National College of Business Administration & Economics Lahore

SOME PROPERTIES OF WEIGHTED REFLECTED WEIBULL DISTRIBUTION

BY

SYED WASIM ABBAS

**MASTER OF PHILOSOPHY
IN
STATISTICS**

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SOME PROPERTIES OF WEIGHTED REFLECTED WEIBULL DISTRIBUTION

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SYED WASIM ABBAS

A dissertation submitted to
Faculty of Social Sciences

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF PHILOSOPHY
IN
STATISTICS

February, 2014

بسم الله الرحمن الرحيم

IN THE NAME OF ALLAH

THE MOST BENEFICENT

AND THE MERCIFUL

NATIONAL COLLEGE OF BUSINESS ADMINISTRATION & ECONOMICS LAHORE

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Dissertation Committee:

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Chairman

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Member

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Member

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Rector
National College of Business
Administration & Economics

DECLARATION

It is to declare that this research work has not been submitted for obtaining similar degree from any other university/college.

SYED WASIM ABBAS
February, 2014

DEDICATED TO

Dedicated to my parents

for their countless efforts

and lot of sacrifices

in developing me

what I am now.

ACKNOWLEDGEMENT

It is with profound feeling of gratitude that I express to Dr. Munir Ahmad for his inordinate help, keen interest skilled advice, constructive criticism and valuable guidance during the course of my research. Whenever I came across any difficulty, his sympathetic behavior and vast knowledge enabled me to get through.

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis entitled **"Some Properties of Weighted Reflected Weibull Distribution"** has been carried out and completed by **Syed Wasim Abbas** under my supervision during her **M.Phil. Statistics** Programme.

(Dr. Munir Ahmad)
Supervisor and Rector
National College of Business
Administration and Economics

SUMMARY

The concept of weighted distribution is widely used to model the situations where equal probability sampling cannot take place. This concept is applied in researches related to probability, statistics and mathematics in the field of ecology, biology, survival, biomedical and reliability studies. The concept of weighted distribution can be traced from the study of the effect of methods of ascertainment upon estimation of frequencies by Fisher (1934). Later on Rao (1965) extended the basic idea of Fisher and introduced weighted distributions to model situations in which samples are drawn by unequal probability sampling. The situations in which sample is selected through some stochastic mechanism but the observations are recorded with some weight function and such observations are needed to model by weighted distributions. Weighted distributions provide a unifying approach for the problem solving where observations fall in the non-experimental, non-replicated and non-random categories.

Weibull distribution is an important and well known distribution which attracted statisticians, working in various fields of applied statistics as well as theory and methods in modern statistic due to its number of special features and ability to fit to data related to various fields like as life testing, biology, ecology, economics, hydrology, engineering and business administration. Cohen (1973) gave the idea of reflected Weibull (RW) distribution by originating in the linear transformation of the classical Weibull variate X as, $Y-\gamma=-(X-\gamma)=\gamma-X$ and used the distribution to model the age distribution of life insurance policy holders.

In this dissertation, weighted reflected Weibull (WRW) distribution is derived. Some basic theoretical properties of the distributions, including cumulative density function, central moments, skewness and kurtosis, moment and cumulant generating function, mode and median, are studied. Shannon entropy and information generating function of WRW distribution are derived. Reliability measures, including survival function, hazard function, reverse hazard rate function, Mills ratio and mean inactivity time of WRW distribution are also obtained. Parameters are estimated through using method of moments and methods of maximum likelihood estimation along with practical example are driven. Fisher information matrix of the WRW distribution is given and characterization of the distribution has been made through using conditional moments. At the end a modified form of the RW distribution is introduced by restricting the range of the variate X from 0 to γ, also some distributional properties of this modified distribution are given.

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CHAPTER 1

INTRODUCTION

1.1 WEIGHTED DISTRIBUTIONS

Weighted distributions are applied in the perspective of unequal probability sampling, such as actuarial sciences, biostatistics, biomedicine, ecology and survival data analysis. The concept of weighted distribution can be traced from the study of the effect of methods of ascertainment upon estimation of frequencies by Fisher (1934). Later on Rao (1965) extended the basic idea of Fisher and gave the idea of weighted distribution to model the distributions arise in practice when observations from a sample are recorded with unequal probability. Weighted distributions provide a unifying approach for the problem solving where observations fall in the non-experimental, non-replicated and non-random categories.

Consider a non-negative random variable X having the random sample $(x_1, x_2, x_3, \ldots, x_n)$ taken from the population. In such studies, as discussed above, it is not possible to have simple random sample. So the distribution of the observed random variable will differ by the actual random variable, and in turn, this will generate bias. In order to meet the situation we have to use weighted distributions.

Let $f(x;\delta)$ be the probability density function of the random variable X and δ be the unknown parameter, and let the observation x of random variable X recorded by the investigator with probability proportional to $w(X,\alpha)$ such that,

$$P(\text{Recording} \mid X = x) = w(X;\alpha)$$

whereas $w(X,\alpha)$ be the non-negative weight function, then weighted distribution may be expressed as:

$$g(x;\delta,\alpha) = \frac{w(X,\alpha) f(x;\delta)}{E\left[w(X;\alpha)\right]}. \tag{1.1}$$

Patil and Ord (1976) used the concept of weighted distribution and gave the idea of m^{th} order size biased distribution with the weight function

$w(X) = x^m$, and that was called moment distribution. When $m = 1$ it is called size biased of order 1 or length biased distribution, whereas for $m = 2$ it is called the area biased distribution.

Patil and Rao (1978) discussed the concept of weighted distributions by using natural example of weighted binomial distribution to model the human families and estimation of the wildlife family size. Gupta and Keating (1986) described the relationship between reliability measures of original and size-biased distribution. Arnold and Nagaraja (1991) gave the idea of bivariate weighted distribution whereas Jain and Nanda (1995) extended this idea and discussed multivariate aspect of weighted distribution.

Gove (2003) used the idea of size-biased distribution in modeling forest data. Mir and Ahmad (2009) derived generalized forms of size-biased discrete distributions and discussed the practical applications in the field of Medical, Zoology, and Accidental studies. Das and Roy (2011) applied the concept of size-biased sampling in the field of environmental studies. Dara (2011) studied a number of moment (weighted) distributions and obtained their some of the reliability measures. Iqbal and Ahmad (2012) found compound scale mixtures of limiting distribution of GLP type VII distribution with different continuous and moment distributions. Hasnain (2013) introduced a new family of distributions named as exponentiated moment exponential (EME) distribution and developed its properties. Iqbal et al. (2013) found a more general class for EME distribution and built up different properties including characterization through conditional moments.

1.2 THE REFLECTED WEIBULL DISTRIBUTION

Cohen (1973) gave the idea of reflected Weibull distribution by originating in the linear transformation of the classical Weibull variate X as,

$$Y - \gamma = -(X - \gamma) = \gamma - X, \tag{1.2}$$

or

$$Y = 2\gamma - X,$$

therefore, the reflected Weibull distribution by reflecting the classical Weibull distribution about variate axis at $x = \gamma$ is,

$$f_{Rw}\left(x\,|\,\gamma,\beta,\delta\right)=\frac{\delta}{\beta}\left(\frac{\gamma-x}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right], \tag{1.3}$$

where $x \le \gamma,\ \beta > 0,\ \delta > 0$, having the distribution function,

$$F_{Rw}\left(x\,|\,\gamma,\beta,\delta\right)=\begin{cases} exp\left[-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right] & \text{for } x<\gamma \\ 1 & \text{for } x\ge\gamma \end{cases} \tag{1.4}$$

The distribution (1.2) has been recognized as the type–III maximum distribution. Cohen (1973) derived this modified form of Weibull distribution and stated that for $\delta > \delta_0$ where (δ_0= 3.60) the function (1.2) is positive skewed, and this is closely similar to Pearson type VI distribution for certain combinations of parameter values. The distribution is applicable in life testing and reliability analysis as well as in environmental studies. He (1973) applied the distribution to model the age distribution of life insurance policy holders. The family of univariate skewed-normal probability distribution, an extension of symmetric normal distribution to a general class of asymmetrical distributions, was originally proposed by Azzalini (1985). Ali and Woo (2006) defined various skew-symmetric reflected distributions including skew-symmetric reflected Weibull distribution and derived its cumulative functions and moments. Holland (2011) fitted reflected Weibull distribution to the observed data for a specified historical period.

When the distribution having the ranges γ to ∞, the shape of the distribution is usually positively skewed and it is suitable for the positively skewed sample data for the small shape parameter of the distribution. However when the data set have negatively skewed behavior we want to fit the same distribution that is not suitable directly as it is a positively skewed distribution. To overcome this issue it is desired to develop a reflected form of the same distribution so that it may fit better on the negatively skewed data set.

In this research, a linear weight function is used, as the distribution is reflected about the variate at $x=\gamma$, therefore the suitable linear weight function used is,

$$w(X)=\gamma-x, \tag{1.5}$$

where,

$$E\left[w(X)\right]=\beta\Gamma\left(\frac{1}{\delta}+1\right). \qquad (1.6)$$

By using (1.6) and (1.3) in (1.1), we obtain a weighted reflected Weibull (WRW) distribution,

$$g(x)=c(\gamma-x)^{\delta}\,exp\left[-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right], \qquad (1.7)$$

where $c=\dfrac{\delta}{\beta^{\delta+1}\Gamma\left(\frac{1}{\delta}+1\right)}$ and $x\le\gamma,\ \beta>0,\ \delta>0$

The new distribution is useful in the field of forestry, ecology, and population sciences.

The aims of this dissertation are to:

i) derive weighted reflected Weibull (WRW) distribution.

ii) derive some basic characteristics of the WRW distribution.

iii) develop Shannon entropy of WRW distribution.

iv) derive estimates of the WRW distribution parameters through different approaches.

v) a) characterize the WRW distribution through conditional moments.

b) apply the WRW distribution on the real life data.

CHAPTER 2

LITERATURE REVIEW

2.1 WEIGHTED DISTRIBUTIONS

The concept of weighted distribution has been used as a tool for the selection of appropriate model for observed data, during last 25 years. The concept is most appropriate when sampling frame is not available and simple random sampling is not possible. The study of weighted distributions starts from Fisher (1934) to study the effect of methods of ascertainment upon estimation of frequencies. Later on, Rao (1965) saw the need for a unifying approach that can be used for various sampling situations, and can be modeled by using the weighted distributions.

Scheaffer (1972) used the idea of moment distribution and gave the concept of size biased sampling and estimators of average unit size were presented for the cases in which the underlying size distribution was gamma, Weibull or log-normal. Patil and Ord (1976) inter-related size biased sampling to the weighted distributions and identified the situations where the classical models are subjected to size-biased sampling in preference to the usual random sampling, but the models maintain their form. Rao (1977) gave practical example of the weighted binomial distribution to effectively understand the concept of Fisher (1934) idea of the methods of ascertainment, in the classroom exercise. Patil and Rao (1978) worked in size biased sampling and derived size biased forms of several distributions and results were applied to the analysis of data relating to human populations and wildlife management.

Rao (1985) made a review of some of the previous work over the weighted distributions, and also discussed the situations in which this type of distributions applies. He (1985) also analyzed data related to natural sciences and human population. Van Deusen and Paul (1986) used size-biased Weibull distribution for fitting the data of diameter at breast height arising from the horizontal point sampling, method of maximum likelihood (ML) estimation and method of moments (MOM) used for parameter estimation. Gupta and Keating (1986) made characterizations and established relationships between reliability measures of size-biased distributions. Statish and Greenhouse (1988) applied weighted distribution theory to analyze data in Meta analysis.

Gupta and Kirmani (1990) developed the relationship between weighted (size-biased) distributions and the original distributions through reliability measures. Arnold and Nagaraja (1991) defined the properties of bivariate weighted distributions and its realization in the renewal process. Jain and Nanda (1995) extended this idea to multivariate case and derived multivariate Poisson negative hyper-geometric distribution. Larose and Dey (1996) derived the weighted distributions using Bayesian approach. Gupta and Akman (1998) used inferential tools to develop the estimates for the reliability measures of size-biased inverse Gaussian distribution.

