسم الله الرحمن الرحيم

IN THE NAME OF ALMIGHTY ALLAH

THE MOST BENEFICENT, COMPASSIONATE

AND THE MOST MERCIFUL

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Dissertation Committee:

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National College of Business
Administration & Economics

DECLARATION

This is to certify that this research work has not been submitted for obtaining similar degree from any other university / college.

UZMA YASMEEN
December, 2013

DEDICATED
TO

My Parents,

and specially to my sweet and nice brother (late)

Shafique-ur-Rehman

His deep affection and

consideration to me

is unforgettable

where time matters nothing,

His physically attachment with him

is a bond Immortal?

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis entitled **"ESTIMATORS IN SAMPLING FOR LINEAR AND NON-LINEAR RELATIONSHIP"** has been carried out and completed by **Uzma Yasmeen** under my supervision during her **M.Phil. Statistics** programme.

(Prof. Dr. Muhammad Hanif)
Supervisor

SUMMARY

In this dissertation, we have developed new estimators of the finite population mean based on the information from two transformed auxiliary variables by using a simple random sampling without replacement (SRSWOR) method. We have derived the expressions for the mean square error as well as the bias of the new estimators up to the first order of approximation. We have obtained the conditions under which these estimators are more efficient than the existing estimators. Empirical studies have also been conducted to show the improvement in efficiency of the proposed estimators over the existing estimators.

Introduction has been given in Chapter 1. A brief history of the use of auxiliary information in the estimation of population mean has been discussed in Chapter 2, whereas Chapter 3 presents some estimators already available in the literature along with the expressions of their mean square errors.

The important part of this dissertation starts from Chapter 4 where we have developed a new ratio-cum-ratio estimator and product-cum-product estimator of the finite population mean by using information of two transformed auxiliary variables. In Chapter 5, we have developed a generalized ratio-cum-ratio estimator of the finite population mean. In Chapter 6, we have proposed four new estimators of the finite population mean based on two auxiliary variables.

In Chapter 7, empirical studies have been conducted for examining the performance of proposed estimators with the available using the data of several populations from different sources.

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CHAPTER 1

INTRODUCTION

1.1 SURVEY SAMPLING

Sample surveys are used to collect data on various matters like business, income, education, health, employment etc. Sampling strategy may be divided into two parts. First is the selection method, by which units are drawn from a finite population. Secondly, the estimation procedure, by which inferences is drawn regarding the population parameters. The Inferences may be divided into enumerative and analytical inferences. Enumerative inference describes finite population while analytical inference explains the population characteristics of the underlying distribution. Enumerative inference concerns by means of estimate of some population parameters like means, totals, percentages and ratios. For example, we may be interested to regress household earnings on variables like number of working middle-aged, education level of family head etc. Regression imply explanatory model. Certain model gives its own probability arrangement, which guided directions in favor of more inferences used analytical and enumerative inferences. In different areas of statistics similar method is used.

Earlier it was thought the best way of drawing a sample was a purposive selection method which was based on the researcher's knowledge of population regarding the closely correlated characteristic. Neyman's (1934) paper was the beginning of a new era. Neyman (1934) found that the use of a random selection procedure had its basis on a sound scientific theory, which made the possibility to predict the validity of the estimates. The use of optimum allocation in stratified sampling scheme was suggested by Neyman (1934). Stephan (1948) says these words about sampling:

> "*Any scientific observation, whether statistical or not, is based on sampling.*"

1.2 AUXILIARY INFORMATION

In the survey sampling, the auxiliary information is very common for estimation of population parameters like population mean, ratio, product, total etc. In survey sampling the history of auxiliary information is as old as the history of survey sampling. The population of England estimated by the first statistical scientist Graunt (1662) based on classical ratio estimate with auxiliary information.

> *"Auxiliary information is any information that is not directly related to the target of estimation. Often direct information is not available in sufficient quantity but auxiliary information is available at a fraction of the cost of the direct information. Sometimes no special effort is required because auxiliary data for one quantity is the direct information for another".*[Kish (1965)]

The auxiliary information may in actual fact be available in numerous types. Tripathi (1970) and Das (1988) recommended in the following four ways:

i) *"The values of one or more auxiliary variables may be available a priori only for some units of a finite population.*

ii) *"Values of one or more parameters of auxiliary variables i.e. population mean(s), population proportion(s), variance, coefficient(s) of variation, Coefficient of skewness may be known. In other words one or more parameters are known."*

iii) *"The exact values of the parameters are not known but their estimated values are known." And*

iv) *"The values of one or more auxiliary variables may be known for all units of a finite population."*

According to Tripathi (1970, 1973, and 1976), the auxiliary information in survey sampling may perhaps utilized in the following four ways, i.e.

i) *"At the planning and designing stage of a survey, the auxiliary information may be used, For example, in stratifying the population, the strata can be made according to supplementary information."*

ii) *"The supplementary information may be used for the purpose of estimation i.e. ratio, regression, difference and product estimators etc."*

iii) *"The auxiliary information may be used while selecting the sample i.e. Probabilities proportional to size sampling.*

iv) *"The auxiliary information may be used in mixed ways i.e. combining at least two of the above."*

Laplace (1820) used the information on auxiliary variable with total no of birth registered during previous year for the estimation of population of France. For the estimation of the mean area of leaves on the plant, Watson (1937) used regression of a leaf area on leaf weight.

1.3 EXPONENTIAL-TYPE ESTIMATORS

Many ratio and regression-type estimators are available in literature for situations when there is a linear relationship between the auxiliary variable and the variable under study. The exponential-type estimators are used in which situation when there is a weak linear relationship between the auxiliary and study variables. The exponential type estimators may be more useful as compared to ratio and regression type estimators in case of weak linear correlation. In the last two decades, many exponential-type estimators have been developed by many researchers utilizing the auxiliary information for single phase sampling as well as for two phase sampling scheme.

1.4 OBJECTIVE OF THE STUDY

The major objectives of the study are:

- To develop simple and generalized ratio-cum-ratio and product-cum-product type estimators in the exponential form using two auxiliary variables.
- To study the gain in the efficiency of the new estimators over some available estimators both theoretically and empirically.

1.5 STATEMENT OF THE PROBLEM

Many statisticians are working for efficient estimation methods to reduce the error in estimates. The development carries on in the form of simple ratio estimators as well as exponential-type estimators. The available exponential-type estimators are relatively simple and most of these are based upon information of a single auxiliary variable. In the present study, our task is to propose some new exponential estimators using information on two auxiliary variables.

The efficiencies of the estimators have been increased by using auxiliary variable in the literature. The estimators based on the auxiliary information are used when the variable of interest and auxiliary variable have a linear or exponential relationship. Practically, it is possible that one auxiliary variable may be exponentially related and other auxiliary variable may be linearly related with variable of interest. In this study, we have developed ratio-cum-exponential type estimators for such situations.

1.6 NOTATIONS

$$\left.\begin{aligned} &\theta = \frac{1}{n} - \frac{1}{N} = \frac{N-n}{Nn}, g = \frac{n}{N-n} \\ &\bar{e}_y = \frac{\bar{y}-\bar{Y}}{\bar{Y}}, \bar{e}_z = \frac{\bar{z}-\bar{Z}}{\bar{Z}}, \bar{e}_x = \frac{\bar{x}-\bar{X}}{\bar{X}}, \end{aligned}\right\} \tag{1.6.1}$$

$$\left.\begin{aligned} &E(\bar{e}_x) = 0, E(\bar{e}_y) = 0, E(\bar{e}_z) = 0, \\ &E(\bar{e}_y) = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{S_y^2}{\bar{Y}^2} = \theta C_y^2, \\ &E(\bar{e}_x) = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{S_x^2}{\bar{X}^2} = \theta C_x^2, \\ &E(\bar{e}_z) = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{S_z^2}{\bar{Z}^2} = \theta C_z^2, \\ &E(\bar{e}_x\bar{e}_y) = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{S_{yx}}{\bar{X}\bar{Y}} = \theta\rho_{xy} C_x C_y, \\ &E(\bar{e}_x\bar{e}_z) = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{S_{xz}}{\bar{X}\bar{Z}} = \theta\rho_{xz} C_x C_z, \\ &E(\bar{e}_y\bar{e}_z) = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{S_{yz}}{\bar{Y}\bar{Z}} = \theta\rho_{yz} C_y C_z. \end{aligned}\right\} \tag{1.6.2}$$

$$K_{ij} = \rho_{ij}\frac{C_i}{C_j}, \tag{1.6.3}$$

where $i = x, y, z$ and $j = x, y, z$. Also, let

$$x_i^* = (1+g)\bar{X} - gx_i, \tag{1.6.4}$$

and

$$z_i^* = (1+g)\bar{Z} - gz_i, \tag{1.6.5}$$

where $i = 1,\ 2, \ldots, N,$ then

$$\bar{x}^* = (1+g)\bar{X} - g\bar{x}, \tag{1.6.6}$$

and

$$\bar{z}^* = (1+g)\bar{Z} - g\bar{z}, \tag{1.6.7}$$

are also unbiased estimators of $\bar{X}$ and $\bar{Z}$ respectively.

CHAPTER 2

LITERATURE REVIEW

The literature of survey sampling describes a great variety of techniques for using auxiliary information in order to obtain improved estimators for estimating some most common population parameters such as population total, population mean, population proportion, population ratio etc. The work of Neyman (1938) may be referred to as one of the pioneer works where auxiliary information has been used.

The use of auxiliary information at the estimation stage started with the efforts of Cochran (1940). He developed the ratio estimator to estimate the population means or total of the study variate Y by using the available supplementary information on an auxiliary variate X, positivity correlated with Y.

If there is a negative correlation between the auxiliary variate X and the study variable Y, then the product method of estimation is quite effective which was studied by Robson (1957) and Murthy (1964).

Searls (1964) utilized the co-efficient of variation of the variable of interest at the estimation stage. Motivated by Searls (1964) method, Sen (1978), Sisodia and Dwivedi (1981) and Upadhyaya and Singh (1984) used coefficient of variation of auxiliary variables in the estimation of finite population mean. The co-efficient of kurtosis at the estimation stage was first utilized by Singh et al. (1973) and later, it was used by Searls and Intarapanich (1990), Upadhyaya and Singh (1999), Singh (2003) and Singh et al. (2004) for estimating the finite population mean.

Mohanty (1967) found that the use of a second available auxiliary source may increase the precision of the estimates of population mean. He combined the regression and the ratio methods of estimation for the estimation of the finite population mean with the help of two auxiliary sources.

Srivastava (1970) provided a new general family of the ratio estimators for the estimation of the finite population mean of the study variable using a single auxiliary variable. A general family of the chain-ratio estimators of population mean was developed by Srivastava et al. (1990) using two auxiliary variables.

Chand (1975) developed two chain ratio type estimators of finite population mean using two auxiliary variables. A chain-ratio-to regression type estimator was developed by Kiregyera (1980) using information on two auxiliary variables. Kiregyera (1980) studied the relative performance of the new estimator to the simple mean estimator, the ratio estimator and the chain-ratio type estimator suggested by Chand (1975).

Srivenkaramana (1980) presented a dual-to-ratio estimator for the population mean of the study variable. The dual estimator was developed by using a transformed on the auxiliary variable. Srivenkaramana found that the use of the transformed auxiliary variable may increase the efficiency of the estimators.

Two new estimators of the population of the variable under study were developed by Kiregyera (1984). Both estimators were constructed using two auxiliary variables and their efficiency was studied under a super-population model and empirically.

Following the technique of Kiregyera (1984), three new estimators were developed by Mukerjee et al. (1987). These estimators were also extended to the case when information on multi-auxiliary variables was available.

Using prior information on the co-efficient of correlation between the auxiliary and the study variables, Singh (1987) developed a regression type estimator for the population mean. The properties of the new estimator were studied and it was found that the new estimator had improvement in efficiency over the conventional regression estimator.

A general theory for the estimation of a general function of the population parameters was provided by Tripathi et al. (1988) by utilizing a general function of the supplementary parameter. They estimated the variance of the study variable for a bivariate population by using the general results obtained from the general function of the population parameters.

For a simple random sampling scheme, Tripathi and Khattree (1989) analyzed the estimation of population mean using information on multi-auxiliary variables. The results were extended to a two-occasion case by Tripathi (1989).

Bahl and Tuteja (1991) were the first to suggest exponential type estimators for the estimation of population mean of the variable under study using auxiliary information.

Tripathi and Singh (1992) developed a generalized transformation for dealing with the situations of both positive and negative correlations simultaneously. They obtained a wide class of the unbiased product-type estimators of population mean which also included the estimators suggested by Rao (1983), Singh (1987) and Rao (1987) etc.

Using known information on the co-efficient of variation of the variable under study, Tracy and Singh (1997, 1998) suggested a new class of estimators for the finite population mean. They found that the new estimator had improvement in efficiency over the mean per unit estimator, the conventional ratio estimator, the product estimator, the Srivastava's (1970) estimator and the estimator suggested by Sen (1978).

When the population mean of the first auxiliary variable is not known and information is available on second auxiliary variable, a regression-type estimator for the population mean of the study variable was constructed by Roy (2003). This estimator was more efficient than the common two-phase ratio type and regression type estimators with two auxiliary variables, recommended by Mohanty (1967), Chand (1975), Kiregyera (1980, 84) and Sahoo et al. (1993). Bajwa (2003) modified Mohanty (1967) estimator in single phase and two phase sampling.

Singh and Vishwakarma (2007) suggested exponential ratio-type and product type estimators for two-phase sampling based on a single auxiliary variable.

In single and two- phase sampling, Haq (2008) proposed general family of estimators by using two auxiliary attributes. Generalized ratio and regression estimator have developed by Ahmed (2008) with multi auxiliary variables for estimating the population mean.

An empirical study conducted, by Tahir (2008) for comparisons the performance of Chand (1975), Singh and Espejo (2007). Swain (2012) suggested generalized classes in the occurrence of two auxiliary variables with SRSWOR for modified ratio-type and regression –cum-ratio type estimators used for finite population mean of study variable.

Using available information on two auxiliary variables, Hanif et al. (2009) developed a new estimator obtained by combining the regression type estimator with the ratio and product type estimators. They obtained the expression for the mean square error of their new suggested estimator to the first order of approximation. They also made efficiency comparisons of their new suggested estimator with the Samiuddin and Hanif (2006) estimator. It was observed that the new estimator performed better than the available estimators.

Sharma and Tailor (2010) proposedratio-cum-dual to ratio estimator for finite population mean. In the class of estimators an asymptotic optimum estimator is identified connected its mean squared error method. They reviewed the properties of their proposed estimator over other estimators.

Using the weighting, transformation and the model-based approaches, Nazuk et al. (2010) developed some new ratio estimators for the population mean. It was found that these estimators were more efficient than the Chakrabarty (1979) estimator, Singh and Singh (1997) estimator, the Singh (2002) estimator and the Singh et al. (2006) estimator.

For situations when the study variable is of qualitative type, Singh et al. (2010) proposed a new family of estimators and obtained their bias and mean square error. In order to show the superiority of the new estimator over some available estimators, they also conducted an empirical study.

Shukla et al. (2012) developed new families of estimators for the population mean of the variable under study utilizing two auxiliary variables. They constructed three different classes of estimators of the population mean and obtained their optimum mean square errors and also conducted an empirical study.

Noor-ul-Amin and Hanif (2012) proposed some exponential type estimators for estimating the population mean in two-phase sampling using information on two auxiliary variables.

For known population parameters of the auxiliary variable, Subramani and Kumarapandiyan (2012) studied the problem of estimation of population mean. They developed strategies for improving the performance of the available ratio estimators and illustrated the results with the help of numerical examples.

Onyeka et al. (2013) developed six new estimators for the ratio of a finite population. These estimators were developed using transformation on auxiliary variable in simple random sampling. They also studied the properties of the new transformed estimators.

Jeelani et al. (2013) studied estimation of mean by utilizing the linear combination of the population co-efficient of skewness and the quartile deviation of the auxiliary variable. They found that their new modified ratio estimators performed better than the available estimators. In order to support the results, they carried out an empirical study.

For estimation of finite population mean of the study variable Choudhury and Singh (2013) suggested a class of product-cum-dual to product estimators. This asymptotical optimum estimator has also been identified along with its approximate bias and mean square error.

Swain (2013) studied various modified ratio and product type estimators and compared their mean square errors and biases. To compare the efficiencies of the estimators, Swain (2013) also provided numerical illustrations using some natural population.

CHAPTER 3

SOME AVAILABLE ESTIMATORS OF POPULATION MEAN

3.1 INTRODUCTION

In this Chapter, different estimators using single and two auxiliary variables for estimating population mean in single-phase sampling have been presented. These estimators have been discussed along with their mean square error and biases.

Let $=\bar{Y}\left(1+\bar{e}_{y_2}\right)\exp\left[\alpha_{FD}\frac{1-1-\bar{e}_{x_1}}{1+1+\bar{e}_{x_1}}+\beta_{FD}\frac{1-1-\bar{e}_{z_2}}{1+1+\bar{e}_{z_2}}\right]$ be the total number of units in the population. A random sample of size $E\left[\bar{e}_{y_2}\left(\bar{e}_{z_2}-\bar{e}_{z_1}\right)\right]$ $=\theta_2\rho_{yz}C_yC_z-\theta_1\rho_{yz}C_yC_z$ is selected by using a simple random sampling without replacement method. The study variable and the two auxiliary variables are denoted by $C_j=\frac{S_j}{\bar{j}}$, $t=\bar{Y}\left(1+\bar{e}_{y_2}\right)\exp\left[\alpha\frac{-\bar{e}_{x_1}}{2+\bar{e}_{x_1}}+\beta\frac{\bar{e}_{z_2}-\bar{e}_{z_1}}{2+\bar{e}_{z_1}+\bar{e}_{z_2}}\right]$

$\alpha=2H_{yx}$

and $\frac{\partial MSE}{\partial\beta}=\frac{2\beta}{4}\left(\theta_2-\theta_1\right)C_z{}^2-\left(\theta_2-\theta_1\right)H_{yz}C_z{}^2=0$ with their sample means $\bar{y}$, $\bar{x}$ and $\bar{z}$ respectively.

3.2 SOME AVAILABLE ESTIMATORS

3.2.1 The Mean per Unit Estimator

The mean per unit estimator is given by

$$\bar{y}=\frac{1}{n}\sum_{i=1} y_i\,. \tag{3.2.1}$$

The variance of $\bar{y}$ is given by

$$\mathrm{Var}(\bar{y})=\theta\bar{Y}^2C_y^2. \tag{3.2.2}$$

3.2.2 The Classical Ratio Estimator

The classical ratio estimator for estimating the finite population mean, suggested by Cochran (1940), is given as:

$$t_1 = \bar{y}\left[\frac{\bar{X}}{\bar{x}}\right]. \tag{3.2.3}$$

The mean square error of t_1 is given as:

$$MSE(t_1) \approx \bar{Y}^2\theta\left[C_y^2 + C_x^2\left(1 - 2K_{yx}\right)\right]. \tag{3.2.4}$$

3.2.3 The Product Estimator

The product estimator of population mean, proposed by Robson (1957), is given by

$$t_2 = \bar{y}\left[\frac{\bar{x}}{\bar{X}}\right]. \tag{3.2.5}$$

The mean square error equation of the estimator t_2 is:

$$MSE(t_2) \approx \bar{Y}^2\theta\left[C_y^2 + C_x^2\left(1 + 2K_{yx}\right)\right]. \tag{3.2.6}$$

3.2.4 The Classical Regression Estimator

The classical regression estimator is given by

$$t_3 = \bar{y} + b_{yx}\left(\bar{X} - \bar{x}\right). \tag{3.2.7}$$

The mean square error of t_3 is given by

$$MSE(t_3) \approx \bar{Y}^2\theta\left[C_y^2\left(1 - \rho_{xy}^2\right)\right]. \tag{3.2.8}$$

3.2.5 Mohanty's (1967) Estimator

Mohanty (1967) proposed an estimator using two auxiliary variables:

$$t_4 = \left[\bar{y} + \beta_{yx}\left(\bar{X} - \bar{x}\right)\right]\frac{\bar{Z}}{\bar{z}}. \tag{3.2.9}$$

The mean square error of t_4 is

$$MSE\left(t_4\right) = \theta\bar{Y}^2\left[C_y^2\left(1-\rho_{xy}^2\right)+\left(C_z - C_y\rho_{yz}\right)^2 - C_y^2\,\rho_{yz}^2 + 2C_y\,C_z\,\rho_{xz}\,\rho_{xy}\right]. \tag{3.2.10}$$

5.2.6 Srivastava's (1967) Estimator

Srivastava (1967) suggested a generalized form of the Classical ratio estimator. The estimator is

$$t_5 = \left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha}, \tag{3.2.11}$$

where α is a constant.

