# Signaling with evidence

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**Abstract**

We model the interaction between an investor and a developer as a game with observable actions where the advancements of the project are private information of the developer. We introduce a verifiability assumption that restricts the set of messages the developer can send to the investor. We find that it is optimal for the investor to play a cutoff strategy in any equilibrium of this game and the developer would send the lowest message possible that makes the game continue. We also show that truth-telling is not an equilibrium in our model.

## 1 Introduction

The communication between an investor(she) who funds a project and a developer(he) who has access to the production technology is key to the project's success. Even when both agents benefit from a completed project, conflict of interest may arise throughout the production process. When reporting the status of the project to the investor, the developer might have incentives to lie about the accomplishments to keep the investment ongoing for a longer time. For example, directors of a movie may report to the investors that most of the scenes have been shot even though only a couple of scenes were taken, just to keep the investors providing grants for every stage of the film process. To improve the communication quality between the economic agents, the investor would require the developer to provide

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physical evidence supporting every message they send about the status of the project. For instance, movie investors may ask to watch the scenes that the director claims have already been shot before giving them post-production grants.

Providing physical evidence to support the communication between the agents is a natural condition that economic agents would apply in a communication setting. The last example, together with many other real-life applications, emphasizes the importance of a verifiability condition in a signaling game, where the sender has limitations on the signals or messages he can send. Additionally, most investments take a dynamic form rather than a static one: the economic agents frequently communicate their private information throughout the investment period. Thus, it is of great importance to study the implications of such a natural condition: verifiability on the interaction between economic agents. We model this interaction between the investor and the developer where both agents get positive benefits only if the project is completed.[1] The novelty of the model is in the way we model a verifiability assumption that forces the developer to provide physical evidence to support the messages he sends to the investor about the status of the project. This idea is clarified in the model by restricting the set of messages that the developer can send to be bounded by the cumulative advancements attained by the time he sends the message.[2]

For another application, consider the manufacturing process of a vaccine.[3] On one side, the government does not know any details about the manufacturing process of each stage. However, they cover the cost of production every month. On the other side, the pharmaceutical company owns the production technology. The outcome during the production process is private information for the company. To know if they should keep funding the project, the government asks for regular reports about the accomplished stage of production, and to ameliorate their understanding of the production process, they would ask to see the reports or every stage the company claims to be attained. [4] The pharmaceutical company decides to show the outcome of some of the stages they completed, and because of the government requirements, they need to demonstrate their work. Upon receiving the message, the gov-

[1]Later, in the extension section, we consider other forms of utility for the investor and the developer.

[2]Note that the project has to be verifiable by its nature, that is, it should be feasible for the developer to send physical evidence of completing a portion of the project. For example, a movie is verifiable as one can watch the completed scenes. However, a computer processor is not since one can only test it when it is ready to use.

[3]The production of a vaccine has clearly ordered stages: Laboratory research, pre-clinical trials, clinical trials, and the manufacturing stage.

[4]For instance, whether the company finalized the laboratory research, or started the clinical trial, etc. ...

ernment will continue funding the project or quit. However, they can find it optimal to hide some achievements: for instance, they might send only the laboratory report even though they completed the pre-clinical trials because they know that new experts are joining the company and their work might reflect better achievements in the future, so the investor (the government here) would be more optimistic about continuing the project after the future reports. In this paper, we will discuss whether such a scenario is plausible in equilibrium.

Formally, we model the above interaction as a two-player multi-stage game with observable actions, where both players benefit only from a completed project. Every period, an advancement in the project is realized by a move of nature and is observed privately by the developer. Then, he chooses the message he sends to the investor, who, upon receiving it, chooses to continue the game and pay the production cost or stop. The verifiability assumption that we study limits the messages that the developer can send every period by the cumulative advancements achieved. Notice that most investments that take a dynamic aspect are affected by several factors- like the difficulty of the project or the effectiveness of the technology used in the manufacturing process- that are revealed over time. More importantly, it is common for those factors to be divulged to the person developing the project. Thus, we consider the case where the state of the world, representing the difficulty of the project, changes with time, and it is private information of the developer.

We discuss how the verifiability assumption allows for a very simple yet intuitive class of Perfect Bayesian Equilibria where the developer only wants to show the minimum cumulative advancement that makes the investor continue funding the project each period. Saving current work to reveal it in the future helps the developer to improve his chances of making the investor continue in succeeding stages. In addition, since both agents get a positive benefit only from a completed project, we show that the developer has no incentives to lie in the last period. However, we show that reporting truthfully in every period of the game is not an equilibrium strategy for the developer: He prefers to lie in the first stage of the game to infer a better jump in the advancements of the project in the second stage, increasing the probability of project completion. Moreover, we show that in every equilibrium of this game, the investor will play a cutoff strategy in all the periods: that is, conditioning on receiving a message of an incomplete project, the investor continues if she gets a message high enough, that is above a certain cutoff and stops otherwise. That result is inferred from the verifiability assumption that is equivalent to a model where any message below the cumulative advancements in the project, i.e., the developer's type, is costless, but any message above that value is infinitely costly. This will allow the developer to send any message below his type but none above, which will make the investor's optimal strategy the cutoff strategy stated briefly above.

This paper contributes to the literature on signaling introduced by Spence (1973) and later discussed by Crawford and Sobel (1982). However, each paper studies an extreme signaling cost. In Spence (1973), signaling cost is modeled as negatively correlated with the sender's productivity and could support truthtelling as an equilibrium behavior, while Crawford and Sobel (1982) eliminated such signaling costs and showed that the sender always partitions the support of his private information and reports only an element of the partition where his type lies in. The verifiability assumption that we introduce in this paper is equivalent to having a cost function that is in between the two extremes since it imposes a specific signaling cost that depends on the sender's type.

We also contribute to the literature on voluntary disclosure initiated by Grossman and Hart (1980), Grossman (1981), and Milgrom (1981). The above papers considered a static model and obtained an "unraveling result" where, in equilibrium, all types reveal their true state. We show that this is no longer the case in a dynamic model with our specific verifiability assumption that forces the sender to disclose, at most, his type. Shin (2003) and Hart, Kremer, and Perry (2017) considered dynamic voluntary disclosure, which is closely related to what we discuss in that paper. Both papers discussed truth-learning equilibria.

Finally, our paper also contributes to the literature on sequential investment. Bar-Ilan and Strange (1998) discussed a model of two-stage sequential investment where each stage requires some time to complete. However, in our discussion, the investor gets information about the investment through another agent, the developer, which will affect his decision about continuing the investment or stopping it.

The paper is organized as follows: In Section 2, we introduce the model. Section 3 is split into two sub-sections. In Section 3.1, we discuss a class of equilibria of the game, as well as a sketch of its proof. In Section 3.2, we discuss further properties of the equilibria we identify. Finally, in Section 4, we discuss a potential extension of the model where the developer gets paid by transfers from the investor depending on the message they send. Detailed proofs can be found in the Appendix.

# 2 Model

## 2.1 Environment

We introduce a model of project development. There are two agents: an investor ($I$) who can fund the project and a developer ($D$) who has access to the production technology. At time $t = 0$, the investor decides whether to start the investment and move to the following period or not. If the investment starts, the agents interact in a two-stage game to complete the project, where advancement is realized as private information for the developer during every period of the game. After working for one period, the developer gets to know the difficulty of the project that will affect the realization of the advancements.

The difficulty of the project is a random variable $\theta \in \Theta \subseteq \mathbb{R}_+$ realized with probability $p(\theta)$ at the beginning of period $t = 2$. The value of $\theta$ is private information of the developer, and it affects the advancements in the project hereafter.

We model an advancement in the project in period $t \in \{1, 2, 3\}$ as a random variable $x_t$ with distribution $f_t$ and support $X_t := [0, k_t] \in \mathbb{R}_+$. Note that for $t = 2, 3$, $f_t$ depends on the realization of $\theta$, so we will denote it by $f_{t,\theta}$. We define $\bar{x}$ as the completion level, i.e., the project is considered completed if the sum of advancements when the game ends is at least $\bar{x}$. We assume that the project can be completed in a single period with positive probability, i.e. $k_t > \bar{x}$ for $t \in \{1, 2, 3\}$. Strategies of the developer for $t \in \{1, 2\}$ are functions $\sigma_t : H^t \to \Delta(M_t)$, where $H^1 = X_1$ and $H^2 = X_1 \times M_1 \times \Theta \times X_2$ represent the history that the developer observe in period $t \in \{1, 2\}$. $M_1 = [0, k_1]$ and $M_2 = [0, k_1 + k_2]$ are the set of messages in period $t \in \{1, 2\}$ respectively. The verifiability assumption forces $\sigma_t(m_t|h^t) = 0$ if $\sum_{i=1}^{t} x_i < m_t$. The investor's strategies, $s_t(m_t)$, is the probability that the investor continues at time $t$ after observing $m_t$. More formally, they are functions $s_t : M^t \to (0, 1)$ where $M^1 = M_1$ and $M^2 = M_1 \times M_2$.
The timing of the game is as follows:

$t = 0$ At the zeroth period, the investor decides whether to start the development and plays ($s_0 = 1$) or not ($s_0 = 0$). If the investor plays ($s_0 = 1$), she pays the starting cost, $c_0$, and the following period ($t = 1$) starts. Otherwise, the game ends.

$t = 1$ In the first period,

1. $x_1$ is realized according to $f_1$.
2. The developer observes $x_1$ and decides on a message $m_1 \in [0, x_1]$ to send to the investor. Equivalently, the developer chooses $\sigma_1 : H^1 \to \Delta(M_1)$.
3. The investor observes $m_1$, forms beliefs $\lambda_1(h^1|m_1)$ for every $h^1 \in H^1$, then decides on $s_1(m_1)$. If the investor continues to the next period ($t = 2$), she pays $c_1$. Otherwise, she pays nothing, and the game ends.

$t = 2$ In the second period,

1. The state of the world $\theta$ occurs according to $p(\theta)$ and $x_2$ is realized according to $f_{2,\theta}$.
2. The developer sees $\theta$ and $x_2$ and decides on a message $m_2 \in [m_1, x_1 + x_2]$ to send to the investor. Equivalently, the developer chooses $\sigma_2 : H^2 \to \Delta(M_2)$. It is without loss of generality WLOG that we restrict attention to $m_2 \geq m_1$ as the status of the project can only increase (weakly).
3. The investor observes $m_2$, forms beliefs $\lambda_2(h^2|m_1, m_2)$ for every $h^2 \in H^2$, and decides on $s_2(m_1, m_2)$. If the investor continues to the next period ($t = 3$), she pays $c_2$. Otherwise, she pays nothing, and the game ends.

$t = 3$ In the third period, the last advancement $x_3$ is realized according to $f_{3,\theta}$ and the game ends.

Note that the game ends either when the investor decides to end it or when the players get to period $t = 3$.[5] If the game ends at time $\tau$, a completed project ($\sum_{i=1}^{\tau} x_i \geq \bar{x}$) will give payoffs of $R > 0$ to the investor and $D > 0$ to the developer; otherwise they both get zero payoffs.

We denote by $u_t : M^t \times \{c, b\} \to \mathbb{R}$ the investor's utility when the history of messages is $m^t \in M^t$ and decides to continue ($c$) or to stop ($b$).[6] The investor's utility at time $t$ is then $u_t(m^t) = s_t(m^t)u_t(m^t, c) + (1 - s_t(m^t))u_t(m^t, b)$. We denote by $w_t : H^t \times M_t \to \mathbb{R}$ the developer's utility at time $t$ for a given history $h^t \in H^t$ when he sends message $m_t \in M_t$. The detailed equations of the utilities are shown in the next section.

We assume that $f_{t,\theta}(\cdot)$ satisfies the monotone likelihood property (MLP) in $(\theta, x_t)$ for $t \in \{2, 3\}$. We consider the difficulty of the project to be the state of the world. A higher $\theta$ reflects that the project is less difficult. In this context, we say that the more difficult the project is, the lower the probability of getting a high realization in the following periods. As an example, a change in the production technology throughout the production process might arise from the availability of new production techniques that would make development easier (or more difficult). In the leading example of producing a vaccine, one can think that after a month of development, the pharmaceutical company hires a new team of experts, and that will impact the laboratory work as well as the clinical trials later on.

The developer's main goal in this model is to complete a project, so he tries to use the messages he sends to keep the project going and thus increase the probability of project

[5]The developer will never choose to end the game on their own unless the project is completed.

[6]For notational convenience, we use $b$ for stop or break the game.

completion. To do so, the developer has incentives to lie above and below the true status of the project, especially because the difficulty is changing through the production process. For instance, he may want to lie above his type in the second period to increase the investor's belief that the state of the world is high. This would make the investor more optimistic about the last period's advancement because the distribution of $x_t$ is assumed to satisfy MLP in $(\theta, x_t)$ for the last two periods. Moreover, the developer also has incentives to lie below his true type earlier in the game. He may want to hide some of the advancements in the project to show a bigger increase in the second period. By doing so, through the MLP, using the same logic as above, the investor becomes more optimistic about the probability that the project will be completed in the future. The verifiability assumption has implications for the ways that the developer can lie to the investor. Whenever the investor gets a message $m_t$ from the developer about the completed advancements in the project, she believes that at least $m_t$ has been completed. One can think of this assumption as imposing a particular cost function for messages in a signaling game: sending messages below the sender's type (the status of the project) is costless, but sending messages above his type is infinitely costly.