Oluyede (2000) derived inequalities for the reliability measures of size-biased and the original distributions. Navarro et al. (2001) made characterization of the original and the size-biased distribution using reliability measures. Gove (2003) discussed the uses of size-biased distributions in forest sciences, to model the basal area diameter at breast height. Gove (2003) made parameter estimation using ML and MOM estimation approaches for Weibull distribution under size-biased and area-biased sampling. Sunoj and Maya (2006) established relationships among weighted and original distributions in the perspective of repairable system and also developed characterizations for the sized-biased and the original distribution. Akman et al. (2007) used a simple test to detect the size-biased sampling. Ghitany and Al-Mutairi (2008) developed size-biased Poisson lindley distribution, parameters were estimated and simulation study was used to compare the estimates for large and small sample sizes. Shen et al. (2009) used semi-parametric transformations to model the length biased data. Hussain and Ahmed (2009) defined misclassification in the size-biased modified power series distributions and discussed its real life application. Mir and Ahmad (2009) derived generalized forms of size-biased distributions for binomial, Poisson, negative binomial and logarithmic series distributions and discussed some practical applications for these distributions. Mir and Ahmad (2009), Mir (2009, 2011) derived size-biased Geeta distribution and size-biased Consul distribution respectively as a well as its real life applications were discussed.

Das and Roy (2011) developed size-biased form of generalized Rayleigh distribution and apply the results to the environmental data. Das and Roy (2011) developed size-biased form of weighted Weibull distribution, and this was used to model the rainfall data from India. Dara (2011) derived reliability measures for size-biased forms of various moment distributions as the special cases of moment distributions. Tzavelasa and Panagiotakos (2013) discussed statistical inference for the size-biased Weibull distribution. Dara and Ahmad (2013) derived structural properties and some measures of reliability for the size-biased gamma distribution.

Pandya et al. (2013) discussed Bayes estimates of size-biased Weibull distribution and the results were compared for different priors. Ahmed et al. (2013) derived a new class of length-biased gamma distribution and discussed its structural properties.

Iqbal and Ahmad (2013) found compound scale mixtures of limiting distribution of GLP type VII distribution with different continuous and moment distributions. Hasnain and Ahmad (2013) introduced a new family of distributions named exponentiated moment exponential (EME) distribution and developed its properties. Iqbal et al. (2013) found a more general class for EME distribution and derived different properties including characterization through conditional moments.

2.2 REFLECTED WEIBULL DISTRIBUTION

Cohen (1973) originated the linear transformation (1.2) to the variate X of the classical Weibull distribution and gave the idea of reflected Weibull distribution (1.3). Ali and Woo (2006) defined various skew-symmetric reflected distributions including skew-symmetric reflected Weibull distribution and used special mathematical functions to derive its cumulative distribution function and moments. Ali et al. (2009) used a real number as Laplace kernel and developed skew reflected double Weibull, Pareto, beta prime and generalized uniform distributions. Ali et al. (2010) derived some skew symmetric inverse reflected distributions including reflected Weibull distribution and developed some of its basic properties. They (2010) found the ML estimates of the parameters of skew symmetric inverse reflected distributions with Fisher information matrices, as well as practical applications were illustrated. Holland (2011) fitted reflected Weibull distribution to the observed data for a specified historical period.

CHAPTER 3

DISTRIBUTIONAL PROPERTIES OF SOME CONTINUOUS DISTRIBUTIONS

3.1 THE REFLECTED WEIBULL (RW) DISTRIBUTION

Cohen (1973) made the linear transformation (1.2) to the variate X of classical Weibull distribution and developed reflected Weibull distribution about variate axis at $x = \gamma$ is,

$$f_{Rw}\left(x \mid \gamma, \beta, \delta\right) = \frac{\delta}{\beta}\left(\frac{\gamma - x}{\beta}\right)^{\delta - 1} exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right],$$

where $x \le \gamma, \beta > 0, \delta > 0.$

3.1.1

Figure 3.1 shows the graph of RW density function for various values of parameters.

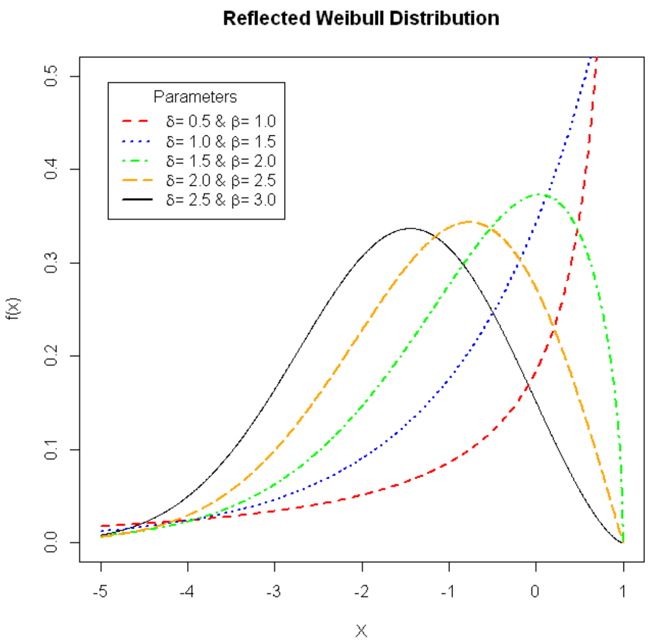

Fig. 3.1: pdf of Reflected Weibull Distribution ($\gamma = 1$)

The cumulative density function of the RW distribution is,

$$F_{Rw}(x\,|\,\gamma,\beta,\delta)=\begin{cases} exp\left[-\left(\dfrac{\gamma-x}{\beta}\right)^{\delta}\right] & \text{for } x<\gamma \\ 1 & \text{for } x\geq\gamma \end{cases}$$

3.1.2 Graph of Cumulative RW Distribution

Figure 3.2 shows the graph of cumulative density function for the different values of parameters.

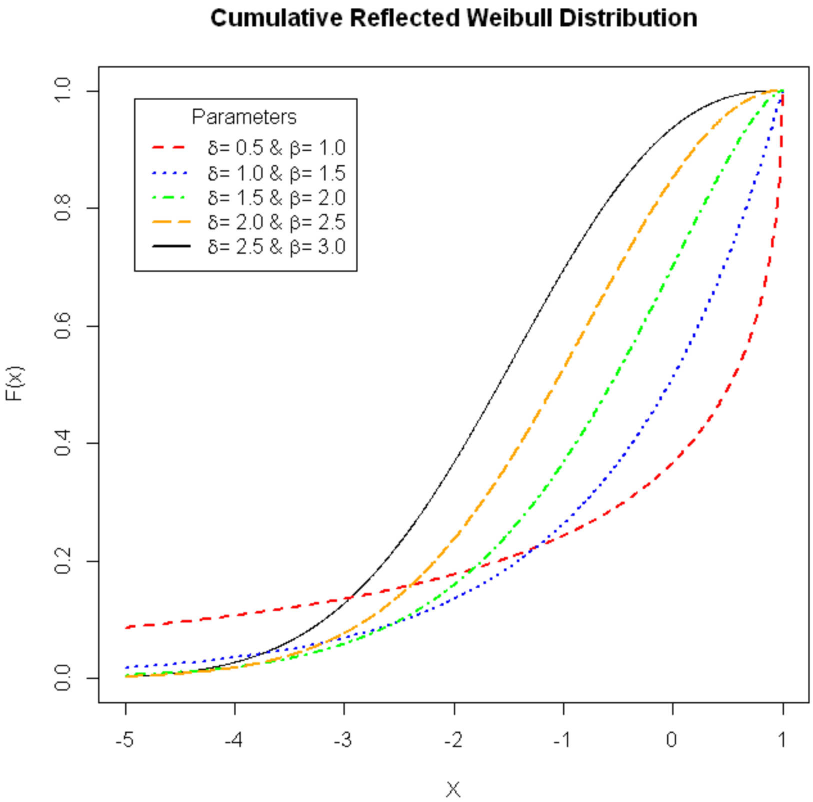

Fig. 3.2: cdf of Reflected Weibull Distribution ($\gamma=1$)

Reflected Weibull distribution is widely applicable in life testing and reliability analysis in insurance companies and environmental studies. Cohen (1973) applied the distribution to model the age distribution of life insurance policy holders. Holland (2011) fitted RW distribution to the observed data for a specified historical period.

3.2 MAIN RESULTS OF (RW) DISTRIBUTION

Cohen (1973) derived some basic properties of RW distribution.

3.2.1 Some Basic Results of RW Distribution

The mean of RW distribution derived by Cohen (1973) is,

$$\mu_1' = \gamma - \beta\,\Gamma_1,$$

where

$$\Gamma_m = \Gamma\left(\frac{m}{\delta}+1\right), \tag{3.2}$$

and

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt, \tag{3.3}$$

$\Gamma(\alpha)$ is the gamma function.

The median of RW distribution is,

$$= \gamma - \beta(\ln 2)^{\frac{1}{\delta}}$$

The mode value of the RW distribution is derived as,

$$= \gamma - \beta\left(1 - \frac{1}{\delta}\right)^{\frac{1}{\delta}}$$

The variance of the RW distribution is,

$$= \beta^2\left[\Gamma_2 - \Gamma_1^2\right] \tag{3.4}$$

The sign of skewness is reversed if any distribution is reflected about its vertical axis. Distribution (1.3) is positive and negative skewed for $\delta > \delta_0$ and $\delta < \delta_0$ respectively,

$$\beta_1 = \frac{\left[\Gamma_3 - 3\Gamma_2\Gamma_1 + 2\Gamma_1^3\right]^2}{\left[\Gamma_2 - \Gamma_1^2\right]^3}. \tag{3.5}$$

The expression for the measure of coefficient of kurtosis for RW distribution is,

$$\beta_2 = \frac{\Gamma_4 - 4\Gamma_3\Gamma_1 + 6\Gamma_2\Gamma_1^2 - 3\Gamma_1^4}{\left[\Gamma_2 - \Gamma_1^2\right]^2}. \tag{3.6}$$

The results (3.4), (3.5) and (3.6) are same as classical Weibull distribution of three parameters.