The mean square error of t_5 is given by

$$MSE\left(t_5\right) = \theta\bar{Y}^2\left[C_y^2 + \alpha^2 C_x^2 - 2\alpha C_x C_y \rho_{xy}\right]. \tag{3.2.12}$$

3.2.7 Rao's (1968) Estimator

Rao (1968) proposed the following estimator is

$$t_6 = \bar{y} - W\frac{\bar{y}}{\bar{X}}\left(\bar{x} - \bar{X}\right), \tag{3.2.13}$$

where W is a suitable constant.

The mean square of the Rao's (1968) estimator is

$$MSE(t_6) = \theta \bar{Y}^2 \left(C_y^2 + W^2 C_x^2 - 2W \rho_{xy} C_x C_y \right). \qquad (3.2.14)$$

The estimator t_6 is identical to the mean per unit estimator for $W = 0$.

3.2.8 Walsh's (1970) Estimator

Walsh (1970) suggested the following estimator

$$t_7 = \bar{y} \left[\frac{\bar{X}}{\bar{X} + b\left(\bar{x} - \bar{X}\right)} \right], \qquad (3.2.15)$$

where b is a constant is to be determined.

The mean square error of t_7 is given by

$$MSE\left(t_7\right) = \theta \bar{Y}^2 C_y^2 \left(1 - \rho_{xy}^2\right). \qquad (3.2.16)$$

3.2.9 Sahai's (1979) Estimator

Sahai (1979) proposed the following estimator of population mean

$$t_8 = \bar{y} \left[\frac{\alpha \bar{X} + (1-\alpha)\bar{x}}{\alpha \bar{x} + \left(1-\alpha\right)\bar{X}} \right], \qquad (3.2.17)$$

where α is a constant.

The mean square error of the Sahai's (1979) estimator is given as:

$$MSE\left(t_8\right) = \theta \bar{Y}^2 \left[C_y{}^2 + \left(1-2\alpha\right)^2 C^2{}_x + 2\left(1-2\alpha\right) C_y C_x \rho_{xy} \right]. \qquad (3.2.18)$$

3.2.10 Srivenkataramana's (1980) Estimator

Srivenkataramana (1980) suggested a dual-to-classical ratio estimator as:

$$t_9 = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right), \qquad (3.2.19)$$

where

$$\bar{x}^* = (1+g)\bar{X} - g\bar{x}. \qquad (1.6.6)$$

The mean square error of t_9 is:

$$MSE(t_9) \approx \bar{Y}^2\theta\left[C_y^2 + gC_x^2\left(g - 2K_{yx}\right)\right]. \qquad (3.2.20)$$

The bias of t_9 is:

$$Bias(t_9) \approx -\bar{Y}g\theta\rho_{yx}C_yC_x. \qquad (3.2.21)$$

3.2.11 Bandyopadhyay's (1980) Estimator

Bandyopadhyay (1980) suggested a dual-to-product estimator. This estimator is given by

$$t_{10} = \bar{y}\left(\frac{\bar{Z}}{\bar{z}^*}\right), \qquad (3.2.22)$$

where

$$\bar{z}^* = (1+g)\bar{Z} - g\bar{z}. \qquad (1.6.7)$$

The mean square error of t_{10} is:

$$MSE(t_{10}) \approx \bar{Y}^2\theta\left[C_y^2 + gC_z^2\left(g + 2K_{yz}\right)\right]. \qquad (3.2.23)$$

The bias of t_{10} is:

$$Bias(t_{10}) \approx \bar{Y}g\theta\rho_{yz}C_yC_z. \qquad (3.2.24)$$

3.2.12 Das and Tripathi (1980) Estimator

An estimator proposed by Das and Tripathi (1980) is given as:

$$t_{11} = \frac{\bar{y} - p\left(\bar{x} - \bar{X}\right)}{\left[\bar{x} - q\left(\bar{x} - \bar{X}\right)\right]^{\alpha}} \bar{X}. \tag{3.2.25}$$

where p and q are constants.

The mean square error of this estimator is given by

$$MSE(t_{11}) \approx \theta\left[\bar{Y}^2 C_y^2 + \beta^2 \bar{X}^2 C_x^2 - 2\beta \bar{X}\bar{Y} C_x C_y \rho_{xy}\right]. \tag{3.2.26}$$

3.2.13 Prasad's (1989) Estimator

Prasad (1989) developed the following modification of ratio estimator following the lines of Searls (1964):

$$t_{12} = \frac{k\bar{y}}{\bar{x}} \bar{X}, \tag{3.2.27}$$

where k is a suitable constant. The optimum value of k is given by

$$k = k_{opt} = \frac{\left\{1 + \theta\left(C_x^2 - \rho_{xy} C_x C_y\right)\right\}}{\left\{1 + \theta\left(C_y^2 + 3C_x^2 - 4\rho_{xy} C_x C_y\right)\right\}}. \tag{3.2.28}$$

The minimum mean square error of t_3 is given by

$$MSE_{\min}\left(t_{12}\right) = \bar{Y}^2 \left[1 - \frac{\left\{1 + \theta\left(C_x^2 - \rho_{xy} C_x C_y\right)\right\}^2}{\left\{1 + \theta\left(C_y^2 + 3C_x^2 - 4\rho_{xy} C_x C_y\right)\right\}}\right]. \tag{3.2.29}$$

3.2.14 Naik and Gupta (1991) Estimators

Naik and Gupta (1991) suggested a general class of estimators for estimating the population mean. Their general estimator is given by

$$t_{13} = \bar{y}\left[\frac{\bar{X} + a\left(\bar{x} - \bar{X}\right)}{\bar{X} + b\left(\bar{x} - \bar{X}\right)}\right]^{p}, \tag{3.2.30}$$

where

$$p\left(b - a\right) = k_1 \text{, and } k_1 = \frac{C_y}{C_x}\rho_{xy}.$$

The mean square error of this estimator is:

$$MSE(t_{13}) \approx \theta\bar{Y}^2\left[C_y^2 + K_1^2 C_x^2 - 2K_1 C_y C_x \rho_{xy}\right]. \tag{3.2.31}$$

3.2.15 Bahl and Tuteja (1991) Estimators

Bahl and Tuteja (1991) proposed an exponential ratio-type estimator as:

$$t_{14} = \bar{y}\exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right]. \tag{3.2.32}$$

The mean square error of exponential ratio-type estimator t_{14} is:

$$MSE\left(t_{14}\right) \approx \bar{Y}^2\theta\left[C_y^2 + \frac{C_x^2}{4} - \rho_{xy}C_y C_x\right]. \tag{3.2.33}$$

The bias of t_{14} is:

$$Bias(t_{14}) \approx \frac{\bar{Y}\theta C_x^2}{8}\left[3 - 4K_{yx}\right]. \tag{3.2.34}$$

Bahl and Tuteja (1991) also proposed an exponential-product-type estimator as:

$$t_{15} = \bar{y}\exp\left[\frac{\bar{z}-\bar{Z}}{\bar{Z}+\bar{z}}\right]. \tag{3.2.35}$$

The mean square error of t_{15} is:

$$MSE(t_{15}) \approx \bar{Y}^2\theta\left[C_y^2 + \frac{C_z^2}{4} + K_{yz}C_z^2\right]. \tag{3.2.36}$$

The bias of t_{15} is:

$$Bias(t_{15}) \approx \frac{\bar{Y}\theta C_z^2}{8}\left[4K_{yz} - 1\right]. \tag{3.2.37}$$

3.2.16 Singh et al. (2005) Estimator

Singh et al. (2005) proposed a dual-to-ratio cum product estimator as:

$$t_{16} = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\left(\frac{\bar{Z}}{\bar{z}^*}\right). \tag{3.2.38}$$

The mean square error of t_{16} is:

$$MSE(t_{16}) \approx \bar{Y}^2\theta\left[C_y^2 + gC_x^2\left(g - 2gK_{zx} - 2K_{yx}\right) + gC_z^2\left(g + 2K_{yz}\right)\right]. \tag{3.2.39}$$

The bias of t_{16} is:

$$Bias(t_{16}) \approx \bar{Y}g\theta\left[C_z^2\left(K_{yz} - gK_{xz}\right) - K_{yx}C_x^2\right]. \tag{3.2.40}$$

3.2.17 Samiuddin and Hanif (2006) Estimator

Samiuddin and Hanif (2006) suggested a new estimator with two auxiliary variables as:

$$t_{17} = \left(\frac{\bar{y}}{\bar{x}}\bar{X}\right)\left(\frac{\bar{y}}{\bar{z}}\bar{Z}\right). \tag{3.2.41}$$

The mean square error of t_{17} is given as:

$$MSE(t_{17}) \approx \theta\bar{Y}^2\left[C_y^2 + C_x^2 + C_z^2 - 2C_xC_y\rho_{xy} - 2C_yC_z\rho_{yz} + 2C_xC_z\rho_{xz}\right]. \tag{3.2.42}$$

The generalized chain ratio estimator suggested by Samiuddin and Hanif (2006) is given as:

$$t_{18} = a\left(\bar{y}\frac{\bar{X}}{\bar{x}}\right) + (1-a)\left(\bar{y}\frac{\bar{Z}}{\bar{z}}\right), \tag{3.2.43}$$

where a is constant.

The optimum value of a is given by

$$a_{opt} = \frac{C_yC_x\rho_{xy} - C_yC_z\rho_{yz} - C_xC_z\rho_{xz} + C_z^2}{C_x^2 + C_z^2 - 2C_xC_z\rho_{xz}}. \tag{3.2.44}$$

The minimum mean square error of the estimator t_{18} may be written as:

$$MSE(t_{18}) \approx \theta\bar{Y}^2\left[\left(C_y^2 + C_z^2 - 2\rho_{yz}C_yC_z\right) - \frac{\left(C_xC_y\rho_{xy} - C_yC_z\rho_{yz} - C_xC_z\rho_{xz} + C_z^2\right)^2}{C_x^2 + C_z^2 - 2C_xC_z\rho_{xz}}\right]. \tag{3.2.45}$$

3.2.18 Hanif et al. (2009) Estimator

Hanif et al. (2009) proposed a modification of the Singh and Espejo (2003) estimator as:

$$t_{19} = \left\{\bar{y} + k_1\left(\bar{X} - \bar{x}\right)\right\}\left\{k_2\frac{\bar{Z}}{\bar{z}} + (1-k_2)\frac{\bar{z}}{\bar{Z}}\right\}, \tag{3.2.46}$$

where k_1 and k_2 are constants. The optimum values of k_1 and k_2 are given by

$$k_1 = \frac{S_Y\left(\rho_{XY} - \rho_{XZ}\rho_{YZ}\right)}{S_X\left(1-\rho_{XZ}^2\right)} = \beta_{XY.Z}, \tag{3.2.47}$$

and

$$k_2 = \frac{1}{2}\left\{1 + \frac{\bar{Z}}{\bar{Y}} \cdot \frac{S_Y\left(\rho_{YZ} - \rho_{XY}\rho_{XY}\right)}{S_X\left(1-\rho_{XZ}^2\right)}\right\} = \frac{1}{2}\left\{1 + \frac{\bar{Z}}{\bar{Y}}\beta_{YZ.X}\right\}. \tag{3.2.48}$$

The mean square error of t_{19} is given by

$$MSE\left(t_{19}\right) \approx \theta\bar{Y}^2C_y^2\left(1-\rho_{y.xz}^2\right). \tag{3.2.49}$$

3.2.19 Singh et al. (2009) Estimator

Singh et al. (2009) proposed an exponential-type estimator by following Kadilar and Cingi (2006). The Singh et al. (2009) estimator is given as:

$$t_{20} = \bar{y}\exp\left[\frac{(a\bar{X}+b)-(a\bar{x}+b)}{(a\bar{X}+b)+(a\bar{x}+b)}\right], \tag{3.2.50}$$

where a and b are suitable constants.

The mean square error of t_{20} is

$$MSE\left(t_{20}\right) \approx \bar{Y}^2\theta\left[C_y^2 + C_x^2\gamma\left(\gamma - 2K_{yx}\right)\right], \tag{3.2.51}$$

and the bias of t_{20} is

$$Bias\left(t_{20}\right) \approx \bar{Y}\theta C_x^2\gamma\left[\gamma + K_{yx}\right], \tag{3.2.52}$$

where $\gamma = \dfrac{a\bar{X}}{2(a\bar{X}+b)}$.

3.2.20 Sharma and Tailor (2010) Estimator

Sharma and Tailor (2010) proposed a dual-to-ratio type exponential estimator as:

$$t_{21} = \bar{y}\exp\left[\frac{\bar{x}^* - \bar{X}}{\bar{x}^* + \bar{X}}\right]. \tag{3.2.53}$$

The mean square error of t_{21} is:

$$MSE(t_{21}) \approx \bar{Y}^2\theta\left[C_y^2 + gC_x^2\left(\frac{g}{4} - K_{yx}\right)\right]. \tag{3.2.54}$$

The bias of t_{21} is:

$$Bias(t_{21}) \approx -\frac{\bar{Y}g\theta C_x^2}{2}\left[\frac{g}{2} + K_{yx}\right]. \tag{3.2.55}$$

Sharma and Tailor (2010) also suggested the following dual-to-product type exponential estimator

$$t_{22} = \bar{y}\exp\left[\frac{\bar{Z} - \bar{z}^*}{\bar{Z} + \bar{z}^*}\right]. \tag{3.2.56}$$

The mean square error of t_{22} is:

$$MSE(t_{22}) \approx \bar{Y}^2\theta\left[C_y^2 + gC_z^2\left(\frac{g}{4} + K_{yz}\right)\right]. \tag{3.2.57}$$

The bias of t_{22} is:

$$Bias(t_{22}) \approx \frac{\bar{Y}g\theta C_z^2}{8}\left[3g + 4K_{yz}\right]. \tag{3.2.58}$$

3.2.21 Upadhyaya et al. (2011) Estimators

Generalized exponential ratio-type estimator suggested by Upadhyaya et al. (2011) is given as:

$$t_{23} = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+(a-1)\bar{x}}\right]. \tag{3.2.59}$$

The mean square error of t_{23} is:

$$MSE\left(t_{23}\right) \approx \bar{Y}^2\theta\left[C_y^2 + \frac{C_x^2}{a^2}\left(1-2aK_{yx}\right)\right]. \tag{3.2.60}$$

The bias of t_{23} is:

$$Bias\left(t_{23}\right) \approx \frac{\bar{Y}\theta C_x^2}{2a^2}\left[2a\left(1-K_{yx}\right)-1\right]. \tag{3.2.61}$$

Upadhyaya et al. (2011) also proposed a generalized exponential product-type estimator as:

$$t_{24} = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\left(b-1\right)\bar{x}}\right]. \tag{3.2.62}$$

The mean square error of t_{24} is:

$$MSE\left(t_{24}\right) \approx \bar{Y}^2\theta\left[C_y^2 + \frac{C_x^2}{b^2}\left(1+2bK_{yx}\right)\right]. \tag{3.2.63}$$

The bias of t_{24} is:

$$Bias\left(t_{24}\right) \approx \frac{\bar{Y}\theta C_x^2}{2b^2}\left[3-2b+2bK_{yx}\right]. \tag{3.2.64}$$

3.2.22 Tailor et al. (2012) Estimator

Tailor et al. (2012) dual-to-ratio cum product estimator is given as:

$$t_{25} = \bar{y}\left(\frac{\bar{x}^* + \rho_{xz}}{\bar{X} + \rho_{xz}}\right)\left(\frac{\bar{Z} + \rho_{xz}}{\bar{z}^* + \rho_{xz}}\right). \tag{3.2.65}$$

The mean square error of t_{25} is:

$$MSE(t_{25}) \approx \bar{Y}^2\theta\Big[C_y^2 + g\lambda_1 C_x^2\left(g\lambda_1 - 2K_{yx}\right) + \lambda_2 g C_z^2\left(\lambda_2 g + 2(K_{yz} - \lambda_1 g K_{xz}\right)\Big]. \tag{3.2.66}$$

The bias of t_{25} is:

$$Bias(t_{25}) \approx \bar{Y}g\theta\Big[g\lambda_2 C_z^2\left(\lambda_2 + K_{yz}\right) - \lambda_1 C_x^2\left(K_{yx} + \lambda_2 g K_{xz}\right)\Big], \tag{3.2.67}$$

where

$$\lambda_1 = \frac{\bar{X}}{\bar{X} + \rho_{xz}} \text{ and } \lambda_2 = \frac{\bar{Z}}{\bar{Z} + \rho_{xz}}.$$

3.2.23 Yadav (2012) Estimator

Yadav (2012) proposed a ratio-cum-dual to ratio estimator as:

$$t_{26} = \bar{y}\left[\alpha \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) + (1-\alpha)\exp\left(\frac{\bar{x}^* - \bar{X}}{\bar{x}^* + \bar{X}}\right)\right]. \tag{3.2.68}$$

The mean square error of t_{26} is:

$$MSE(t_{26}) \approx \theta\bar{Y}^2\left[C_y^2 + \alpha^2\frac{C_x^2}{4} + \alpha K_{yx} C_x^2\right]. \tag{3.2.69}$$

The bias of t_{26} is:

$$Bias\left(\mathrm{t}_{26}\right) \approx \bar{Y}\theta\left[\alpha\frac{C_x^2}{8}+\frac{\alpha}{2}K_{yx}C_x^2\right], \tag{3.2.70}$$

where

$$\alpha=\frac{2K_{yx}+g}{1+g}.$$

CHAPTER 4

RATIO AND PRODUCT TYPE EXPONENTIAL ESTIMATORS OF POPULATION MEAN

4.1 INTRODUCTION

In this chapter, dual to ratio-cum-ratio type and dual to product-cum-product type estimators of population mean under simple random sampling without replacement (SRSWOR). The proposed estimators are helpful to estimate the finite population mean. Two auxiliary variables are used for the development of dual to ratio-cum-ratio type estimators when correlation between variable of interest (say 'y') and auxiliary variables (say 'x' and 'z') is positive. The dual to product-cum-product type estimator is applicable when correlation between variable of interest (say 'y') and auxiliary variables (say 'x' and 'z') is negative. The mean square error equations of the proposed estimators have been obtained along with their biases. The conditions under which the proposed estimators perform better than the available estimators have also been obtained.