We solve for Perfect Bayesian Equilibrium, $\left((s_0^*, s_1^*, s_2^*)\ ;\ (\sigma_1^*, \sigma_2^*)\ ;\ (\lambda_1^*, \lambda_2^*)\right)$ where sequential rationality is satisfied for both players in all $t \in \{0, 1, 2, 3\}$ and beliefs are constructed using Bayes' rule, on the equilibrium path. We will focus on off-equilibrium beliefs that are natural extensions of on-equilibrium beliefs, given the verifiability assumption. We will explicitly state them in the following section.

# 3 Equilibrium Analysis

The novelty of the model is embodied in the verifiability assumption. The developer is limited by the set of possible messages he can send to the investor throughout the stages of the game. One might question why the developer would send anything but the truth at each stage of the game, especially when he gets positive revenue only by completion of the project. We argue that with the state of the world changing in the second stage, the developer still finds it optimal to lie in the early stages of the game and report truthfully in the second stage. That is to save some of the accomplishments from the first period to the second period to reflect a better state of the world. Even though the investor knows about the way the developer would lie to them, she builds her beliefs accordingly and is still willing to continue in the game if she gets a message high enough. More importantly, the verifiability assumption makes the problem interesting to begin with: later in this section, we discuss a case of the model without the verifiability assumption, and we show that relaxing this condition will cause the development not to start to begin with.

The finite horizon of the game has direct implications on the behavior of the developer. Since both agents get positive utility only if the project is completed, the developer has no strict preference for lying in the second period. In fact, we will see that the benefit of lying will depend on the possibility of influencing beliefs in the following periods. As there are no additional beliefs to make after the report in the second period, the developer will find it optimal to tell the truth.

Recall the example of the government trying to invest in the production of a vaccine. The government needs the verifiability assumption to ensure that the company reports at least a fraction of what they have accomplished so far. However, the difficulty of producing the vaccine can change with time: for example, the pharmaceutical company may learn from the laboratory research that the clinical trials will be more (or less) difficult than expected. As the pharmaceutical company gets positive utility only if they produce a fully functioning vaccine, they might find it optimal to hide the laboratory research outcomes. Instead, they wait to get some outcomes of the pre-clinical trials before reporting their true accomplishments in the production process. However, if the difficulty of the project does not change over time, reporting truthfully at all stages might still be an optimal strategy. Moreover, the government, which also gets positive utility only from a fully functioning vaccine, is willing to continue in the manufacturing process only if they can still cover the cost of production when they believe that the project can be completed with sufficient probability.

In what follows, we explore how powerful the verifiability assumption that we introduced is in a dynamic signaling game. In the first claim, we show that the verifiability assumption implies that it is WLOG to restrict attention to pure strategy for the investor. Moreover, we show that on any equilibrium of the game, after any history of incomplete project, the investor plays a cutoff strategy, where she is willing to continue in the game if the message is high enough. That is, the message she receives is greater than a certain cutoff.

***Claim* 1.** *The investor plays a cutoff strategy for any history in any equilibrium of this game.*

In the proof of Claim 1, we make no assumptions about the developer's strategy in either period. Although the result is interesting, the intuition behind it is very simple. Assume that in period $t$, the investor stops at message $m_t$, which is sent with a strictly positive probability by the developer. Because of the verifiability assumption, the developer is allowed to assign a positive probability for sending any message that is below the cumulative advancements in the project (the current status of the project). Thus, if the investor is stopping at a message $m_t$ but is willing to continue at a lower message $m'_t$, then the developer would assign zero probability for sending $m_t$ and strictly positive probability to message $m'_t$. This leads to a contradiction since we assumed that the message $m_t$ is sent with strictly positive probability. Thus, it has to be that the investor will not continue for any message below $m_t$. Therefore, in equilibrium, the investor will play a cutoff strategy. That is, for any history $h^t$ such that the project is incomplete, there will be a cutoff $\bar{m}_t$, such that the investor will continue for any message that is higher than the cutoff and stop for any message lower than the cutoff.

## 3.1 Main result

In Proposition 1, we formalize a class of equilibria where the investor plays a cutoff strategy as discussed in Claim 1, and the developer reports truthfully in the second stage but lies in the first stage of the game.

To begin with, we propose strategies of both players that will help to build Proposition 1.

We start with the investor's strategies. For period 0, the investor starts the project. Period 1 and period 2 are as follows. First, the developer will stop for any message such that the project is completed, i.e., $m_t \geq \bar{x}$. Otherwise, by Claim 1, for any time $t \in \{1, 2\}$ and any message $m_t < \bar{x}$, the investor will play a cutoff strategy. This effectively partitions the message space $M_t$ in three intervals: stop if $m_t < \bar{m}_t$, continue if $m_t \in [\bar{m}_t, \bar{x})$, stop if $m_t \geq \bar{x}$. That is, for messages in the lowest interval, the investor stops with an incomplete job; for messages in the middle interval, the investor continues hoping to get the job completed in the future; for messages in the upper interval, the investor stops because the project is completed.

The developer strategies are different between the period 1 and period 2. In the second period, the developer will tell the truth for all possible histories. That is, given the current status of the project $x_1 + x_2$, the developer will send a message $m_2 = x_1 + x_2$. The strategy of the first period is more involved. If the current status of the project, $x_1$, is lower than the cutoff $\bar{m}_1$, the developer has no message he can send to make the project continue. Thus, all messages available to the developer will lead the project to stop. Then, he will be indifferent between all messages he can send, so he might as well tell the truth by informing the investor exactly his type.[7] If the current status of the project $x_1$ lies in the region

[7]This need not be the case. In fact, the only thing that matters is that the developer sends a message that informs the investor that the current status is in $[0, \bar{m}_1)$. As the only thing that the developer can

$[\bar{m}_1, \bar{x})$, the developer will send the message $\bar{m}_1$, the lowest message that gets the project continued. Finally, if the project is completed, the developer's payoff is constant and equal to $D$, regardless of what happens in the rest of the game, so he might as well tell the truth.[8]

Formally, we define the following strategy for the developer, $(\sigma^*_{\bar{m}_1}, \sigma^*_2)$, where, for a fixed $\bar{m}_1$, $\sigma^*_2 : X_1 \times M_1 \times \Theta \times X_2 \to \Delta(M_2)$ is defined as,

$$\sigma^*_2(\cdot|x_1, m_1, \theta, x_2) \equiv \delta_{x_1+x_2},$$

where $\delta_{x_1+x_2}$ is the delta Dirac distribution that assigns a point mass at $x_1 + x_2$.
$\sigma^*_{\bar{m}_1} : X_1 \to \Delta(M_1)$ is defined as

$$\sigma^*_{\bar{m}_1}(\cdot|x_1) = \begin{cases} \delta_{x_1} & \text{if } x_1 \in [0, \bar{m}_1) \cup [\bar{x}, k_1] \\ \delta_{\bar{m}_1} & \text{if } x_1 \in [\bar{m}_1, \bar{x}). \end{cases}$$

We also define the following strategy for the investor, $(1, s^*_{\bar{m}_1}, s^*_{\bar{m}_2})$, where
$s^*_{\bar{m}_1} : M_1 \to \{0, 1\}$ is defined as

$$s^*_{\bar{m}_1}(m_1) = \begin{cases} 0 & \text{if } m_1 \in [0, \bar{m}_1) \cup [\bar{x}, k_1] \\ 1 & \text{if } m_1 \in [\bar{m}_1, \bar{x}) \end{cases}$$

and, $s^*_{\bar{m}_2} : M_1 \times M_2 \to \{0, 1\}$ is defined as

$$s^*_{\bar{m}_2}(m_2) = \begin{cases} 0 & \text{if } m_2 \in [0, \bar{m}_2) \cup [\bar{x}, k_1 + k_2] \\ 1 & \text{if } m_2 \in [\bar{m}_2, \bar{x}) \end{cases}$$

The investor's beliefs are defined as follows. For the first period, if $m_1 \in [0, \bar{m}_1) \cup [\bar{x}, k_1]$, the first period beliefs $\lambda^*_1$ assigns a point mass to $x_1 = m_1$. That is,

$$\lambda^*_1(\cdot|m_1) = \delta_{m_1} \quad \text{if } m_1 \in [0, \bar{m}_1) \cup [\bar{x}, k_1]$$

If $m_1 < \bar{m}_1$, the density of $\lambda^*_1$ is given by

$$d\lambda^*_1(x_1|m_1) = \begin{cases} \frac{f(x_1)}{F(\bar{x}) - F(m_1)} & \text{if } m_1 \in [\bar{m}_1, \bar{x}) \ \& \ x_1 \in [m_1, \bar{x}) \\ 0 & \text{if } m_1 \in [\bar{m}_1, \bar{x}) \ \& \ x_1 \notin [m_1, \bar{x}) \end{cases}$$

do is send messages in this region, all other specifications are payoff equivalent. For simplicity, we impose truthtelling in this region.

[8]While the developer is indifferent between telling the truth or not in this region, by lying, he may impose an additional unnecessary cost to the investor, so telling the truth is Pareto Optimal in this region, while lying can be Pareto inefficient.

For the second period, the density of $\lambda_2^*$ is given by

$$d\lambda_2^*(x_1,\theta,x_2|m_1,m_2) = \begin{cases} 0 & \text{if } m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1] \text{ \& } x_1 \neq m_1 \\ \frac{p(\theta)f_{2,\theta}(x_2)\sigma_2^*(x_1,\theta,x_2|m_1,m_2)}{\int_H p(\theta)f_{2,\theta}(x_2)\sigma_2^*(x_1,\theta,x_2|m_1,m_2)dxd\theta} & \text{if } m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1] \text{ \& } x_1 = m_1 \\ \frac{f(x_1)p(\theta)f_{2,\theta}(x_2)\sigma_2^*(x_1,\theta,x_2|m_1,m_2)}{\int_{A(m_1)} f(x_1)p(\theta)f_{2,\theta}(x_2)\sigma_2^*(x_1,\theta,x_2|m_1,m_2)dxd\theta} & \text{if } m_1 \in [\bar{m}_1,\bar{x}) \text{ \& } x_1 \in [m_1,\bar{x}) \\ 0 & \text{if } m_1 \in [\bar{m}_1,\bar{x}) \text{ \& } x_1 \notin [m_1,\bar{x}) \end{cases}$$

where $A(m_1) \equiv \{(x_1,\theta,x_2) \in X_1 \times \Theta \times X_2 | x_1 \in [m_1,\bar{x})\}$, and $H \equiv \Theta \times X_2$

Using the defined strategies and beliefs, we now write the relevant utility functions for the developer and investor for the relevant histories and periods. At period $t$ and history $h^2 = (x_1,m_1,\theta,m_2) \in H^2$, the developer's utility of sending message $m_2 \in M_2$ is

$$w_2(x_1,m_1,\theta,x_2,m_2) = \left[s_{\bar{m}_2}^*(m_1,m_2)(1-F_{3,\theta}(\bar{x}-x_1-x_2)) + (1-s_{\bar{m}_2}^*(m_1,m_2))\mathbb{1}_{x_1,x_2\geq\bar{x}}\right]D$$

At period $t=1$ and history $h^1 = x_1 \in H^1$, the developer's utility of sending message $m_1$ is

$$w_1(x_1,m_1) = s_{\bar{m}_1}^*(m_1)\int_{\theta,x_2} w_2(x_1,m_1,\theta,x_2,x_1+x_2)f_{2,\theta}(x_2)p(\theta)dx_2d\theta + (1-s_{\bar{m}_1}^*(m_1))\mathbb{1}_{x_1\geq\bar{x}}D$$

At period $t=2$ and history $m^2 = (m_1,m_2) \in M^2$, the investor's utility of continuing ($c$) or stopping ($b$) is

$$u_2(m_1,m_2,c) = -c_2 + R\int_{x_1,\theta,x_2}[1-F_{3,\theta}(\bar{x}-x_1-x_2)]\lambda_2^*(x_1,\theta,x_2|m_1,m_2)dxd\theta$$
$$u_2(m_1,m_2,b) = R\int_{x_1+x_2\geq\bar{x}}\lambda_2^*(x_1,\theta,x_2|m_1,m_2)dxd\theta$$

Moreover, with a slight abuse of notation, define

$$u_2(m_1,m_2) = \max\{u_2(m_1,m_2,c),u_2(m_1,m_2,b)\}$$

At period $t=1$ and history $m^1 = m_1 \in M^2$, the investor's utility of continuing ($c$) or stopping ($b$) is

$$u_1(m_1,c) = -c_1 + \int_{x_1,\theta,x_2} u(m_1,x_1+x_2)\lambda_1^*(x_1|m_1)p(\theta)f_{2,\theta(x_2)}d\theta dx$$
$$u_1(m_1,b) = R\int_{x_1\geq\bar{x}}\lambda_1^*(x_1|m_1)dx$$

Finally, we let

$$\mu_1(m_1) = u_1(m_1,c) - u_1(m_1,b)$$
$$\mu_2(m_1,m_2) = u_2(m_1,m_2,c) - u_2(m_1,m_2,b)$$

We call $(s^*_{\bar{m}_1}, s^*_{\bar{m}_2})$ a cutoff strategy for the investor with cutoffs $\bar{m}_1$ and $\bar{m}_2$, where $\bar{m}_1$ is implicitly defined as $\mu_1(m_1) < 0$ for $m_1 < \bar{m}_1$ and $\mu_1(m_1) \geq 0$ for $m_1 \geq \bar{m}_1$; and for a fixed $\bar{m}_2$ that is implicitly defined as $\mu_2(\bar{m}_1, \bar{m}_2) = 0$.