3.2.2 Survival Function

The survival function is an important measure in reliability studies, therefore by definition, the survival function for the RW distribution is,

$$S(x) = 1 - F(x) \ , \tag{3.7}$$

$$= 1 - exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right].$$

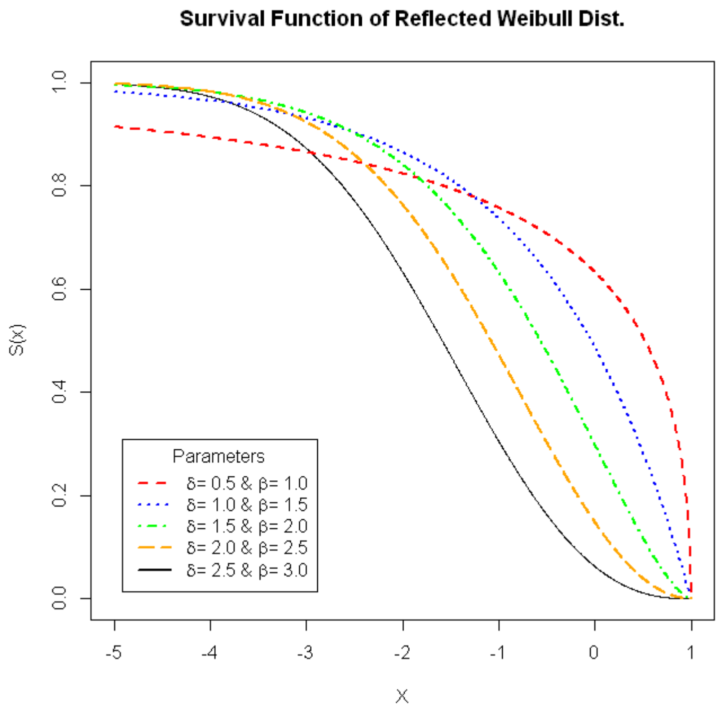

Fig. 3.3: Survival Function RW Distribution ($\gamma = 1$)

3.2.3 Hazard Rate Function

The hazard rates are also called the failure rates, it is the current chance of failure for the population that has not been yet failed. It is very useful in reliability analysis of life time data. Hazard function of RW distribution is defined as,

$$h(x) = f(x) / S(x), \tag{3.8}$$

$$= \frac{\frac{\delta}{\beta}\left(\frac{\gamma - x}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right]}{1 - exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right]}.$$

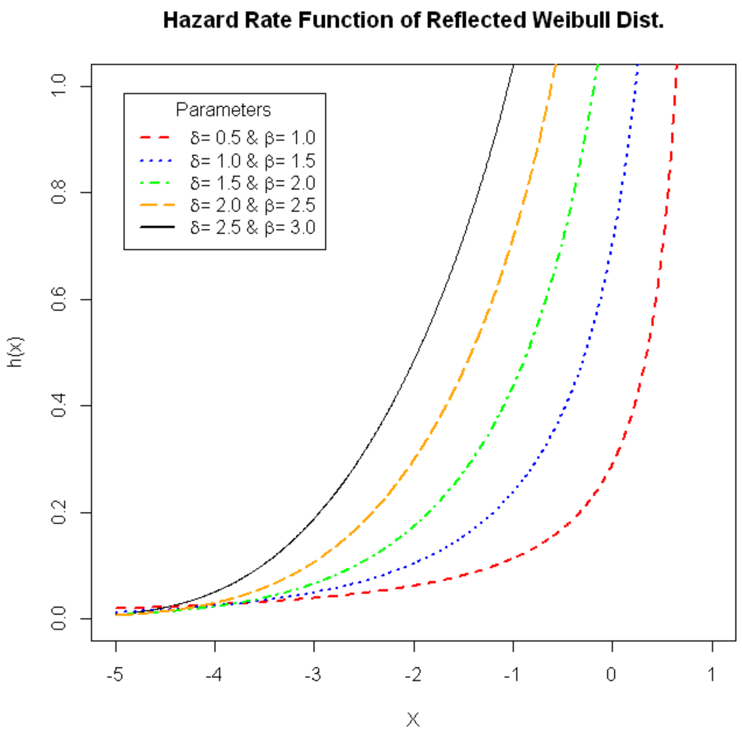

Fig. 3.4: Hazard Function of RW Distribution ($\gamma = 1$)

3.3 WEIGHTED WEIBULL DISTRIBUTIONS

3.3.1 Length and Area Biased Weibull Distribution

Gove (2003) derived the length-biased and area-biased forms of the Weibull distribution. He (2003) derived the estimates of parameter for these weighted Weibull distributions for the two and three parameter cases.

Three parameters size-biased and area-biased Weibull distribution,

$$f(x;\theta) = \mu_\alpha'^{-1} x^\alpha \left(\frac{\gamma}{\beta}\right)\left(\frac{x-\varsigma}{\beta}\right)^{\gamma-1} \exp\left(-\left(x-\varsigma/\beta\right)^\gamma\right), \qquad (3.9)$$

$\alpha = 1\ \&\ 2$ for size-biased & area-biased respectively,

where $\mu_\alpha' = \int_{-\infty}^{\infty} x^\alpha \left(\frac{\gamma}{\beta}\right)\left(\frac{x-\varsigma}{\beta}\right)^{\gamma-1} \exp\left(-\left(x-\varsigma/\beta\right)^\gamma\right) dx.$

Gove (2003) used first three moments of size-biased distribution and used the relationship to find the estimates through MOM,

$$\mu_{\alpha,1}^{*\prime} = \frac{\mu'_{\alpha+1}}{\mu'_{\alpha}} \quad \mu_{\alpha,2}^{*\prime} = \frac{\mu'_{\alpha+2}}{\mu'_{\alpha}} \text{ and } \quad \mu_{\alpha,3}^{*\prime} = \frac{\mu'_{\alpha+3}}{\mu'_{\alpha}}. \tag{3.10}$$

Solution is found through setting raw moments equal to the sample moments simultaneously. He derived ML estimates of the parameters numerically through Newton Raphson technique.

3.3.2 Length-Biased Weighted Weibull Distribution

Das and Roy (2011) developed the weighted form of the three parameters Weibull distribution and then developed its length biased form. By using the weight function $w(x) = x^{c\beta}$ in (1.1) to the three parameters Weibull distribution, the pdf of weighted Weibull distribution was derived as,

$$f_1(x) = \frac{c}{\alpha^{c+c\beta}\Gamma(\beta+1)} x^{c+c\beta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^c\right]. \; x>0, c>0, \alpha>0, \beta>0 \tag{3.11}$$

After this again using the weight function $w(x) = x$ in (1.1) for (3.12) the length-biased form of the weighted Weibull distribution was observed as,

$$f_2(x) = \frac{c}{\alpha^{c+c\beta+1}\Gamma\left(\beta+\frac{1}{c}+1\right)} x^{c+c\beta} \exp\left[-\left(\frac{x}{\alpha}\right)^c\right]. \; x>0, c>0, \alpha>0, \beta>0 \tag{3.12}$$

For estimation of parameters in length biased weighted Weibull distribution, the method of moments and Newby's method is used and the distribution is fitted to the rainfall data to justify its significance.

3.3.3 Moment Weibull Distribution

Dara (2012) generated the moment Weibull distribution by applying the weight function $w(x) = x^m$ in (1.1) to the pdf of the single parameter classical Weibull distribution,

$$g_0(x) = \frac{\beta x^{\beta-1+m} \exp\left(-x^{\beta}\right)}{\Gamma\left(1+\frac{m}{\beta}\right)}, x > 0; \beta > 0. \tag{3.13}$$

She (2012) also derived basic properties and reliability measures for the special case $\left(m=1\right)$ of moment Weibull distribution.

CHAPTER 4

WEIGHTED REFLECTED WEIBULL DISTRIBUTION

4.1 INTRODUCTION

The idea of weighted distribution was first delivered by Fisher (1934) from the study of the effect of methods of ascertainment upon estimation of frequencies. Afterward Rao (1965) reshaped the idea and developed a unifying approach as a problem solving of various sampling situations that can be modeled through weighted distributions. Patil and Rao (1978) worked in size biased sampling and derived size biased forms of several distributions and apply the results to model human populations and wildlife management. Cohen (1973) derived the reflected from of classical Weibull distribution (1.3) by transforming linearly to the variate X as (1.2). The distribution is positive skewed and is closely similar to the Pearson type VI distribution for certain combination of parameters values. The distribution is widely applicable in the field of life testing, reliability analysis environmental studies and as well as in modeling human population. He (1973) modeled the age distribution of life insurance policy holders using this distribution.

4.2 WEIGHTED REFLECTED WEIBULL (WRW) DISTRIBUTION

Here the weighted form of the reflected Weibull distribution is derived as (1.7) by applying the linear weight function $w(X) = \gamma - x$, to meet the situation of unequal probability sampling.

$$g(x) = c\left(\gamma - x\right)^{\delta} exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right], \tag{4.1}$$

where $c = \dfrac{\delta}{\beta^{\delta+1}\Gamma\left(\dfrac{1}{\delta}+1\right)}$ and $x \le \gamma,\ \beta > 0,\ \delta > 0$

4.2.1 Graph of WRW Distribution

Figure 4.1 shows the graph of WRW density function for various values of parameters.

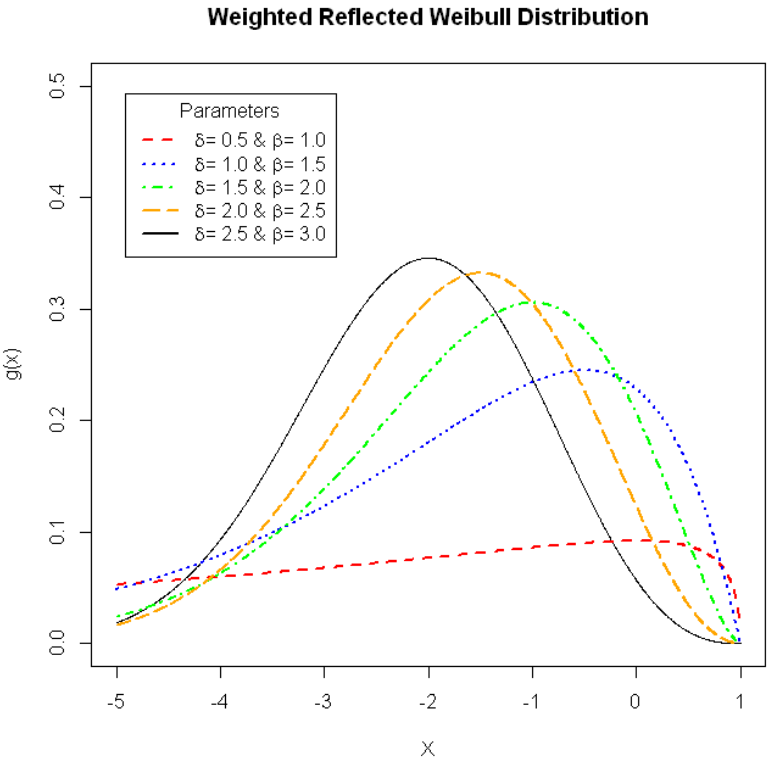

Fig. 4.1: pdf of WRW Distribution ($\gamma = 1$)

4.3 THE DISTRIBUTION FUNCTION OF WRW DISTRIBUTION

The distribution function of the weighted form of the reflected Weibull distribution is defined as;

$$G(x) = \frac{\delta}{\beta^2 \Gamma\left(\frac{1}{\delta}+1\right)} \int_{-\infty}^{\gamma} (\gamma - t)\left(\frac{\gamma - t}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{\gamma - t}{\beta}\right)^{\delta}\right] dt,$$

applying transformation $\left(\frac{\gamma - t}{\beta}\right)^{\delta} = z,$ and after simplification, (4.2)

$$G(x) = \frac{\Gamma\left(\left(\frac{1}{\delta}+1\right), \left(\frac{\gamma - x}{\beta}\right)^{\delta}\right)}{\Gamma\left(\frac{1}{\delta}+1\right)}, \quad \text{where } x \le \gamma,\ \beta > 0,\ \delta > 0 \qquad (4.3)$$

and

$$\Gamma(a, x) = \int_{x}^{\infty} t^{a-1} e^{-t} dt, \text{ is the incomplete gamma function.} \qquad (4.4)$$

4.3.1 Graph of Cumulative WRW Distribution

Figure 4.2 shows the graph of cumulative WRW density function for various values of parameters.