Consider a finite population S of size N consisting of units $S_1, S_2, ..., S_N$. The population mean of the study variable Y, is $\bar{Y}$ whereas $\bar{X}$ and $\bar{Z}$ are the population means of auxiliary variables X and Z respectively. The sample is drawn by the simple random sampling without replacement (SRSWOR) of size $n(n < N)$ from the population and we assume that $\bar{x}, \bar{y}$ and $\bar{z}$ are the sample means of the variables X, Y and Z respectively.

4.2 PROPOSED ESTIMATORS

We propose ratio-cum-ratio and product-cum-product estimators by using above mentioned transformation on two auxiliary variables (x, z).

(i) Proposed Estimator-I

The proposed ratio-cum-ratio type estimator of population mean in the exponential form is

$$t_{n1} = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\exp\left[\frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right], \tag{4.2.1}$$

where

$$\bar{x}^* = (1+g)\bar{X} - g\bar{x}, \tag{1.6.6}$$

and

$$\bar{z}^* = (1+g)\bar{Z} - g\bar{z}. \tag{1.6.7}$$

(ii) Proposed Estimator-II

The product -cum -product type estimator of the population mean in the exponential form is

$$t_{n2} = \bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right)\exp\left[\frac{\bar{Z} - \bar{z}^*}{\bar{Z} + \bar{z}^*}\right]. \tag{4.2.2}$$

4.2.1 Bias of the Proposed Estimator-I

Using the notations (1.6.1), we may write the estimator (4.2.1) as:

$$t_{n1} = \bar{Y}\left(1+\bar{e}_y\right)\left[\frac{(N-n)\bar{X}\left(1+\bar{e}_x\right)}{(N-n)\bar{X}}\right]\exp\left[\frac{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n) - \bar{Z}}{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n) + \bar{Z}}\right],$$

or

$$t_{n1} = \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left[\frac{-\bar{Z}g\bar{e}_z}{2\bar{Z} - \bar{Z}g\bar{e}_z}\right],$$

or

$$= \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left[\frac{-\bar{Z}g\bar{e}_z}{2\bar{Z}\left(1-\frac{1}{2}g\bar{e}_z\right)}\right].$$

After simplification, we get

$$t_{n1} = \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left(-\frac{1}{2}g\bar{e}_z\left(1-\frac{1}{2}g\bar{e}_z\right)^{-1}\right),$$

or

$$t_{n1} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left(-\frac{1}{2}g\bar{e}_z-\frac{1}{4}g\bar{e}_z\right). \qquad (4.2.3)$$

Expanding the exponential function in (4.2.3) and ignoring the terms with power three or greater as:

$$t_{n1} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\left(1-\frac{1}{2}g\bar{e}_z-\frac{2}{8}g^2\bar{e}_z^2+\frac{1}{8}g^2\bar{e}_z^2\right),$$

or

$$t_{n1} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\left(1-\frac{1}{2}g\bar{e}_z-\frac{1}{8}g^2\bar{e}_z^2\right). \qquad (4.2.4)$$

After simplification, we get

$$t_{n1} \approx \bar{Y}+\bar{Y}\bar{e}_y\left(1-g\bar{e}_x\right)\left(1-\frac{1}{2}g\bar{e}_z-\frac{1}{8}g^2\bar{e}_z^2\right),$$

or

$$t_{n1} \approx \bar{Y}+\bar{Y}\bar{e}_y\left(1-g\bar{e}_x-\frac{1}{2}g\bar{e}_z-\frac{1}{8}g^2\bar{e}_z^2+\frac{1}{2}g^2\bar{e}_x\bar{e}_z\right),$$

or

$$t_{n1} \approx \bar{Y}\left(1+\bar{e}_y-g\bar{e}_x-\frac{1}{2}g\bar{e}_z-\frac{1}{8}g^2\bar{e}_z^2+\frac{1}{2}g^2\bar{e}_x\bar{e}_z-g\bar{e}_y\bar{e}_x-\frac{1}{2}g\bar{e}_y\bar{e}_z\right),$$

or

$$t_{n1}-\bar{Y} \approx \bar{Y}\left(\bar{e}_y-g\bar{e}_x-\frac{1}{2}g\bar{e}_z-\frac{1}{8}g^2\bar{e}_z^2+\frac{1}{2}g^2\bar{e}_x\bar{e}_z-g\bar{e}_y\bar{e}_x-\frac{1}{2}g\bar{e}_y\bar{e}_z\right). \qquad (4.2.5)$$

Taking expectation on both sides of (4.2.5) and using (1.6.2), we get

$$E(t_{n1}-\bar{Y}) \approx \bar{Y}\left(-\frac{1}{8}g^2C_z^2+\frac{1}{2}g^2C_xC_z\rho_{xz}-gC_yC_x\rho_{yx}-\frac{1}{2}gC_yC_z\rho_{yz}\right).$$

After simplifications we get the bias of t_{n1} as:

$$Bias(t_{n1}) \approx \frac{g}{2}\bar{Y}\left(-\frac{1}{4}gC_z^2 + gC_xC_z\rho_{xz} - 2C_xC_y\rho_{xy} - C_yC_z\rho_{yz}\right). \quad (4.2.6)$$

4.2.2 Mean Square Error of the Proposed Estimator-I

Squaring and taking expectation on both sides of (4.2.5), we have

$$E\left(t_{n1} - \bar{Y}\right)^2 \approx \bar{Y}^2 E\left(\bar{e}_y^2 + g^2\bar{e}_x^2 + \frac{1}{4}g^2\bar{e}_z^2 - 2g\bar{e}_y\bar{e}_x + g^2\bar{e}_x\bar{e}_z - g\bar{e}_y\bar{e}_z\right). \quad (4.2.7)$$

Using (1.6.2) in (4.2.7) and after simplification, we get

$$MSE(\mathrm{t}_{n1}) \approx \bar{Y}^2\theta\left[C_y^2 + g^2C_x^2 + \frac{1}{4}g^2C_z^2 - 2gC_yC_x\rho_{yx} + g^2C_xC_z\rho_{xz} - gC_yC_z\rho_{yz}\right]. \quad (4.2.8)$$

4.2.3 Bias of the Proposed Estimator-II

Using (1.6.1) in (4.2.2), the proposed estimator t_{n2} may be written as:

$$t_{n2} = \bar{Y}\left(1+\bar{e}_y\right)\left[\frac{(N-n)\bar{X}}{(N-n)\bar{X}\left(1+\bar{e}_x\right)}\right]\exp\left[\frac{\bar{Z} - (N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)}{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n) + \bar{Z}}\right]. \quad (4.2.9)$$

Expanding the right hand side of (4.2.9) and neglecting the terms with power two or greater, we get

$$t_{n2} = \bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left[\frac{\bar{Z}g\bar{e}_z}{2\bar{Z} - \bar{Z}g\bar{e}_z}\right],$$

or

$$= \bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left[\frac{\bar{Z}g\bar{e}_z}{2\bar{Z}\left(1-\frac{1}{2}g\bar{e}_z\right)}\right].$$

After simplification, we get

$$t_{n2} = \bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left(\frac{1}{2}g\bar{e}_z\left(1-\frac{1}{2}g\bar{e}_z\right)^{-1}\right),$$

or

$$t_{n2} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left(+\frac{1}{2}g\bar{e}_z+\frac{1}{4}g^2\bar{e}_z^2\right). \tag{4.2.10}$$

Expanding the series up to first order, re-writing (4.2.10) as:

$$t_{n2} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x+g^2\bar{e}_x^2\right)\exp\left(+\frac{1}{2}g\bar{e}_z+\frac{1}{4}g^2\bar{e}_z^2\right). \tag{4.2.11}$$

Expanding the exponential function in (4.2.11) and ignoring the terms with power three or greater as:

$$t_{n2} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\left(1+\frac{1}{2}g\bar{e}_z+\frac{2}{8}g^2\bar{e}_z^2+\frac{1}{8}g^2\bar{e}_z^2\right),$$

or

$$t_{n2} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x+\frac{1}{2}g\bar{e}_z+\frac{3}{8}g^2\bar{e}_z^2+\frac{1}{2}g^2\bar{e}_x\bar{e}_z\right),$$

or

$$t_{n2} \approx \bar{Y}\left(1+\bar{e}_y+g\bar{e}_x+\frac{1}{2}g\bar{e}_z+\frac{3}{8}g^2\bar{e}_z^2+\frac{1}{2}g^2\bar{e}_x\bar{e}_z+g\bar{e}_y\bar{e}_x+\frac{1}{2}g\bar{e}_y\bar{e}_z\right),$$

or

$$t_{n2}-\bar{Y} \approx \bar{Y}\left(\bar{e}_y+g\bar{e}_x+\frac{1}{2}g\bar{e}_z+\frac{3}{8}g^2\bar{e}_z^2+\frac{1}{2}g^2\bar{e}_x\bar{e}_z+g\bar{e}_y\bar{e}_x+\frac{1}{2}g\bar{e}_y\bar{e}_z\right). \tag{4.2.12}$$

Taking expectation on both sides of (4.2.12) and using (1.6.2), we get

$$E\left(t_{n1}-\bar{Y}\right)\approx\bar{Y}\left(\frac{3}{8}g^2C_z^2+\frac{1}{2}g^2C_xC_z\rho_{xz}+gC_yC_x\rho_{yx}+\frac{1}{2}gC_yC_z\rho_{yz}\right).$$

After simplifications we get bias of t_{n2} as:

$$E\left(t_{n1}-\bar{Y}\right)\approx\bar{Y}\left(\frac{3}{8}g^2C_z^2+\frac{1}{2}g^2C_xC_z\rho_{xz}+gC_yC_x\rho_{yx}+\frac{1}{2}gC_yC_z\rho_{yz}\right),$$

or

$$Bias\left(t_{n2}\right)\approx\frac{g}{2}\bar{Y}\left(\frac{3}{4}gC_z^2+gC_xC_z\rho_{xz}+2C_xC_y\rho_{xy}\right). \tag{4.2.13}$$

4.2.4 Mean Square Error of Proposed Estimator-II

Squaring and taking expectation on both sides of (4.2.12) and using (1.6.2), we have

$$E\left(t_{n2}-\bar{Y}\right)^2\approx\bar{Y}^2E\left(\bar{e}_y^2+g^2\bar{e}_x^2+\frac{1}{4}g^2\bar{e}_z^2+2g\bar{e}_y\bar{e}_x+g^2\bar{e}_x\bar{e}_z+g\bar{e}_y\bar{e}_z\right). \tag{4.2.14}$$

On simplification, we get

$$MSE\left(\mathrm{t}_{n2}\right)\approx\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2+2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}+gC_yC_z\rho_{yz}\right). \tag{4.2.15}$$

4.3 EFFICIENCY COMPARISON

In this section, the theoretical comparisons between some available estimators of population mean and proposed estimators have been made with respect to mean square error. The exponential ratio-cum-ratio estimator has been compared with $\bar{y}$, t_{10}, t_{14}, t_{17} and t_{21}. The exponential product-cum-product estimator has been compared with $\bar{y}$, t_{11}, t_{18} and t_{22}.

The proposed estimator t_{n1} is more efficient than the mean per unit estimator if

$$MSE\left(t_{n1}\right)\le Var\left(\bar{y}\right). \tag{4.3.1}$$

Using (4.2.8) and (3.2.1) in (4.3.1), we have

$$\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2-2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}-gC_yC_z\rho_{yz}\right)\le\theta\bar{Y}^2C_y^2,$$

or

$$\rho_{yx}\ge\frac{g}{2C_yC_x}\left[C_x^2+\frac{1}{4}C_z^2+C_xC_z\rho_{xz}-C_yC_z\rho_{yz}\right]. \tag{4.3.2}$$

The proposed estimator t_{n1} is more efficient than the Srivenkataramana's (1980) estimator t_{10} if

$$MSE\left(t_{n1}\right)\le MSE\left(t_{10}\right). \tag{4.3.3}$$

Using (4.2.8) and (3.2.23) in (4.3.3), we get

$$\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2-2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}-gC_yC_z\rho_{yz}\right)$$
$$\le\bar{Y}^2\theta\left[C_y^2+g^2C_x^2-2gC_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx}\ge\left(\frac{4g^2-1}{4}\right)C_x^2+g^2\left(\frac{1}{4}C_z^2+C_xC_z\rho_{xz}-\frac{C_yC_z\rho_{yz}}{g}\right). \tag{4.3.4}$$

The proposed estimator t_{n1} is more efficient than the Samiuddin and Hanif (2006) estimator t_{14} if

$$MSE\left(t_{n1}\right)\le MSE\left(t_{14}\right). \tag{4.3.5}$$

Using (4.2.8) and (3.2.33) in (4.3.5), we get

$$\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2-2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}-gC_yC_z\rho_{yz}\right)$$
$$\le\theta\bar{Y}^2\left[C_y^2+C_x^2+C_z^2-2C_xC_y\rho_{xy}-2C_yC_z\rho_{yz}+2C_xC_z\rho_{xz}\right],$$

or

$$\rho_{yx} \ge \frac{C_x^2\left(g^2-1\right)+C_z^2\left(\frac{1}{4}g^2-1\right)-C_yC_z\rho_{yz}\left(g-2\right)+C_xC_z\rho_{xz}\left(g^2-2\right)}{2C_yC_x}. \tag{4.3.6}$$

The proposed estimator t_{n1} is more efficient than the Bahl and Tuteja (1991) estimator t_{17} if

$$MSE\left(t_{n1}\right) \le MSE\left(t_{17}\right). \tag{4.3.7}$$

Using (4.2.8) and (3.2.42) in (4.3.7), we get

$$\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2-2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}-gC_yC_z\rho_{yz}\right)$$
$$\le \bar{Y}^2\theta\left[C_y^2+\frac{1}{4}C_x^2-C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx} \ge \frac{g^2\left(C_x^2+\frac{1}{4}C_z^2+C_xC_z\rho_{xz}\right)-\frac{1}{4}C_x^2}{C_yC_x\left(2g-1\right)}. \tag{4.3.8}$$

The proposed estimator t_{n1} is more efficient than the Sharma and Tailor (2010) estimator t_{21} if

$$MSE\left(t_{n1}\right) \le MSE\left(t_{21}\right). \tag{4.3.9}$$

Using (4.2.8) and (3.2.54) in (4.3.9), we get

$$\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2-2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}-gC_yC_z\rho_{yz}\right)$$
$$\le \bar{Y}^2\theta\left[C_y^2+gC_x^2\left(\frac{g}{4}-K_{yx}\right)\right],$$

or

$$\rho_{yx} \ge \frac{C_z}{\rho_{yx}}\left[\rho_{yz}-\frac{g}{C_y}\left(\rho_{xz}+\frac{3C_x}{4C_z}+\frac{C_z}{4C_x}\right)\right]. \tag{4.3.10}$$

The proposed estimator t_{n2} is more efficient than the mean per unit estimator $\bar{y}$ if

$$MSE\left(t_{n2}\right) \le MSE\left(\bar{y}\right). \tag{4.3.11}$$

Using (4.2.15) and (3.2.1) in (4.3.11), we have

$$\bar{Y}^2\theta\left(C_y^2 + g^2C_x^2 + \frac{1}{4}g^2C_z^2 + 2gC_yC_x\rho_{yx} + g^2C_xC_z\rho_{xz} + gC_yC_z\rho_{yz} \right) \le \theta\bar{Y}^2C_y^2,$$

or

$$\rho_{yx} \ge \frac{g}{2C_yC_x}\left[C_x^2 + \frac{1}{4}C_z^2 + C_xC_z\rho_{xz} + \frac{C_yC_z\rho_{yz}}{g} \right]. \tag{4.3.12}$$

The proposed estimator t_{n2} is more efficient than the Bandyopadhyay's (1980) estimator t_{11} if

$$MSE\left(t_{n2}\right) \le MSE\left(t_{11}\right). \tag{4.3.13}$$

Using (4.2.15) and (3.2.26) in (4.3.13) we have,

$$\begin{aligned}&\bar{Y}^2\theta\left(C_y^2 + g^2C_x^2 + \frac{1}{4}g^2C_z^2 + 2gC_yC_x\rho_{yx} + g^2C_xC_z\rho_{xz} + gC_yC_z\rho_{yz} \right)\\ &\qquad \le \bar{Y}^2\theta\left[C_y^2 + gC_z^2(\mathrm{g} + 2K_{yz}) \right],\end{aligned}$$

or

$$\rho_{yx} \ge \left(\frac{gC}{C}\rho_{xz} - \frac{C_z}{C_x}\rho_{yz} - \frac{3}{8}gC_z^2 + \frac{gC_y}{C_x} \right). \tag{4.3.14}$$

The proposed estimator t_{n2} is more efficient than the Bahl and Tuteja (1991) estimator t_{18} if

$$MSE\left(t_{n2}\right) \le MSE\left(t_{18}\right). \tag{4.3.15}$$

Using (4.2.15) and (3.2.45) in (4.3.15) we have,

$$\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2+2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}+gC_yC_z\rho_{yz}\right)$$
$$\leq \bar{Y}^2\theta\left[C_y^2+\frac{C_z^2}{4}+C_yC_z\rho_{yz}\right],$$

or

$$\rho_{yx}\geq\frac{1}{2gC_yC_x}\left[g^2C_x^2+\frac{1}{4}C_z^2\left(g^2-1\right)+g^2C_xC_z\rho_{xz}+C_yC_z\rho_{yz}\left(g-1\right)\right].$$

(4.3.16)

The proposed estimator t_{n2} is more efficient than the Sharma and Tailor (2010) estimator t_{22} if

$$MSE\left(t_{n2}\right)\leq MSE\left(t_{22}\right). \tag{4.3.17}$$

Using (4.2.15) and (3.2.57) in (4.3.17), we get

$$\bar{Y}^2\theta\left(C_y^2+g^2C_x^2+\frac{1}{4}g^2C_z^2+2gC_yC_x\rho_{yx}+g^2C_xC_z\rho_{xz}+gC_yC_z\rho_{yz}\right).$$
$$\leq \bar{Y}^2\theta\left[C_y^2+gC_z^2\left(\frac{g}{4}+K_{yz}\right)\right],$$

or

$$\rho_{yx}\geq\frac{g\left(C_z\rho_{xz}+C_x\right)}{2C_y}. \tag{4.3.18}$$

When the condition (4.3.2), (4.3.4), (4.3.6), (4.3.8) and (4.3.10) is satisfied, the proposed dual-to-ratio cum ratio type exponential estimator t_{n1} will be more efficient as compare to $\bar{y}$, t_{10}, t_{14}, t_{17} and t_{21} respectively. When the condition (4.3.12), (4.3.14), (4.3.16) and (4.3.18) is satisfied, the proposed dual-to-product cum product type exponential estimator t_{n2} will be more efficient as compare to $\bar{y}$, t_{11}, t_{18} and t_{22} respectively.

CHAPTER 5

GENERALIZED EXPONENTIAL TYPE ESTIMATORS OF POPULATION MEAN

5.1 INTRODUCTION

In this chapter, generalized dual-to-exponential type estimator of population mean for simple random sampling without replacement (SRSWOR) has been developed. Two auxiliary variables are used for the development of the generalized dual-to-exponential type estimator of population mean. The mean square error equation of the proposed estimator has been obtained along with bias. The conditions under which the proposed generalized estimator perform better than the available estimators have also been obtained.

