# Part 1: From Erdős–Rényi Graphs to High-Dimensional Phase Transitions

*Linial–Peled Phase Transitions in Random Simplicial Complexes*

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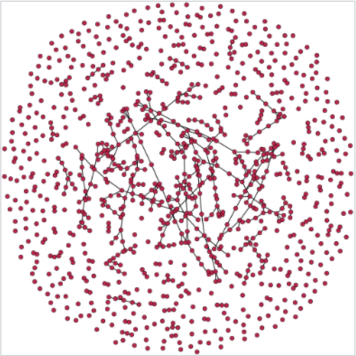
1st Doctoral Candidate Symposium
*BeyondTheEdge Network*
May 27, 2025

# Outline

# Erdős–Rényi Model: Basics

- Introduced by Paul Erdős and Alfréd Rényi (1959), and independently by Gilbert.
- Two common variants:
  - $\mathbf{G(n, M)}$: choose uniformly at random from all graphs with $n$ vertices and $M$ edges.
  - $\mathbf{G(n, p)}$: for each of the $\binom{n}{2}$ edges, include it independently with probability $p$.
- Key phenomenon: *Phase transitions* around critical probabilities (e.g. $\frac{1}{n}$ for cycles, $\frac{\ln n}{n}$ for connectivity).

# A Visual Example of the Transition



An Erdős–Rényi–Gilbert graph with 1000 vertices at the critical edge probability $p = 1/(n-1)$, showing a large component and many small ones

Figure: An Erdős–Rényi–Gilbert graph with 1000 vertices at the critical edge probability $p = \frac{1}{n-1}$, showing a large component and many small ones (Wikipedia). *(Image: Wikipedia)*

# Phase Transitions in $G(n, p)$

- $p = \frac{1}{n}$:
  - A cycle appears **almost surely**.
  - The *giant component* emerges.
- $p = \frac{\ln n}{n}$:
  - Threshold for graph connectivity (above it, the graph is connected a.a.s.).
- Typically these transitions are *continuous* (second-order) in the 1D (graph) case:
  - The size of the giant component grows continuously from 0 to $\Theta(n)$.

# A Quick Recall: Homology and Betti Numbers

- **Homology** provides a way to measure “holes” in a topological space (or a simplicial complex).
- In low dimensions:
  - $H_0$ measures connected components.
  - $H_1$ measures loops or “cycles” (in a graph, these are cycles in the usual sense).
  - $H_2$ and higher capture higher-dimensional holes.
- The **Betti number** $\beta_d$ is the dimension of the $d$-th homology group $H_d$.
  - $\beta_d$ counts (roughly) the number of independent $d$-dimensional holes.
- In random complexes, we ask: *When do these Betti numbers vanish or become large?*

# Random Simplicial Complexes: $Y_d(n, p)$

- In dimension $d = 1$, $Y_1(n, p) \equiv G(n, p)$.
- For $d \geq 2$:
  - We have a full $(d-1)$-dimensional skeleton on $n$ vertices.
  - Each possible $d$-face (set of $d+1$ vertices) is included with probability $p$, independently.
- This model generalizes Erdős–Rényi graphs to *higher-order interactions*.

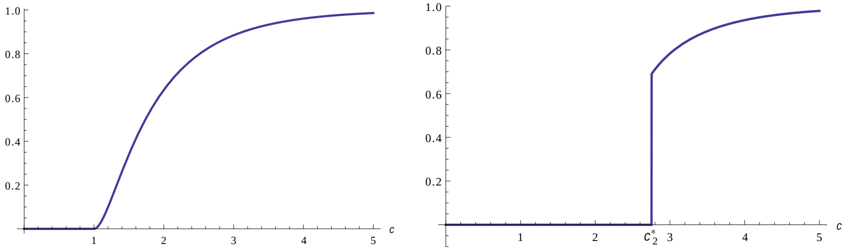
# Intuition: Phase Transitions in High Order

- Just as graphs have thresholds for connectivity or cycles, high-dimensional complexes exhibit:
  1. **Threshold for top homology** to appear (i.e. when $\beta_d \neq 0$).
  2. **Threshold for collapsibility** (akin to being acyclic in higher dimensions).
- **Surprise:** in dimension $d \geq 2$, the transition for certain properties (like the "shadow" density) becomes *first order* (a discontinuous jump), rather than the smoother transitions seen in 1D.
- Key references:
  - Linial–Meshulam (2006),
  - Linial–Peled (2016).

# The "Shadow" of a Complex

- In 1D (graphs), the *shadow* is the set of edges not in the graph but that would create a cycle if added.
- In higher dimensions, $\mathrm{SH}_{\mathbb{R}}(Y)$ is the set of $d$-faces not in $Y$ whose addition increases the $d$-th homology dimension.
- **Key result (Linial–Peled)**:
  - For $d \geq 2$, the density of the shadow jumps from near 0 to a positive constant at the critical probability.
  - This is a **first-order phase transition**.

# Figures: Density of the Shadow (Linial–Peled)




**Left:** Shadow density in the graph case $d = 1$.
**Right:** Shadow density for $d = 2$. Notice the *discontinuous* jump at the critical threshold $c > c_d^*$.

# Breakout room

**Potential questions for pairwise breakout:**

- What other generalizations from graph theory to higher-dimensional complexes have you seen or would you like to see?
- How might classical random graph results (like thresholds for connectivity or diameter) translate into higher-dimensional analogs?
- What are other possible "giant" structures (besides homology cycles) in random simplicial complexes?
- Any real-world contexts (beyond standard networks) where these high-dimensional models might be fruitful?

# References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