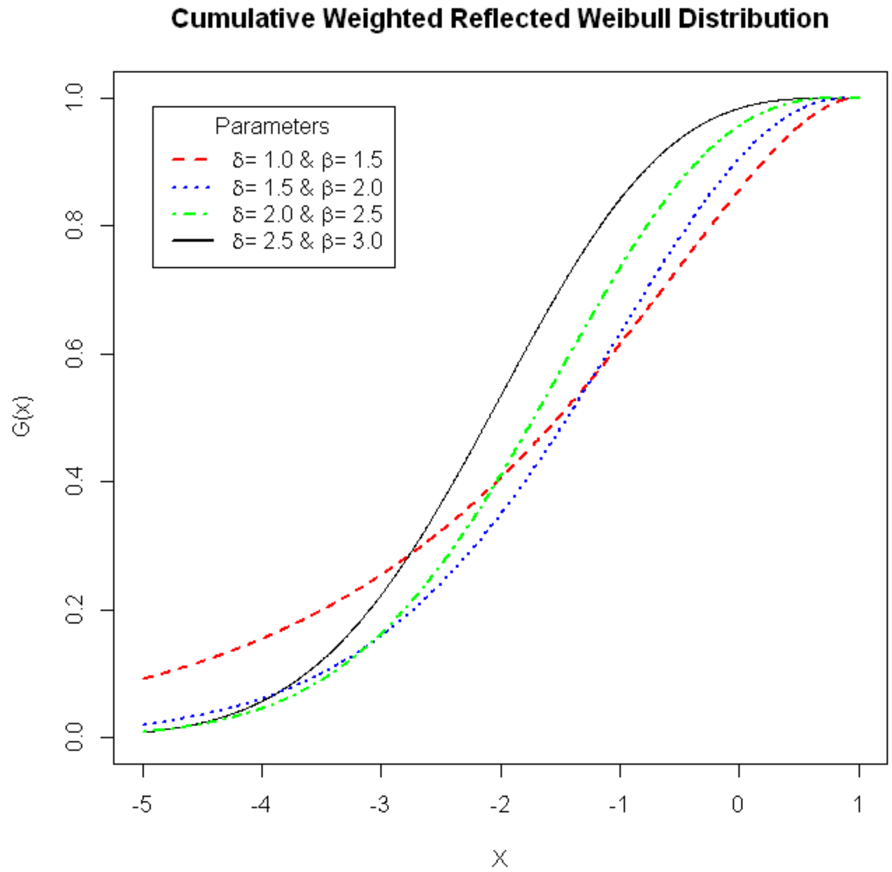

Fig. 4.2: cdf of WRW Distribution ($\gamma = 1$)

4.4 r^{th} MOMENT ABOUT ORIGIN OF WRW DISTRIBUTION

By definition of the r^{th} moment about origin we have,

$$\mu'_r = \frac{\delta}{\beta^2 \Gamma\left(\frac{1}{\delta}+1\right)} \int_{-\infty}^{\gamma} x^r (\gamma - x) \left(\frac{\gamma - x}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right] dx,$$

applying transformation (4.2) we obtain,

$$=\frac{1}{\Gamma\left(\frac{1}{\delta}+1\right)}\int_0^\infty\left(\gamma-\beta z^{\frac{1}{\delta}}\right)^r z^{\frac{1}{\delta}} exp\left[-z\right]dz,$$

using $\left(a-b\right)^r=\sum_{n=0}^{r}\left(-1\right)^n\binom{r}{n}a^{r-n}b^n$ and after simplification,

$$\mu_r'=c\sum_{n=0}^{r}\left(-1\right)^n\binom{r}{n}\gamma^{r-n}\beta^n\Gamma_{n+1}, \tag{4.6}$$

where $\Gamma_m=\Gamma\left(\frac{m}{\delta}+1\right)$ & $c^{-1}=\Gamma\left(\frac{1}{\delta}+1\right)$ (4.7)

4.4.1 Moments about Zero of WRW Distribution

From (4.6) using $r=1, 2, 3$ and 4 we get the first four raw moments as,

$$=c\left[\gamma\Gamma_1-\beta\Gamma_2\right] \quad \text{(Mean of WRW distribution)} \tag{4.8}$$

$$=c\left[\gamma^2\Gamma_1-2\gamma\beta\Gamma_2+\beta^2\Gamma_3\right] \tag{4.9}$$

$$=c\left[\gamma^3\Gamma_1-3\gamma^2\beta\Gamma_2+3\gamma\beta^2\Gamma_3-\beta^3\Gamma_4\right] \tag{4.10}$$

$$=c\left[\gamma^4\Gamma_1-4\gamma^3\beta\Gamma_2+6\gamma^2\beta^2\Gamma_3-4\gamma\beta^3\Gamma_4+\beta^4\Gamma_5\right] \tag{4.11}$$

where Γ_m and c^{-1} are defined in (4.7)

4.4.2 Central Moments of WRW Distribution

The central moments of WRW distribution may obtain by using relationship with raw moments (4.8) to (4.11).

$$\mu_1=0$$

$$\mu_2=c\left[\gamma^2\Gamma_1-2\gamma\beta\Gamma_2+\beta^2\Gamma_3\right]-c^2\left[\gamma\Gamma_1-\beta\Gamma_2\right]^2 \tag{4.12}$$

$$\mu_3 = c\left[\gamma^3\Gamma_1 - 3\gamma^2\beta\Gamma_2 + 3\gamma\beta^2\Gamma_3 - \beta^3\Gamma_4\right]$$
$$-3c^2\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right]\left[\gamma\Gamma_1 - \beta\Gamma_2\right] + 2c^3\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^3 \tag{4.13}$$

$$\mu_4 = c\left[\gamma^4\Gamma_1 - 4\gamma^3\beta\Gamma_2 + 6\gamma^2\beta^2\Gamma_3 - 4\gamma\beta^3\Gamma_4 + \beta^4\Gamma_5\right]$$
$$-4c^2\left[\gamma^3\Gamma_1 - 3\gamma^2\beta\Gamma_2 + 3\gamma\beta^2\Gamma_3 - \beta^3\Gamma_4\right]\left[\gamma\Gamma_1 - \beta\Gamma_2\right]$$
$$+6c^3\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right]\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^2 - 3c^4\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^4 \tag{4.14}$$

4.4.3 Coefficient of Variation

By definition the coefficient of variation (CV) for the WRW distribution is,

$$CV = \frac{\sqrt{c\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right] - c^2\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^2}}{c\left[\gamma\Gamma_1 - \beta\Gamma_2\right]}. \tag{4.15}$$

4.4.4 Skewness and Kurtosis

The coefficient of skewness α_1 for WRW distribution may be define as,

$$\alpha_1 = \frac{c\left[\gamma^3\Gamma_1 - 3\gamma^2\beta\Gamma_2 + 3\gamma\beta^2\Gamma_3 - \beta^3\Gamma_4\right] - 3c^2\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right]\left[\gamma\Gamma_1 - \beta\Gamma_2\right] + 2c^3\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^3}{\left[c\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right] - c^2\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^2\right]^{\frac{3}{2}}}, \tag{4.16}$$

and coefficient kurtosis α_2 is,

$$\alpha_2 = \frac{c\left[\gamma^4\Gamma_1 - 4\gamma^3\beta\Gamma_2 + 6\gamma^2\beta^2\Gamma_3 - 4\gamma\beta^3\Gamma_4 + \beta^4\Gamma_5\right] - 4c^2\left[\gamma^3\Gamma_1 - 3\gamma^2\beta\Gamma_2 + 3\gamma\beta^2\Gamma_3 - \beta^3\Gamma_4\right]\left[\gamma\Gamma_1 - \beta\Gamma_2\right] + 6c^3\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right]\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^2 - 3c^4\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^4}{\left[c\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right] - c^2\left[\gamma\Gamma_1 - \beta\Gamma_2\right]^2\right]^2}. \tag{4.17}$$

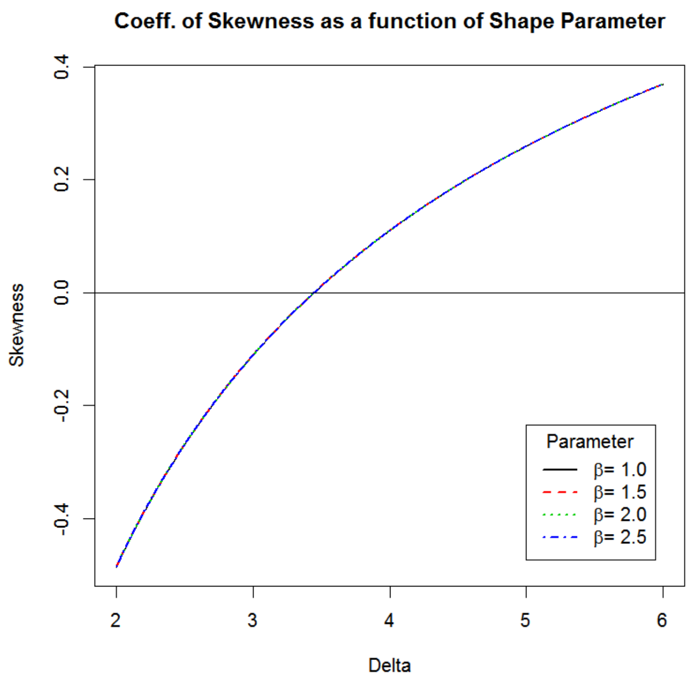

Fig. 4.3: Curve of α_1 as a Function of δ for the WRW Distribution

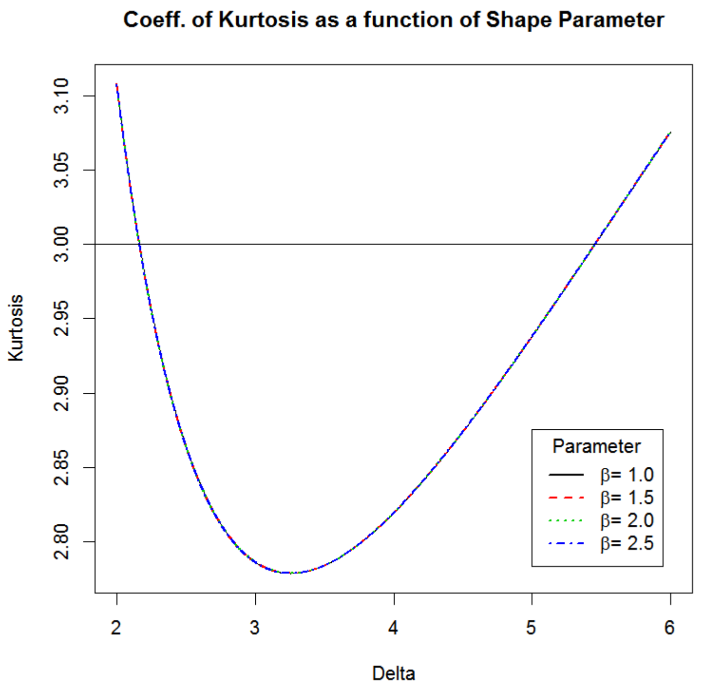

Fig. 4.4: Curve of α_2 as a Function of δ for the WRW Distribution

The distribution has $\beta_1 = 0$ and $\beta_2 = 2.7812$ at $\delta = 3.4475$, i.e. at $\beta_1 = 0$ and $\beta_2 < 3$ therefore the distribution follows Pearson type-II distribution.

4.5 MOMENT GENERATING FUNCTION OF WRW DISTRIBUTION

Moment generating function is an efficient tool to generate moments from the distribution, therefore by the definition of mgf.,

$$M_x(t) = E\left(e^{tx}\right),$$

$$= \sum_{a=0}^{\infty} \frac{t^a}{a!} E\left[x^a\right],$$

From (4.6) substituting in (4.18),

$$= c \sum_{a=0}^{\infty} \frac{t^a}{a!} \sum_{n=0}^{a} (-1)^n \binom{a}{n} \gamma^{a-n} \beta^n \Gamma_{n+1}. \quad (4.18)$$

This gives the k^{th} moment about origin of WRW distribution as,

$$\mu'_k = c \sum_{n=0}^{k} (-1)^k \binom{k}{n} \gamma^{k-n} \beta^n \Gamma_{n+1}. \quad (4.19)$$

4.6 CHARACTERISTIC FUNCTION

By definition the characteristic function of the WRW distribution is,

$$E\left(e^{itx}\right) = \int_{-\infty}^{\infty} e^{itx} f(x) dx,$$

$$= c \sum_{a=0}^{\infty} \frac{it^a}{a!} \sum_{n=0}^{a} (-1)^n \binom{a}{n} \gamma^{a-n} \beta^n \Gamma_{n+1}. \quad (4.20)$$

4.7 CUMULANT GENERATING FUNCTION

By definition the cumulant generating function of the distribution is,

$$K_x(t) = \ln\left[M_x(t)\right],$$

$$= \ln\left[c\sum_{a=0}^{\infty}\frac{t^a}{a!}\sum_{n=0}^{a}(-1)^n\binom{a}{n}\gamma^{a-n}\beta^n\Gamma_{n+1}\right]. \quad (4.21)$$

This gives cumulants for the WRW distribution.