Consider a finite population S of size N consisting of units $S_1, S_2, \ldots\ldots, S_N$. The population mean of the study variable Y is $\bar{Y}$, whereas $\bar{X}$ and $\bar{Z}$ are the population means of auxiliary variables X and Z respectively. The sample is drawn by the simple random sampling without replacement (SRSWOR) of size $n(n<N)$ from the population and we assume that $\bar{x}, \bar{y}$ and $\bar{z}$ are the sample means of the variables X, Y and Z respectively.

5.2 PROPOSED GENERALIZED ESTIMATOR

5.2.1 Proposed Estimator

We propose a generalized dual-to-exponential type estimator of population mean of the study variable using information on the two auxiliary variables (x, z).

$$t_{ng} = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\upsilon_1} \exp\left[\upsilon_2 \frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right], \qquad (5.2.1)$$

where υ_1 and υ_2 are suitable constants and

$$\bar{x}^* = (1+g)\bar{X} - g\bar{x}, \qquad (1.6.6)$$

and

$$\bar{z}^* = (1+g)\bar{Z} - g\bar{z}. \tag{1.6.7}$$

5.2.2 Some Special Cases of the Proposed Estimator

When $\upsilon_1 = \upsilon_2 = 0$, the proposed estimator t_{ng} reduces to the mean per unit estimator $\bar{y}$.

When $\upsilon_1 = \upsilon_2 = 1$, the proposed estimator t_{ng} reduces to the proposed dual-to-ratio cum exponential type estimator

$$t_{n1} = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\exp\left[\frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right]. \tag{4.2.1}$$

When $\upsilon_1 = \upsilon_2 = -1$, the proposed estimator t_{ng} reduces to the proposed dual-to-product cum exponential type estimator

$$t_{n2} = \bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right)\exp\left[\frac{\bar{Z} - \bar{z}^*}{\bar{Z} + \bar{z}^*}\right]. \tag{4.2.2}$$

When $\upsilon_1 = 1$ and $\upsilon_2 = 0$, the proposed estimator t_{ng} reduces to the Srivenkataramana's (1980) dual-to-ratio type estimator

$$t_{10} = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right). \tag{3.2.22}$$

When $\upsilon_1 = -1$ and $\upsilon_2 = 0$, the proposed estimator t_{ng} reduces to the Bandyopadhyay's (1980) dual-to-product type estimator

$$t_{11} = \bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right). \tag{3.2.25}$$

When $\upsilon_1 = 0$ and $\upsilon_2 = 1$, the proposed estimator t_{ng} reduces to the Sharma and Tailor (2010) dual-to-ratio type exponential estimator

$$t_{21} = \bar{y}\exp\left[\frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right]. \tag{3.2.53}$$

When $\upsilon_1 = 0$ and $\upsilon_2 = -1$, the proposed estimator t_{ng} reduces to the Sharma and Tailor (2010) dual-to-ratio type exponential estimator

$$t_{22} = \bar{y} \exp\left[\frac{\bar{Z} - \bar{z}^*}{\bar{Z} + \bar{z}^*}\right]. \tag{3.2.56}$$

When $\upsilon_1 = 0$ and $\upsilon_2 = \upsilon_2$, the proposed estimator t_{ng} reduces to the estimator

$$t_{ng(1)} = \bar{y} \exp\left[\upsilon_2 \frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right]. \tag{5.2.2}$$

5.2.3 Bias of the Proposed Estimator

Using (1.6.1), (1.6.6) and (1.6.7), we may write the estimator (5.2.1) as:

$$t_{ng} = \bar{Y}\left(1+\bar{e}_y\right)\left[\frac{(N-n)\bar{X}\left(1+\bar{e}_x\right)}{(N-n)\bar{X}}\right]^{\upsilon_1} \exp\left[\upsilon_2 \frac{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)-\bar{Z}}{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)+\bar{Z}}\right]. \tag{5.2.3}$$

On simplification, (5.2.3) may be written as:

$$t_{ng} = \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)^{\upsilon_1} \exp\left[\upsilon_2 \frac{-\bar{Z}g\bar{e}_z}{2\bar{Z}-\bar{Z}g\bar{e}_z}\right],$$

or

$$= \bar{Y}(1+\bar{e}_y)\left(1-g\bar{e}_x\right)^{\upsilon_1} \exp\left[\upsilon_2 \frac{-\bar{Z}g\bar{e}_z}{2\bar{Z}(1-\frac{1}{2}g\bar{e}_z)}\right],$$

or

$$= \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)^{\upsilon_1} \exp\left(-\frac{1}{2}g\bar{e}_z\upsilon_2\left(1-\frac{1}{2}g\bar{e}_z\right)^{-1}\right),$$

or

$$t_{ng} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)^{\upsilon_1} \exp\left(-\frac{1}{2}\upsilon_2 g\bar{e}_z - \frac{1}{4}\upsilon_2 g\bar{e}_z\right). \tag{5.2.4}$$

Expanding the series up to first order, re-writing (5.2.4) as:

$$t_{ng} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1-\upsilon_1 g\bar{e}_x+\frac{\upsilon_1\left(\upsilon_1-1\right)}{2}g^2\bar{e}_x^2\right)\exp\left(-\frac{1}{2}\upsilon_2 g\bar{e}_z-\frac{1}{4}\upsilon_2 g\bar{e}_z\right),$$

or

$$t_{ng} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1-\upsilon_1 g\bar{e}_x+\frac{\upsilon_1\left(\upsilon_1-1\right)}{2}g^2\bar{e}_x^2\right)\exp\left(-\frac{1}{2}g^2\bar{e}_z^2-\frac{1}{4}g^2 e_z^2\right). \tag{5.2.5}$$

Expanding the exponential function in (5.2.5) and ignoring the terms with power three or greater as:

$$t_{ng} \approx \bar{Y}\left(1+\bar{e}_y\right)\left(1-\upsilon_1 g\bar{e}_x+\frac{\upsilon_1\left(\upsilon_1-1\right)}{2}g^2\bar{e}_x\right)$$
$$\left(1-\frac{1}{2}\upsilon_2 g\bar{e}_z-\frac{2}{8}\upsilon_2 g^2\bar{e}_z^2+\frac{1}{8}\upsilon_2^2 g^2 e_z^2\right),$$

or

$$t_{ng} \approx \bar{Y}\left(1+\bar{e}_y\right)\left[1-\upsilon_1 g\bar{e}_x+\frac{\upsilon_1(\upsilon_1-1)g^2}{2}\bar{e}_x^2-\frac{1}{2}\upsilon_2 g\bar{e}_z\right.$$
$$\left.+\frac{\upsilon_2(\upsilon_2-1)g^2}{8}\bar{e}_z^2+\frac{1}{2}\upsilon_1\upsilon_2 g^2\bar{e}_x\bar{e}_z\right],$$

or

$$t_{ng} \approx \bar{Y}\left\{1+\bar{e}_y-\upsilon_1 g\bar{e}_x-\frac{1}{2}\upsilon_2 g\bar{e}_z+\frac{\upsilon_1\left(\upsilon_1-1\right)}{2}g^2\bar{e}_x^2+\frac{\upsilon_2\left(\upsilon_2-2\right)}{8}g^2\bar{e}_z^2\right.$$
$$\left.+\frac{1}{2}\upsilon_1\upsilon_2 g^2\bar{e}_x\bar{e}_z-\upsilon_1 g\bar{e}_x\bar{e}_z-\frac{1}{2}\upsilon_1 g e_y e_z\right\},$$

or

$$\left(t_{ng}-\bar{Y}\right) \approx \bar{Y}\left\{\bar{e}_y-\upsilon_1 g\bar{e}_x-\frac{1}{2}\upsilon_2 g\bar{e}_z+\frac{\upsilon_1\left(\upsilon_1-1\right)}{2}g^2\bar{e}_x^2+\frac{\upsilon_2\left(\upsilon_2-2\right)}{8}g^2\bar{e}_z^2\right.$$
$$\left.+\frac{1}{2}\upsilon_1\upsilon_2 g^2\bar{e}_x\bar{e}_z-\upsilon_1 g\bar{e}_x\bar{e}_z-\frac{1}{2}\upsilon_1 g e_y e_z\right\}. \tag{5.2.6}$$

Taking expectation on both sides of (5.2.6) and using (1.6.2), we get

$$Bias\left(t_{ng}\right) \approx \bar{Y}\theta\left\{\frac{\upsilon_1(\upsilon_1 - 1)}{2}g^2C_x^2 + \frac{\upsilon_2(\upsilon_2 - 2)}{8}g^2C_z^2\right.$$
$$\left. + \frac{1}{2}\upsilon_1\upsilon_2 g^2 C_x C_z \rho_{xz} - \upsilon_1 g C_x C_z \rho_{xz} - \frac{1}{2}\upsilon_2 g C_y C_z \rho_{yz}\right\}.$$

After simplification, we get the bias of t_{ng} as:

$$Bias\left(t_{ng}\right) \approx \frac{\bar{Y}\theta g^2}{2}\left[\upsilon_1(\upsilon_1 - 1)C_x^2 + \frac{\upsilon_2(\upsilon_2 - 2)}{4}C_z^2\right.$$
$$\left. + \upsilon_1\upsilon_2 C_x C_z \rho_{xz} - \frac{C_z}{g}\left(\upsilon_1 C_x \rho_{xz} - \upsilon_2 C_y \rho_{yz}\right)\right]. \quad (5.2.7)$$

5.2.4 Mean Square Error of the Proposed Estimator

Squaring and taking expectation on both sides of (5.2.6), we have

$$E\left(t_{ng} - \bar{Y}\right)^2 \approx \bar{Y}^2 E\left[\bar{e}_y^2 + \upsilon_1^2 g^2 \bar{e}_x^2 + \frac{1}{4}\upsilon_2^2 g^2 \bar{e}_z^2\right.$$
$$\left. - 2\upsilon_1 g \bar{e}_y \bar{e}_x + \upsilon_1\upsilon_2 g^2 \bar{e}_x \bar{e}_z - \upsilon_2 g \bar{e}_y \bar{e}_z\right]. \quad (5.2.8)$$

Using (1.6.2) in (5.2.8) and simplifying, we get

$$MSE\left(\mathrm{t}_{ng}\right) \approx \bar{Y}^2\theta\left[C_y^2 + \upsilon_1^2 g^2 C_x^2 + \frac{1}{4}\upsilon_2^2 g^2 C_z^2 - 2\upsilon_1 g C_y C_x \rho_{yx}\right.$$
$$\left. + \upsilon_1\upsilon_2 g^2 C_x C_z \rho_{xz} - \upsilon_2 g C_y C_z \rho_{yz}\right]. \quad (5.2.9)$$

Differentiating (5.2.9) with respect to υ_1 and equating to zero, we get

$$\frac{\partial MSE\left(t_{ng}\right)}{\partial \upsilon_1} \approx \bar{Y}^2\theta\left(2\upsilon_1 g^2 C_x^2 - 2g C_y C_x \rho_{yx} + \upsilon_2 g^2 C_x C_z \rho_{xz}\right) = 0.$$

After simplification, we have

$$2\upsilon_1 g^2 C_x^2 - 2gC_yC_x\rho_{yx} + \upsilon_2 g^2 C_xC_z\rho_{xz} = 0,$$

or

$$\upsilon_1 = \frac{1}{g}K_{yx} - \frac{\upsilon_2}{2}K_{zx}. \tag{5.2.10}$$

Differentiating (5.2.9) with respect to υ_2 and equating to zero, we get

$$\frac{\partial MSE(t_{ng})}{\partial \upsilon_2} \approx \bar{Y}^2\theta\left(\frac{1}{2}\upsilon_2 g^2 C_z^2 - gC_yC_z\rho_{yz} + \upsilon_1 g^2 C_xC_z\rho_{xz}\right) = 0,$$

or

$$\upsilon_2 = \frac{2}{g}K_{yz} - 2\upsilon_1 K_{xz}. \tag{5.2.11}$$

Solving (5.2.10) and (5.2.11), we get the optimum value of υ_1 and υ_2 as:

$$\upsilon_{1(opt)} = \frac{K_{yx} - K_{zx}K_{yz}}{g\left(1-\rho_{xz}^2\right)} = \frac{A}{g}, \tag{5.2.12}$$

and

$$\upsilon_{2(opt)} = \frac{2\left(K_{yz} - K_{xz}K_{yx}\right)}{g\left(1-\rho_{xz}^2\right)} = \frac{B}{g}, \tag{5.2.13}$$

where

$$A = \frac{K_{yx} - K_{zx}K_{yz}}{\left(1-\rho_{xz}^2\right)}, \text{ and } B = \frac{2\left(K_{yz} - K_{xz}K_{yx}\right)}{\left(1-\rho_{xz}^2\right)}.$$

Using (5.2.12) and (5.2.13) in (5.2.9), the minimum mean square error is given by

$$\min MSE(\mathrm{t}_{ng}) \approx \bar{Y}^2\theta\Big[C_y^2 + A^2C_x^2 + \frac{1}{4}B^2C_z^2 - 2AC_yC_x\rho_{yx} + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\Big]. \tag{5.2.14}$$

5.3 EFFICIENCY COMPARISON

In this section, the theoretical comparisons between some available estimators of population mean and the proposed estimator have been made with respect to the mean square error. The proposed generalized dual-to-ratio type exponential estimator has been compared with $\bar{y}$, t_1, t_{10}, t_{14}, t_{17}, t_{21} and t_{25}.

The proposed estimator t_{ng} is more efficient than the mean per unit estimator $\bar{y}$ if

$$MSE\left(t_{ng}\right) \le Var\left(\bar{y}\right). \qquad (5.3.1)$$

Using (5.2.14) and (3.2.1) in (5.3.1), we have

$$\bar{Y}^2\theta\left(C_y^2 + A^2C_x^2 + \frac{1}{4}B^2C_z^2 - 2AC_yC_x\rho_{yx} + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\right) \le \bar{Y}^2\theta C_y^2,$$

or

$$\rho_{yx} \ge \frac{1}{2AC_yC_x}\left(A^2C_x^2 + \frac{1}{4}B^2C_z^2 + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\right). \qquad (5.3.2)$$

The proposed estimator t_{ng} is more efficient than the Srivenkataramana's (1980) estimator t_9 if

$$MSE\left(t_{ng}\right) \le MSE\left(t_9\right). \qquad (5.3.3)$$

Using (5.2.14) and (3.2.20) in (5.3.3), we get

$$\bar{Y}^2\theta\left(C_y^2 + A^2C_x^2 + \frac{1}{4}B^2C_z^2 - 2AC_yC_x\rho_{yx} + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\right)$$
$$\le \bar{Y}^2\theta\left[C_y^2 + g^2C_x^2 - 2gC_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx} \ge \frac{1}{2(A-g)C_yC_x}\left(C_x^2 + \frac{1}{4}B^2C_z^2 + ABC_xC_z\rho_{xz} - BgC_yC_z\rho_{yz}\right). \qquad (5.3.4)$$

The proposed estimator t_{ng} is more efficient than the Samiuddin and Hanif (2006) estimator t_{17} if

$$MSE\left(t_{ng}\right) \le MSE\left(t_{17}\right). \tag{5.3.5}$$

Using (5.2.14) and (3.2.42) in (5.3.5), we get

$$\bar{Y}^2\theta\left(C_y^2 + A^2C_x^2 + \frac{1}{4}B^2C_z^2 - 2AC_yC_x\rho_{yx} + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\right)$$
$$\le \theta\bar{Y}^2\left[C_y^2 + C_x^2 + C_z^2 - 2C_xC_y\rho_{xy} - 2C_yC_z\rho_{yz} + 2C_xC_z\rho_{xz}\right],$$

or

$$\rho_{yx} \ge \frac{1}{2C_yC_x\left(A-1\right)}\left(C_x^2\left(A^2-1\right) + C_z^2\left(\frac{1}{4}B^2-1\right) + C_xC_z\rho_{xz}\left(AB-2\right)\right). \tag{5.3.6}$$

The proposed estimator t_{ng} is more efficient than the Bahl and Tuteja (1991) estimator t_{15} if

$$MSE\left(t_{ng}\right) \le MSE\left(t_{15}\right). \tag{5.3.7}$$

Using (5.2.14) and (3.2.36) in (5.3.7), we get

$$\bar{Y}^2\theta\left(C_y^2 + A^2C_x^2 + \frac{1}{4}B^2C_z^2 - 2AC_yC_x\rho_{yx} + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\right)$$
$$\le \bar{Y}^2\theta\left[C_y^2 + \frac{1}{4}C_x^2 - C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx} \ge \frac{BC_yC_z\rho_{yz} - C_x^2\left(A^2 - \frac{1}{4}\right) + \frac{1}{4}B^2C_z^2}{C_yC_x\left(1-2A\right)}. \tag{5.3.8}$$

The proposed estimator t_{ng} is more efficient than the Sharma and Tailor (2010) estimator t_{21} if

$$MSE\left(t_{ng}\right) \le MSE\left(t_{21}\right). \tag{5.3.9}$$

Using (5.2.14) and (3.2.54) in (5.3.9), we get

$$\bar{Y}^2\theta\left(C_y^2 + A^2C_x^2 + \frac{1}{4}B^2C_z^2 - 2AC_yC_x\rho_{yx} + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\right)$$
$$\leq \bar{Y}^2\theta\left[C_y^2 + gC_x^2\left(\frac{g}{4} - K_{yx}\right)\right],$$

or

$$\rho_{yx} \geq \frac{ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz} + \left(A^2 - 4g^2\right)C_x^2 + \frac{1}{4}C_z^2}{C_yC_x\left(2A - g\right)}. \tag{5.3.10}$$

The proposed estimator t_{ng} is more efficient than the Tailor et al. (2012) estimator t_{25} if

$$MSE\left(t_{ng}\right) \leq MSE\left(t_{25}\right). \tag{5.3.11}$$

Using (5.2.14) and (3.2.66) in (5.3.11), we get

$$\bar{Y}^2\theta\left(C_y^2 + A^2C_x^2 + \frac{1}{4}B^2C_z^2 - 2AC_yC_x\rho_{yx} + ABC_xC_z\rho_{xz} - BC_yC_z\rho_{yz}\right)$$

$$\leq \bar{Y}^2\theta\left[C_y^2 + g\lambda_1C_x^2\left(g\lambda_1 - 2K_{yx}\right) + \lambda_2gC_z^2\left(\lambda_2g + 2(K_{yz} - \lambda_1gK_{xz}\right)\right].$$

or,

$$\rho_{yx} \geq \frac{1}{2C_yC_x\left(A - g\lambda_1\right)}\left\{C_y\rho_{yz}\left(BC_z - \frac{2}{C_z}\right) - C_x^2\left(A^2 - g^2\lambda_1^2\right)\right.$$
$$\left. - C_z^2\left(\frac{1}{4}B - g\lambda_2^2\right) - C_x\rho_{xz}\left(ABC_z - \lambda_1\frac{g}{C_z}\right)\right\}. \tag{5.3.12}$$

When the condition (5.3.2), (5.3.4), (5.3.6), (5.3.8) , (5.3.10) and (5.3.12) is satisfied, the proposed generalized dual-to-exponential type estimator will be more efficient as compared to $\bar{y}$, t_{10}, t_{14}, t_{17}, t_{21} and t_{25} respectively.

CHAPTER 6

ESTIMATORS OF POPULATION MEAN USING TWO AUXILIARY VARIABLES

6.1 INTRODUCTION

In this Chapter, we have developed four new estimators of finite population mean using information on two auxiliary variables for simple random sampling without replacement (SRSWOR). The mean square error equations of the proposed estimators have been obtained along with their biases. The conditions under which the proposed estimators perform better than the available estimators have also been obtained.