**Proposition 1.** $\big((\sigma^*_{\bar{m}_1}, \sigma^*_2); (1, s^*_{\bar{m}_1}, s^*_{\bar{m}_2}), (\lambda^*_1, \lambda^*_2)\big)$*, where* $\bar{m}_1 < \bar{x}$ *and* $\bar{m}_2 < \bar{x}$ *satisfy*

$$\begin{cases} u_1(m_1, c) \geq u_1(m_1, b) & \forall m_1 \geq \bar{m}_1 \\ u_1(m_1, c) < u_1(m_1, b) & \forall m_1 < \bar{m}_1 \\ u_2(\bar{m}_1, \bar{m}_2, c) = u_2(\bar{m}_1, \bar{m}_2, b) \end{cases}$$

*is an equilibrium of the above game.*

We prove the proposition using backward induction. Notice that whenever the project is completed, the developer will disclose this fact, and the investor will stop. What remains to characterize is the equilibrium for histories in which the project is not completed. We do so in a series of claims, starting with the behavior of the developer and the investor in the second period in Claims 2 and 3.

***Claim* 2.** *Suppose that the investor starts the development in period* $t = 0$ *and plays* $s^*_{\bar{m}_1}$ *for a fixed* $\bar{m}_1$*. Then* $\sigma^*_2$ *is the developer's best response for* $s^*_{\bar{m}_2}$*.*

Claim 2 suggests that it is optimal for the developer to report truthfully the status of the project if he knows that the investor is playing a cutoff strategy. The intuition behind such behavior for the developer is that he will get positive utility only if the project is completed. Thus, the main goal of the developer at this point of the game is to send a message that makes the investor continue to the following period. Since there are no further stages in the game, the developer has no incentives to lie about the true state. He is then indifferent between reporting truthfully and sending any other message that makes the investor continue to the next period. Therefore, truthtelling in $t = 2$ is optimal.

***Claim* 3.** *Assuming the behavior of the developer in both periods, then for a fixed* $\bar{m}_1$*,* $\mu_2(\bar{m}_1, m_2) = u_2(\bar{m}_1, m_2, c) - u_2(\bar{m}_1, m_2, b)$ *is continuous and increasing in* $m_2$ *and the cutoff for the investor in the second period is defined by* $\mu_2(\bar{m}_1, \bar{m}_2) = 0$

Suppose the developer reports the status of the project truthfully in period $t = 2$. The investor, in that case, knows exactly how much is left for the project to be completed. Thus, a higher message $m_2$ has two effects: first, it reflects that the project is closer to completion.[9] Second, given that the developer is reporting truthfully in period $t = 2$, a higher message

[9] As, given $\sigma^*_2$, the investor is sure that $m_2 = x_1 + x_2$.

$m_2$ leads to a higher belief about the state of the world.[10] To get the second implication, we show that the investor's beliefs in the second period, $\lambda_2^*(x_1, \theta, x_2|m_1, m_2)$ satisfies the monotone likelihood property in $(m_2, \theta)$ for $m_2 < \bar{x}$. We call this result Lemma 1, and its detailed proof is in the Appendix. Since the state of the world changes in $t = 2$ and affects the distribution of the advancements in periods $t = 2$ and $t = 3$ and $f_{t,\theta}$ is assumed to satisfy MLP in $(\theta, x_t)$ for $t = 2, 3$, then a higher $\theta$ increases the probability of getting a higher $x_3$ in period $t = 3$. Thus, the investor's utility is increasing in the message that she receives from the developer at $t = 2$. Since the utility of the investor is increasing (and continuous), the investor's cutoff in the second stage is well-defined.[11]

Next, we present an optimal strategy for the developer in the first stage of the game, taking into consideration that the investor will only play a cutoff strategy in equilibrium.

***Claim* 4.** *Suppose that the investor's strategy is $(1, s^*_{\bar{m}_1}, s^*_{\bar{m}_2})$ where $\bar{m}_1 \in M_1$ is fixed. Suppose also that the developer plays according to $\sigma_2^*$ in period $t = 2$. Then, $\sigma^*_{\bar{m}_1}$ is a developer's best response for $s^*_{\bar{m}_1}$ in period $t = 1$.*

Claim 4 suggests that when the developer is reporting truthfully, and the investor is playing a cutoff strategy in the second period, it is optimal for the developer to send the lowest message possible that makes the investor choose to continue rather than stop in the first period. Knowing that the cutoff for the investor at $t = 1$ is $\bar{m}_1$, the developer is indifferent between all messages greater than $\bar{m}_1$. In particular, he will send $\bar{m}_1$ when the status is $x_1 \in [\bar{m}_1, \bar{x})$. This is possible because of the verifiability assumption since it allows for any message below the current status of the project to be sent at no cost. If instead the advancement $x_1$ is less than $\bar{m}_1$, any message that the developer can send will make the investor stop. Thus, conditioning on the project not being completed in the first period, an optimal strategy for the developer is to report truthfully when $x_1 < \bar{m}_1$, and send $\bar{m}_1$ otherwise.

***Claim* 5.** *Assume that the developer plays $\sigma^*$ and the investor plays $s^*_{\bar{m}_2}$ in period $t = 2$ for a fixed $\bar{m}_2$. Then, $BR_D\left(BR_I(\sigma_{\bar{m}_1})\right) = \sigma_{\bar{m}_1}$ if $\bar{m}_1 < \bar{x}$ and satisfies the following inequalities*

$$(*)\begin{cases} \forall m_1 \geq \bar{m}_1 & \mu_1(m_1, c) \geq 0 \\ \forall m_1 < \bar{m}_1 & \mu_1(m_1, c) < 0 \end{cases}$$

Claims 4 and 5 characterize the cutoff of the first stage of the game. In other words, we first show that by taking the other player's strategy as given, each player wants to play a

[10] Higher believes in first-order stochastic dominance sense.

[11] We assume the cutoff to be the minimum $m_2$ satisfying $u_2(\bar{m}_1, m_2, c) - u_2(\bar{m}_1, m_2, b) = 0$.

cutoff strategy of their own. Claim 5 tells us that they choose the same cutoffs so that we have an equilibrium. The four claims combined imply the result of Proposition 1.

## 3.2 Properties of the class of equilibria

In this section, we first show that there is a continuum of cutoff equilibria that are parametrized by the first-period cutoff. We also show how the second-period cutoff depends on the first-period cutoff and the consequences on the player's expected utility. Finally, we show two interesting results. The first one is that in our setting truthtelling in every period is not an equilibrium, even when both players benefit from a completed project. The second one is that if we remove the verifiability assumption so that the developer is free to send any message they want, the investor would not have started funding the project at all.

In Proposition 1, the cutoffs $\bar{m}_1$ and $\bar{m}_2$ were fixed. In the following claim, we identify the set of equilibrium cutoffs $\bar{m}_1$ for period $t = 1$. To begin with, we define the utility of the investor that she would have if she could directly observe the realization of $x_1$, assuming that she could also directly observe $x_1 + x_2$ in the second period.[12] We denote this utility by $u_{FI}$.

$$u_{FI}(x_1, c) = -c_1 + \mathbb{1}_{x_1 \geq \bar{x}} R + (1 - \mathbb{1}_{x_1 \geq \bar{x}} R) \int_\theta \Big[ \int_{\bar{m}_2 - x_1}^{\bar{x} - x_1} [R(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) - c_2] f_{2,\theta}(x_2) dx_2 + \int_{x_2 \geq \bar{x} - x_1} R f_{2,\theta}(x_2) dx_2 \Big] p(\theta) d\theta$$

***Claim* 6.** *For any $\bar{m}_1 \in [\bar{m}_1(c_0), x_1^0]$, there is an equilibrium of the form*

$$\big((\sigma^*_{\bar{m}_1}, \sigma^*_2) \ ; \ (1, s^*_{\bar{m}_1}, s^*_{\bar{m}_2}) \ ; \ (\lambda^*_1, \lambda^*_2)\big)$$

*where $\bar{m}_2$ satisfies $\mu_2(\bar{m}_1, \bar{m}_2) = 0$, $x_1^0$ is defined implicitly as $u_{FI}(x_1^0) = 0$ and $\bar{m}_1(c_0)$ is defined by*

$$c_0 = u(\bar{m}_1(c_0), c)(F_1(\bar{x}) - F(\bar{m}_1(c_0))) + R(1 - F(\bar{x}))$$

Notice that, given that the developer is playing $(\sigma^*_{\bar{m}_1}, \sigma^*_2)$, the investor's utility $u_1(m_1, \cdot)$ in period $t = 1$ when she receives message $m_1$ is the expectation of $u_{FI}(\cdot)$ over the set $X_1$ using probability measure $\lambda_1(x_1|m_1)$.

To prove Claim 6, we first show that $u_{FI}$ is increasing in $x_1$ for $x_1 < \bar{x}$. Since we define $x_1^0$ in the statement of Claim 6 implicitly as $u_{FI}(x_1^0) = 0$, then $u_{FI}(x_1, c) \geq 0$ for

[12]The investor directly observing $x_1 + x_2$ is equivalent to the developer playing $\sigma^*_2$.

$x_1 \geq x_1^0$ and $u_{FI}(x_1, c) < 0$ for $x_1 < x_1^0$. Next, we show that any $\bar{m} \in [\bar{m}_1(c_0), x_1^0]$ can be a equilibrium cutoff. That is, $s^*_{\bar{m}}$ is an equilibrium strategy for the investor in the first period. To do so, we fix $\bar{m} \in [\bar{m}_1(c_0), x_1^0]$ as well as the strategy of the developer to be $(\sigma^*_{\bar{m}}, \sigma^*_2)$ and the beliefs $(\lambda^*_1, \lambda^*_2)$. For any $m'_1 < \bar{m}$, the investor assign $\Pr(x_1 = m'_1) = 1$. Therefore, $\mu_1(m'_1) = u_{FI}(m'_1, c)$. Moreover, $u_{FI}(m'_1, c) < 0$ since $m'_1 < x_1^0$. Thus, the investor will stop at this message. However, if $m'_1 > \bar{m}$, then given $\sigma^*_{\bar{m}}$, the investor's utility if she wants to continue in the game $u_1(m'_1, c)$ would be the expected $u_{FI}$ for $x_1 \in [m'_1, \bar{x})$. Although the function $u_{FI}(m'_1, c)$ is negative for $m_1 \in [\bar{m}_1(c_0), x_1^*]$, the investor's utility for receiving $m'_1 \geq \bar{m}$ is still positive. That is because $u_1(m'_1, c)$ is the expected $u_{FI}(x_1, c)$ for $x_1 \in [m_1, \bar{x})$. Moreover, $u_1(m'_1, c) > 0$ when $[m'_1 \in [x_1^0, \bar{x})$ since $u_{FI}$ is positive in this region. Therefore, for a fixed $\bar{m}_1 \in [\bar{m}_1(c_0), x_1^0]$ as well as the developer's strategy $(\sigma^*_{\bar{m}_1}, \sigma^*_2)$ and beliefs $(\lambda^*_1, \lambda^*_2)$, the cutoff strategy $s^*_{\bar{m}_1}$ will be optimal for the investor.

In the rest of the proof, we show that any $\bar{m} \notin [\bar{m}_1(c_0), x_1^0]$ cannot be an equilibrium cutoff for period $t = 1$. The logic is similar to the one used in the first part of the proof. Fix $\bar{m} in(x_1^0, \bar{x})$. For $m''_1 \in (x_1^0, \bar{m})$, the investor's utility is strictly positive since it is equal to $u_{FI}(m''_1, c)$. This implies that the investor will continue with the game for some message lower than $\bar{m}$. Thus $\bar{m}$ cannot be an equilibrium cutoff at $t = 1$ when $\bar{m} > x_1^0$. Moreover, if $\bar{m} < \bar{m}_1(c_0)$, the expected utility for the investor at time $t = 0$ will be negative, so she will not start the project.

So far, we have characterized a class of equilibria where the developer sends the lowest message that makes the investment proceed, and the investor plays a cutoff strategy in both periods of the game. We also identified the set of possible cutoffs for the first period. In the following claim, we discuss how the cutoff in the second period depends on the message that the investor receives in the first period. The claim will have consequences for the developer's utility among different cutoff equilibria.

***Claim 7.*** *$\bar{m}_2$ is increasing in $\bar{m}_1$*

In the proof of Claim 7, we show that $\mu_2(\cdot, \cdot)$ is increasing in $m_2$ and decreasing in $m_1$. Since the cutoffs are defined such that $\mu_2(\bar{m}_1, \bar{m}_2) = 0$, we use the implicit function theorem to get the result. The idea is that for a higher cutoff in the first period, the investor would require a higher cutoff in the second period in order to continue in the game. The reason for such an increase is that the investor cares not only about the sum of accomplishments in periods $t = 1$ and $t = 2$ but also about the magnitude of $x_2$. That is because of the assumption that $f_{\theta,t}$ satisfies the MLP in $(\theta, x_t)$ for $t = 2, 3$. Thus, a higher $x_2$ implies higher beliefs about the state of the world $\theta$, and so a higher probability of getting a high $x_3$, which increases the probability of project completion.