4.8 FOURIER TRANSFORM OF WRW DISTRIBUTION

The Fourier transform is a mathematical transformation developed by Joseph Fourier, used to transform signals within a time/spatial domain to frequency domain. The resulting function is known as the Fourier transform or the frequency spectrum of the function f. It is widely applicable in physics and engineering (Hazewinkel, 2001).

The Fourier transform of the function $f(t)$ is defined as,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \quad (4.22)$$

Using (1.7) "the expression for f (t)" in (4.22),

$$= c\int_{-\infty}^{\gamma}(\gamma - t)^{\delta}.\exp\left\{-\left(\frac{\gamma - t}{\beta}\right)^{\delta}\right\}e^{-i\omega t}dt,$$

$$\text{where } c = \frac{\delta}{\beta^{\delta+1}\Gamma\left(\frac{1}{\delta}+1\right)},$$

$$= c\sum_{k=0}^{\infty}\frac{(i\omega)^k}{k!}\int_{-\infty}^{\gamma}(\gamma - t)^{\delta}\exp\left\{-\left(\frac{\gamma - t}{\beta}\right)^{\delta}\right\}t^k dt,$$

applying transformation (4.2),

$$= c\frac{\beta^{\delta+1}}{\delta}\sum_{k=0}^{\infty}\frac{(i\omega)^k}{k!}\int_{0}^{\infty}e^{-z}z^{\frac{1}{\delta}}\left(\gamma - \beta z^{\frac{1}{\delta}}\right)^k dz,$$

after simplification the Fourier transform for WRW distribution

$$=\frac{1}{\Gamma\left(\frac{1}{\delta}+1\right)}\sum_{k=0}^{\infty}\frac{(i\omega)^k}{k!}\sum_{a=0}^{k}(-1)^a\binom{k}{a}\beta^a\gamma^{k-a}\Gamma\left(\frac{a+1}{\delta}+1\right). \qquad (4.23)$$

4.9 MODE OF WRW DISTRIBUTION

The mode of the WRW distribution is the value $\hat{x}$ at which $g(x)'=0$ and $g(x)''<0$.

Taking log on both sides of (1.7), differentiating with respect to x and equating to zero,

$$\frac{d}{dx}\ln[g(x)]=0,$$

$$\frac{\delta}{\gamma-x}=\frac{\delta(\gamma-x)^{\delta-1}}{\beta^{\delta}},$$

$$\hat{x}=\gamma-\beta. \qquad \text{for } \gamma\geq\beta \qquad (4.24)$$

Again differentiating $w.r.t\ x$ to check minima,

$$\frac{d^2}{dx^2}\log[g(x)]<0,$$

$$-\left[\frac{\delta}{(\gamma-x)^2}+\frac{\delta(\delta-1)(\gamma-x)^{\delta-2}}{\beta^2}\right]<0$$

hence $\hat{x}=\gamma-\beta$ is the mode value of WRW distribution.

4.10 MEDIAN OF WRW DISTRIBUTION

m is the median of WRW distribution at which $2G(m)-1=0$ using (4.3) to get the median value we have,

$$G(m) = \frac{\Gamma\left(\left(\frac{1}{\delta}+1\right), \left(\frac{\gamma - m}{\beta}\right)^{\delta}\right)}{\Gamma\left(\frac{1}{\delta}+1\right)} - \frac{1}{2} = 0. \qquad (4.25)$$

$\beta > 0, \; \delta > 0$

This is the function for median of the WRW distribution which can be solved numerically for different values of parameters. Following is the table of median values for different sets of parameters by using R software.

Table 4.1

$\gamma = 10$		**Median**
δ	β	
1	1	8.3227
2	1	8.9125
2	2	7.8248
2	3	6.7373
3	3	6.9813
4	3	7.0484

CHAPTER 5

ENTROPY AND RELIABILITY MEASURES

5.1 ENTROPY

Entropy is an important tool to explore variety in distributions at particular instants in time (e.g., market shares) and to investigate fruition processes over time (e.g., technical change). Shannon (1948) gives the idea of entropy, since it's the measure of uncertainty in different fields of areas like engineering, information theory and other sciences.

5.1.1 Shannon Entropy

The Shannon (1948) entropy of a continuous random variable X , whose density function is $f(x)$, is defined as,

$$H(x) = -E\left[\ln\left(g(x)\right)\right].$$

Using (1.7) we have,

$$= -\left[\ln c + \delta E\left\{\ln\left(\gamma - x\right)\right\} - \left(\delta - 1\right)\ln\beta - E\left\{\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right\}\right],$$

$$= -\left[\ln c + \delta I_1 - \left(\delta - 1\right)\ln\beta - I_2\right], \quad (5.1)$$

Now from (5.1)

$$I_1 = \int_{-\infty}^{\gamma} \ln\left(\gamma - x\right) g\left(x\right) dx,$$

Again using (1.7) and applying transformation (4.2)

$$= c\frac{\beta^2}{\delta}\left[\ln\beta\Gamma\left(\frac{1}{\delta} + 1\right) + \frac{1}{\delta}\int_0^{\infty} z^{\frac{1}{\delta}} \ln z \; exp\left[-z\right] dz\right],$$

using following formula from Table of Integrals, Series and Products,

$$\int_0^{\infty} x^{\nu-1} e^{-\mu x} \ln x dx = \frac{1}{\mu^{\nu}} \Gamma(\nu)\left[\psi(\nu) - \ln(\mu)\right]$$

where $\psi(\nu) = \frac{\partial}{\partial \nu} \ln \Gamma(\nu)$

(See 4.352.1 Gradshteyn and Ryzhik 2007)

$$= c\frac{\beta^2}{\delta}\left[\ln\beta\Gamma\left(\frac{1}{\delta}+1\right)+\frac{1}{\delta}\Gamma\left(\frac{1}{\delta}+1\right)\left(\psi\left(\frac{1}{\delta}+1\right)\right)\right],$$

after simplification,

$$I_1 = \left[\ln\beta + \frac{1}{\delta}\psi\left(\frac{1}{\delta}+1\right)\right]. \tag{5.2}$$

Now from (5.1),

$$I_2 = c\int_{-\infty}^{\gamma}\left((\gamma - x)\left(\frac{\gamma - x}{\beta}\right)^{2\delta-1} exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right]\right),$$

again using (1.7) and making transformation (4.2),

$$= \frac{1}{\Gamma\left(\frac{1}{\delta}+1\right)}\int_0^{\infty} t^{\frac{1}{\delta}+2-1} e^{-t} dt,$$

$$I_2 = \left(\frac{1}{\delta}+1\right). \tag{5.3}$$

Using (5.2) and (5.3) in (5.1) and after simplification,

$$H(x) = -\left[\ln c + \psi\left(\frac{1}{\delta}+1\right) + \ln\beta - \left(\frac{1}{\delta}+1\right)\right]. \tag{5.4}$$

where $\beta > 0,\ \delta > 0$

This is the Entropy function of WRW distribution.

5.2 INFORMATION GENERATING FUNCTION

The information generating function provides the moments of self-information of the probability density function by taking the derivatives at certain place.

The information generating function for WRW distribution is defined as,

$$T(u) = E\left[g(x)\right]^{u-1}. \tag{5.5}$$

Using (1.7) in (6.15)

$$= \int_{-\infty}^{\infty} \left(g(x)\right)^{u} dx,$$

$$= \int_{-\infty}^{\infty} c^{u}\left(\gamma - x\right)^{\delta u} \exp\left[-\left\{\left(\frac{\gamma - x}{\beta}\right)^{\delta} u\right\}\right] dx,$$

using transformation $u\left(\frac{\gamma - x}{\beta}\right)^{\delta} = z$, we get,

$$= c^{u}\int_{0}^{\infty}\left(\beta\left(\frac{z}{u}\right)^{\frac{1}{\delta}}\right)^{\delta u} e^{-z}\frac{\beta}{\delta}\left(\frac{z}{u}\right)^{\frac{1}{\delta}-1}\frac{1}{u}dz,$$

after simplification we get,

$$T(u) = \left(\frac{\delta}{\beta}\right)^{u-1}\frac{\Gamma\left(\frac{1}{\delta}+u\right)}{u^{\frac{1}{\delta}+u}\left(\Gamma\left(\frac{1}{\delta}+1\right)\right)^{u}}. \tag{5.6}$$

This is the expression for the information generating function of WRW distribution, taking first derivative of these moments is the entropy.

5.3 RELIABILITY MEASURES

In life testing analysis reliability measures are too necessary, it is the probability of a system or item that it will work properly without failure within a specific time period under specific conditions. It is very important characteristic of any product required by the consumers from manufacturers.

"Reliability is the probability of success or the probability that the system will perform its intended function under specified design limits." (Pham, 2006)

5.3.1 Survival Function of WRW Distribution

Survival function measures the time to the occurrence of a certain event, such that failure or death of a system/component. It is the probability that the component will survive longer than t,

Mathematically it can be expressed as the probability of system's success within time 0 to t,

$$S(t) = P(T > t),\ t \geq 0$$

where random variable T denotes failure time.

Therefore by definition the survival function of WRW distribution is,

$$S(x) = 1 - G(x),$$

from (4.3) we have,

$$S(x) = \frac{\Gamma\left(\frac{1}{\delta}+1\right) - \Gamma\left(\left(\frac{1}{\delta}+1\right), \left(\frac{\gamma - x}{\beta}\right)^{\delta}\right)}{\Gamma\left(\frac{1}{\delta}+1\right)}. \qquad (5.7)$$

where $x \leq \gamma,\ \beta > 0,\ \delta > 0$

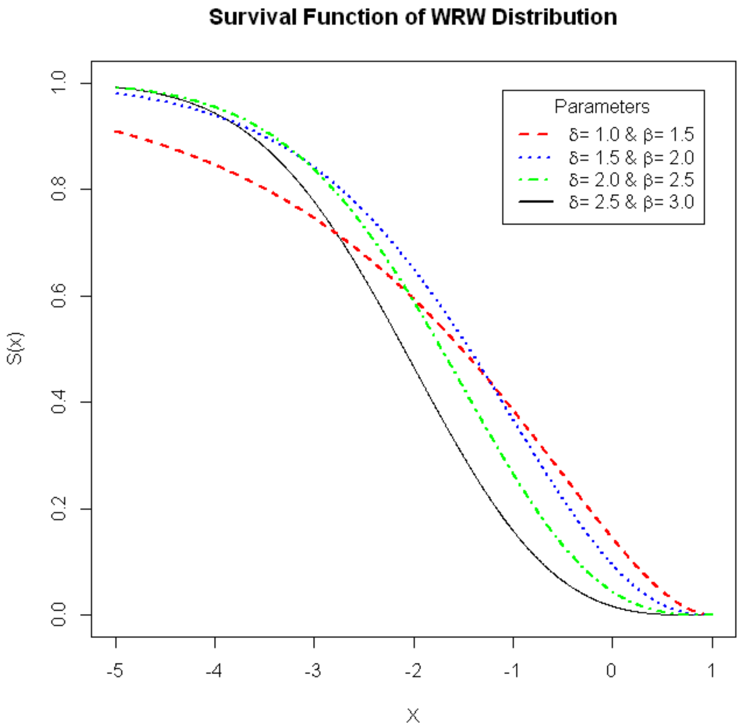

Fig. 5.1: Graph of Survival Function of WRW Distribution $(\gamma = 1)$

5.3.2 Hazard Function

Barlow (1963) gave the idea of hazard rate function. It is the conditional failure rate and defined as the probability of failure of a component during a time interval t, assuming that component has survived to the time t. This is very important tool in reliability measures and widely used in actuarial sciences, survival analysis, economics, demography and engineering.