6.2 PROPOSED ESTIMATORS

Using information on two auxiliary variables (x, z), we propose the following estimators of finite population mean.

i) $$t_{nl1} = \omega\bar{y} + (1-\omega)\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\exp\left[\frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right], \tag{6.2.1}$$

ii) $$t_{nl2} = \mu\bar{y} + (1-\mu)\bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right)\exp\left[\frac{\bar{Z} - \bar{z}^*}{\bar{Z} + \bar{z}^*}\right], \tag{6.2.2}$$

iii) $$t_{nl3} = \delta\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\exp\left[\frac{\bar{Z} - \bar{z}}{\bar{z} + \bar{Z}}\right] + (1-\delta)\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)\exp\left[\frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right], \tag{6.2.3}$$

and

iv) $$t_{nl4} = \varepsilon\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)\exp\left[\frac{\bar{z} - \bar{Z}}{\bar{z} + \bar{Z}}\right] + (1-\varepsilon)\bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right)\exp\left[\frac{\bar{Z} - \bar{z}^*}{\bar{z}^* + \bar{Z}}\right], \tag{6.2.4}$$

where ω, μ, δ and ε are suitable constants.

6.2.1 Biases of the Proposed Estimators

Using the notations (1.1.6), we may write the estimator (6.2.1) as:

$$t_{nl1} = \omega\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\omega\right)\bar{Y}\left(1+\bar{e}_y\right)\left[\frac{N\bar{X}-n\bar{x}}{(N-n)\bar{X}}\right]\exp\left[\frac{\dfrac{N\bar{Z}-n\bar{z}}{N-n}-\bar{Z}}{\dfrac{N\bar{Z}-n\bar{z}}{N-n}+\bar{Z}}\right],$$

or

$$t_{nl1} = \omega\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\omega\right)\bar{Y}\left(1+\bar{e}_y\right)\left[\frac{(N-n)\bar{X}\left(1+\bar{e}_x\right)}{(N-n)\bar{X}}\right]$$
$$\exp\left[\frac{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)-\bar{Z}}{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)+\bar{Z}}\right],$$

or

$$t_{nl1} = \omega\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\omega\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left[\frac{-\bar{Z}g\bar{e}_z}{2\bar{Z}-\bar{Z}g\bar{e}_z}\right],$$

or

$$t_{nl1}=\omega\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\omega\right)\bar{Y}(1+\bar{e}_y)\left(1-g\bar{e}_x\right)\exp\left[\frac{-\bar{Z}g\bar{e}_z}{2\bar{Z}\left(1-\frac{1}{2}g\bar{e}_z\right)}\right]. \quad (6.2.5)$$

After simplification, we get

$$t_{nl1} = \omega\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\omega\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left(-\frac{1}{2}g\bar{e}_z\left(1-\frac{1}{2}g\bar{e}_z\right)^{-1}\right),$$

or

$$t_{nl1} \approx \omega\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\omega\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left(-\frac{1}{2}g\bar{e}_z-\frac{1}{4}g^2\bar{e}_z^2\right). \quad (6.2.6)$$

Expanding the exponential function in (6.2.6) and ignoring the terms with power greater than two, we have

$$t_{nl1} \approx \omega\bar{Y}+\omega\bar{Y}\bar{e}_y+\left(1-\omega\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\left(1-\frac{1}{2}g\bar{e}_z-\frac{1}{8}g^2\bar{e}_z^2\right),$$

or

$$t_{nl1} \approx \omega\bar{Y} + \omega\bar{Y}\bar{e}_y + (1-\omega)\bar{Y}\left(1+\bar{e}_y\right)\left(1 - \frac{1}{2}g\bar{e}_z - \frac{1}{8}g^2\bar{e}_z^2 - g\bar{e}_x + \frac{1}{2}g^2\bar{e}_z\bar{e}_x\right),$$

or

$$t_{nl1} \approx \omega\bar{Y} + \omega\bar{Y}\bar{e}_y + (1-\omega)\left(\bar{Y} + \bar{Y}\bar{e}_y\right)\left(1 - \frac{1}{2}g\bar{e}_z - \frac{1}{8}g^2\bar{e}_z^2 - g\bar{e}_x + \frac{1}{2}g^2\bar{e}_z\bar{e}_x\right),$$

or

$$t_{nl1} \approx \omega\bar{Y} + \omega\bar{Y}\bar{e}_y + (1-\omega)\bar{Y}\left[1 - \frac{1}{2}g\bar{e}_z - \frac{1}{8}g^2\bar{e}_z^2 - g\bar{e}_x + \frac{1}{2}g^2\bar{e}_z\bar{e}_x + \bar{e}_y - \frac{1}{2}g\bar{e}_y\bar{e}_z - g\bar{e}_y\bar{e}_x\right]. \tag{6.2.7}$$

After simplification, we get

$$t_{nl1} - \bar{Y} \approx \bar{Y}\left\{-\frac{1}{2}g\bar{e}_z - \frac{1}{8}g^2\bar{e}_z^2 - g\bar{e}_x + \frac{1}{2}g^2\bar{e}_z\bar{e}_x + \bar{e}_y - \frac{1}{2}g\bar{e}_y\bar{e}_z - g\bar{e}_y\bar{e}_x + \frac{1}{2}\omega g\bar{e}_z + \omega\frac{1}{8}g^2\bar{e}_z^2 + \omega g\bar{e}_x - \frac{1}{2}\omega g^2\bar{e}_z\bar{e}_x + \frac{1}{2}\omega g\bar{e}_y\bar{e}_z + \omega g\bar{e}_y\bar{e}_x\right\}. \tag{6.2.8}$$

Taking expectation on both sides of (6.2.8) and using (1.1.6), we get

$$E\left(t_{nl1} - \bar{Y}\right) \approx \bar{Y}\theta\left\{-\frac{1}{8}g^2C_z^2 + \frac{1}{2}g^2C_zC_x\rho_{xz} - \frac{1}{2}gC_yC_z\rho_{yz} - gC_yC_x\rho_{yx} + \frac{\omega}{8}g^2C_z^2 - \frac{1}{2}\omega g^2C_zC_x\rho_{zx} + \frac{1}{2}\omega gC_yC_z\rho_{yz} + \omega gC_yC_x\rho_{yx}\right\}. \tag{6.2.9}$$

After simplifications we get bias of t_{nl1} as:

$$Bias\left(t_{nl1}\right) \approx \bar{Y}\theta g^2\frac{(\omega-1)}{2}\left[\frac{C_z^2}{4} - C_zC_x\rho_{xz} + \frac{C_yC_z\rho_{yz}}{g} - \frac{2C_yC_x\rho_{yx}}{g}\right]. \tag{6.2.10}$$

Using the notations (1.1.6), we may write the estimator (6.2.2) as:

$$t_{nl2} = \mu\bar{Y}\left(1+\bar{e}_y\right) + (1-\mu)\bar{Y}\left(1+\bar{e}_y\right)\left[\frac{(N-n)\bar{X}}{N\bar{X}-n\bar{x}}\right]\exp\left[\frac{\bar{Z} - \frac{N\bar{Z}-n\bar{z}}{N-n}}{\frac{N\bar{Z}-n\bar{z}}{N-n} + \bar{Z}}\right],$$

or

$$t_{nl2}=\left\{\mu\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left[\frac{(N-n)\bar{X}}{(N-n)\bar{X}\left(1+\bar{e}_x\right)}\right]\right.$$

$$\left.\times\exp\left[\frac{\bar{Z}-(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)}{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)+\bar{Z}}\right]\right\},$$

$$t_{nl2}=\mu\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left[\frac{\bar{Z}g\bar{e}_z}{2\bar{Z}-\bar{Z}g\bar{e}_z}\right],$$

or

$$=\mu\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left[\frac{\bar{Z}g\bar{e}_z}{2\bar{Z}\left(1-\frac{1}{2}g\bar{e}_z\right)}\right].\quad(6.2.11)$$

After simplification, we get

$$t_{nl2}=\mu\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left(\frac{1}{2}g\bar{e}_z\left(1-\frac{1}{2}g\bar{e}_z\right)^{-1}\right),$$

or

$$t_{nl2}\approx\mu\bar{Y}\left(1+\bar{e}_y\right)+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left(\frac{1}{2}g\bar{e}_z+\frac{1}{4}g^2\bar{e}_z^2\right).$$

or

$$t_{nl2}\approx\mu\bar{Y}+\mu\bar{Y}\bar{e}_y+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left(\frac{1}{2}g\bar{e}_z+\frac{1}{4}g^2\bar{e}_z^2\right).\,(6.2.12)$$

Expanding the exponential function in (6.2.14) and ignoring the terms with power greater than two, we have

$$t_{nl2}\approx\mu\bar{Y}+\mu\bar{Y}\bar{e}_y+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\left(1+\frac{1}{2}g\bar{e}_z+\frac{3}{8}g^2\bar{e}_z^2\right),$$

or

$$t_{nl2}\approx\mu\bar{Y}+\mu\bar{Y}\bar{e}_y+\left(1-\mu\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+\frac{1}{2}g\bar{e}_z+\frac{3g^2}{8}\bar{e}_z^2+g\bar{e}_x+\frac{1}{2}g^2\bar{e}_z\bar{e}_x\right),$$

or

$$\approx \mu\bar{Y} + \mu\bar{Y}\bar{e}_y + (1-\mu)\left(\bar{Y} + \bar{Y}\bar{e}_y\right)\left(1 + \frac{1}{2}g\bar{e}_z + \frac{3}{8}g^2\bar{e}_z^2 + g\bar{e}_x + \frac{1}{2}g^2\bar{e}_z\bar{e}_x\right),$$

or

$$\approx \mu\bar{Y} + \mu\bar{Y}\bar{e}_y + (1-\mu)\bar{Y}\left[1 + \frac{1}{2}g\bar{e}_z + \frac{3}{8}g^2\bar{e}_z^2 + g\bar{e}_x \right.$$
$$\left. + \frac{1}{2}g^2\bar{e}_z\bar{e}_x + \bar{e}_y + \frac{1}{2}g\bar{e}_y\bar{e}_z + g\bar{e}_y\bar{e}_x\right]. \qquad (6.2.13)$$

After simplification, we get

$$t_{nl2} - \bar{Y} \approx \bar{Y}\left\{\frac{1}{2}g\bar{e}_z + \frac{3}{8}g^2\bar{e}_z^2 + g\bar{e}_x + \frac{1}{2}g^2\bar{e}_z\bar{e}_x + \bar{e}_y + \frac{1}{2}g\bar{e}_y\bar{e}_z + g\bar{e}_y\bar{e}_x\right.$$
$$\left. - \frac{1}{2}\mu g\bar{e}_z - \mu\frac{3}{8}g^2\bar{e}_z^2 - \mu g\bar{e}_x - \frac{1}{2}\mu g^2\bar{e}_z\bar{e}_x - \frac{1}{2}\mu g\bar{e}_y\bar{e}_z - \mu g\bar{e}_y\bar{e}_x\right\}.$$
$$(6.2.14)$$

Taking expectation on both sides of (6.2.9), using the notations (1.1.6), we get

$$E\left(t_{nl2} - \bar{Y}\right) \approx \bar{Y}\theta\left\{\frac{3}{8}g^2C_z^2 + \frac{1}{2}g^2C_zC_x\rho_{xz} + \frac{1}{2}gC_yC_z\rho_{yz} + gC_yC_x\rho_{yx}\right.$$
$$\left. - \frac{3\mu}{8}g^2C_z^2 - \frac{1}{2}\mu g^2C_zC_x\rho_{zx} - \frac{1}{2}\mu gC_yC_z\rho_{yz} - \mu gC_yC_x\rho_{yx}\right\}. (6.2.15)$$

After simplifications we get bias of t_{nl2} as:

$$Bias\left(t_{nl2}\right) \approx \bar{Y}\theta g^2\frac{(1-\mu)}{2}\left[\frac{3C_z^2}{4} + C_zC_x\rho_{xz} + \frac{1}{g}\left(C_yC_z\rho_{yz} + 2C_yC_x\rho_{yx}\right)\right].$$
$$(6.2.16)$$

Using the notations (1.1.6), we may write the estimator (6.2.3) as:

$$t_{nl3} = \delta\bar{Y}\left(1+\bar{e}_y\right)\exp\left[\frac{\bar{Z}-\bar{z}}{\bar{Z}+\bar{z}}\right]\left(\frac{\bar{X}}{\bar{x}}\right)$$
$$+(1-\delta)\bar{Y}\left(1+\bar{e}_y\right)\left[\frac{N\bar{X}-n\bar{x}}{(N-n)\bar{X}}\right]\exp\left[\frac{\frac{N\bar{Z}-n\bar{z}}{N-n} - \bar{Z}}{\frac{N\bar{Z}-n\bar{z}}{N-n} + \bar{Z}}\right],$$

or

$$t_{nl3}=\delta\bar{Y}\left(1+\bar{e}_y\right)\exp\left[\frac{\bar{Z}-\bar{Z}\left(1+\bar{e}_z\right)}{\bar{Z}+\bar{Z}\left(1+\bar{e}_z\right)}\right]\frac{\bar{X}}{\bar{X}\left(1+\bar{e}_x\right)}$$

$$+\left(1-\delta\right)\bar{Y}\left(1+\bar{e}_y\right)\left[\frac{(N-n)\bar{X}\left(1+\bar{e}_x\right)}{(N-n)\bar{X}}\right]\exp\left[\frac{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)-\bar{Z}}{(N-n)\bar{Z}(1-g\bar{e}_z)/(N-n)+\bar{Z}}\right],$$

or

$$=\delta\bar{Y}\left(1+\bar{e}_y\right)\exp\left[\frac{-\bar{Z}\bar{e}_z}{2\bar{Z}\left(1+\frac{\bar{e}_z}{2}\right)}\right]\left(1-\bar{e}_x\right)$$

$$+\left(1-\delta\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left[\frac{-\bar{Z}g\bar{e}_z}{2\bar{Z}\left(1-\frac{1}{2}g\bar{e}_z\right)}\right],$$

or

$$t_{nl3}=\delta\bar{Y}\left(1+\bar{e}_y\right)\exp\left[-\frac{\bar{e}_z}{2}\left(1-\frac{\bar{e}_z}{2}\right)\right]\left(1-\bar{e}_x\right)$$

$$+\left(1-\delta\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left(-\frac{1}{2}g\bar{e}_z-\frac{1}{4}g^2\bar{e}_z^2\right),$$

or

$$t_{nl3}\approx\delta\bar{Y}\left(1+\bar{e}_y\right)\exp\left[-\frac{\bar{e}_z}{2}+\frac{\bar{e}_z^2}{4}\right]\left(1-\bar{e}_x\right)$$

$$+\left(1-\delta\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\exp\left(-\frac{1}{2}g\bar{e}_z-\frac{1}{4}g^2\bar{e}_z^2\right). \quad (6.2.17)$$

Expanding the exponential function and re-writing (6.2.17) as:

$$t_{nl3}\approx\delta\bar{Y}\left(1+\bar{e}_y\right)\left[1-\frac{\bar{e}_z}{2}-\frac{\bar{e}_z^2}{8}\right]\left(1-\bar{e}_x\right)$$

$$+\left(1-\delta\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1-g\bar{e}_x\right)\left(1-\frac{1}{2}g\bar{e}_z-\frac{1}{8}g^2\bar{e}_z^2\right),$$

or

$$\approx \bar{Y}\left\{-\delta\bar{e}_x - \frac{\delta}{2}\bar{e}_z - \frac{1}{8}\delta\bar{e}_z^2 + \frac{\delta}{2}\bar{e}_x\bar{e}_z - \delta\bar{e}_y\bar{e}_x - \frac{\delta}{2}\bar{e}_z\bar{e}_y + 1 - \frac{g}{2}\bar{e}_z \right.$$
$$-\frac{g^2}{8}\bar{e}_z^2 - g\bar{e}_x + \frac{g^2}{2}\bar{e}_z\bar{e}_x + \bar{e}_y - \frac{g}{2}\bar{e}_y\bar{e}_z - g\bar{e}_y\bar{e}_x + \frac{\delta}{2}g\bar{e}_z$$
$$\left. +\frac{\delta}{8}g^2\bar{e}_z^2 + \delta g\bar{e}_x - \frac{g^2}{2}\delta\bar{e}_z\bar{e}_x + \frac{1}{2}\delta g\bar{e}_y\bar{e}_z + \delta g\bar{e}_y\bar{e}_x \right\}. \qquad (6.2.18)$$

After simplification, we have

$$t_{nl3} - \bar{Y} \approx \bar{Y}\left\{-\bar{e}_y\bar{e}_x\left(g + \delta - \delta g\right) - \frac{1}{2}\bar{e}_y\bar{e}_z\left(g + \delta - \delta g\right) \right.$$
$$\left. -\frac{1}{8}\bar{e}_z^2\left(g^2 - g^2\delta + \delta\right) + \frac{1}{2}\bar{e}_z\bar{e}_x\left(\delta + g^2 - \delta g^2\right)\right\}. \qquad (6.2.19)$$

Taking expectation on both sides of (6.2.19), using the notations (1.1.6), we get

$$E\left(t_{nl3} - \bar{Y}\right) \approx \bar{Y}\theta\left\{C_y C_x \rho_{yx}\left(\delta g - \delta - g\right) + \frac{1}{2}C_y C_z \rho_{yz}\left(\delta g - \delta - g\right) \right.$$
$$\left. -\frac{1}{8}C_z^2\left(g^2 - g^2\delta + \delta\right) + \frac{1}{2}C_z C_x \rho_{zx}\left(\delta + g^2 - \delta g^2\right)\right\}. \quad (6.2.20)$$

After simplifications we get bias of t_{nl3} as:

$$Bias\left(t_{nl3}\right) \approx \bar{Y}\theta\left\{\left(\delta g - \delta - g\right)\left(C_y C_x \rho_{yx} + \frac{1}{2}C_y C_z \rho_{yz}\right) \right.$$
$$\left. -\frac{1}{8}C_z^2\left(g^2 - g^2\delta + \delta\right) + \frac{1}{2}C_z C_x \rho_{zx}\left(\delta + g^2 - \delta g^2\right)\right\}. \quad (6.2.21)$$

Using the notations (1.1.6), we may write the estimator (6.2.4) as:

$$t_{nl4} = \delta\bar{Y}\left(1 + \bar{e}_y\right)\exp\left[\frac{\bar{Z}\left(1 + \bar{e}_z\right) - \bar{Z}}{\bar{Z} + \bar{Z}\left(1 + \bar{e}_z\right)}\right]\frac{\bar{X}\left(1 + \bar{e}_x\right)}{\bar{X}}$$
$$+\left(1 - \delta\right)\bar{Y}\left(1 + \bar{e}_y\right)\left[\frac{(N-n)\bar{X}}{N\bar{X} - n\bar{x}}\right]\exp\left[\frac{\bar{Z} - \dfrac{N\bar{Z} - n\bar{z}}{N-n}}{\dfrac{N\bar{Z} - n\bar{z}}{N-n} + \bar{Z}}\right],$$

or

$$= \varepsilon\bar{Y}\left(1+\bar{e}_y\right)\exp\left[\frac{\bar{Z}\bar{e}_z}{2\bar{Z}\left(1+\frac{\bar{e}_z}{2}\right)}\right]\left(1+\bar{e}_x\right)$$

$$+\left(1-\varepsilon\right)\bar{\mathrm{Y}}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left[\frac{\bar{Z}g\bar{e}_z}{2\bar{Z}\left(1-\frac{1}{2}g\bar{e}_z\right)}\right].$$

$$t_{nl4} \approx \varepsilon\bar{Y}\left(1+\bar{e}_y\right)\exp\left[\frac{\bar{e}_z}{2}\left(1-\frac{\bar{e}_z}{2}\right)\right]\left(1+\bar{e}_x\right)$$

$$+\left(1-\varepsilon\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\exp\left(\frac{1}{2}g\bar{e}_z+\frac{1}{4}g^2\bar{e}_z^2\right). \quad (6.2.22)$$