The most interesting consequence of Claim 7 is that it lets us order the developer's expected payoffs. The lower the equilibrium cutoff in the first period is, the lower the cutoff in the second period. This automatically tells us that the developer's expected utility is decreasing in the value of the first-period cutoff. The reason for this is that the developer's expected utility is proportional to the probability of completion. The higher the cutoff in the first period is, the smaller the continuation interval in both the first and the second periods. Therefore, an increase in the cutoff in the first period will decrease the probability of completion.

For the investor, recall that her utility in the first period is the integral of an increasing function on $x_1$, $u_{FI}$, that represents the utility she would get if she directly observed $x_1$. The highest possible first-period cutoff $\bar{m}_1$ is given by the value of $x_1^0$ at which $u_{FI}$ crosses the horizontal axis. As such, the investor would act for all values $x_1 \in [x_1^0, \bar{x})$ and not act for all values $x_1 \in [0, x_1^0)$ (incomplete project is stopped) or $x_1 \in [\bar{x}, k^t]$ (complete project is stopped). The highest possible cutoff for $\bar{m}_1$ effectively partitions the space $X_1$ in exactly these three sets. For lower equilibrium first-period cutoffs, the investor continues for some values of $x_1$ she would not continue if she had full information (that is, values of $u_{FI}$ that are negative). Therefore, her utility goes down the lower the first-period cutoff is. As follows, the investor's utility is increasing in the first-period equilibrium cutoffs.

***Claim* 8.** *Truth-telling every period is not an equilibrium strategy for the developer*

Using Claim 1, the investor plays a cutoff strategy on equilibrium. Having this in mind, we assume that the developer reports truthfully in the second period. Now assume that the developer reports truthfully in the first period as well. Let $m_1' < m_1''$ be two messages in period $t = 1$ such that the investor is willing to continue in both, i.e., $s^*(m_1') = s^*(m_1'') = 1$. First, we show that the cutoff in the second period is increasing in the message of the first period; that is, $\bar{m}_2(m_1)$ is increasing in $m_1$. Second, we use that the developer's utility in the following period is decreasing in the message he sent in the first period, i.e., $w_2(x_2, m_1'', \theta, x_2, m_2) < w_2(x_2, m_1', \theta, x_2, m_2)$. Therefore, since $s^*(m_1') = s^*(m_1'') = 1$, we get that $w_1(x_1, m_1'') < w_1(x_1, m_1')$. Thus, the developer is better off sending the lowest message in the first period, which will make the investor continue. Therefore, truthtelling in every period is not an equilibrium of this game.

Next, we will show that without the verifiability assumption, the developer prefers not to start the project under the following assumption:

**Assumption 3.1.** *If the only information that the investor can learn on its own is whether the job is done each period or not, then she prefers not to start the project.*

The intuition behind Assumption 3.1 is that if, in any period, the investor could observe whether the project was completed or not but no additional information, such as how big the advancements were, then the investor will not start the development at all. This assumption is relevant since it implies that the information released by the developer is useful. Without it, the developer could release no information at all, and the project would start, rendering the problem trivial.

***Claim* 9.** *Suppose that the developer is allowed to send any message at any history and suppose assumption 3.1 holds. Then, the investor does not start the project in equilibrium.*

The idea is very simple. Without the verifiability assumption, the developer can send any message he wants. So, if there exists a message that makes the investor continue, all developers without a full project would prefer to send that message and get the project continued with the hope that it will be completed in the future. In other words, there cannot be a separation between incomplete projects. But this implies that either all incomplete projects get continued or all incomplete projects get stopped. Assumption 3.1 guarantees that regardless of what is the case, the investor prefers not to start the development.

An economic consequence of Claim 9 is that requiring evidence during the development process of a project can allow for projects to be completed with positive probability when they would not be completed at all otherwise. As a consequence, it translates a feature of the product (that is partially verifiable) to the probability of the project's success.

# 4 Discussion

In this paper, we modeled the interaction between an investor and a developer as a game where both players get positive utility only when the project is completed. We introduced a verifiability assumption to the signaling game by limiting the set of messages the developer can send every period. Key features of the game, other than the verifiability assumption, were the finite horizon of the game (three-stage game) and the form of agents' utilities. These features led to the main result of the paper presented by the class of equilibria we discussed in the previous section. Note that a change in the key features listed above would change the developer's incentive to lie about the status of the project.

For future research, we consider a contract where the investor still covers the cost of production and gets positive utility only upon completion of the project. However, the developer gets paid through transfers, paid by the investor each period. The transfers are functions of the messages the developer sends every period. The agents agree on the transfers at the beginning of the investment. Thus, the investor chooses the transfers she pays to the developer, which maximizes her expected utility at $t = 0$.

Notice that if the outside option of the developer is public information, the problem is trivialized: The investor can pay the developer their outside option every period, as long as she can afford it. The developer is indifferent about whether to continue with the project or not in every period. Alternatively, we consider the above case where the developer's outside option is private information.

General Conjecture: The analysis done in Claim 1 should still hold with the new setting. Moreover, in the advanced stages of the game, where the completion of the project is close enough, the investor will find it very costly to stop the project and will pay the developer high transfers -since she does not know his outside option- to avoid early termination of the contract. Notice that a key assumption here is that the investor gets a negative payoff, which is the sum of the continuation costs if the contract is stopped before the completion of the project. However, the developer gets a strictly positive payoff in that scenario. Therefore, the developer's "power" increases with time until he reaches the point where the investor terminates the contract because she believes, with a high enough probability, that the project is completed. Thus, the verifiability assumption plays a crucial role in reducing the "milking" that the developer tries to do using the messages he sends every period. This idea would be better understood in an infinite horizon model, where there is no "due date" to terminate the contract.

More specifically, we consider the outside option of the developer at a certain period to be the accomplishments of the project in the previous period. Therefore, a completed project is owned by the investor, but a partially completed project is owned by the developer. To illustrate the idea, consider the example of an author who is writing a book and wants to publish it in a publishing company. The company covers the costs and pays the author transfers to finish writing the book. However, once the book is complete and the author gets all the payment, the company has all the copyrights.

It is worth mentioning some preliminary results we have obtained for the special case in which the outside option is private information of the developer each period, and it is modeled as a random variable that is i.i.d. over time. In this case, the model becomes analogous to a model of multidimensional screening, where the investor must screen over both the current status of the project as well as the value of the outside option. The presence of the outside option generates incentives for the developer to leave the project at any point in time when his outside option is high enough. In our specification, the investor has no way to screen high values of outside option from low values. Therefore, any developer of a low outside option value can argue that he is of high outside option value and extracts all potential surplus from the investor, which is a striking difference from the case in which the outside option was known. We leave the sketch of the proof for the second period in the appendix, which we believe generalizes to the first period in expectations.

Appendix

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# 5 Appendix

## 5.1 Lemmas

**Lemma 1.** *Assume $f_{t,\theta}$ satisfies MLP in $(\theta, x_1)$ for $t \in \{2,3\}$ and that the developer's strategy is $(\sigma^*_{\bar{m}_1}, \sigma^*_2)$. Then, $d\lambda^*(x_1,\theta,x_2|m_1,m_2)$ satisfies MLP in $(x_2,\theta)$.*

*Proof.* To prove Lemma 1, fix $\bar{m}_1$, $\sigma^*_{\bar{m}_1}$ and $\sigma^*_2$. Recall that the density of $\lambda^*_2$ is given by

$$d\lambda^*(x_1,\theta,x_2|m_1,m_2) = \begin{cases} 0 & \text{if } m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1] \ \& \ x_1 \neq m_1 \\ \frac{p(\theta)f_{2,\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m_2)}{\int_H p(\theta)f_{2,\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m_2)dxd\theta} & \text{if } m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1] \ \& \ x_1 = m_1 \\ \\ \frac{f(x_1)p(\theta)f_{2,\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m_2)}{\int_{A(m_1)} f(x_1)p(\theta)f_{2,\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m_2)dxd\theta} & \text{if } m_1 \in [\bar{m}_1,\bar{x}) \ \& \ x_1 \in [m_1,\bar{x}) \\ 0 & \text{if } m_1 \in [\bar{m}_1,\bar{x}) \ \& \ x_1 \notin [m_1,\bar{x}) \end{cases}$$

where $A(m_1) \equiv \{(x_1,\theta,x_2) \in X_1 \times \Theta \times X_2 | x_1 \in [m_1,\bar{x})\}$, and $H \equiv \Theta \times X_2$

Let $m''_2 > m'_2$. If $m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1]$ & $x_1 \neq m_1$ or if $m_1 \in [\bar{m}_1,\bar{x})$ & $x_1 \notin [m_1,\bar{x})$, then $\frac{d\lambda^*(x_1,\theta,x_2|m_1,m''_2)}{d\lambda^*(x_1,m_1,\theta,m'_2-x_1,m'_2)}$ has undetermined value. However, given $\bar{m}_1$ and $\sigma^*_{\bar{m}_1}$, those are zero probability events. Moreover, if $m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1]$ & $x_1 = m_1$, then

$$\begin{aligned} \frac{d\lambda^*(x_1,\theta,x_2|m_1,m''_2)}{d\lambda^*(x_1,\theta,x_2|m_1,m'_2)} &= \frac{\frac{p(\theta)f_{2,\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m''_2)}{\int_H f(x_1)p(\theta)f_{2,\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m''_2)dxd\theta}}{\frac{p(\theta)f_{2,\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m'_2)}{\int_H f(x_1)p(\theta)f_{\theta}(x_2)\sigma^*_2(x_1,\theta,x_2|m_1,m'_2)dxd\theta}} \\ &= \frac{f_{2,\theta}(m''_2-x_1)}{f_{2,\theta}(m'_2-x_1)} \frac{\int_H f(x_1)p(\theta)f_{2,\theta}(m'_2-x_1)dxd\theta}{\int_H f(x_1)p(\theta)f_{2,\theta}(m''_2-x_1)dxd\theta} \end{aligned}$$

The second fraction is constant in $\theta$ and the first fraction is increasing in $\theta$ since $f_{t,\theta}$ satisfies MLP in $(\theta,x_2)$. The proof is analogous if $m_1 \in [\bar{m}_1,\bar{x})$ & $x_1 \in [m_1,\bar{x})$. □

**Lemma 2.** *Assume that the developer's strategy is $(\sigma^*_{\bar{m}_1}, \sigma^*_2)$. Then, $d\lambda^*(x_1|m_1)$ satisfies MLP in $(x_1,m_1)$ for $m_1 \in [0,\bar{x})$.*

*Proof.* Recall the beliefs of the investor in period $t = 1$ are given by

$$\lambda^*(\cdot|m_1) = \delta_{m_1} \quad \text{if } m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1]$$

if $m_1 \in [0,\bar{m}_1) \cup [\bar{x},k_1]$, while if $m_1 \in [\bar{m}_1,\bar{x})$ its density is given by

$$d\lambda^*(x_1|m_1) = \begin{cases} \frac{f(x_1)}{F(\bar{x})-F(m_1)} & \text{if } m_1 \in [\bar{m}_1,\bar{x}) \ \& \ x_1 \in [m_1,\bar{x}) \\ 0 & \text{if } m_1 \in [\bar{m}_1,\bar{x}) \ \& \ x_1 \notin [m_1,\bar{x}) \end{cases}$$

Let $m_1' < m_1'' < \bar{x}$. We will split the proof depending on the value of $m_1'$ and $m_1''$ compared to $\bar{m}_1$.

**Case 1:** if $m_1' < m_1'' < \bar{m}_1$, then

$$\frac{d\lambda^*(x_1|m_1'')}{d\lambda^*(x_1|m_1')} = \begin{cases} \frac{0}{0} & \text{if } x_1 < m_1' \\ 0 & \text{if } x_1 = m_1' \\ \frac{0}{0} & \text{if } x_1 \in (m_1', m_1'') \\ \infty & \text{if } x_1 = m_1'' \\ \frac{0}{0} & \text{if } x_1 > m_1'' \end{cases}$$

**Case 2:** if $m_1' < \bar{m}_1 \leq m_1''$, then

$$\frac{d\lambda^*(x_1|m_1'')}{d\lambda^*(x_1|m_1')} = \begin{cases} \frac{0}{0} & \text{if } x_1 < m_1' \\ 0 & \text{if } x_1 \in [m_1', m_1'') \\ \frac{0}{0} & \text{if } x_1 \in (m_1', m_1'') \\ \infty & \text{if } x_1 \in [m_1'', k_1] \end{cases}$$

**Case 3:** if $\bar{m}_1 \leq m_1' < m_1''$, then

$$\frac{d\lambda^*(x_1|m_1'')}{d\lambda^*(x_1|m_1')} = \begin{cases} \frac{0}{0} & \text{if } x_1 < m_1' \\ 0 & \text{if } x_1 \in [m_1', m_1'') \\ \frac{0}{0} & \text{if } x_1 \in (m_1', m_1'') \\ \frac{F(\bar{x})-F_1(m_1')}{F(\bar{x})-F_1(m_1'')} & \text{if } x_1 \in [m_1'', k_1] \end{cases}$$

Note that, given $\sigma^*_{\bar{m}_1}$, the value for which the ration is undetermined are zero probability events, thus we can exclude them from the analysis as, when integrating using either $\lambda^*(\cdot|m'')$ or $\lambda^*(\cdot|m'')$, those values $x_1$ will not affect the expectation for each distribution, so we can exclude them from the analysis. Then, notice that in the tree cases, for the regions in which the rations are well defined, they go up as $x_1$ goes up. Therefore $\lambda^*(x_1|m_1)$ satisfies MLP in $(x_1, m_1)$. □