Mathematically hazard function is expressed as,

$$h(x) = \frac{g(x)}{S(x)}.$$

From (1.7) and (5.7) we derive the expression of hazard function for the WRW distribution as,

$$h(x)=\frac{\delta(\gamma-x)\left(\frac{\gamma-x}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right]}{\beta^2\left\{\Gamma\left(\frac{1}{\delta}+1\right)-\Gamma\left(\left(\frac{1}{\delta}+1\right),\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right)\right\}}. \tag{5.8}$$

where $x \le \gamma,\ \beta > 0,\ \delta > 0$

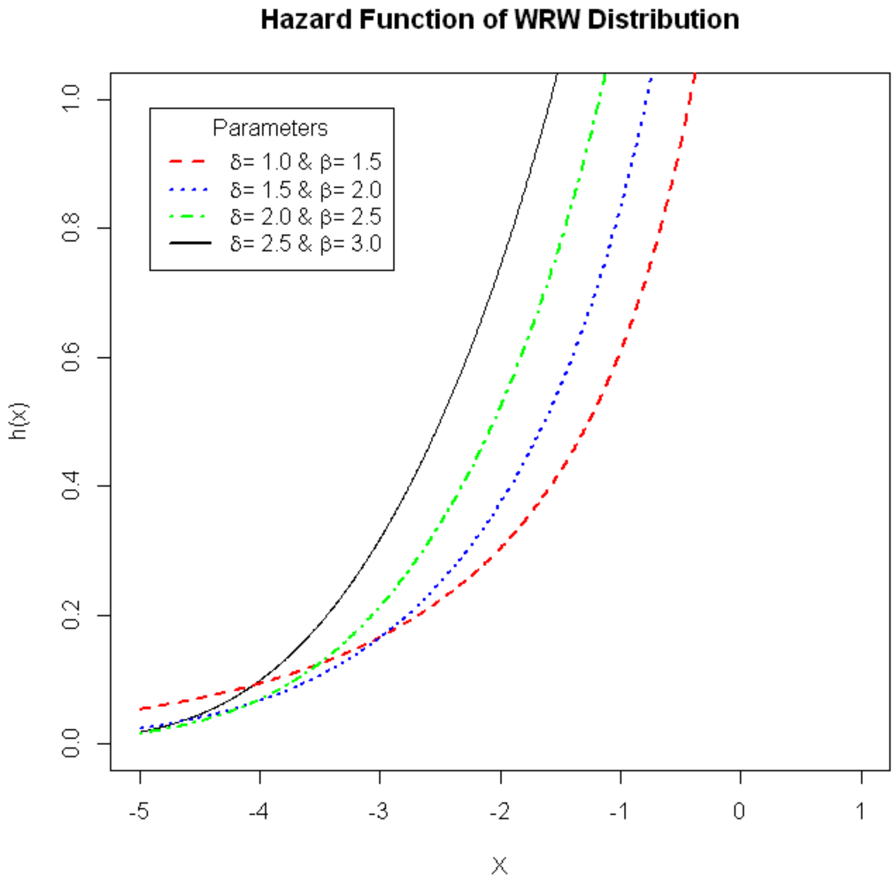

Fig. 5.2: Graph of Failure Rates for WRW Distribution ($\gamma = 1$)

Figure 5.2 shows that the hazard rate function of WRW distribution is monotonically increasing. As well as the value of the scale parameter decreases the increasing rate of hard rate increases rapidly. The hazard rates of this WRW distribution will be useful in modeling failure rates of engineering / electrical components and life length of insured persons.

5.3.3 Reverse Hazard Rate Function

It is very useful measure in Forensic sciences and medical studies to model hidden failures and waiting time.

By definition it is the ratio of probability density function and cumulative density function,

$$r(x) = \frac{g(x)}{G(x)}.$$

From (1.7) and (4.3) we have the expression for reverse hazard rates of WRW distribution as,

$$r(x) = \frac{\delta(\gamma - x)\left(\frac{\gamma - x}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right]}{\beta^2 \Gamma\left(\left(\frac{1}{\delta}+1\right),\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right)}. \tag{5.9}$$

where $x \leq \gamma,\ \beta > 0,\ \delta > 0$

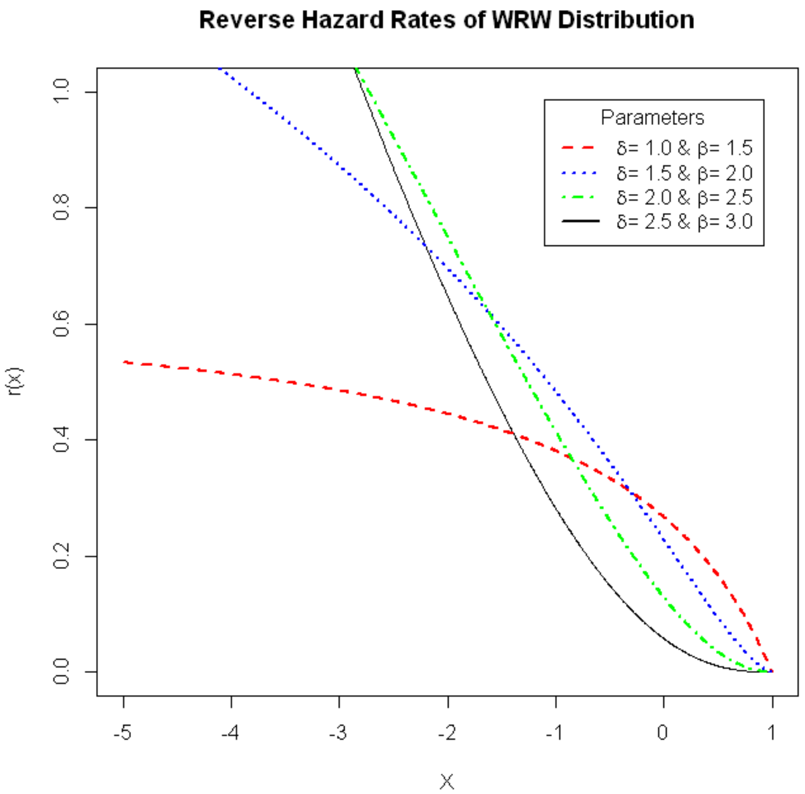

Fig. 5.3: Reverse Hazard Rate Function of WRW Distribution ($\gamma = 1$)

5.3.4 The Mills Ratio

The mills ratio is introduced by John P. Mills. This is defined as the inverse of hazard rate function and mathematically expressed as,

$$m(x) = \frac{S(x)}{g(x)}.$$

The Mills ratio of WRW distribution is illustrated by using (5.7) and (1.7) as,

$$= \frac{\beta^{\delta+1}\left\{\Gamma\left(\frac{1}{\delta}+1\right) - \Gamma\left(\left(\frac{1}{\delta}+1\right),\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right)\right\}}{\delta(\gamma-x)^{\delta}\,exp\left[-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right]}. \tag{5.10}$$

where $x \le \gamma,\ \beta > 0,\ \delta > 0$

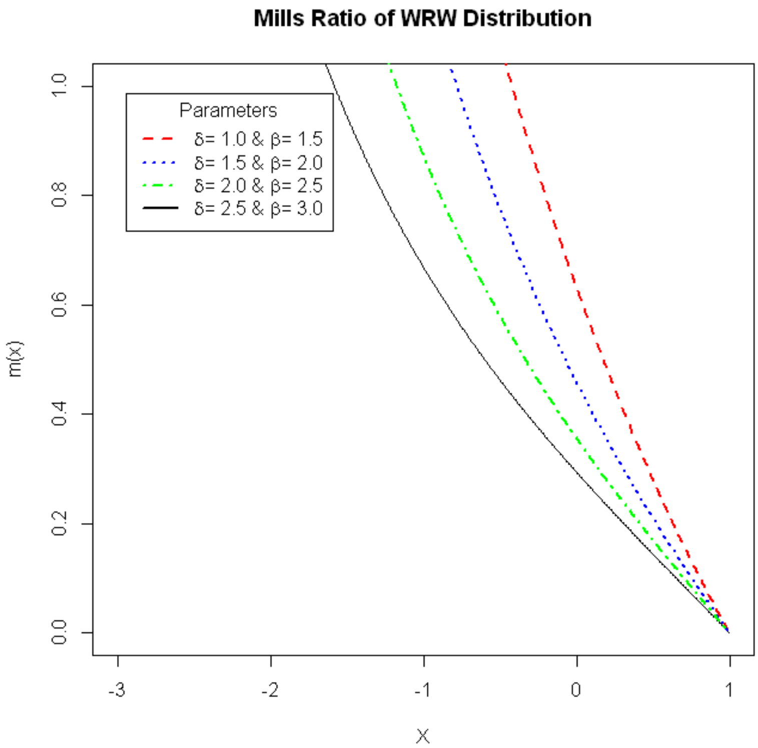

Fig. 5.4: Mills Ratio of WRW Distribution ($\gamma = 10,\ \delta = 2$)

5.3.5 Mean Inactivity Time

The interval time elapsed after an event occurs until the time of its observance is called the mean inactivity time. It is very useful tool in medical studies to predict the exact time of the occurrence of an event. Dara (2011) derived mean inactivity time for various moment distributions.

By the definition of mean inactivity time,

$$mt(x) = \frac{\int_{-\infty}^{x} G(t)dt}{G(x)},$$

from (4.3) we have the mean inactivity time for WRW distribution,

$$mt(x) = \frac{\delta \int_{-\infty}^{x} \int_{-\infty}^{t} (\gamma - x)\left(\frac{\gamma - x}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right] dx\, dt}{\beta^2 \Gamma\left(\left(\frac{1}{\delta}+1\right), \left(\frac{\gamma - x}{\beta}\right)^{\delta}\right)}. \tag{5.11}$$

where $x \leq \gamma,\ \beta > 0,\ \delta > 0$.

CHAPTER 6

ESTIMATION OF PARAMETERS AND CHARACTERIZATION OF WRW DISTRIBUTION

6.1 ESTIMATION OF PARAMETERS

When samples are selected from a population represented by a probability density function $f(x \mid \theta_1, \theta_2, ..., \theta_k)$ then in this situation knowledge about θ yields very important information about the whole population. Therefore it is necessary to apply a method of estimating point estimator of parameter θ (Casella and Berger, 2011).

Here in this research we have applied method of moments and method of maximum likelihood estimator to find parameter estimates of WRW distribution.

6.2 METHOD OF MOMENTS

Let $X_1, X_2, ..., X_n$ be a random sample from a population having values, $x_1, x_2, ..., x_n$ with pdf $g(x \mid \beta, \delta, \gamma)$. Estimators by the method of moments are found by equating first k population moments to the corresponding sample moments and solving the system of equations simultaneously.

By equating the raw moments of the WRW distribution to the sample moments,

$$\mu_1' = \frac{\sum x}{n}, \tag{6.1}$$

$$\mu_2' = \frac{\sum x^2}{n}, \tag{6.2}$$

$$\mu_3' = \frac{\sum x^3}{n}, \tag{6.3}$$

using (4.8) in (6.1) we have,

$$\frac{\left[\gamma\Gamma_1 - \beta\Gamma_2\right]}{\Gamma_1} = \frac{\sum x}{n}, \tag{6.4}$$

and using (4.9) in (6.2) we have,

$$\frac{\left[\gamma^2\Gamma_1 - 2\gamma\beta\Gamma_2 + \beta^2\Gamma_3\right]}{\Gamma_1} = \frac{\sum x^2}{n}, \tag{6.5}$$

in the same way using (4.10) in (6.3) we obtain the equation,

$$\frac{\left[\gamma^4\Gamma_1 - 4\gamma^3\beta\Gamma_2 + 6\gamma^2\beta^2\Gamma_3 - 4\gamma\beta^3\Gamma_4 + \beta^4\Gamma_5\right]}{\Gamma_1} = \frac{\sum x^3}{n}, \tag{6.6}$$

Solving equations (6.4), (6.5) and (6.6) simultaneously, we may obtain the parameter estimates for γ, β and δ δ. The system of equations can only numerically be solved.

6.3 PRACTICAL EXAMPLE

Here are some practical examples, showing the practical aspects of the study as well as the implementation of the methods of moment.