Expanding the exponential function up to the first order, re-writing (6.2.22) as:

$$t_{nl4} \approx \varepsilon\bar{Y}\left(1+\bar{e}_y\right)\left[1+\frac{\bar{e}_z}{2}-\frac{\bar{e}_z^2}{4}+\frac{\bar{e}_z^2}{8}\right]\left(1+\bar{e}_x\right)$$

$$+\left(1-\varepsilon\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\left(1+\frac{1}{2}g\bar{e}_z+\frac{1}{4}g^2\bar{e}_z^2+\frac{1}{8}g^2\bar{e}_z^2\right),$$

or

$$t_{nl4} \approx \varepsilon\bar{Y}\left(1+\bar{e}_y\right)\left[1+\frac{\bar{e}_z}{2}-\frac{\bar{e}_z^2}{8}\right]\left(1+\bar{e}_x\right)$$

$$+\left(1-\varepsilon\right)\bar{Y}\left(1+\bar{e}_y\right)\left(1+g\bar{e}_x\right)\left(1+\frac{1}{2}g\bar{e}_z+\frac{3}{8}g^2\bar{e}_z^2\right),$$

or

$$t_{nl4} \approx \bar{Y}\left\{\varepsilon\bar{e}_x+\frac{\varepsilon}{2}\bar{e}_z-\frac{1}{8}\varepsilon\bar{e}_z^2+\frac{\varepsilon}{2}\bar{e}_x\bar{e}_z+\varepsilon\bar{e}_y\bar{e}_x+\frac{\varepsilon}{2}\bar{e}_z\bar{e}_y+1+\frac{g}{2}\bar{e}_z\right.$$

$$+\frac{3g^2}{8}\bar{e}_z^2+g\bar{e}_x+\frac{g^2}{2}\bar{e}_z\bar{e}_x+\bar{e}_y+\frac{1}{2}g\bar{e}_y\bar{e}_z+g\bar{e}_x\bar{e}_y-\frac{\varepsilon}{2}g\bar{e}_z$$

$$\left.-\frac{3\varepsilon}{8}g^2\bar{e}_z^2-\varepsilon g\bar{e}_x-\frac{g^2}{2}\varepsilon\bar{e}_z\bar{e}_x-\frac{1}{2}\varepsilon g\bar{e}_y\bar{e}_z-\varepsilon g\bar{e}_y\bar{e}_x\right\}. \quad (6.2.23)$$

After simplification, we have

$$t_{nl4}-\bar{Y}\approx \bar{Y}\left\{\bar{e}_y\bar{e}_x\left(+\varepsilon+g-\varepsilon g\right)+\frac{1}{2}\bar{e}_y\bar{e}_z\left(+\varepsilon+g-\varepsilon g\right)-\frac{1}{8}\bar{e}_z^2\left(\varepsilon-3g^2-3g^2\varepsilon\right)\right.$$
$$\left.+\frac{1}{2}\bar{e}_z\bar{e}_x\left(\varepsilon+g^2-\varepsilon g^2\right)+\bar{e}_y+\frac{\bar{e}_z}{2}\left(\varepsilon+g-g\varepsilon\right)+\bar{e}_x\left(\varepsilon+g-g\varepsilon\right)\right\}. \tag{6.2.24}$$

Taking expectation on both sides of (6.2.9), using the notations (1.1.6), we get

$$E\left(t_{nl4}-\bar{Y}\right)\approx \bar{Y}\left\{C_yC_x\rho_{yx}\left(+\varepsilon+g-\varepsilon g\right)+\frac{1}{2}C_yC_z\rho_{yz}\left(\varepsilon+g-\varepsilon g\right)\right.$$
$$\left.-\frac{1}{8}C_z^2\left(\varepsilon-3g^2-3g^2\varepsilon\right)+\frac{1}{2}C_zC_x\rho_{zx}\left(\varepsilon+g^2-\varepsilon g^2\right)\right\}. \tag{6.2.25}$$

After simplifications we get the bias of t_{nl4} as:

$$Bias\left(t_{nl4}\right)\approx \bar{Y}\theta\left\{\left(\varepsilon+g-\varepsilon g\right)\left(C_yC_x\rho_{yx}+\frac{1}{2}C_yC_z\rho_{yz}\right)\right.$$
$$\left.-\frac{1}{8}C_z^2\left(\varepsilon-3g^2-3g^2\delta\right)+\frac{1}{2}C_zC_x\rho_{zx}\left(\varepsilon+g^2-\varepsilon g^2\right)\right\}. \tag{6.2.26}$$

6.2.2 Mean Square Errors of the Proposed Estimators

Squaring and taking expectation on both sides of (6.2.8), we have

$$E\left(t_{nl1}-\bar{Y}\right)^2\approx \bar{Y}^2E\left\{\bar{e}_y^2+\frac{1}{4}g^2\bar{e}_z^2\left(1-\omega\right)^2+\left(1-\omega\right)^2g^2\bar{e}_x^2\right.$$
$$\left.-\left(1-\omega\right)g\bar{e}_y\bar{e}_z+\left(1-\omega\right)^2g^2\bar{e}_x\bar{e}_z-2\left(1-\omega\right)g\bar{e}_y\bar{e}_x\right\}. \tag{6.2.27}$$

Using (1.1.6) in (6.2.29) and after simplification, we get

$$MSE\left(t_{nl1}\right)\approx \bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2\left(1-\omega\right)^2+\left(1-\omega\right)^2g^2C_x^2-\left(1-\omega\right)gC_yC_z\rho_{yz}\right.$$
$$\left.+\left(1-\omega\right)^2g^2C_xC_z\rho_{xz}-2\left(1-\omega\right)gC_yC_x\rho_{yx}\right\}. \tag{6.2.28}$$

Differentiating (6.2.28) with respect to ω and equating to zero, we get

$$\frac{\partial MSE(t_{nl1})}{\partial\omega}=\left\{-\frac{1}{2}g^2C_z^2(1-\omega)-2(1-\omega)g^2C_x^2+gC_yC_z\rho_{yz}\right.$$
$$\left.-2(1-\omega)g^2C_xC_z\rho_{xz}+2gC_yC_x\rho_{yx}\right\}=0,$$

or

$$(\omega-1)\left(\frac{1}{2}g^2C_z^2+2g^2C_x^2+2g^2C_xC_z\rho_{xz}\right)=-\left(2gC_yC_x\rho_{yx}+gC_yC_z\rho_{yz}\right),$$

or

$$\omega=1-\frac{\left(2gC_yC_x\rho_{yx}+gC_yC_z\rho_{yz}\right)}{2\left(\frac{1}{4}g^2C_z^2+g^2C_x^2+g^2C_xC_z\rho_{xz}\right)},$$

or

$$\omega_{opt}=1-\frac{\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)}{2g\left(\frac{1}{4}C_z^2+C_x^2+C_xC_z\rho_{xz}\right)}=\omega_o. \tag{6.2.29}$$

Using (6.2.29) in (6.2.28), the minimum mean square error of t_{nl1} is:

$$MSE(t_{nl1})\approx\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2(1-\omega_o)^2+(1-\omega_o)^2g^2C_x^2\right.$$
$$\left.-(1-\omega_o)gC_yC_z\rho_{yz}+(1-\omega_o)^2g^2C_xC_z\rho_{xz}-2(1-\omega_o)gC_yC_x\rho_{yx}\right\}. \tag{6.2.30}$$

Squaring and taking expectation on both sides of (6.2.16), we have

$$E\left(t_{nl2}-\bar{Y}\right)^2\approx\bar{Y}^2E\left\{\bar{e}_y^2+\frac{1}{4}g^2\bar{e}_z^2(1-\mu)^2+(1-\mu)^2g^2\bar{e}_x^2\right.$$
$$\left.+(1-\mu)g\bar{e}_y\bar{e}_z+(1-\mu)^2g^2\bar{e}_x\bar{e}_z+2(1-\mu)g\bar{e}_y\bar{e}_x\right\}. \tag{6.2.31}$$

Using (1.1.6) in (6.2.31) and after simplification, we get

$$MSE(t_{nl2})\approx\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2(1-\mu)^2+(1-\mu)^2g^2C_x^2+(1-\mu)gC_yC_z\rho_{yz}\right.$$
$$\left.+(1-\mu)^2g^2C_xC_z\rho_{xz}+2(1-\mu)gC_yC_x\rho_{yx}\right\}. \tag{6.2.32}$$

Differentiating (6.2.32) with respect to μ, we get

$$\frac{\partial MSE(t_{nl2})}{\partial\mu}=\left\{-\frac{1}{2}g^2C_z^2(1-\mu)-2(1-\mu)g^2C_x^2-gC_yC_z\rho_{yz}\right.$$
$$\left.-2(1-\mu)g^2C_xC_z\rho_{xz}-2gC_yC_x\rho_{yx}\right\}=0,$$

or

$$-(1-\mu)g^2\left(\frac{1}{2}C_z^2+2C_x^2+2C_xC_z\rho_{xz}\right)=2gC_yC_x\rho_{yx}+gC_yC_z\rho_{yz},$$

or

$$\mu=1+\frac{\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)}{g\left(\frac{1}{2}C_z^2+2C_x^2+2C_xC_z\rho_{xz}\right)},$$

or

$$\mu_{opt}=1+\frac{\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)}{2g\left(\frac{1}{4}C_z^2+C_x^2+C_xC_z\rho_{xz}\right)}=\mu_0. \tag{6.2.33}$$

Using (6.2.33) in (6.2.32) and after simplification, we get

$$\min MSE(t_{nl2})\approx\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2(1-\mu_o)^2+(1-\mu_o)^2g^2C_x^2\right.$$
$$\left.+(1-\mu_o)gC_yC_z\rho_{yz}+(1-\mu_o)^2g^2C_xC_z\rho_{xz}+2(1-\mu_o)gC_yC_x\rho_{yx}\right\},$$

or

$$\min MSE(t_{nl2})\approx\bar{Y}^2\theta\left\{C_y^2+g^2(1-\mu_o)^2\left(C_z^2+C_x^2+C_xC_z\rho_{xz}\right)\right.$$
$$\left.+(1-\mu_o)\left(gC_yC_z\rho_{yz}+2gC_yC_x\rho_{yx}\right)\right\}. \tag{6.2.34}$$

Squaring and taking expectation on both sides of (6.2.18), we have

$$E\left(t_{nl3}-\bar{Y}\right)^2\approx\bar{Y}^2E\left[\bar{e}_y+\bar{e}_x(\delta g-\delta-g)+\frac{1}{2}\bar{e}_z(\delta g-\delta-g)\right]^2,$$

or

$$E\left(t_{nl3}-\bar{Y}\right)^2 \approx \bar{Y}^2\left\{\bar{e}_y^2+\bar{e}_x^2\left(\delta+g-\delta g\right)^2+\frac{1}{4}\bar{e}_z^2\left(\delta g-\delta-g\right)^2\right.$$
$$\left.+2\bar{e}_y\bar{e}_x\left(\delta g-\delta-g\right)+\bar{e}_y\bar{e}_z\left(\delta g-\delta-g\right)+\bar{e}_z\bar{e}_x\left(\delta g-\delta-g\right)^2\right\}, \quad (6.2.35)$$

or

$$MSE\left(t_{nl3}\right)\approx \bar{Y}^2\left\{C_y^2+C_x^2\left(\delta g-\delta-g\right)^2\right.$$
$$+\frac{1}{4}C_z^2\left(\delta g-\delta-g\right)^2+2C_yC_x\rho_{yx}\left(\delta g-\delta-g\right)$$
$$\left.+C_yC_z\rho_{yz}\left(\delta g-\delta-g\right)+C_zC_x\rho_{zx}\left(\delta g-\delta-g\right)^2\right\}.$$

After simplification, we get the mean square error of t_{nl3} as:

$$MSE\left(t_{nl3}\right)\approx \bar{Y}^2\theta\left\{C_y^2+C_x^2\left(\delta g-\delta-g\right)^2\right.$$
$$+\frac{1}{4}C_z^2\left(\delta g-\delta-g\right)^2+2C_yC_x\rho_{yx}\left(\delta g-\delta-g\right)$$
$$\left.+C_yC_z\rho_{yz}\left(\delta g-\delta-g\right)+C_zC_x\rho_{zx}\left(\delta g-\delta-g\right)^2\right\}, \quad (6.2.36)$$

or

$$MSE\left(t_{nl3}\right)\approx \bar{Y}^2\theta\left[C_y^2+\tau^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\tau\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right], \quad (6.2.37)$$

where $\tau=\delta g-\delta-g$.

Differentiating (6.2.37) with respect to τ and equating to zero, we have

$$2\tau\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)=0.$$

On simplification, we get the minimum value of τ as:

$$\tau_{opt}=\frac{-2C_yC_x\rho_{yx}-C_yC_z\rho_{yz}}{2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)}=\tau_0. \quad (6.2.38)$$

Using (6.2.38) in (6.2.37), the minimum mean square error of the estimator t_{nl3} is given by

$$MSE_{\min}\left(t_{nl3}\right) \approx \bar{Y}^2\theta\left[C_y^2 + \tau_0^2\left\{C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right\}\right.$$
$$\left. + \tau_0\left(2C_y C_x \rho_{yx} + C_y C_z \rho_{yz}\right)\right]. \qquad (6.2.39)$$

Squaring and taking expectation on the both sides of (6.2.24), we get

$$E\left(t_{nl4} - \bar{Y}\right)^2 \approx \bar{Y}^2 E\left\{\bar{e}_y^2 + \bar{e}_x^2\left(\varepsilon - \varepsilon g + g\right)^2 + \frac{1}{4}\bar{e}_z^2\left(\varepsilon - \varepsilon g + g\right)^2\right.$$
$$\left. + 2\bar{e}_y\bar{e}_x\left(\varepsilon - \varepsilon g + g\right) + \bar{e}_y\bar{e}_z\left(\varepsilon - \varepsilon g + g\right) + 2\bar{e}_z\bar{e}_x\left(\varepsilon - \varepsilon g + g\right)^2\right\}.$$
$$(6.2.40)$$

Using (1.6.6) in (6.2.40), we have

$$MSE\left(t_{nl4}\right) \approx \bar{Y}^2\theta\left\{C_y^2 + C_x^2\left(\varepsilon - \varepsilon g + g\right)^2\right.$$
$$+ \frac{1}{4}C_z^2\left(\varepsilon - \varepsilon g + g\right)^2 + 2C_y C_x \rho_{yx}\left(\varepsilon - \varepsilon g + g\right)$$
$$\left. + C_y C_z \rho_{yz}\left(\varepsilon - \varepsilon g + g\right) + C_z C_x \rho_{zx}\left(\varepsilon - \varepsilon g + g\right)^2\right\},$$

or

$$MSE\left(t_{nl4}\right) = \bar{Y}^2\theta\left[C_y^2 + \lambda^2\left\{C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right\} + \lambda\left(2C_y C_x \rho_{yx} + C_y C_z \rho_{yz}\right)\right],$$
$$(6.2.41)$$

where $\lambda = \varepsilon - \varepsilon g + g$.

Differentiating (6.2.41) with respect to λ and equating to zero, we have

$$\frac{\partial MSE\left(t_{nl4}\right)}{\partial\lambda} = 2\lambda\left(C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right) + \left(2C_y C_x \rho_{yx} + C_y C_z \rho_{yz}\right) = 0,$$

or

$$\lambda_{opt} = \frac{-2C_y C_x \rho_{yx} - C_y C_z \rho_{yz}}{2\left(C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right)} = \lambda_o. \qquad (6.2.42)$$

Using (6.2.42) in (6.2.41), the minimum mean square error of the estimator t_{nl4} is given by

$$\min MSE\left(t_{nl4}\right)=\bar{Y}^2\theta\left[C_y^2+\lambda_o^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}\right.$$
$$\left.+\lambda_o\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]. \qquad (6.2.43)$$

6.3 COMPARISON OF THE PROPOSED ESTIMATORS WITH SOME AVAILABLE ESTIMATORS

In this section, the theoretical comparisons between some available estimators of population mean and proposed estimators have been made with respect to mean square error. The proposed estimators t_{nl1} and t_{nl3} have been compared with $\bar{y}$, t_1, t_9, t_{14}, t_{17}, t_{21} and t_{26}. The proposed estimators t_{nl2} and t_{nl4} have been compared with $\bar{y}$, t_{10}, t_{15} and t_{22}.

