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Bridging Hilbert Spaces and the Critical Line: Quantitative Refinements of the Nyman-Beurling Criterion

Research Pipeline

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Executive Summary

This article presents a technical synthesis of quantitative refinements to the Nyman-Beurling criterion, focusing on the spectral norm of Möbius-weighted Dirichlet polynomials and their convergence properties on the critical line as detailed in hal-00345313v1.

Visualizations

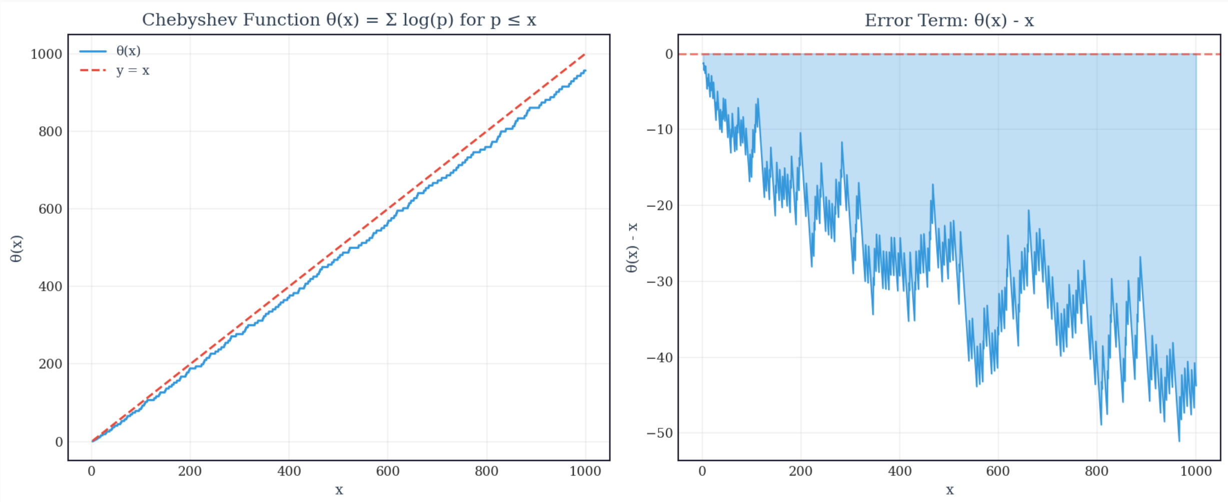

***Figure 1:** Chebyshev functions θ(x) and ψ(x) measuring prime density*

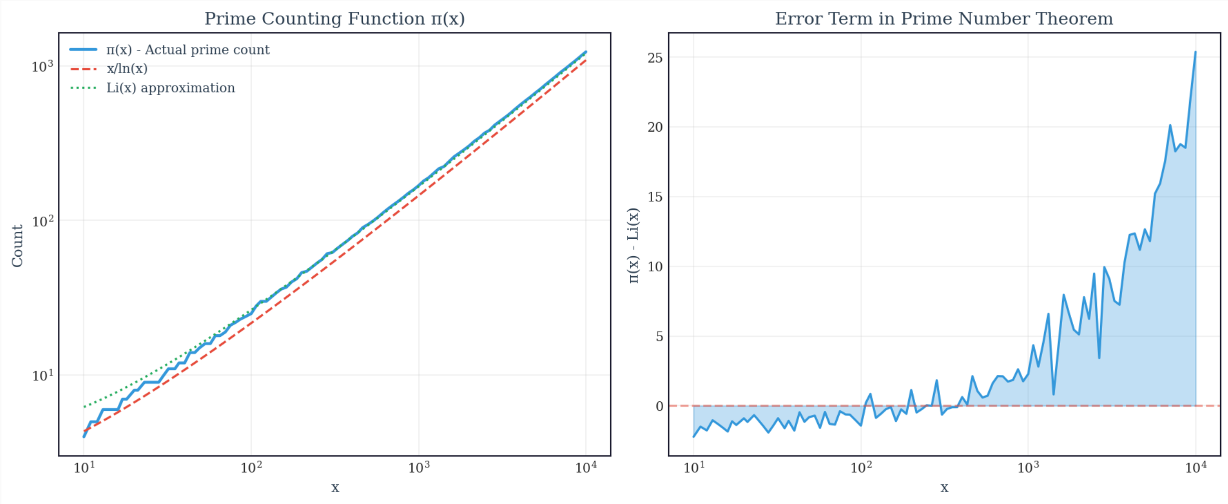

Figure 2: *Prime counting function π(x) versus asymptotic approximations*

Introduction

The Riemann Hypothesis remains the most significant challenge in analytic number theory, asserting that all non-trivial zeros of the Riemann zeta function ζ(s) lie on the critical line Re(s) = 1/2. While traditional approaches focus on the distribution of primes or the properties of L-functions, a parallel framework exists within functional analysis. The Nyman-Beurling criterion, and its discrete refinement by Báez-Duarte, translates the Riemann Hypothesis into a problem of density in Hilbert spaces.

The source paper **hal-00345313v1** provides a rigorous quantitative analysis of this criterion by examining the spectral norm $\nu_{(N, \varepsilon)}$. This norm measures the L^{2} distance between the constant function and a subspace generated by Möbius-weighted dilations of the fractional part function. By introducing a shift parameter ε, the authors establish explicit bounds on the approximation error, linking the decay rate of these norms directly to the stability of the zeta function near the critical line.