**Lemma 3.** *The developer's utility is decreasing in the first message $m_1$ he sends that makes the investor continue.*

*Proof.* We want to show show $w_1(x_1, \cdot)$ is decreasing in $m_1$. Let $m_1' < m_1''$ such that $s^*_{\bar{m}_1}(m_1'') = s^*_{\bar{m}_1}(m_1') = 1$. Recall that

$$w_2(x_1, m_1, \theta, x_1, x_1 + x_2) = \left[s^*_{\bar{m}_2}(m_1, m_2)(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) + (1 - s^*_{\bar{m}_2}(m_1, m_2))\mathbb{1}_{x_1, x_2 \geq \bar{x}}\right] D$$

$$w(x_1, m_1) = s^*_{\bar{m}_1}(m_1) \int_{\theta, x_2} w_2(x_1, m_1, \theta, x_2, x_1 + x_2) f_{2,\theta}(x_2) p(\theta) dx_2 d\theta$$

Then, assuming $s^*_{\bar{m}_1}(\cdot)$, we get

$$w_2(x_1, m_1'', \theta, x_1, m_2) = \begin{cases} 0 & \text{if } m_2 < \bar{m}_2(m_1') \\ 0 & \text{if } \bar{m}_2(m_1') \leq m_2 < \bar{m}_2(m_1'') \\ \big[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)\big] D & \text{if } \bar{m}_2(m_1'') \leq m_2 < \bar{x} \\ D & \text{if } m_2 \geq \bar{x} \end{cases}$$

$$w_2(x_1, m_1', \theta, x_1, m_2) = \begin{cases} 0 & \text{if } m_2 < \bar{m}_2(m_1') \\ \big[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)\big] D & \text{if } \bar{m}_2(m_1') \leq m_2 < \bar{m}_2(m_1'') \\ \big[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)\big] D & \text{if } \bar{m}_2(m_1'') \leq m_2 < \bar{x} \\ D & \text{if } m_2 \geq \bar{x} \end{cases}$$

Clearly, $w_2(x_1, m_1, \theta, x_1, m_2)$ is decreasing in $m_1$. Thus, the developer prefer to deviate to the lowest message $m_1$ such that $s^*_{\bar{m}_1}(m_1) = 1$.[13] □

**Lemma 4.** *$u_{FI}(x_1, c)$ is increasing in $x_1$ where*

$$\begin{aligned} u_{FI,1}(x_1, c) = -c_1 + \mathbb{1}_{x_1 \geq \bar{x}} R + (1 - \mathbb{1}_{x_1 \geq \bar{x}}) \int_\theta \Big[ & \int_0^{\bar{m}_2 - x_1} 0 f_{2,\theta}(x_2) dx_2 \\ & + \int_{\bar{m}_2 - x_1}^{\bar{x} - x_1} [R(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) - c_2] f_{2,\theta}(x_2) dx_2 \\ & + \int_{x_2 \geq \bar{x} - x_1} R f_{2,\theta}(x_2) dx_2 \Big] p(\theta) d\theta \end{aligned}$$

*Proof.* Define the function $\tilde{u}_1$ as follows

$$\tilde{u}_1(x_1, x_2, \theta) = \begin{cases} 0 & \text{if } x_2 \in [0, \bar{m}_2 - x_1) \\ R(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) - c_2 & \text{if } x_2 \in [\bar{m}_2 - x_1, \bar{x} - x_1) \\ R & \text{if } x_2 \in [\bar{x} - x_1, k_2] \end{cases}$$

First, notice that $R(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) - c_2$ is increasing in $x_2$. It follows that $\tilde{u}_1$ is increasing in $x_2$. The reason is that, by definition of $\bar{m}_1$, $R(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) - c_2$ is greater than zero in $[\bar{m}_2 - x_1, \bar{x} - x_1)$, which is clearly smaller than $R$.

[13] This analysis fails if the state of the world does not change. Because in that case, $u_2(m_1, m_2)$ is independent of $m_1$.

Moreover, notice that $R(1-F_{3,\theta}(\bar{x}-x_1-x_2))-c_2$ is also increasing in $x_1$. It is direct to see that the function $u_{FI}$ in increasing in $x_1$. As $x_1$ goes up to $x_1'$, some values of $x_2$ that were originally in $[0,\bar{m}_2-x_1)$ leave the interval $[0,\bar{m}_2-x_1')$ to go to the interval $[\bar{m}_2-x_1',\bar{x}-x_1')$. By doing so, the value of the integrand function $\tilde{u}_1$ for that $x_2$ moves from 0 to $R(1-F_{3,\theta}(\bar{x}-x_1'-x_2))-c_2 \geq R(1-F_{3,\theta}(\bar{x}-x_1-x_2))-c_2 \geq 0$. Meanwhile some values $x_2$ that were originally in $[\bar{m}_2-x_1,\bar{x}-x_1)$ move to $[\bar{x}-x_1',k_2]$. So the value of the integrand function moves from $R(1-F_{3,\theta}(\bar{x}-x_1-x_2))-c_2$ to $R$, which is greater.

As all values of $x_2$ now integrate the same or a higher value, $u_{FI}$ is increasing in $x_1$. □

**Lemma 5.** *Assuming the behavior of the developer in both periods, $u(m_1,c)$ is continuous and increasing in $m_1$.*

*Proof.* To proof this claim, notice that

$$u_1(m_1,c)=\int_x u_{FI}(x_1,c)d\lambda_1^*(x_1|m_1)dx$$

where $u_{FI}$ is defined in lemma 4. By lemma 4, $u_{FI}(x_1,c)$ is increasing in $x_1$. By lemma 2, $d\lambda_1^*$ is MLP in $(x_1,m_1)$. Therefore, $u_1(m_1,c)$ is increasing in $m_1$. □

**Lemma 6.** *$\mu_1(m_1)=u_1(m_1,c)-u_1(m_1,b)$ is increasing in $m_1$, where*

$$\begin{aligned} u_1(m_1,c)= & -c_1+\int_{\theta,\bar{m}_2<x_1+x_2<\bar{x}}\big[R[1-F_{3,\theta}(\bar{x}-x_1-x_2)]-c_2\big]d\lambda_1^*(x_1|m_1)p_\theta f_{2,\theta(x_2)}d\theta dx \\ & +\int_{\theta,x_1+x_2>\bar{x}} R d\lambda_1^*(x_1|m_1)p_\theta f_{2,\theta(x_2)}d\theta dx \end{aligned}$$

*and*

$$u_1(m_1,b)=\int_{x_1\geq\bar{x}} R d\lambda_\theta^*(x_1|m_1)dx$$

*Proof.* Notice that

$$\begin{aligned} u_1(m_1,c)= & -c_1+\int_{\theta,\bar{m}_2<x_1+x_2<\bar{x}}\big[R[1-F_{3,\theta}(\bar{x}-x_1-x_2)]-c_2\big]d\lambda_1^*(x_1|m_1)p_\theta f_{2,\theta(x_2)}d\theta dx \\ & +\int_{\theta,x_1+x_2>\bar{x}} R d\lambda_1^*(x_1|m_1)p_\theta f_{2,\theta(x_2)}d\theta dx \\ = & -c_1+\int_\theta\int_{x_2<\bar{x}}\int_{x_1=\bar{m}_2-x_2}^{\bar{x}-x_2}[R(1-F_{3,\theta}(\bar{x}-x_1-x_2))-c_2]d\lambda_1^*(x_1|m_1)p_\theta f_{2,\theta(x_2)}dx d\theta \\ & +\int_{x_1\geq\bar{x}} R d\lambda_1^*(x_1|m_1)dx+\int_\theta\int_{x_1<\bar{x}} R d\lambda^*(m_1|x_1)p_\theta\big[1-F_{2,\theta}(\bar{x}-x_1)\big]dx d\theta \end{aligned}$$

It follows that $\mu_1(m_1) = u_1(m_1, c) - u_1(m_1, d)$ becomes

$$\mu_1(m_1) = -c_1 + \int_\theta \int_{x_2<\bar{x}} \int_{x_1=\bar{m}_2-x_2}^{\bar{x}-x_2} [R(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) - c_2] d\lambda_1^*(x_1|m_1) p_\theta f_{2,\theta(x_2)} dx d\theta$$
$$+ \int_\theta \int_{x_1<\bar{x}} R d\lambda^*(x_1|m_1) p_\theta \big[1 - F_{2,\theta}(\bar{x} - x_1)\big] dx d\theta$$

Since $f_{3,\theta}$ satisfies MLP in $(\theta, x_3)$, then $[1-F_{3,\theta}(\bar{x}-x_1-x_2)]$ in increasing in $x_1$. Together with the result of lemma 2, we get that the first term of the equation above is increasing in $m_1$. Since $f_{2,\theta}$ satisfies MLP in $(\theta, x_2)$, then $[1 - F_{2,\theta}(\bar{x} - x_1)]$ in increasing in $x_1$. Together with the result of lemma 2, we get that the second term of the equation above is increasing in $m_1$. Therefore, since $\mu_1(x_1)$ is increasing in $m_1$ □

## 5.2 Claims

***Proof of Claim 1***. We will do the proof for period $t = 2$. The proof for $t = 1$ is similar. For a fixed history $h^2 = (x_1, \theta, x_2)$, assume that on equilibrium, the developer sends message $m_2'$ with positive probability, i.e., $\sigma^*(m_2'|h^2) > 0$. By the verifiability assumption, $x_1 + x_2 \geq m_2'$. WLOG, assume $x_1+x_2 < \bar{x}$.[14] For any $m_2 \in [0, x_1+x_2]$, if $s^*(m_1, m_2) = 0$, then $w_2(h^2, m_2) = 0$. Otherwise, if $s^*(m_1, m_2) = 1$, then $w_2(h^2, m_2) > 0$.
Assume $s^*(m_1, m_2') = 0$. If $\exists\ m_2'' \in [0, m_2')$ such that $s^*(m_1, m_2'') = 1$. Notice that by the verifiability assumption, the developer can send $m_2''$ since it is less than $x_1 + x_2$. However, if $\sigma^*(m_2''|h^2) > 0$, then it has to be that $\sigma^*(m_2'|h^2) = 0$. Which is a contradiction with $\sigma^*(m_2'|h^2) > 0$. Thus, in equilibrium $\exists\ m_2^*$, such that,

$$s^*(m_2) = \begin{cases} 0 & \text{if } m_2 \in [m_2^*, \bar{x}) \\ 1 & \text{if } m_2 \notin [m_2^*, \bar{x}) \end{cases}$$

□

From claim 1 we conclude the behavior of the investor in this game that are driven because of the verifiability assumption and the way the developer gets payed; which makes him always prefer a strategy that will lead the investor to continue rather than stop the investment whenever the project is not completed.

Fixing the investor's behavior in equilibrium to be $(1, s^*_{\bar{m}_1}, s^*_{\bar{m}_2})$ fo some $\bar{m}_1$ and $\bar{m}_2$. We proceed to study how the developer would behave in equilibrium, then we characterize the thresholds that the investor will play using backward induction.

The following claim shows that given the investor's equilibrium strategy, the developer finds it optimal to report truthfully in the second to last period of the game.

[14] Otherwise, the developer would report truthfully and the investor would stop.

***Proof of Claim 2.*** Let $\bar{m}_1$ be fixed. Suppose the investor plays $(1, s^*_{\bar{m}_1}, s^*_{\bar{m}_2})$, where $\mu_2(\bar{m}_1, \bar{m}_2) = 0$. We will show that a developer's best response in the second period is to play $\sigma^*_2$. Recall that the developer's utility in $t = 2$ is

$$w_2(x_1, m_1, \theta, x_1, x_2) = \left[s^*_{\bar{m}_2}(m_1, m_2)(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) + (1 - s^*_{\bar{m}_2}(m_1, m_2))\mathbb{1}_{x_1, x_2 \geq \bar{x}}\right] D$$

We will split the proof into two cases depending on the value of $x_1 + x_2$.
**Case 1:** $x_1 + x_2 \geq \bar{x}$: Regardless of the strategy of the investor, the utility of the developer is equal to $D$. Therefore, reporting truthfully, i.e., playing $\sigma^*_2$, is optimal.
**Case 2:** $x_1 + x_2 < \bar{x}$: In this case, using the verifiability assumption,

$$w_2(x_1, m_1, \theta, x_1, x_2) = \begin{cases} \left[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)\right] D & \text{if } m_2 \in [\bar{m}_2, \bar{x}) \\ 0 \quad \text{otherwise} \end{cases}$$

If $x_1 + x_2 \geq \bar{m}_2$, then the developer is indifferent between reporting the truth and sending any message $m_2 \in [\bar{m}_2, x_1 + x_2]$, each strictly preferred to any message in $m'_2 \in [0, \bar{m}_2]$
If $x_1 + x_2 < \bar{m}_2$, the developer is indifferent between all the messages $m_2 \in [0, x_1 + x_2]$
Therefore, a developer's best response for a cutoff strategy in the second period is $\sigma^*_2$. □

***Proof of Claim 3.*** Let $\bar{m}_1$ be fixed. Suppose that the investor starts the development in period $t = 0$ and plays $s^*_{\bar{m}_1}$ for a fixed $\bar{m}_1$. Suppose also that the developer plays according to $\sigma^*_{\bar{m}_1}$ in period 1.