The proposed estimator t_{nl1} is more efficient than the mean per unit estimator if

$$\min MSE\left(t_{nl1}\right)\le Var\left(\bar{y}\right). \qquad (6.3.1)$$

Using (6.2.30) and (3.2.1) in (6.3.1), we have

$$\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2\left(1-\omega_0\right)^2+\left(1-\omega_0\right)^2g^2C_x^2-\left(1-\omega_0\right)gC_yC_z\rho_{yz}\right.$$
$$\left.+\left(1-\omega_0\right)^2g^2C_xC_z\rho_{xz}-2\left(1-\omega_0\right)gC_yC_x\rho_{yx}\right\}\le\theta\bar{Y}^2C_y^2,$$

or

$$\rho_{yx}\le\frac{C_yC_z\rho_{yz}+\left(\omega_0-1\right)g\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)}{2C_yC_x}. \qquad (6.3.2)$$

The proposed estimator t_{nl1} is more efficient than the classical ratio estimator t_1 if

$$\min MSE\left(t_{nl1}\right)\le MSE\left(t_1\right). \qquad (6.3.3)$$

Using (6.2.30) and (3.2.4) in (6.3.3), we have

$$\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2\left(1-\omega_0\right)^2+\left(1-\omega_0\right)^2g^2C_x^2-\left(1-\omega_0\right)gC_yC_z\rho_{yz}\right.$$
$$\left.+\left(1-\omega_0\right)^2g^2C_xC_z\rho_{xz}-2\left(1-\omega_0\right)gC_yC_x\rho_{yx}\right\}$$
$$\le\bar{Y}^2\theta\left[C_y^2+C_x^2-2C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx}\le\frac{\left(1-\omega_0\right)C_yC_z\rho_{yz}-\left(1-\omega_0\right)^2g^2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)+C_x^2}{C_yC_x\left(\omega_0g-g+2\right)}.$$

(6.3.4)

The proposed estimator t_{nl1} is more efficient than the Srivenkataramana's (1980) estimator t_9 if

$$\min MSE\left(t_{nl1}\right)\le MSE\left(t_9\right). \tag{6.3.5}$$

Using (6.2.30) and (3.2.20) in (6.3.5), we get

$$\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2\left(1-\omega_0\right)^2+\left(1-\omega_0\right)^2g^2C_x^2-\left(1-\omega_0\right)gC_yC_z\rho_{yz}\right.$$
$$\left.+\left(1-\omega_0\right)^2g^2C_xC_z\rho_{xz}-2\left(1-\omega_0\right)gC_yC_x\rho_{yx}\right\}$$
$$\le\bar{Y}^2\theta\left[C_y^2+g^2C_x^2-2gC_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx}\le\frac{\left(1-\omega_0\right)C_yC_z\rho_{yz}-g\left(1-\omega_0\right)^2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)+gC_x^2}{\left(\omega_0+1\right)C_yC_x}. \tag{6.3.6}$$

The proposed estimator t_{nl1} is more efficient than the Bahl and Tuteja (1991) estimator t_{14} if

$$\min MSE\left(t_{nl1}\right)\le MSE\left(t_{14}\right). \tag{6.3.7}$$

Using (6.2.30) and (3.2.33) in (6.3.7), we get

$$\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2(1-\omega)^2+(1-\omega)^2g^2C_x^2-(1-\omega)gC_yC_z\rho_{yz}\right.$$
$$\left.+(1-\omega)^2g^2C_xC_z\rho_{xz}-2(1-\omega)gC_yC_x\rho_{yx}\right\}$$
$$\le\bar{Y}^2\theta\left[C_y^2+\frac{1}{4}C_x^2-C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx}\le\frac{(1-\omega)gC_yC_z\rho_{yz}-(\omega-1)^2g^2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)}{2C_yC_x\left(\omega g-g+\frac{1}{2}\right)}. \qquad (6.3.8)$$

The proposed estimator t_{nl1} is more efficient than the Samiuddin and Hanif (2006) estimator t_{17} if

$$\min MSE(t_{nl1})\le MSE(t_{17}). \qquad (6.3.9)$$

Using (6.2.30) and (3.2.42) in (6.3.9), we get

$$\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2(1-\omega_0)^2+(1-\omega_0)^2g^2C_x^2-(1-\omega_0)gC_yC_z\rho_{yz}\right.$$
$$\left.+(1-\omega_0)^2g^2C_xC_z\rho_{xz}-2(1-\omega_0)gC_yC_x\rho_{yx}\right\}$$
$$\le\theta\bar{Y}^2\left[C_y^2+C_x^2+C_z^2-2C_xC_y\rho_{xy}-2C_yC_z\rho_{yz}+2C_xC_z\rho_{xz}\right],$$

or

$$\rho_{yx}\le\frac{C_xC_z\rho_{zx}\left(g^2(1-\omega_0)^2-2\right)-\left(g^2(1-\omega_0)^2-1\right)\left(\frac{1}{4}C_z^2+C_x^2\right)}{2(1+\omega_0g-g)C_yC_z}+\frac{\rho_{yz}}{2}.$$
$$(6.3.10)$$

The proposed estimator t_{nl1} is more efficient than the Sharma and Tailor (2010) estimator t_{21} if

$$\min MSE(t_{nl1})\le MSE(t_{21}). \qquad (6.3.11)$$

Using (6.2.30) and (3.2.54) in (6.3.11), we get

$$\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2\left(1-\omega_0\right)^2+\left(1-\omega_0\right)^2g^2C_x^2-\left(1-\omega_0\right)gC_yC_z\rho_{yz}\right.$$
$$\left.+\left(1-\omega_0\right)^2g^2C_xC_z\rho_{xz}-2\left(1-\omega_0\right)gC_yC_x\rho_{yx}\right\}$$
$$\leq\bar{Y}^2\theta\left[C_y^2+gC_x^2\left(\frac{g}{4}-K_{yx}\right)\right],$$

or

$$\rho_{yx}\leq\frac{\left(\omega_0-1\right)g\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)-C_yC_z\rho_{yz}}{C_yC_x}+\frac{C_x}{4C_y\left(\omega_0-1\right)}. \quad (6.3.12)$$

The proposed estimator t_{nl1} is more efficient than the Yadav (2012) estimator t_{26} if

$$\min MSE\left(t_{nl1}\right)\leq MSE\left(t_{26}\right). \quad (6.3.13)$$

Using (6.2.31) and (3.2.54) in (4.3.9), we get

$$\bar{Y}^2\theta\left\{C_y^2+\frac{1}{4}g^2C_z^2\left(1-\omega_0\right)^2+\left(1-\omega_0\right)^2g^2C_x^2-\left(1-\omega_0\right)gC_yC_z\rho_{yz}\right.$$
$$\left.+\left(1-\omega_0\right)^2g^2C_xC_z\rho_{xz}-2\left(1-\omega_0\right)gC_yC_x\rho_{yx}\right\}$$
$$\leq\bar{Y}^2\theta\left[C_y^2+\alpha^2\frac{C_x^2}{4}+\alpha C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx}\leq\frac{\left(1-\omega_0\right)gC_zC_y\rho_{yz}-g^2\left(1-\omega_0\right)^2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)}{C_yC_x\left(\alpha-2g+\omega g\right)}+\frac{\alpha^2C_x^2}{4}. \quad (6.3.14)$$

The proposed estimator t_{nl2} is more efficient than the mean per unit estimator if

$$\min MSE\left(t_{nl2}\right)\leq MSE\left(\bar{y}\right). \quad (6.3.15)$$

Using (6.2.34) and (3.2.1) in (4.3.15), we have

$$\bar{Y}^2\theta\Big\{C_y^2 + g^2(1-\mu_o)^2\left(C_z^2 + C_x^2 + C_xC_z\rho_{xz}\right)$$
$$+(1-\mu_o)\left(gC_yC_z\rho_{yz} + 2gC_yC_x\rho_{yx}\right)\Big\} \le \theta\bar{Y}^2C_y^2$$

or

$$\rho_{yx} \le \frac{g(\mu_0 - 1)\left(C_x^2 + C_z^2 + C_xC_z\rho_{xz}\right) - C_yC_z\rho_{yz}}{2C_xC_y}. \tag{6.3.16}$$

The proposed estimator t_{nl2} is more efficient than the Bandyopadhyay's (1980) estimator t_{10} if

$$\min MSE(t_{nl2}) \le MSE(t_{10}). \tag{6.3.17}$$

Using (6.2.34) and (3.2.23) in (6.3.17) we have,

$$\bar{Y}^2\theta\Big\{C_y^2 + g^2(1-\mu_o)^2\left(C_z^2 + C_x^2 + C_xC_z\rho_{xz}\right)$$
$$+(1-\mu_o)\left(gC_yC_z\rho_{yz} + 2gC_yC_x\rho_{yx}\right)\Big\} \le \bar{Y}^2\theta\left[C_y^2 + gC_z^2(\mathrm{g} + 2K_{yz})\right],$$

or

$$\rho_{yx} \le \frac{gC_z^2 - g(1-\mu_0)^2\left(C_x^2 + C_z^2 + C_xC_z\rho_{xz}\right)}{2C_yC_x(1-\mu_0)}. \tag{6.3.18}$$

The proposed estimator t_{nl2} is more efficient than the Bahl and Tuteja (1991) estimator t_{15} if

$$\min MSE(t_{nl2}) \le MSE(t_{15}). \tag{6.3.19}$$

Using (6.2.34) and (3.2.36) in (6.3.19) we have,

$$\bar{Y}^2\theta\Big\{C_y^2 + g^2(1-\mu_o)^2\left(C_z^2 + C_x^2 + C_xC_z\rho_{xz}\right)$$
$$+(1-\mu_o)\left(gC_yC_z\rho_{yz} + 2gC_yC_x\rho_{yx}\right)\Big\} \le \bar{Y}^2\theta\left[C_y^2 + \frac{C_z^2}{4} + C_yC_z\rho_{yz}\right],$$

or

$$\rho_{yx} \le \frac{\frac{C_z^2}{4} - g(1-\mu_0)^2\left(C_x^2 + C_z^2 + C_x C_z \rho_{xz}\right) + (g - g\mu_0 - 1) C_y C_z \rho_{yz}}{2 C_y C_x (1-\mu_0)}. \tag{6.3.20}$$

The proposed estimator t_{nl2} is more efficient than the Sharma and Tailor (2010) estimator t_{22} if

$$\min MSE(t_{nl2}) \le MSE(t_{22}). \tag{6.3.21}$$

Using (6.2.34) and (3.2.57) in (6.3.21), we get

$$\bar{Y}^2\theta\Big\{C_y^2 + g^2(1-\mu_o)^2\left(C_z^2 + C_x^2 + C_x C_z \rho_{xz}\right) + (1-\mu_o)\left(gC_y C_z \rho_{yz} + 2gC_y C_x \rho_{yx}\right)\Big\} \le \bar{Y}^2\theta\left[C_y^2 + gC_z^2\left(\frac{g}{4} + K_{yz}\right)\right]$$

,
or

$$\rho_{yx} \le \frac{\frac{gC_z^2}{4} - g(1-\mu_0)^2\left(C_x^2 + C_z^2 + C_x C_z \rho_{xz}\right) - g\mu_0 C_y C_z \rho_{yz}}{2 C_y C_x (1-\mu_0)}. \tag{6.3.22}$$

The proposed estimator t_{nl3} is more efficient than the mean per unit estimator if

$$\min MSE(t_{nl3}) \le Var(\bar{y}). \tag{6.3.23}$$

Using (6.2.39) and (3.2.1) in (6.3.23), we have

$$\bar{Y}^2\theta\left[C_y^2 + \tau_0^2\left\{C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right\} + \tau_0\left(2C_y C_x \rho_{yx} + C_y C_z \rho_{yz}\right)\right] \le \theta\bar{Y}^2 C_y^2,$$

or

$$\rho_{yx} \le \frac{-\left[C_y C_z \rho_{yz} + \tau_0\left(C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right)\right]}{2 C_y C_x}. \tag{6.3.24}$$

The proposed estimator t_{nl3} is more efficient than the classical ratio estimator t_1 if

$$\min MSE\left(t_{nl3}\right) \leq MSE\left(t_1\right). \tag{6.3.25}$$

Using (6.2.39) and (3.2.4) in (6.3.25), we have

$$\bar{Y}^2\theta\left[C_y^2+\tau_0^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\tau_0\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]$$
$$\leq \bar{Y}^2\theta\left[C_y^2+C_x^2-2C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx} \leq \frac{\left[C_x^2-\tau_0C_yC_z\rho_{yz}-\tau_0^2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)\right]}{2\left(\tau_0+1\right)C_yC_x}. \tag{6.3.26}$$

The proposed estimator t_{nl3} is more efficient than the Srivenkataramana's (1980) estimator t_9 if

$$\min MSE\left(t_{nl3}\right) \leq MSE\left(t_9\right). \tag{6.3.27}$$

Using (6.2.39) and (3.2.20) in (6.3.27), we get

$$\bar{Y}^2\theta\left[C_y^2+\tau_0^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\tau_0\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]$$
$$\leq \bar{Y}^2\theta\left[C_y^2+g^2C_x^2-2gC_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx} \leq \frac{\left[g^2C_x^2-\tau_0C_yC_z\rho_{yz}-\tau_0^2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)\right]}{2\left(\tau_0+g\right)C_yC_x}. \tag{6.3.28}$$

The proposed estimator t_{nl3} is more efficient than the Bahl and Tuteja (1991) estimator t_{14} if

$$\min MSE\left(t_{nl3}\right) \leq MSE\left(t_{14}\right). \tag{6.3.29}$$

Using (6.2.39) and (3.2.33) in (6.3.29), we get

$$\bar{Y}^2\theta\left[C_y^2+\tau_0^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\tau_0\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]$$
$$\leq \bar{Y}^2\theta\left[C_y^2+\frac{1}{4}C_x^2-C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx}\leq\frac{\frac{C_x^2}{4}-\tau_0C_yC_z\rho_{yz}-\tau_0^2\left(C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)}{\left(2\tau_0+1\right)C_yC_x}. \qquad (6.3.30)$$

The proposed estimator t_{nl3} is more efficient than the Samiuddin and Hanif (2006) estimator t_{17} if

$$\min MSE\left(t_{nl3}\right)\leq MSE\left(t_{17}\right). \qquad (6.3.31)$$

Using (6.2.39) and (3.2.42) in (6.3.31), we get

$$\bar{Y}^2\theta\left[C_y^2+\tau_0^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\tau_0\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]$$
$$\leq\theta\bar{Y}^2\left[C_y^2+C_x^2+C_z^2-2C_xC_y\rho_{xy}-2C_yC_z\rho_{yz}+2C_xC_z\rho_{xz}\right],$$

or

$$\rho_{yx}\leq\frac{\left(\tau_0-1\right)C_x}{2C_y}+\frac{\left(1-\frac{1}{4}\tau_0^2\right)C_z^2-\left(\tau_0^2-2\right)C_zC_x\rho_{zx}-\left(\tau_0+2\right)C_yC_z\rho_{yz}}{2\left(\tau_0+1\right)C_yC_x}.$$
$$(6.3.32)$$

The proposed estimator t_{nl3} is more efficient than the Sharma and Tailor (2010) estimator t_{21} if

$$\min MSE\left(t_{nl3}\right)\leq MSE\left(t_{21}\right). \qquad (6.3.33)$$

Using (6.2.39) and (3.2.54) in (6.3.33), we get

$$\bar{Y}^2\theta\left[C_y^2+\tau_0^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\tau_0\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]$$
$$\le\bar{Y}^2\theta\left[C_y^2+gC_x^2\left(\frac{g}{4}-K_{yx}\right)\right],$$

or

$$\rho_{yx}\le\frac{\dfrac{g^2C_x^2}{4}-\tau_0^2\left(C_x^2+\dfrac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)-\tau_0C_yC_z\rho_{yz}}{\left(2\tau_0+g\right)C_yC_x}. \qquad (6.3.34)$$

The proposed estimator t_{nl3} is more efficient than the Yadav (2012) estimator t_{26} if

$$\min MSE\left(t_{nl3}\right)\le MSE\left(t_{26}\right). \qquad (6.3.35)$$

Using (6.2.39) and (3.2.54) in (6.3.35), we get

$$\bar{Y}^2\theta\left[C_y^2+\tau_0^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\tau_0\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]$$
$$\le\bar{Y}^2\theta\left[C_y^2+\alpha^2\frac{C_x^2}{4}+\alpha C_yC_x\rho_{yx}\right],$$

or

$$\rho_{yx}\le\frac{\dfrac{\alpha^2C_x^2}{4}-\tau_0^2\left(C_x^2+\dfrac{1}{4}C_z^2+C_zC_x\rho_{zx}\right)-\tau_0C_yC_z\rho_{yz}}{\left(2\tau_0-\alpha\right)C_yC_x}. \qquad (6.3.36)$$

The proposed estimator t_{nl4} is more efficient than the mean per unit estimator if

$$\min MSE\left(t_{nl4}\right)\le MSE\left(\bar{y}\right). \qquad (6.3.37)$$

Using (6.2.43) and (3.2.1) in (6.3.37), we have

$$\bar{Y}^2\theta\left[C_y^2+\lambda_o^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\lambda_o\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]\le\theta\bar{Y}^2C_y^2$$

or

$$\rho_{yx} \le \frac{-\lambda_0\left(C_x^2 + \frac{C_z^2}{4} + C_x C_z \rho_{xz}\right) - C_y C_z \rho_{yz}}{2C_x C_y}. \tag{6.3.38}$$

The proposed estimator t_{nl4} is more efficient than the Bandyopadhyay's (1980) estimator t_{10} if

$$\min MSE\left(t_{nl4}\right) \le MSE\left(t_{10}\right). \tag{6.3.39}$$

Using (6.2.43) and (3.2.23) in (6.3.39) we have,

$$\bar{Y}^2\theta\left[C_y^2 + \lambda_o^2\left\{C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right\} + \lambda_o\left(2C_y C_x \rho_{yx} + C_y C_z \rho_{yz}\right)\right]$$
$$\le \bar{Y}^2\theta\left[C_y^2 + gC_z^2(\mathrm{g} + 2K_{yz})\right],$$

or

$$\rho_{yx} \le \frac{g^2 C_z^2 - \lambda_0^2\left(C_x^2 + \frac{C_z^2}{4} + C_x C_z \rho_{xz}\right) - C_y C_z \rho_{yz}\left(\lambda_0 + 2g\right)}{2\lambda_0 C_x C_y}. \tag{6.3.40}$$

The proposed estimator t_{nl4} is more efficient than the Bahl and Tuteja (1991) estimator t_{15} if

$$\min MSE\left(t_{nl4}\right) \le MSE\left(t_{15}\right). \tag{6.3.41}$$

Using (6.2.43) and (3.2.36) in (6.3.41) we have,

$$\bar{Y}^2\theta\left[C_y^2 + \lambda_o^2\left\{C_x^2 + \frac{1}{4}C_z^2 + C_z C_x \rho_{zx}\right\} + \lambda_o\left(2C_y C_x \rho_{yx} + C_y C_z \rho_{yz}\right)\right]$$
$$\le \bar{Y}^2\theta\left[C_y^2 + \frac{C_z^2}{4} + C_y C_z \rho_{yz}\right],$$

or

$$\rho_{yx} \le \frac{\frac{C_z^2}{4} - \lambda_0^2\left(C_x^2 + \frac{C_z^2}{4} + C_x C_z \rho_{xz}\right) - C_y C_z \rho_{yz}\left(\lambda_0 - 1\right)}{2\lambda_0 C_x C_y}. \tag{6.3.42}$$

The proposed estimator t_{nl4} is more efficient than the Sharma and Tailor (2010) estimator t_{22} if

$$\min MSE\left(t_{nl4}\right) \le MSE\left(t_{22}\right). \tag{6.3.43}$$

Using (6.2.43) and (3.2.57) in (6.3.43), we get

$$\bar{Y}^2\theta\left[C_y^2+\lambda_o^2\left\{C_x^2+\frac{1}{4}C_z^2+C_zC_x\rho_{zx}\right\}+\lambda_o\left(2C_yC_x\rho_{yx}+C_yC_z\rho_{yz}\right)\right]$$
$$\le \bar{Y}^2\theta\left[C_y^2+gC_z^2\left(\frac{g}{4}+K_{yz}\right)\right],$$

or

$$\rho_{yx} \le \frac{\frac{g^2C_z^2}{4}-\lambda_0^2\left(C_x^2+\frac{C_z^2}{4}+C_xC_z\rho_{xz}\right)-C_yC_z\rho_{yz}\left(\lambda_0-g\right)}{2\lambda_0C_xC_y}. \tag{6.3.44}$$

When the condition (6.3.2), (6.3.4), (6.3.6), (6.3.8), (6.3.10), (6.3.12) and (4.3.14) is satisfied, the proposed linear ratio type exponential estimator t_{nl1} will be more efficient as compare to $\bar{y}$, t_1, t_9, t_{14}, t_{17}, t_{21} and t_{26} respectively. When the condition (6.3.16), (6.3.18), (6.3.20) and (6.3.22) is satisfied, the proposed linear product type exponential estimator t_{nl2} will be more efficient as compare to $\bar{y}$, t_{10}, t_{15} and t_{22} respectively.

When the condition (6.3.24), (6.3.26), (6.3.28), (6.3.30), (6.3.32), (6.3.34) and (4.3.36) is satisfied, the proposed linear ratio type exponential estimator t_{nl3} will be more efficient as compare to $\bar{y}$, t_1, t_9, t_{14}, t_{17}, t_{21} and t_{26} respectively. When the condition (6.3.38), (6.3.40), (6.3.42) and (6.3.44) is satisfied, the proposed linear product type exponential estimator t_{nl4} will be more efficient as compare to $\bar{y}$, t_{10}, t_{15} and t_{22} respectively.