6.3.1 Distribution of Life Insurance Policy Holders

Here is the frequency distribution of 368 life insurance policy holders from Elderton and Johnson (1969). Method of moments is used to inquire estimates about the parameters of the WRW distribution. The results are,

$\hat{\gamma} = 350$

$\hat{\beta} = 320.703$

$\hat{\delta} = 38.895$

The result shows following expected frequencies along with observed ones as shown below,

Table 6.1
Age distribution of Life insurance policy holders

Age in Years	Observed Frequency	Expected Frequency
05-14	1	1.2
15-24	56	63.2
25-34	167	156.9
35-44	98	98.0
45-54	34	35.0
55-64	9	10.1
65-74	2	2.7
75-84	1	0.7
Total	**368**	**367.8**
Chi-square		**2.01**

6.3.2 Distribution of Masses of Male Students at a College

Data distribution of masses of 180 male students from certain college of Pakistan is given below; method of moments is applied to estimate the parameters of the WRW distribution. The distribution of observed and expected frequency is given below along with estimates of parameters and goodness of fit statistics.

Table 6.2
Frequency Distribution of Body Masses of Male Students at a College (Kgs.)

Mass (Kgs.)	Observed Frequency	Expected Frequency
60 - 64	4	5.2
65 - 69	27	22.6
70 - 74	42	49.5
75 - 79	60	55.6
80 - 84	35	34.3
85 - 89	12	11.4
Total	**180**	**178.5**
Chi-square		**2.64**
$\hat{\gamma}$	97.424	
$\hat{\beta}$	22.021	
$\hat{\delta}$	03.519	

6.4 MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimation is the most famous technique for estimation of parameters. Again if $X_1, X_2, ..., X_n$ be a random sample from a population having pdf $g(x|\beta,\delta,\gamma)$ the likelihood function of WRW distribution may be defined as,

$$L(\gamma,\delta,\beta \mid x_1,x_2,...,x_n)=\prod_{i=1}^{n} g(x_i \mid \gamma,\delta,\beta), \tag{6.7}$$

$$L(\gamma,\delta,\beta)=\frac{\prod_{i=1}^{n}(\gamma-x_i)\delta^n}{\beta^{2n}\Gamma\left(\frac{1}{\delta}+1\right)^n}\frac{\prod_{i=1}^{n}(\gamma-x_i)^{\delta-1}}{\beta^{n(\delta-1)}}exp\left[\frac{-\sum_{i=1}^{n}(\gamma-x_i)^{\delta}}{\beta^{\delta}}\right], \tag{6.8}$$

taking *ln* of (6.8),

$$\ln L(\gamma,\delta,\beta)=\sum_{i=1}^{n}\ln(\gamma-x_i)+n\ln\delta-n\ln\Gamma\left(\frac{1}{\delta}+1\right)-2n\ln\beta$$
$$+(\delta-1)\sum_{i=1}^{n}(\gamma-x_i)-n(\delta-1)\ln\beta-\frac{\sum_{i=1}^{n}(\gamma-x_i)^{\delta}}{\beta^{\delta}}, \tag{6.9}$$

partially differentiating w.r.t β to (6.9),

$$\frac{\partial \ln L}{\partial\beta}=-2n\frac{1}{\beta}-n\frac{(\delta-1)}{\beta}+\frac{\delta\sum_{i=1}^{n}(\gamma-x_i)^{\delta}}{\beta^{\delta+1}},$$

equating to zero we get,

$$\hat{\beta}=\left(\frac{\delta\sum_{i=1}^{n}(\gamma-x_i)^{\delta}}{n(\delta+1)}\right)^{\frac{1}{\delta}}. \tag{6.10}$$

Again partially differentiating w.r.t δ to (6.10),

$$\frac{\partial \ln L}{\partial \delta} = \frac{n}{\delta} + n\psi\left(\frac{1}{\delta}+1\right)\frac{1}{\delta^2} + \sum_{i=1}^{n}(\gamma - x_i) - n\ln\beta$$
$$-\sum_{i=1}^{n}(\gamma - x_i)^{\delta}\ln(\gamma - x_i)\frac{1}{\beta^{\delta}} + \sum_{i=1}^{n}(\gamma - x_i)^{\delta}\frac{\ln\beta}{\beta^{\delta}}, \qquad (6.11)$$

here as,

$x < \gamma$ therefore,

$$\hat{\gamma} = x_{(n)} \qquad (6.12)$$

Here from (6.10) and (6.11) we can find only the numerical estimates of the parameters.

6.5 FISHER'S INFORMATION MATRIX OF WRW DISTRIBUTION

The Fisher's information is that a random variable "X" contains about the parameter λ is given by,

$$I(\lambda) = E\left[\frac{\partial}{\partial\lambda}\log\left(g(x;\lambda)\right)\right]^2. \qquad (6.13)$$

where, if $\log\left[g(x)\right]$ is twice differentiable with respect to λ under some regularity conditions, the Fisher information is given by,

$$I(\lambda) = E_{\lambda}\left[\frac{\partial^2}{\partial\lambda^2}\log\left(g(x;\lambda)\right)\right]. \qquad (6.14)$$

The pdf of WRW distribution as given in (1.7), has the mean (4.8) and variance at (4.12). So, by applying log to (1.7) we have the expression,

$$\log\left[\mathrm{g}(\mathrm{x};\gamma,\beta,\delta)\right] = \log(\delta) - (\delta+1)\log(\beta)$$
$$-\log\Gamma\left(\frac{1}{\delta}+1\right) + \delta\log(\gamma - x) - \left(\frac{\gamma - x}{\beta}\right)^{\delta}. \qquad (6.14)$$

Differentiating (6.14) with respect to γ, β *and* δ we get,

$$\frac{\partial}{\partial\gamma}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{\delta}{\gamma-x}-\frac{\delta}{\beta^{\delta}}\left(\gamma-x\right)^{\delta-1}$$

$$\frac{\partial}{\partial\beta}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{\delta}{\beta}\left(\frac{\gamma-x}{\beta}\right)^{\delta}-\frac{\delta+1}{\beta}$$

$$\frac{\partial}{\partial\delta}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{1}{\delta}-\log\beta+\psi\left(\frac{1}{\delta}+1\right)\frac{1}{\delta^{2}}$$
$$+\ln\left(\gamma-x\right)-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\ln\left(\frac{\gamma-x}{\beta}\right)$$

$$\frac{\partial^{2}}{\partial\gamma^{2}}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=-\frac{\delta}{\left(\gamma-x\right)^{2}}-\frac{\delta\left(\delta-1\right)\left(\gamma-x\right)^{\delta-2}}{\beta^{\delta}}$$

$$\frac{\partial^{2}}{\partial\beta\partial\gamma}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{\delta^{2}}{\beta^{\delta+1}}\left(\gamma-x\right)^{\delta-1}$$

$$\frac{\partial^{2}}{\partial\delta\partial\gamma}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{1}{\gamma-x}-\left(\gamma-x\right)^{\delta-1}\beta^{-\delta}\left[1-\delta\log\beta-\delta\log\left(\gamma-x\right)\right]$$

$$\frac{\partial^{2}}{\partial\beta^{2}}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{\delta+1}{\beta^{2}}\left[1-\frac{\delta\left(\gamma-x\right)^{\delta}}{\beta^{\delta}}\right]$$

$$\frac{\partial^{2}}{\partial\gamma\partial\beta}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{\delta^{2}\left(\gamma-x\right)^{\delta-1}}{\beta^{\delta+1}}$$

$$\frac{\partial^{2}}{\partial\delta\partial\beta}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{\left(\gamma-x\right)^{\delta}}{\beta^{\delta+1}}\left[1-\delta\log\beta-\delta\log\left(\gamma-x\right)\right]-\frac{1}{\beta}$$

$$\frac{\partial^2}{\partial\delta^2}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\psi'\left(\frac{1}{\delta}+1\right)\frac{1}{\delta^4}-\frac{1}{\delta^2}-\frac{2}{\delta^3}\psi\left(\frac{1}{\delta}+1\right)$$
$$-\left(\frac{\gamma-x}{\delta}\right)^{\delta}\left[\log\left(\frac{\gamma-x}{\delta}\right)\right]^2$$

$$\frac{\partial^2}{\partial\gamma\,\partial\delta}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=\frac{1}{\gamma-x}-\frac{1}{\beta}\left(\frac{\gamma-x}{\beta}\right)^{\delta-1}\left[\delta\log\left(\frac{\gamma-x}{\beta}\right)+1\right]$$

$$\frac{\partial^2}{\partial\beta\,\partial\delta}\log\left(g\left(x;\gamma,\beta,\delta\right)\right)=-\frac{1}{\beta}+\left(\frac{\gamma-x}{\beta}\right)^{\delta-1}\left(\frac{\gamma-x}{\beta^2}\right)\left[\delta\log\left(\frac{\gamma-x}{\beta}\right)+1\right]$$

The Fisher's information matrix for the WRW distribution is,

$$I(\gamma,\delta,\beta)=\begin{bmatrix}-E\left[\frac{\partial^2}{\partial\gamma^2}\log\left(g\left(x\right)\right)\right] & -E\left[\frac{\partial^2}{\partial\gamma\,\partial\beta}\log\left(g\left(x\right)\right)\right] & -E\left[\frac{\partial^2}{\partial\gamma\,\partial\delta}\log\left(g\left(x\right)\right)\right]\\ -E\left[\frac{\partial^2}{\partial\beta\,\partial\gamma}\log\left(g\left(x\right)\right)\right] & -E\left[\frac{\partial^2}{\partial\beta^2}\log\left(g\left(x\right)\right)\right] & -E\left[\frac{\partial^2}{\partial\beta\,\partial\delta}\log\left(g\left(x\right)\right)\right]\\ -E\left[\frac{\partial^2}{\partial\delta\,\partial\gamma}\log\left(g\left(x\right)\right)\right] & -E\left[\frac{\partial^2}{\partial\delta\,\partial\beta}\log\left(g\left(x\right)\right)\right] & -E\left[\frac{\partial^2}{\partial\delta^2}\log\left(g\left(x\right)\right)\right]\end{bmatrix}.$$

6.6 CHARACTERIZATION THROUGH CONDITIONAL MOMENTS OF WRW DISTRIBUTION

A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model. There are several functions associated with a probability distribution that uniquely identify it. We call these characterizing functions. Here we are characterizing the WRW distribution through conditional moments by using the characterizing function $u\left(x\right)$.

Su and Huang (2000) have given necessary and sufficient conditions such that there exists a random variable X satisfying the equation,

$$E\left[u\left(x\right)\mid X\geq y\right]=k\left(y\right)r_x\left(y\right). \qquad (6.15)$$

Theorem 6.1

Let X be the random variable on its support $(-\infty, \gamma)$. Then for characterizing the WRW distribution,

If $$E\left[u(x) \mid X \geq t\right] = \frac{c}{\bar{F}(t)} \frac{\beta^{\delta}}{\delta}\left[1-\exp\left\{-\left(\frac{\gamma-t}{\beta}\right)^{\delta}\right\}\right].$$

where $u(x) = (\gamma - x)^{-1}$ and c is a constant.