Recall the utility of the investor in period $t = 2$ depending on the decision of continuing or stopping:

$$u_2(m_1, m_2, c) = -c_2 + R \int_H [1 - F_{3,\theta}(\bar{x} - x_1 - x_2)] \lambda^*(x_1, \theta, x_2 | m_1, m_2) dx d\theta$$
$$u_2(m_1, m_2, b) = R \int_{x_1 + x_2 \geq \bar{x}} \lambda^*(x_1, \theta, x_2 | m_1, m_2) dx d\theta$$

Then,

$$\begin{aligned}\mu_2(m_1, m_2) &= -c_2 + R \int_{x_1 + x_2 < \bar{x}} \int_\theta [1 - F_{3,\theta}(\bar{x} - x_1 - x_2)] \lambda^*(x_1, \theta, x_2 | m_1, m_2) dx d\theta \\ &= -c_2 + R \int_{x_1 + x_2 < \bar{x}} \int_\theta [1 - F_{3,\theta}(\bar{x} - m_2)] \lambda^*(x_1, \theta, x_2 | m_1, m_2) dx d\theta \geq 0\end{aligned}$$

Note that $\mu_2$ is the integral of a continuous function, thus continuous. Now let $m'_2 < m''_2$.

We have that

$$\begin{aligned}
\mu_2(m_1, m_2') &= -c_2 + R\int_{x_1}\int_{x_2}\int_{\theta}(1 - F_{3,\theta}(\bar{x} - m_2'))\lambda^*(x_1, m_1, \theta, m_2' - x_1, m_2')dxd\theta \\
&\leq -c_2 + R\int_{x_1}\int_{x_2}\int_{\theta}(1 - F_{3,\theta}(\bar{x} - m_2''))\lambda^*(x_1, m_1, \theta, m_2' - x_1, m_2')dxd\theta \\
&\leq -c_2 + R\int_{x_1}\int_{x_2}\int_{\theta}(1 - F_{3,\theta}(\bar{x} - m_2''))\lambda^*(x_1, m_1, \theta, m_2'' - x_1, m_2'')dxd\theta \\
&= \mu_2(m_1, m_2'')
\end{aligned}$$

where the first inequality follows from the fact that $(1 - F_{3,\theta}(\bar{x} - m_2))$ is increasing in $m_2$ and the second inequality follows from the that $(1 - F_{3,\theta}(\bar{x} - m_2))$ is increasing in $\theta$ and $\lambda^*$ has the MLP in $(m_2, \theta)$. Therefore $\mu_2$ is continuous in both arguments and increasing in the second. □

For the problem to have meaning, we can assume that for every $m_1$, $\mu(m_1, \cdot)$ changes sign in $[0, \bar{x})$. Claim 3 implies that for every $m_1$, there exists $\bar{m}_2$ that satisfies $\mu_2(m_1, \bar{m}_2) = 0$

From this point on, we fix the behavior of both players in period $t = 2$. The following two claims present the behavior of the players in period $t = 1$, assuming the investor starts the development.

***Proof of Claim 4***. Suppose that the investor's strategy is $(1, s^*_{\bar{m}_1}, s^*_{\bar{m}_2})$ where $\bar{m}_1 \in M_1$ is fixed. Suppose also that the developer plays according to $\sigma_2^*$ in period $t = 2$. If $x_1 \geq \bar{x}$ the developer is indifferent in all messages as his utility is fixed at $D$, so in particular, following $\sigma^*_{\bar{m}_1}$ is optimal. Now assume $x_1 < \bar{x}$. Recall the developer's strategy in period $t = 1$;

$$\begin{aligned}
w(x_1, m_1) &= s^*_{\bar{m}_1}(m_1)\int_{\theta, x_2} w_2(x_1, m_1, \theta, x_2, x_1 + x_2)f_{2,\theta}(x_2)p(\theta)dx_2d\theta + (1 - s^*_{\bar{m}_1}(m_1))0 \\
&= s^*_{\bar{m}_1}(m_1)\int_{\theta}\Bigg(\int_{x_2=0}^{\bar{m}_2 - x_1} 0 f_{2,\theta}(x_2)p(\theta)dx_2 \\
&\quad + \int_{\bar{m}_2 - x_1}^{\bar{x} - x_1} D[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)]f_{2,\theta}(x_2)p(\theta)dx_2 \\
&\quad + \int_{x_2 \geq \bar{x} - x_1} D f_{2,\theta}(x_2)p(\theta)dx_2\Bigg)d\theta \\
&= s^*_{\bar{m}_1}(m_1)D\int_{\theta}\Bigg(\int_{\bar{m}_2 - x_1}^{\bar{x} - x_1}[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)]f_{2,\theta}(x_2)p(\theta)dx_2 \\
&\quad + \int_{x_2 \geq \bar{x} - x_1} f_{2,\theta}(x_2)p(\theta)dx_2\Bigg)d\theta
\end{aligned}$$

Notice that the integral is strictly positive and constant in $m_1$. Therefore, the developer is indifferent between sending any message $m_1$ such that $s^*_{\bar{m}_1}(m_1) = 1$, all of which are preferred to messages $m_1$ such that $s^*_{\bar{m}_1}(m_1) = 0$. In particular, since the strategy of the investor is

$$s^*_{\bar{m}_1}(m_1) = \begin{cases} 0 & \text{if } m_1 \in [0, \bar{m}_1) \cup [\bar{x}, k_1] \\ 1 & \text{if } m_1 \in [\bar{m}_1, \bar{x}) \end{cases}$$

if $x_1 \in [\bar{m}_1, \bar{x})$ it is optimal for the developer to send message $\bar{m}_1$ with probability 1, which he can do by the verifiability assumption. However, if $x_1 < \bar{m}_1$, by the verifiability assumption, the developer will not be able to send any message $m_1$ such that $s^*_{\bar{m}_1}(m_1) = 1$. Thus, he will be indifferent between sending any message $m_1 \leq x_1$. So, reporting truthfully in this case is optimal for the developer. Therefore,

$$\sigma^*_{\bar{m}_1}(\cdot|x_1) = \begin{cases} \delta_{x_1} & \text{if } x_1 \in [0, \bar{m}_1) \cup [\bar{x}, k_1] \\ \delta_{\bar{m}_1} & \text{if } x_1 \in [\bar{m}_1, \bar{x}). \end{cases}$$

is a best response for the developer to $s^*_1$ in period $t = 1$. □

In the following claim 5, we show that on equilibrium, the investor's threshold and the developer's cutoff in the first period are the same.

***Proof of Claim 5.*** Note that the inequalities $(*)$ makes sense because, by lemma 6, $\mu_1(m_1, c)$ in increasing in $m_1$. From Claim1, we know that $\exists \tilde{m}_1$ such that $\forall m_1 < \tilde{m}_1$, the investor chooses to stop the game, and $\forall m_1 \geq \tilde{m}_1$, the investor chooses to continue. This forces $\tilde{m}_1$ to satisfy the inequalities $(*)$. Thus, if $\bar{m}_1$ satisfies $(*)$, then $BR_I(\sigma_{\bar{m}_1}) = s^*_{\bar{m}_1}$. Also, we conclude from Claim 4 that $BR_D(s^*_{\bar{m}_1}) = \sigma_{\bar{m}_1}$. Therefore, when $\bar{m}_1$ satisfies $(*)$, $BR_D\left(BR_I(\sigma_{\bar{m}_1})\right) = \sigma_{\bar{m}_1}$. □

***Proof of Claim 6.*** Let $x^0_1$ be defined implicitly as $u_{FI}(x^0_1) = 0$ and let $\bar{m}_1(c_0)$ be defined by $c_0 = u(\bar{m}_1(c_0), c)(F_1(\bar{x}) - F(\bar{m}_1(c_0))) + R(1 - F(\bar{x}))$. Assume the investor and developer will play with a cutoff strategy at time 1 as discussed before. Then, the expected utility of starting at time zero depends on the cutoff in period one in the following way

$$\begin{aligned} u_0(\bar{m}) &= -c_0 + (F(\bar{x}) - F(\bar{m}))u_1(\bar{m}, c) + (1 - F(\bar{x}))R \\ &= -c_0 + (F(\bar{x} - F(\bar{m})) \int_{x_1 \geq \bar{m}}^{\bar{x}} u_{FI}(x_1, c) \frac{f(x_1)}{F(\bar{x}) - F(\bar{m})} dx_1 + (1 - F(\bar{x}))R \\ &= -c_0 + \int_{x_1 \geq \bar{m}}^{\bar{x}} u_{FI}(x_1, c) f(x_1) dx_1 + (1 - F(\bar{x}))R \end{aligned}$$

Notice that $u_{FI}(\cdot, c)$ is negative below $x_1^0$ and positive above. Then, $u_0(\cdot)$ is increasing from $[0, x_1^0]$ and decreasing from $[x_1^0, \bar{x}]$.

For the problem to be interesting, we assume that $u_0(0) < 0$ and $u_0(x_1^0) > 0$. Since $u_0(.)$ is continuous and increasing in $[0, x_1^0]$, there exists $0 < \bar{m}_1(c_0) < x_1^0$ such that $u_0(\bar{m}_1(c_0)) = 0$.

Now we show that for any $\bar{m} \in [\bar{m}(c_0), x_1^0]$, $(\sigma_{,\bar{m}}(x_1, m_1), s_{,\bar{m}}(m_1))$ is an equilibrium. Fix any $\bar{m} \in [\bar{m}(c_0), x_1^0]$, so that $u_0(\bar{m}) \geq 0$. Suppose that the investor receives a message $m' < \bar{m}$. Given the strategy of the developer, he would assign probability 1 to $x_1 = m' < \bar{m} < x_1^0 < \bar{x}$. Clearly, $u_1(m', b) = 0$. Then, $\mu_1(m') = u_1(m', c) - u_1(m', b) = u_1(m', c) = u_{FI}(m', c) < 0$ so the investor will stop. Alternatively, suppose the investor receives a message $m'' \in [\bar{m}, \bar{x})$. Then, $u_1(m'', b) = 0$ and

$$\mu_1(m'') = u_1(m'', c) - u_1(m'', b) = u_1(m'', c) \geq u_1(\bar{m}, c) \geq u_1(\bar{m}(c_0), c) = \frac{c_0 - R(1 - F(\bar{x}))}{F(\bar{x}) - F(\bar{m}(c_0)} > 0$$

where the first inequality follows lemma 5, the second one by the previous analysis and the last one by definition of $\bar{m}(c_0)$. It follows then that the investor will continue. The fact that the developer will not deviate from the strategy follows from claim 2 and claim 4. Finally, we show that any $\bar{m} \notin [\bar{m}(c_0), x_1^0]$ will not work as a cutoff. Suppose first that $\bar{m} > x_1^0$. Then, let $m'' \in (x_1^0, \bar{m})$. By $u_1(\cdot, c)$ increasing, we would have that the developer would choose to continue, contradicting the equilibrium.

For $\bar{m} < \bar{m}(c_0)$, we have that $u_0(\bar{m}) < 0$ so the investor will prefer not to start the project. □

Notice that for any of cutoff equilibria with $\bar{m} \in (\bar{m}(c_0), x_1^0]$, there exist $\epsilon > 0$ such that it is still profitable to the investor to let the developer lie by $\epsilon$ above. However, this is not the case for $\bar{m}(c_0)$

***Proof of Claim 7***. We proceed in the proof using the implicit function theorem. But first, we will need the following two lemmas:

**Lemma 7.** $\Pr(\theta | x_1, x_2, \theta, m_1, m_2)$ *satisfies MLP in* $(\theta, x_2)$

**Lemma 8.** $\Pr(x_1 | m_1, m_2)$ *satisfies MLP in* $(x_1, m_1)$

Recall that given $\bar{m}_1$, we define $\bar{m}_2$ by the following equation

$$\begin{aligned}
&\mu_2(m_1, mu_2) = u_2(m_1, mu_2, c) - u_2(m_1, mu_2, b) = 0 \\
&- c_2 + R \int_{x_1} \int_{x_2} \int_{\theta} (1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) d\lambda_2^*(x_1, \theta, x_2 | m_1, m_2) dx d\theta = 0 \\
&- c_2 + R \int_{x_1} \int_{x_2} \int_{\theta} (1 - F_{3,\theta}(\bar{x} - m_2)) d\lambda_2^*(x_1, \theta, x_2 | m_1, m_2) dx d\theta = 0
\end{aligned}$$

Proof of lemma 7 is direct from the definition and the assumption that $f_\theta(x_2)$ satisfies MLP in $(\theta, x_2)$.

Proof of lemma 8 is a bit more involved. Let $m_1'' > m_1'$.

$$\frac{\Pr(x_1|m_1'', m_2)}{\Pr(x_1|m_1', m_2)} = \begin{cases} \frac{0}{0} & \text{if } 0 < x_1 < m_1' \\ 0 & \text{if } m_1' \leq x_1 < m_1'' \\ \frac{\int_{x_1 \geq m_1'} f(x_1) \int_\theta p(\theta) f_{2,\theta}(m_2 - x_1) d\theta dx_1}{\int_{x_1 \geq m_1''} f(x_1) \int_\theta p(\theta) f_{2,\theta}(m_2 - x_1) d\theta dx_1} & \text{if } m_1'' \leq x_1 \end{cases}$$

Notice that the region where $x_1 < m_1$ is assigned zero probability because of the verifiability assumption. Also, $\frac{\int_{x_1 \geq m_1'} f(x_1) \int_\theta p(\theta) f_{2,\theta}(m_2 - x_1) d\theta dx_1}{\int_{x_1 \geq m_1''} f(x_1) \int_\theta p(\theta) f_{2,\theta}(m_2 - x_1) d\theta dx_1} \geq 0$. Therefore, $\Pr(x_1|m_1, m_2)$ satisfies MLP in $(x_1, m_1)$.