CHAPTER 7

EMPIRICAL COMPARISON OF THE PROPOSED ESTIMATORS WITH SOME AVAILABLE ESTIMATORS

7.1 INTRODUCTION

In this Chapter, empirical study has been conducted for the efficiency comparison of the proposed estimators with some available estimators using some real populations (given in Appendix). The percentage relative efficiencies have been computed and presented in table 7.1 and table 7.2. We have compared our proposed estimators with the classical ratio and product estimators, dual-to-ratio estimator proposed by Srivenkataramana (1980), dual-to-product estimator suggested by Bandyopadhyay's (1980), the exponential ratio-type and product-type estimators proposed by Bahl and Tuteja (1991),the Samiuddin and Hanif (2006) estimator, the estimator dual-to-ratio type exponential estimator and dual-to-product type exponential estimator suggested by Sharma and Tailor (2010) and the ratio-cum-dual to ratio type estimator proposed by Yadav (2012)have been considered.

The percentage relative efficiencies (PRE) have been computed using the following formula:

$$PRE = \frac{Var(\bar{y})}{MSE(t_*)} \times 100. \tag{7.1.1}$$

7.2 EMPIRICAL RESULTS

The percentage relative efficiencies of ratio type estimators have been computed in Table 7.1. The percentage relative efficiencies of product type estimators have been computed in Table 7.2.

Table 7.1
Percent Relative Efficiency of the Estimators

Estimators	Population		
	1	**2**	**3**
$\bar{y}$	100	100	100
t_1	100.0081	121.48	100.0008
t_9	135.1086	194.4862	102.70
t_{14}	172.37	208	155.35
t_{17}	72.29149	45.054	107.086
t_{21}	116.60	144.3385	101.344
t_{26}	43.33	39.92	46.42691
t_{nr1}	160	231	104.74
t_{nrg}	231	235	563.97
t_{nl1}	225.37	234.96	249.43
t_{nl3}	195.20	212	216.97

Table 7.2
Percentage Relative Efficiencies for the Product Estimators

Estimators	Population			
	4	**5**	**6**	**7**
$\bar{y}$	100	100	100	100
t_2	98.27	113.12	79.60	53.33
t_{10}	120	68.74	46	82.2523
t_{15}	107.30	81.42	76.40	59.025
t_{22}	108.71	88.75	72.82	75.32
t_{n2}	127	*	*	*
$t_{nl2} = t_{nl4}$	142.45	113.46	244.236	124.608

Percentage relative efficiencies of proposed estimators in comparison with Classical ratio estimator, the dual-to- ratio estimator proposed by Srivenkataramana (1980), exponential ratio-type estimator suggested by Bahl and Tuteja (1991) using two auxiliary variables , Samiuddin and Hanif (2006) and ratio-cum-dual to ratio type estimator proposed by Yadav(2012) have been computed using the formula (7.1.1). From Table 7.1, we can see that the proposed estimators are more efficient than all other estimators.

From Table 7.2, we can see that the proposed product type estimators t_{nl2}, t_{nl4} are more efficient than classical product estimator t_2, the dual to product-type estimator t_{10} proposed by Bandyopadhyay's (1980), the exponential product-type estimator by Bahl and Tuteja (1991) and the dual-to product type estimator suggested by Sharma and Tailor(2010).

REFERENCES

1. Ahmed, A. (2008). *A Comparative Study in Two-Phase Sampling*. Unpublished M.Phil. Thesis, National College of Business Administration and Economics, Lahore.

2. Anderson, T.W. (1958). *An Introduction to Multivariate Statistical Analysis*. John Wiley & Sons, Inc., New York.

3. Bahl, S. and Tuteja, R.K. (1991). Ratio and product type exponential estimator. *Infor. and Optimi. Sci.*, 12, 159-163.

4. Bajwa, M. (2003). *Modification of Mohanty's Estimators in One and Two-Phase Sampling*. Unpublished M.Phil. Thesis, National College of Business Administration and Economics, Lahore.

5. Bandyopadhyaya, S. (1980). Improved ratio and product estimators. *Sankhya*, 42(C), 45-49.

6. Chakrabarty, R.P. (1979). Some ratio type estimators. *J. Ind. Soc. Agri. Stat.*, 31, 46-62.

7. Chand, L. (1975). *Some ratio type estimators based on two or more auxiliary variables*. Unpublished Ph.D. Dissertation. Iowa State University, Iowa.

8. Choudhury, S. and Singh, B.K. (2013). A class of product -cum-dual to product estimators of the population mean in survey sampling using auxiliary information. *Asian J. Math. and Statist.*, 6(1), 1-11.

9. Cochran, W.G. (1940). The estimation of the yields of the cereal experiments by sampling for the ratio of grain to total produce. *The J. Agricul. Sci.*, 30, 262-275.

10. Cochran, W.G. (1977). *Sampling Techniques*. New-York: John Wiley and Sons.

11. Das, A.K. (1988). *Contribution to the theory of sampling strategies based on auxiliary information.* Ph.D. Thesis submitted to B.C.K.V., Mohanpur, Nadia, West Bengal, India.

12. Das, A.K. and T.P. Tripathi, T.P. (1980). Sampling strategies for population mean when the coefficient of variation of an auxiliary character is known. *Sankhya*, *C*, 42, 76-86.

13. Graunt, J. (1662). *The Economic Writings of Sir William Petty*, 1899, Cambridge University Press, 2, 314-431.

14. Gujarati, D.N. (2004). *Basic Econometrics*. New-York: McGraw-Hill Companies.

15. Hanif, M., Hamad, N. and Shahbaz, M.Q. (2009). A Modified Regression Type Estimator in Survey Sampling. *World App. Sci. J.*, 7(12), 1559-1561.

16. Haq, I. (2008). *A General Family of Estimators using Information on Two Auxiliary Attributes*. Unpublished M.Phil. Thesis, National College of Business Administration and Economics, Lahore.

17. Jeelani, M.I., Maqbool, S. and Mir, S.A. (2013) modified ratio estimators of population mean using linear combination of co-efficient of skewness and quartile deviation. *Intl. J. Mod. Math. Sci.*, 6(3), 174-183.

18. Kadilar, C. and Cingi, H. (2006). New ratio estimators using correlation coefficient. *Interstat*. http://interstat.Statjournals.net/Year/2006/articles/0603004.pdf.

19. Kiregyera, B. (1980). A chain ratio type estimator in finite population double sampling using two auxiliary variables. *Metrika*, 27, 217-223.

20. Kiregyera, B. (1984). Regression type estimator using two auxiliary variables and model of double sampling from finite populations. *Metrika*, 31, 215-226.

21. Kish, L. (1965). *Survey Sampling*. John Wiley & Sons, New York.

22. Laplace, P.S. (1820). *A philosophical essay on probabilities*. English translation, Dover, 1951.

23. Mohanty, S. (1967). Combination of regression and ratio estimate. *J. Ind. Statist. Assoc.*, 5, 16-19.

24. Mukerjee, R., Rao, T.J. and Vijayan, K. (1987). Regression type estimators using multi-auxiliary information. *Austral. J. Statist.*, 29(3), 244-254.

25. Murthy, M.N. (1964). Product method of estimation. *Sankhya A*, 26, 69-74.

26. Naik, V.D. and Gupta, P.C. (1991). A general class of estimators for estimating population mean using auxiliary information. *Metrika*, 38, 11-17.

27. Nazuk, A., Nadir, S. and Shabbir, J. (2010). *Adjustment of the auxiliary variable(s) for estimation of a finite population mean*. Manuscript seen by the courtesy of the Authors.

28. Neyman, J. (1934). On the two different aspects of the representative method: the method of stratified sampling and the method of purposive selection. *J. Roy. Statist. Soc.*, 97, 558-606.

29. Neyman, J. (1938). Contributions to the theory of sampling human populations. *J. Amer. Statist. Assoc.*, 33, 101-11.

30. Noor-ul-Amin, M. and Hanif, M. (2012). Some exponential estimators in survey sampling. *Pak. J. Statist.*, 28(3), 367-374.

31. Onyeka, A.C., Nlebedim, V.U. and Izunobi, C.H. (2013). Estimation of population ratio in simple random sampling using variable transformation. *Global J. Sci. Front. Res. Math. and Dec. Sci.*, 13(4), Online ISSN: 2249-4626 & Print ISSN: 0975-5896.

32. Prasad, B. (1989). Some improved ratio-type estimators of population mean and ratio in finite population sample surveys. *Commun. Statist. Theo. and Meth.*, 18, 379-392.

33. Rao, P.S.R.S. (1987). Ratio and regression estimates with sub sampling the non- respondents. Paper presented at a special contributed session of the *International Statistical Association Meeting*, Sept., 2-16, Tokyo, Japan.

34. Rao, P.S. (1968). On three procedures of sampling from finite populations. *Biometrika*, 55(2), 438-441.

35. Rao, T.J. (1983). A new class of unbiased product estimators. *Tech. Rep.* No. 15183, Stat. Math. Indian Statistical Institute, Calcutta, India.

36. Robson, D.S. (1957). Application of multivariate polykays to the theory of unbiased ratio-type estimation. *J. Amer. Statist. Assoc.*, 52, 511-522.

37. Roy, D.C. (2003). A regression type estimator in two phase sampling using two auxiliary variables. *Pak. J. Statist.*, 19(3), 281-290.

38. Sahai, A. (1979). An efficient variant of the product and ratio estimators. *Statistica Neerlandica*, 33, 27-35.

39. Sahoo, J., Sahoo, L.N. and Mohanty, S. (1993). A regression approach to estimation in two phase sampling using two auxiliary variables. *Current Science*, 65(1), 73-75.

40. Samiuddin, M. and Hanif, M. (2006). Estimation in two-phase sampling with complete and incomplete information. *Proc. 8th Islamic Countries Conference on Statistical Sciences*, 13, 479-495.

41. Searls, D.T. (1964). The utilization of known coefficient of variation in the estimation procedure. *J. Amer. Statist. Assoc.*, 59, 1125-26.

42. Searls, D.T. and Intarapanich, P. (1990). A note on an estimator for the variance that utilizes the kurtosis. *American Statistician*, 44, 295-296.

43. Sen, A.R. (1978). Estimation of the population mean when the coefficient of variation is known. *Commun. Statist.-Theory Meth.*, 7, 657-672.

44. Sharma, B. and Tailor, R. (2010). A new ratio-cum-dual to ratio estimator of finite population mean in simple random sampling. *Global J. Sci. Frontier Res.*, 10(1), 27-31.

45. Shukla, D., Pathak, S. and Thakur, N.S. (2012). Estimation of population mean using two auxiliary sources in sample surveys. *Statist. in Transi.-New Series*, 13(1), 21-36.

46. Singh, A.K. and Singh, H. P. (1997). A note on the efficiencies of three product-type estimators under a linear model. *J. Ind. Soc. of Agri. Stat.*, 50, 130-134.

47. Singh, A.K., Upadhyaya, L.N. and Singh, H.P. (2006). Comparisons of three product type estimators in small sample. *Statistics in Transition*, 7, 917-928.

48. Singh, G.N. (2002). Empirical studies of generalized classes of ratio and product type estimators under a linear model. *Statistics in Transition*, 5, 701-720.

49. Singh, G.N. (2003). On the improvement of product method of estimation in sample surveys. *J. Ind. Soc. of Agri. Stat.*, 56(3), 267-265.

50. Singh, H.P., Tailor, R., Tailor, R. and Kakran, M. (2004). An improved estimation of population mean using power transformation. *J. Ind. Soc. Agri. Stat.*, 58, 223-230.

51. Singh, H.P. and Espejo, M.R. (2003). On linear regression and ratio-product estimation of a finite population mean. *Statistician*, 52(1), 59-67.

52. Singh, H.P. and Espejo, M.R. (2007). Double sampling ratio-product estimator of a finite population mean in sample surveys. *J. Appl. Statist.*, 34, 71-85.

53. Singh, H.P., Singh, R., Espejo, M.R., Pineda, M.D. and Nadarajan, S. (2005). On the efficiency of a dual to ratio-cum-product estimator in sample surveys. *Math. Proceed. Royal Irish Academy*, 105, A(2), 51-56.

54. Singh, H.P., Tailor, R. and Tailor, R. (2010). On ratio and product methods with certain known population parameters of auxiliary variable in sample surveys. *SORT*, 34(2), 157-180.

55. Singh, J., Pandey, B.N. and Hirano, K. (1973). On the utilization of a known coefficient of kurtosis in the estimation procedure of variance. *Ann. Inst. Stat. Math.*, 25, 51-55.

56. Singh, P. and Vishwakarma, K. (2007). Modified exponential ratio and product estimators for finite population mean in double sampling. *Aust. J. Statist*., 36, 217-225.

57. Singh, P.V. (2003). The class of unbiased dual to ratio estimator. *J. Ind. Soc. Agri. Stat.*, 56(3), 211-221.

58. Singh, R. (1987). *Investigating the efficiencies of some robust estimators in sample survey in the presence of auxiliary information*. Unpublished Ph.D. Thesis submitted to Bihar Uni. India.

59. Singh, R., Chauhan, P., Sawan, N. and Smarandache, F. (2009). Improvement in estimating the population mean using exponential estimator in simple random sampling. *Int. J. Statist. Econ. (BSE)*, 3, 13-18.

60. Sisodia, B.V.S. and Dwivedi, V.K. (1981). A modified ratio estimator using coefficient of variation of auxiliary variable. *J. Ind. Soc. Agri. Stat.*, 33, 13-18.

61. Srivastava, RaniSrivastava, Hare, B.B. and Srivastava, S.R. (1990). A generalized chain ratio type estimator for mean of finite population. *J. Ind. Soc. Agri. Stat.*, 42(1), 108-117.

62. Srivastava, S.K. (1967). An estimator using auxiliary information. *Cal. Statist. Assoc. Bull.*, 16, 121-132.

63. Srivastava, S.K. (1970). A two phase sampling estimator in sample surveys. *Aust. J. Statist.*, 12(1), 23-27.

64. Srivenkataramana, T. (1980). A dual to ratio estimator in sample surveys. *Biometrika*, 67, 199-204.

65. Steel, R.G.D. and Torrie, J.H. (1960). *Principles and Procedures of Statistics.* McGraw-Hill Book Company, New York.

66. Stephan, F.F. (1948). History and uses of modern sampling procedures. *J. Amer. Statist. Assoc.*, 43, 12-39.

67. Subramani, J. and Kumarapandiyan, G. (2012). A class of almost unbiased modified ratio estimators for population mean with known population parameters. *Elixir Statistics*, 44, 7411-7415.

68. Swain, A.K.P.C. (2012). On some classes of modified ratio type and regression type estimators in two phase sampling. *Pak. J. Statist.*, 28(4), 445-466.

69. Swain, A.K.P.C. (2013). On some modified ratio and product type estimators-revisited. *Revista Investigación Operacional*, 34(1), 35-57.

70. Tahir, A. (2008). *Empirical Comparison of Some Estimators for Two-Phase Sampling*. Unpublished M.Phil. Thesis, National College of Business Administration and Economics, Lahore.

71. Tailor, R., Tailor, R., Parmar, R. and Kumar, M. (2012). Dual to ratio cum product estimator using known parameters of auxiliary variables. *J. Reliab. Statist. Stud.*, 5, 65-71.

72. Tracy, D.S. and Singh, H.P. (1997). A family of estimators for population mean using a priori/auxiliary information in samples surveys. *J. App. Statist. Sci.*, 6(1), 1-20.

73. Tracy, D.S. and Singh, H.P. (1998). A modified ratio cum product estimator. *Intl. J. Math. and Statist. Sci.*, 7, 201-212.

74. Tripathi, T.P. (1970). *Contributions to the sampling theory in multivariate information*. Ph.D. Thesis submitted to Punjabi University, Patiala, India.

75. Tripathi, T.P. (1973). Double sampling for inclusion probabilities and regression method of estimation. *J. Ind. Statist. Assoc.*, 10, 33-46.

76. Tripathi, T.P. (1976). On double sampling for multivariate ratio and difference method of estimation. *J. Ind. Statist. Assoc.*, 33, 33-54.

77. Tripathi, T.P. (1989). Optimum estimation of mean vector for dynamic population. Invited paper in the *Proceeding of International Symposium on Optimization and Statistics*, held at AMU. Aligarh.

78. Tripathi, T.P. and Khattree, R. (1989). Simultaneous estimation of several means using multivariate auxiliary information. *Tech. Report* No. 13/89, Stat/Math, ISI, Calcutta.

79. Tripathi, T.P. and Singh, H.P. (1992). A class of unbiased product type estimators for the mean suitable for the positive and negative correlation situations. *Commun. Statist.-Theory Meth.*, 21(2), 507-518.

80. Tripathi, T.P., Singh, H.P. and Upadhyaya, L.N. (1988). A generalized method of estimation in double sampling. *J. Ind. Statist. Assoc.*, 26, 91-101.

81. Upadhyaya, L. N. and Singh, H. P. (1999). Use of transformed auxiliary variable in estimating the finite population mean. *Biom. Jour.*, 41, 627-636.

82. Upadhyaya, L.N. and Singh, H.P. (1984). On the estimation of the population mean with known coefficient of variation. *Biom. Jour.*, 26(8), 915-922.

83. Upadhyaya, L.N., Singh, H.P., Chatterjee, S. and Yadav, R. (2011). Improved ratio and product exponential type estimators. *J. Statist. Theo. and Prac.*, 5(2), 285-302.

84. Walsh, J.E. (1970). Generalization of ratio estimates for population total. *Sankhya*, *A*, 32, 99-106.

85. Watson, D.J. (1937). The estimation of leaf areas. *J. Agri. Sci.*, 27, 474.

86. Yadav, S.K. (2012). Improved exponential ratio cum dual to ratio type estimator of population mean. *ESMSJ*, 2(1), 58-60.

APPENDIX

Table A1:
Source of Population

S#	Source of Population
1	Anderson (1958)
2	Cochran (1977)
3	Population Census Report of Multan district (1998), Pakistan
4	Gujarati (2004)
5	Gujrati (237)
6	Srivastava et al.(1989, page 3922)
7	Steel and Torrie (1960)

Table A2:
Description of variables for Population 1-4

Population	*Y*	*X*	*Z*
1	Head length of second son	Head length of first son	Head breadth of first son
2	Number of "placebo" children	Number of paralytic polio cases in the "not inoculated" group.	Number of paralytic polio cases in the placebo group.
3	Population matric and above	Primary but below matric	Population of both sexes
4	Average miles per gallon	Top speed, miles per	Cubic feet of cab space
5			
6	The measurement of weight of children	Mid arm circumferences of children	Skull circumference of children
7	Log of leaf burn in sec.	Potassium percentage	Chlorine percentage

Table A3:
Parameters of Populations

Parameter	Population						
	1	**2**	**3**	**4**	**5**	**6**	**7**
N	25	34	424	81	30	82	30
n	5	5	10	20	12	25	8
$\bar{Y}$	183.84	4.92	646.215	33.8345	10.637	5.60	0.6860
$\bar{X}$	185.72	2.59	4533.981	112.4568	4.4497	11.90	4.6537
$\bar{Z}$	151.122	2.91	325.0325	98.7654	7.5248	39.80	0.8077
C_y	*0.0546*	*1.0123*	*1.509*	0.2972	0.2178	1.509	0.4803
C_x	0.0526	1.0720	1.342	0.1256	0.1473	1.342	0.2295
C_z	0.0488	1.2319	1.335	0.2258	0.1799	1.335	0.7493
ρ_{yx}	0.7108	0.6430	0.623	-0.69079	-0.4285	0.623	0.1794
ρ_{yz}	0.6932	0.7326	0.907	-0.36831	0.13778	0.907	-0.4996
ρ_{xz}	0.7346	0.6837	0.682	-0.04265	-0.3054	0.682	0.4074