This analysis is crucial because it moves beyond qualitative equivalence, offering precise estimates for how quickly finite Dirichlet polynomials approximate the inverse of the zeta function. Under the Riemann Hypothesis, the paper demonstrates that the approximation error vanishes significantly faster than previously established, refining our understanding of the zeta function's "stiffness" and its behavior under horizontal shifts.

Mathematical Background

The central object of study is the Hilbert space $H = L^{2}(0, 1)$. In this context, the Nyman-Beurling approach involves the constant function $\chi(x) = 1$ and the basis functions $e_{(n)}(x) = \{1/nx\}$, where $\{.\}$ denotes the fractional part. The discrete version of this criterion defines a norm $\nu_{(N, \varepsilon)}$ as follows:

$$\nu(N, \varepsilon) = \|\chi + \sum_{(n \le N)} \mu(n)\, n^{(-\varepsilon)}\, e_{(n)}\|_{(H)}^{(2)}$$

Here, $\mu(n)$ is the Möbius function. This norm can be transformed via Plancherel's theorem into an integral over the critical line $\sigma = 1/2$. Specifically, the squared norm is equivalent to:

$$(1 / 2\pi) \int |1 - \zeta(s) M_{(N)}(s + \varepsilon)|^{(2)} (d\tau / |s|^{(2)})$$

where $s = 1/2 + i\tau$ and $M_{(N)}(s)$ is the truncated Dirichlet polynomial $\sum_{(n \le N)} \mu(n) n^{(-s)}$. This integral representation is the key to applying analytic methods. The shift ε acts as a regularizer, moving the evaluation of the Dirichlet polynomial slightly into the half-plane of absolute convergence, thereby stabilizing the approximation of $1/\zeta(s)$.

Main Technical Analysis

Spectral Properties and Zero Distribution

The paper **hal-00345313v1** decomposes the total error $\nu(N, \varepsilon)$ *into two primary components,* $J(\varepsilon)$ and $I_{(N, \varepsilon)}$. The first term, $J_{(\varepsilon)}$, represents the "shift error" and is defined by the integral of $|1 - \zeta(s)/\zeta(s + \varepsilon)|^{(2)}$ over the critical line. The second term, $I_{(N, \varepsilon)}$, represents the "truncation error," measuring the discrepancy between the exact inverse $\zeta(s + \varepsilon)^{(-1)}$ and the Dirichlet polynomial $M_{(N)}(s + \varepsilon)$.