Having these two lemmas in mind, we proceed to prove our claim using the implicit function theorem, we get

$$\frac{\partial \bar{m}_2}{\partial \bar{m}_1} = -\frac{\frac{\partial u_2}{\partial \bar{m}_1}}{\frac{\partial u_2}{\partial \bar{m}_2}}$$

In the proof of proposition 1, we showed that $\frac{\partial u_2}{\partial \bar{m}_2} \geq 0$. What remains to show is that $\frac{\partial u_2}{\partial \bar{m}_1} \leq 0$. We start by decomposing $\lambda^*$ as follows. For very small intervals of size $dx_1$, $dx_2$ and $d\theta$ around $x_1$,$x_2$ and $\theta$ we have:

$$\begin{aligned} \lambda^*(x_1, m_1, \theta, x_2, m_2) &= \Pr(x_1, x_2, \theta | m_1, m_2) \\ &= \Pr(x_1|m_1, m_2) \Pr(x_2|x_2, m_1, m_2) \Pr(\theta|x_1, x_2, m_1, m_2) \\ &= \frac{f(x_1)\sigma(x_1, m_1) \int_\theta p(\theta) f_{2,\theta}(m_2 - x_1)}{\int_{x_1} f(x_1)\sigma(x_1, m_1) \int_\theta p(\theta) f_\theta(m_2 - x_1)} dx_1 * 1 dx_2 \frac{p(\theta) f_{2,\theta}(m_2 - x_1)}{\int_\theta p(\theta) f_\theta(m_2 - x_1)} d\theta \end{aligned}$$

Then, using the result that $m_2$ will be truthtelling, i.e. $m_2 = x_1 + x_2$, $u_2(m_1, m_2)$ can be written as

$$\mu_2(m_1, m_2) = -c_2 + \int_{x_1} \underbrace{\int_\theta [1 - F_{3,\theta}(\bar{x} - m_2)] \Pr(\theta|x_1, m_2 - x_1, \theta, m_1, m_2) d\theta}_{\equiv A(x_1, m_1, m_2)} \Pr(x_1|m_1 m_2) dx_1$$

We call the term with under-brackets $A(x_1, m_1, m_2)$. Notice that, $1 - F_{3,\theta}(\bar{x} - m_2)$ in increasing in $\theta$ as $F_{3,\theta}$ is decreasing in $\theta$ since it is by assumption that $f_{3,\theta}(.)$ satisfies MLP in $(\theta, .)$ and so it is first order stochastically ordered in the same variables. Since $\Pr(\theta|x_1, m_2 - x_1, \theta, m_1, m_2)$ satisfies MLP in $(\theta, m_2 - x_1)$, we get that $A(x_1, m_1, m_2)$ is increasing in $m_2 - x_1$. Thus, it is increasing in $m_2$ and decreasing in $x_1$ for a fixed $m_1$.

Therefore, since $\Pr(x_1|m_1, m_2)$ satisfies MLP in $(x_1, m_1)$, then

$$\mu_2(m_1, m_2) = -c_2 + \int_{x_2} A(x_1, m_1, m_2) \Pr(x_1|m_1, m_2) dx_1$$

is decreasing in $m_1$. Therefore, $\frac{\partial \bar{m}_2}{\partial \bar{m}_1} = -\frac{\frac{\partial u_2}{\partial \bar{m}_1}}{\frac{\partial u_2}{\partial \bar{m}_2}} \geq 0$, and $m_2$ increases with $m_1$. □

***Proof of Claim 8***. We will show that there is a profitable deviation for the developer from truthtelling in the first period. Assume $\sigma_1(\cdot|x_1) = \delta_{x_1}$ and $\sigma_2(\cdot|x_1, m_1, \theta, x_2) = \delta_{x_1+x_2}$ is an equilibrium strategy for the developer. Then, the investor's beliefs in the second period are

$$d\lambda^*(x_1, x_2, \theta|m_1, m_2) = \begin{cases} \frac{p(\theta) f_{2,\theta}(m_2 - m_1)}{\int_\theta p(\theta) f_{2,\theta}(m_2 - m_1) d\theta} & \text{if } x_1 = m_1 \ \& \ x_2 = m_2 - x_1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that $d\lambda^*(x_1, x_2, \theta|m_1, m_2)$ is satisfies MLP in $(\theta, x_2)$. That follows directly from the assumption that $f_{2,\theta}$ satisfies MLP in $(\theta, x_2)$. The utility of the investor, depending on his choice:

$$u_2(m_1, m_2, c) = R \int_\theta [1 - F_{3,\theta}(\bar{x} - m_2)] d\lambda_2^*(m_1, m_2 - m_1, \theta|m_1, m_2) d\theta$$
$$u_2(m_1, m_2, s) = R \mathbb{1}_{m_2 \geq \bar{x}}$$

Then, assuming $m_2 < \bar{x}$ and the strategy of the developer,

$$\mu_2(m_1, m_2) = R \int_\theta [1 - F_{3,\theta}(\bar{x} - m_2)] d\lambda_2^*(m_1, m_2 - m_1, \theta|m_1, m_2) d\theta$$

Since $[1 - F_{3,\theta}(\bar{x} - m_2)]$ is increasing in $\theta$, and $d\lambda^*(x_1, x_2, \theta|m_1, m_2)$ satisfies MLP in $(\theta, x_2)$, Then, $\mu_2(m_1, m_2)$ is increasing in $m_2$ and decreasing in $m_1$.
Recall that $\bar{m}_2(m_1)$ is defined explicitly as $\mu_2(m_1, \bar{m}_2(m_1)) = 0$. Therefore, using the implicit function theorem, we get that $\bar{m}_2(m_1)$ is increasing in $m_1$. Next, we show $w_1(x_1, \cdot)$ is decreasing in $m_1$. Let $m_1' < m_1''$ such that $s^*_{\bar{m}_1}(m_1'') = s^*_{\bar{m}_1}(m_1') = 1$. Recall that

$$w_2(x_1, m_1, \theta, x_1, x_1 + x_2) = \left[s^*_{\bar{m}_2}(m_1, m_2)(1 - F_{3,\theta}(\bar{x} - x_1 - x_2)) + (1 - s^*_{\bar{m}_2}(m_1, m_2)) \mathbb{1}_{x_1, x_2 \geq \bar{x}}\right] D$$

$$w(x_1, m_1) = s^*_{\bar{m}_1}(m_1) \int_{\theta, x_2} w_2(x_1, m_1, \theta, x_2, x_1 + x_2) f_{2,\theta}(x_2) p(\theta) dx_2 d\theta$$

Then, assuming $s^*_{\bar{m}_1}(\cdot)$, we get

$$w_2(x_1, m_1'', \theta, x_1, m_2) = \begin{cases} 0 & \text{if } m_2 < \bar{m}_2(m_1') \\ 0 & \text{if } \bar{m}_2(m_1') \leq m_2 < \bar{m}_2(m_1'') \\ \left[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)\right] D & \text{if } \bar{m}_2(m_1'') \leq m_2 < \bar{x} \\ D & \text{if } m_2 \geq \bar{x} \end{cases}$$

$$w_2(x_1, m_1', \theta, x_1, m_2) = \begin{cases} 0 & \text{if } m_2 < \bar{m}_2(m_1') \\ \big[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)\big] D & \text{if } \bar{m}_2(m_1') \le m_2 < \bar{m}_2(m_1'') \\ \big[1 - F_{3,\theta}(\bar{x} - x_1 - x_2)\big] D & \text{if } \bar{m}_2(m_1'') \le m_2 < \bar{x} \\ D & \text{if } m_2 \ge \bar{x} \end{cases}$$

Clearly, $w_2(x_1, m_1, \theta, x_1, m_2)$ is decreasing in $m_1$. Thus, the developer prefer to deviate to the lowest message $m_1$ such that $s^*_{\bar{m}_1}(m_1) = 1$. Therefore, it is profitable for the developer to deviate from truthtelling in period $t = 1$.[15] □

***Proof of Claim 9***. Assumption 3.1 can be written in terms of the investor's expected utility of starting the project when he gets no additional information except whether the project is completed in every period. That is, the following expression is negative

$$\tilde{u}_0 = -c_0 + \int_{x_1 \ge \overline{x}} R f_1(x_1) dx_1 + \max\Big\{0, -c_1 + \int_{x_1 < \overline{x}} \Big( \int_{x_2 \ge \overline{x} - x_1} R f_2(x_2) dx_2 + \max\Big\{0, -c_2 + \int_{x_2 < \overline{x} - x_1} (R(1 - F_3(\overline{x} - x_1 - x_2)) f_2(x_2) dx_2) \Big) f_1(x_1) dx_1 \Big\}$$

Suppose that in equilibrium the developer starts the project. We will see that the expected payoff of the developer is identical to $\tilde{u}_0$. Notice that whenever the job is done, there is no reason for the developer not to release this information as his payoff is fixed to $D$. So WLOG we assume this is the case.

If the job is done in the first period, the investor gets paid $R$. This event happens with probability $\Pr(x_1 \ge \overline{x})$. If the job is not done, two things can happen in equilibrium:

1. The developer stops for all messages that get sent with positive probability.

2. The developer continues for some messages that are sent with positive probability.

$$A = \{x_1 \in [0, \overline{x}) | s(m_1) = 1 \ \forall \ m_1 \text{ such that } \sigma_1(m_1|x_1) > 0\}$$
$$B = \{x_1 \in [0, \overline{x}) | s(m_1) = 0 \ \forall \ m_1 \text{ such that } \sigma_1(m_1|x_1) > 0\}$$

If it is the first case, then $B = [0, \overline{x})$, as all messages that are sent get stopped, so the set of $x_1$ that get stopped and incomplete is equal to $[0, \overline{x})$. If it is the second case, we argue that now $A = [0, \overline{x})$. That is, all types that are incomplete will only send messages $m_1$ that are continued. To see this, notice that the utility of the developer is

$$w(x_1, m_1) = s(m_1) \int_{\theta, x_2} w_2^*(x_1, m_1, \theta, x_2) f_{2,\theta}(x_2) p(\theta) dx_2 d\theta + (1 - s(m_1))0$$

[15] This analysis fails if the state of the world does not change. Because in that case, $u_2(m_1, m_2)$ is independent of $m_1$. Truthtelling will remain an equilibrium.

Because the job can get done in any period, for all $x_1 < \overline{x}$, there exist a strictly positive measure set of $x_2$ such that $x_1 + x_2 > \overline{x}$. But then, for a positive measure set of $x_2$, $w_2^*(x_1, m_1, \theta, x_2)$ has to be strictly positive. By assumption, there exist (possibly non unique) message $m_1$ such that $s(m_1)$ gets continued. But then, any type $x_1$ such that $x_1 < \overline{x}$ will only send messages that gets continued. As there are no restriction on who can send this messages, we must have that $A = [0, \overline{x})$.

Therefore, in any equilibrium two things can happen: for all types $x_1$ with the job incomplete the investor stops for all messages or continues for all messages sent with positive probability. If it is the first case, the value of continuing has to be zero, which is the left expression in the first max operator of $\tilde{u}_0$. If it is the second case, we repeat the argument to get that again the investor gets either zero or the second term inside the interior max. But then, the utility of the investor is precisely $\tilde{u}_0$ which is negative, so she should not have started.

□

## 5.3 Contract

In this case the developer gets paid only by transfers. At any point $t$, the developer's outside option is given by $v_t$ with distribution $G(v)$ and density $g(v)$. Unlike the case studied in the main body in which the developer would never stops a project, it is now possible that the developer would choose to do so. We allow for this case without further extending notation by expanding the message set with the message $m_t =$ "*out*" which imposes $s(\cdot, m_t) = 0$.

By the revelation principle for this game (Sugaya and Wolitzky (2021)), it is enough to restrict attention to truthtelling mechanisms in which at any point the investor ask the developer about her current type (the status of the project and her outside option) and the investor truthfully reveals them. We let $x_t$ be the period $t$ advancement of the project, $v_t$ be period $t$ outside option, $m_t$ be period $t$ developer's report about the current status of the project and $n_t$ be period $t$ developer's report about his outside option.

To simplify notation, we let $\epsilon_t$ be an indicative function for completion at time $t$. If at time $t = 1$ or $t = 2$ the process continues, the investors pays continuation transfer $p_t$ to the developer as well a cost $c_t > 0$ of development. If at time $t = 1$ or $t = 2$, any of the two parties decide to stop the game, the developer gets time payoff $v_t$ plus any promised transfers the investor promised him, $q_t$, which is considered a prize or punishment depending on the status of the project when the production ends. Payoffs for the investor will be $R$ if the project gets completed when the game ends, 0 if not, minus all promised payments. We impose limited liability, so that $p_t, q_t \geq 0$.

Relevant histories for this game for the following $t$ are

0.
  - Developer: no history
  - Investor: no history
1.
  - Developer: $h_1^d = (x_1, v_1)$
  - Investor: $h_1^i = (m_1, n_1)$
2.
  - Developer: $h_2^d = (x_1, v_1, m_1, n_1, x_2, v_2)$
  - Investor: $h_2^i = (m_1, n_1, m_2, n_2)$
3.
  - Developer: $h_3^d = (x_1, v_1, m_1, n_1, x_2, v_2, m_2, n_2, x_3)$
  - Investor: $h_3^i = (m_1, n_1, m_2, n_2)$

## 5.4 Utilities

$p_t$ and $q_t$ will be functions of investor histories at time $t$ plus $\epsilon_t$. That is $p_t(h_t^i, \epsilon)$. For clarity, the last coordinate will always be $\epsilon_t$.