Proof:

$$E\left[(\gamma-x)^{-1} \mid X \geq t\right] = \frac{1}{\bar{F}(t)} \int_{t}^{\gamma} (\gamma-x)^{-1} g(x) dx,$$

from (1.7),

$$= \frac{c}{\bar{F}(t)} \int_{t}^{\gamma} (\gamma-x)^{\delta-1} . \exp\left[-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right] dx.$$

Using transformation (4.2) we have,

$$= \frac{c}{\bar{F}(t)} \int_{0}^{\left(\frac{\gamma-t}{\beta}\right)^{\delta}} \left(\beta z^{\frac{1}{\delta}}\right)^{\delta-1} e^{-z} \frac{\beta}{\delta} z^{\frac{1}{\delta}-1} dz,$$

after simplification,

$$= \frac{c}{\bar{F}(t)} \frac{\beta^{\delta}}{\delta} \int_{0}^{\left(\frac{\gamma-t}{\beta}\right)^{\delta}} e^{-z} dz,$$

$$E\left[(\gamma-x)^{-1} \mid X \geq t\right] = \frac{c}{\bar{F}(t)} \frac{\beta^{\delta}}{\delta}\left[1-\exp\left\{-\left(\frac{\gamma-t}{\beta}\right)^{\delta}\right\}\right].$$

Conversly,

$$\frac{c}{\bar{F}(t)}\int_t^{\gamma}(\gamma-u)^{-1}f(u)du=\frac{c}{\bar{F}(t)}\frac{\beta^{\delta}}{\delta}\left[1-\exp\left\{-\left(\frac{\gamma-t}{\beta}\right)^{\delta}\right\}\right].$$

Differentiating w.r.to t both sides we get,

$$-(\gamma-t)^{-1}f(t)=\frac{\beta^{\delta}}{\delta}\left[-\exp\left\{-\left(\frac{\gamma-t}{\beta}\right)^{\delta}\right\}\frac{\delta}{\beta}\left(\frac{\gamma-t}{\beta}\right)^{\delta-1}\right],$$

after simplification,

$$f(t)=(\gamma-t)^{\delta}\exp\left\{-\left(\frac{\gamma-t}{\beta}\right)^{\delta}\right\}, \tag{6.16}$$

Theorem 6.2

Let X be the random variable on its support $(-\infty,\gamma)$. Then for characterizing the WRW distribution,

If $E\left[u(x)\,|\,X\geq t\right]=\dfrac{c}{\bar{F}(t)}\dfrac{\beta^{2\delta+1}}{\delta}\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\left(\frac{\gamma-t}{\beta}\right)^{\delta\left(\frac{1}{\delta}+n+2\right)}}{\frac{1}{\delta}+n+2}.$

where $u(x)=(\gamma-x)^{\delta}$ and c is a constant.

Proof:

$$E\left[(\gamma-x)^{\delta}\,|\,X\geq t\right]=\frac{1}{\bar{F}(t)}\int_t^{\gamma}(\gamma-x)^{\delta}g(x)dx, \tag{6.17}$$

from (1.7),

$$=\frac{c}{\bar{F}(t)}\int_t^{\gamma}(\gamma-x)^{2\delta}.\exp\left[-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right]dx.$$

Using transformation $\left(\frac{\gamma - x}{\beta}\right)^{\delta} = z$ we have,

$$= \frac{c}{\bar{F}(t)} \int_{0}^{\left(\frac{\gamma-t}{\beta}\right)^{\delta}} \left(\beta z^{\frac{1}{\delta}}\right)^{2\delta} e^{-z} \frac{\beta}{\delta} z^{\frac{1}{\delta}-1} dz,$$

after simplification,

$$= \frac{c}{\bar{F}(t)} \frac{\beta^{2\delta+1}}{\delta} \int_{0}^{\left(\frac{\gamma-t}{\beta}\right)^{\delta}} z^{\frac{1}{\delta}+1} e^{-z} dz,$$

$$= \frac{c}{\bar{F}(t)} \frac{\beta^{2\delta+1}}{\delta} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{0}^{\left(\frac{\gamma-t}{\beta}\right)^{\delta}} z^{\frac{1}{\delta}+n+1} dz,$$

$$E\left[(\gamma - x)^{\delta} \mid X \geq t\right] = \frac{c}{\bar{F}(t)} \frac{\beta^{2\delta+1}}{\delta} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left(\frac{\gamma-t}{\beta}\right)^{\delta\left(\frac{1}{\delta}+n+2\right)}}{\frac{1}{\delta}+n+2}.$$

Conversly,

$$\frac{c}{\bar{F}(t)} \int_{t}^{\gamma} (\gamma - u)^{\delta} f(u) du = \frac{c}{\bar{F}(t)} \frac{\beta^{2\delta+1}}{\delta} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left(\frac{\gamma-t}{\beta}\right)^{\delta\left(\frac{1}{\delta}+n+2\right)}}{\frac{1}{\delta}+n+2}$$

Differentiating w.r.to t both sides we get,

$$-(\gamma - t)^{\delta} f(t) = -\beta^{2\delta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\gamma-t}{\beta}\right)^{\delta\left(\frac{1}{\delta}+n+2\right)-1}$$

$$-(\gamma-t)^{\delta} f(t) = -(\gamma-t)^{\delta}(\gamma-t)^{\delta}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\gamma-t}{\beta}\right)^{n\delta}$$

after simplification,

$$f(t) = (\gamma-t)^{\delta}\exp\left[-\left(\frac{\gamma-t}{\beta}\right)^{\delta}\right] \tag{6.18}$$

6.7 MODIFIED REFLECTED WEIBULL DISTRIBUTION

If X is a reflected Weibull in (1.3) where $x<\gamma$ and X_{δ} is the random variable with PDF, $f(x_{\delta}) = c' f(x)$, where $0<x<\gamma$ and $c' = 1\Big/\left[1-\exp\left(-\left(\gamma/\beta\right)^{\delta}\right)\right]$ then, $f(x_{\delta})$ is also a PDF.

Therefore the PDF of modified reflected Weibull (MRW) distribution is,

$$f(x_{\delta}) = \frac{\delta}{\beta^{\delta}\left[1-\exp\left(-\left(\gamma/\beta\right)^{\delta}\right)\right]}(\gamma-x)^{\delta-1}\exp\left\{-\left(\frac{\gamma-x}{\beta}\right)^{\delta}\right\}. \tag{6.19}$$

where $0<x<\gamma$ and $\delta>0$, $\beta>0$.

6.7.1 Graph of Modified Reflected Weibull Distribution

Figure 6.1 shows the graphs of MRW distribution for various combinations of parameters.

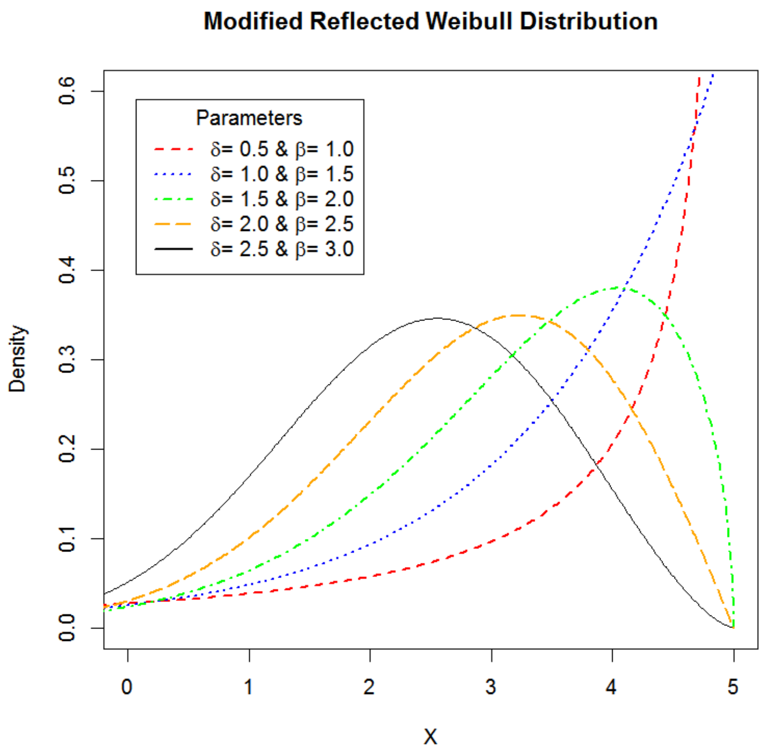

Fig. 6.1: Graph of MRW Distribution

6.8 CUMULATIVE DENSITY FUNCTION OF MRW DISTRIBUTION

Cumulative density function of MRW distribution is defined as,

$$F(x_\delta) = P(X \le x),$$

$$F(x_\delta) = \left[\frac{\exp\left\{-\left(\frac{\gamma - x}{\beta}\right)^{\delta}\right\} - \exp\left\{-\left(\frac{\gamma}{\beta}\right)^{\delta}\right\}}{1 - \exp\left\{-\left(\frac{\gamma}{\beta}\right)^{\delta}\right\}} \right], \qquad (6.20)$$

where $0 < x < \gamma$ and $\delta > 0$, $\beta > 0$.

6.8.1 Graph of Cumulative MRW Distribution

Following figure 6.2 shows the cumulative density function of MRW distribution for various values of Parameters,

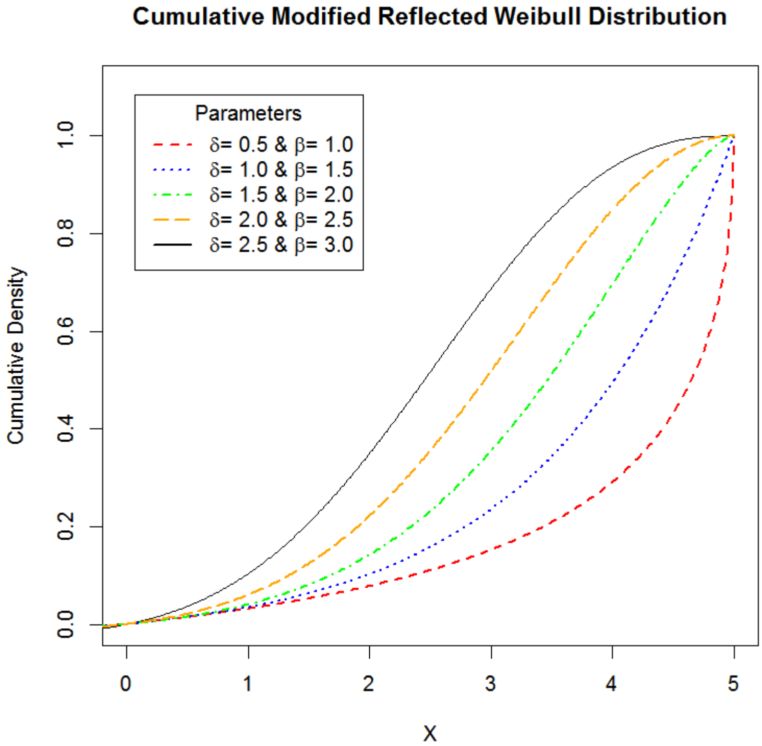

Fig. 6.2: Graph of Cumulative MRW Distribution

6.9 DISTRIBUTIONAL PROPERTIES OF MRW DISTRIBUTION

6.9.1 kth Moment about Origin of MRW Distribution

$$\mu_k' = c\sum_{i=0}^{k}(-1)^i\binom{k}{i}\beta^i\gamma^{k-i}\gamma\left[\left(\frac{i}{\delta}+1\right),\left(\frac{\gamma}{\beta}\right)^{\delta}\right], \tag{6.21}$$

where $c=\dfrac{1}{\left[1-\exp\left(-\left(\gamma/\beta\right)^{\delta}\right)\right]}$ and, $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt,$

is the lower incomplete gamma function.

Substituting $r =$ 1, 2, 3, and 4, we obtain the first four raw moments of MRW distribution as,

$$\mu_1' = c\left[\gamma - \beta\gamma_1\right] \tag{6.22}$$

$$\mu_2' = c\left[\gamma^2 - 2\beta\gamma\gamma_1 + \beta^2\gamma_1\right] \tag{6.23}$$

$$\mu_3' = c\left[\gamma^3 - 3\beta\gamma^2\gamma_1 + 3\beta^2\gamma\gamma_2 - \beta^3\gamma_3\right] \tag{6.24}$$

$$\mu_4' = c\left[\gamma^4 - 4\beta\gamma^3\gamma_1 + 6\beta^2\gamma^2\gamma_2 - 4\beta^3\gamma\gamma_3 + \beta^4\gamma_4\right] \tag{6.25}$$

where $\gamma_m = \gamma\left[\left(\frac{m}{\delta}+1\right),\left(\frac{\gamma}{\beta}\right)^{\delta}\right]$

At $\delta = 3.60235$ $\beta_1 = 0$ and $\beta_2 = 2.7168$, i.e. at $\beta_1 = 0$ and $\beta_2 < 3$ Therefore the distribution (6.19) follows Pearson type II distribution.

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