### 5.4.1 Period $t = 3$

$$w_3(h_3^d) = \epsilon_3 q_3(m_1, n_1, m_2, n_2, 1) + (1 - \epsilon_3) q_3(m_1, n_1, m_2, n_2, 0)$$

$$u_3(h_3^i) = \epsilon_3 (R - q_3(m_1, n_1, m_2, n_2, 1)) + (1 - \epsilon_3)(-q_3(m_1, n_1, m_2, n_2, 0))$$

### 5.4.2 Period $t = 2$

$$w_2(m_2, n_2 | h_2^d) = s(h_2^i) \Big[ \int_{x_3} [w_3(h_3^d) + p_2(h_2^i)] f_3(x_3) dx_3 \Big] + (1 - s(h_2^i)) \Big[ v_2 + q_2(h_2^i, \epsilon_2) \Big]$$

$$\hat{w}_2(x_1, v_1, m_1, n_1, x_1, v_2) = \max_{m_2, n_2} w_2(m_2, n_2 | h_2^d)$$

$$\begin{aligned} u_2(s | h_2^i) =& s(h_2^i) \Big[ \int_{x_1, x_2, x_3, v_1, v_2} [u_3(h_3^i) - p_2(h_2^i) - c_2] \lambda_2(x_1, v_1, x_2, v_2 | h_2^i) dx dv \Big] \\ &+ (1 - s(h_2^i)) \Big[ \int_{x_1, x_2, v_1, v_2} [\epsilon_2(-q_2(h_2^i, 1)) + (1 - \epsilon_2)(-q_2(h_2^i, 0))] \lambda_2(x_1, v_1, x_2, v_2 | h_2^i) \Big] dx dv \end{aligned}$$

$$\hat{u}_2(h_2^i) = \max_{s \in \{0,1\}} u_2(s | h_2^i)$$

#### 5.4.3 Period $t = 1$

$$w_1(m_1, n_1|h_1^d) = s(h_1^i)\Big[\int_{x_2,v_2,m_2,n_2}[\sigma(m_2,n_2|h_2^d)w_2(m_2,n_2|h_2^d) + p_1(h_1^d)]f_2(x_2)g(v_2)dx_2dv_2\Big]$$
$$+ (1 - s(h_1^i))\Big[v_1 + q_1(h_1^i, \epsilon_1)\Big]$$
$$\hat{w}_1(x_1, v_1) = \max_{m_1,n_1} w_1(h_1^d)$$

$$u_1(h_2^i, s) = s(h_1^i)\Big[\int_{x_1,x_2,v_1,v_2,m_2,n_2}\sigma(m_2,n_2|h_2^d)[\hat{u}_2(h_2^i) - p_1(h_1^i) - c_1]f(x_2)g(v_2)\lambda_1(x_1,v_1|h_1^i)dxdv\Big]$$
$$+ (1 - s(h_1^i))\Big[\int_{x_1}[\epsilon_1(-q_1(h_1^i,1)) + (1-\epsilon_1)(-q_1(h_1^i,0))]\lambda_1(x_1,v_1|h_1^i)dxdv\Big]$$

## 5.5 Period $t = 2$ analysis

Because of the incentive compatibility constraints, the developer will tell the truth in equilibrium, so beliefs can be replaced by point mass at each history $h_2^i$.

#### 5.5.1 If continued

Fix an investor history at $h_2^i = (m_1, n_1, m_2, n_2)$. Under a truthtelling strategy, we have $h_2^i = (m_1, n_1, m_2, n_2) = (x_1, v_1, x1 + x_2, v_2)$. Suppose that at $h_2^i$ the investor wants the developer to continue. That means that we must have that the value of continuation is greater than the value of stopping for the investor, which requires

$$-p_2(h_2^i) - c_2 + (1 - F_3(\overline{x} - x_1 - x_2))(R - q_3(h_2^i, 1)) + F_3(\overline{x} - x_1 - x_2)(-q_3(h_2^i, 0))$$
$$\geq \epsilon_2(R - q_2(h_2^i, 1)) + (1 - \epsilon_2)(-q_2(h_2^i, 0))$$

##### 5.5.1.1 Case A. $x_1 + x_2 \geq \overline{x}$

The previous condition becomes

$$-p_2(h_2^i) - c_2 + (R - q_3(h_2^i, 1)) \geq (R - q_2(h_2^i, 1)) \leftrightarrow -c_2 \geq p_2(h_2^i) + q_3(h_2^i, 1)) - q_2(h_2^i, 1)$$

For the developer to want to continue in this case we need the following inequalities. First, it has to be the case that continuing is better than stopping, which implies that

$$p_2(h_2^i) + q_3(h_2^i, 1) \geq v_2 + q_2(h_2^i, 1)$$

Notice that the right hand side is increasing in $v_2$. So if it holds for the largest value of $v_2$ that we wants to be continued for given history $(x_1, v_1, x_1 + x_2)$, we are good as all higher types we don't want them to continue, all lower types will want to continue. Abusing notation, call this value of $\overline{v}_2(h_2^i)$. Expanding the histories notation we get

$$p_2(h_2^i) + q_3(h_2^i, 1) = \overline{v}_2(h_2^i) + q_2(h_2^i, 1) \leftrightarrow \overline{v}_2(h_2^i) = p_2(h_2^i) + q_3(h_2^i, 1) - q_2(h_2^i, 1)$$

We also need that the developer wants to tell the truth. First, deviation to another history $(x_1, v_1, m', n')$ that also gets continue cannot be better

$$p_2(x_1, v_1, x_1 + x_2, v_2) + q_3(x_1, v_1, x_1 + x_2, v_2, 1) \geq p_2(x_1, v_1, m', n') + q_3(x_1, v_1, m', n', 1)$$

Combining the previous two conditions, we get

$$-c_2 \geq v_2$$

This can never be the case, so if the job is done, development has to be stopped.

#### 5.5.1.2 Case B. $x_1 + x_2 < \overline{x}$

The condition for the developer becomes

$$-p_2(h_2^i) - c_2 + (1 - F_3(\overline{x} - x_1 - x_2))(R - q_3(h_2^i, 1)) + F_3(\overline{x} - x_1 - x_2)(-q_3(h_2^i, 0)) \geq (-q_2(h_2^i, 0))$$

For the developer to want to continue in this case we need the following inequalities. First, it has to be the case that continuing is better than stopping, which implies that

$$p_2(h_2^i) + (1 - F_3(\overline{x} - x_1 - x_2))q_3(h_2^i, 1) + F_3(\overline{x} - x_1 - x_2)q_3(h_2^i, 0) \geq v_2 + q_2(h_2^i, 0)$$

Notice that the right hand side is increasing in $v_2$. For history $(x_1, v_1, x_1 + x_2)$ let $\hat{v}_2(h_2^i)$ be the highest type $v_2$ that the investor wants to continue, which, by continuity, implies that

$$p_2(h_2^i) + (1 - F_3(\overline{x} - x_1 - x_2))q_3(h_2^i, 1) + F_3(\overline{x} - x_1 - x_2)q_3(h_2^i, 0) = \hat{v}_2(h_2^i) + q_2(h_2^i, 0)$$

Higher types will not want to get continued and lower types will want to get continue. Replacing in the condition of the investor we have,

$$-c_2 + (1 - F_3(\overline{x} - x_1 - x_2))R - v_2(h_2^i) \geq 0$$

or, by continuity and expanding the history

$$(1 - F_3(\overline{x} - x_1 - x_2))R - c_2 = v_2(h_2^i 2)$$

Notice that the LHS is constant in $v_1$, and only depends on the message $m_2 = x_1 + x_2$. Therefore, conditioning on $m_2 = x_1 + x_2$, the cutoff value is constant in the first message $m_1 =$

$x_1$ and in the first message $n_1 = v_1$. In words, to know if for type $h_2^i = (x_1, v_1, x_1 + x_2, v_2)$ development needs to be continued, we only need to check if $v_2$ is greater or not from $\hat{v}_2(h_2^i 2) = (1 - F_3(\overline{x} - x_1 - x_2))R - c_2$. If it is, it is too costly to keep the developer producing. More over, notices that $\hat{v}_2$ is increasing in $m_2 = x_1 + x_2$.

Now, notice that because of the constant relationship between $p_2$, $q_2$ and $q_3$ given by the definition of $\hat{v}_2(h_2^i) = p_2 + q_3 - q_2 = (1 - F_3(\overline{x} - x_1 - x_2))R - c_2$, the higher $q_2$ the higher the payment the principal has to make (as he is continuing he pays $p_2$ and $q_3$), therefore we want $q_2$ for values that get stopped equal to the minimum (say $= 0$). So, we get $q_2(x_1, v_1, x_1 + x_2, v_2(h_2^i)) = 0$

We now need to make sure he does not want to lie. First, we need to make sure he does not want to deviate to other values $(m', n')$ that gets them stopped. That is,

$$p_2(x_1, v_1, x_1 + x_2, v_2) + (1 - F_3(\overline{x} - x_1 - x_2))q_3(x_1, v_1, x_1 + x_2, v_2, 1)F_3(\overline{x} - x_1 - x_2)q_3(x_1, v_1, x_1 + x_2, v_2, 0) \geq v_2 + q_2(x_1, v_1, m', n', 0)$$

From the previous condition for type $\hat{v}_2(h_2^i)$, replacing we get that for all $(m', n')$ such that the alternative is to stop we have

$$0 = q_2(x_1, v_1, x_1 + x_2, \hat{v}_2(h_2^i), 0) \geq q_2(x_1, v_1, m', n', 0)$$

Second, we need to make sure that deviating for messages that also get continued $(m', n')$ it is not optimal.

$$p_2(x_1, v_1, x_1 + x_2, v_2) + (1 - F_3(\overline{x} - x_1 - x_2))q_3(x_1, v_1, x_1 + x_2, v_2, 1) + F_3(\overline{x} - x_1 - x_2)q_3(x_1, v_1, x_1 + x_2, v_2, 0) \geq$$
$$p_2(x_1, v_1, m', n') + (1 - F_3(\overline{x} - x_1 - x_2))q_3(x_1, v_1, m', n', 1) + F_3(\overline{x} - x_1 - x_2)q_3(x_1, v_1, m', n', 0)$$

By definition of $v_2(h_2^i)$ for $h_2^i = (x_1, v_1, x_1 + x_2, v_2)$ and $h_2^i = (x_1, v_1, m', n')$, we have that the previous condition becomes

$$\hat{v}_2(x_1, v_1, x_1 + x_2) \geq \hat{v}_2(x_1, v_1, m')$$

By the verifiability assumption, the deviations that can happen are those for which $m'$ are lower than $x_1 + x_2$. Because we saw that $\hat{v}_2$ is increasing in $m_2$ and only values lowers than $\hat{v}_2$ get continued, we have $\hat{v}_2(x_1, v_1, x_1 + x_2) \geq \hat{v}_2(x_1, v_1, m') \geq n'$ for any feasible $(m', n')$ that also gets continued. Then, he never wants to deviate to alternative values that get continued.

Notice that the only way for a type $(x_1, v_1, x_1 + x_2, v_2)$ with $v_t < \hat{v}_2(x_1, v_1, x_1 + x_2)$ not to deviate to reporting $\hat{v}_2(x_1, v_1, x_1 + x_2)$ is that payments $p_2$ and $q_3$ are constant in $n_2$ for all $n_2 < \hat{v}_2(x_1, v_1, x_1 + x_2)$. Therefore, the developer extracts all profits from the developer.

### 5.5.2 If stopped

Fix an investor history at $h_2^i = (m_1, n_1, m_2, n_2)$. Under a truthtelling strategy, we have $h_2^i = (m_1, n_1, m_2, n_2) = (x_1, v_1, x_2, v_2)$. Suppose that at $h_2^i$ the investor wants to stop development. That means that we must have that the value of continuation is smaller than the value of stopping for the investor, which requires

$$-p_2(h_2^i) - c_2 + (1 - F_3(\overline{x} - x_1 - x_2))(R - q_3(h_2^i, 1)) + F_3(\overline{x} - x_1 - x_2)(-q_3(h_2^i, 0)) \\ \leq \epsilon_2(R - q_2(h_2^i, 1)) + (1 - \epsilon_2)(-q_2(h_2^i, 0))$$

Suppose that $x_1 + x_2 \geq \overline{x}$. The previous condition becomes

$$-p_2(h_2^i) - c_2 + (R - q_3(h_2^i, 1)) \leq (R - q_2(h_2^i, 1)) \leftrightarrow -c_2 \leq p_2(h_2^i) + q_3(h_2^i, 1)) - q_2(h_2^i, 1)$$

Because the developer is paying $q_2$ in this region, he wants to set it up as small as possible. Then $q_2(\epsilon = 1) = 0$. Now notice that only way for the previous to be violated in the equilibrium path is if $p_2 + q_3$ are negative, but because of the outside option for the developer $v_2 \geq 0$, then the agent will never pick that option, so it would not be on the equilibrium path. Therefore, we have that if the job is done, the game always stops. Which implies that the payment is zero if the game is stopped, as there is nothing the developer can do about it.