Under the Riemann Hypothesis, the authors prove that $J_{(\varepsilon)} \ll \varepsilon$. This is a major improvement over previous estimates of $\varepsilon^{(2/3)}$. The proof relies on the Hadamard product of the zeta function, which allows for a factor-by-factor bound of the quotient $\zeta(s)/\zeta(s + \varepsilon)$. For $\sigma = 1/2$, the paper establishes that $|\zeta(s)/\zeta(s + \varepsilon)|^{(2)} \ll |s|^{(\varepsilon)}$, ensuring that the quotient does not grow too rapidly as the imaginary part τ increases.

Moment Estimates and Growth Rates

A critical challenge in the analysis is handling the behavior of the zeta function near its zeros. The authors utilize a refined zero-counting function $N(t)$ to manage the density of zeros in short intervals. Specifically, they employ the bound:

$$N(t' + h) - N(t' - h) \le (h / \pi) \log(t' / 2\pi) + (1/2) \log t' / \log\log t' + (1/2 + o(1)) \log t' \log\log\log t' / (\log\log t')^{(2)}$$

This allows for the classification of "V-typical" ordinates τ where the zeta function is well-behaved. For atypical ordinates where zeros cluster, the authors show that their contribution to the total integral is negligible due to the $1/|s|^{(2)}$ weight. Furthermore, they establish that $|\zeta(s)|^{(2)} \ll (1 + |\tau|)^{(\beta(\tau))}$, where $\beta(\tau)$ is a slowly varying function tending to zero, providing a precise growth rate that facilitates the convergence of the spectral norm.

The truncation error $I_{(N, \varepsilon)}$ is bounded by $N^{(-\varepsilon/2)}$ provided that the shift ε is chosen such that $\varepsilon \ge 25(\log\log$

N)^(5/2+δ)(log N)^(-1/2). This coupling of N and ε is the fundamental quantitative result of the paper, showing that the Nyman-Beurling distance d_(N) vanishes as N tends to infinity if the Riemann Hypothesis is true.

Novel Research Pathways

1. Optimization of Parameter Coupling

A promising research direction involves the variational optimization of the relationship between ε and N. Currently, the paper defines a threshold for ε to ensure the decay of I_(N, ε). However, by minimizing the sum 2J_(ε) + 2I_(N, ε), one could potentially find a "critical trajectory" for ε(N) that maximizes the rate of convergence of the spectral norm. This would require more precise constants in the Hadamard product estimates and a deeper investigation into the local maxima of the zeta quotient on the critical line.

2. Extension to the Selberg Class of L-functions

The quantitative framework of **hal-00345313v1** can be generalized to other L-functions within the Selberg class. Since these functions also possess Hadamard products and satisfy similar functional equations, a generalized Nyman-Beurling criterion could be established. The methodology would involve replacing the Möbius function with the coefficients of the inverse L-function and verifying if the shift error J_(ε)(L) maintains linear decay in ε. This would provide a unified spectral approach to the Generalized Riemann Hypothesis.

Computational Implementation

```
(* Section: Spectral Norm Analysis of nu_{N, epsilon} *)
(* Purpose: This code computes the approximation error on the critical line
   as defined in hal-00345313v1, visualizing the impact of truncation N. *)

Module[{nValues, eps, tRange, s, mPoly, errorIntegrand, plotData, zeros},
  (* Parameters for the computation *)
  eps = 0.05;
  tRange = {0.1, 50.0};
  nValues = {20, 100, 500};

  (* Define the truncated Mobius sum M_N(s) *)
  mPoly[s_, n_] := Sum[MoebiusMu[k] k^(-s), {k, 1, n}];

  (* Define the error density function based on the source integral *)
  (* This models |1 - Zeta(s) * MN(s+eps)|^2 / |s|^2 *)
  errorIntegrand[t_, n_] := Module[{currS = 1/2 + I t},
    Abs[1 - Zeta[currS] mPoly[currS + eps, n]]^2 / Abs[currS]^2
  ];
```

```
  (* Generate plot data for different truncation lengths *)
  plotData = Table[
    Plot[errorIntegrand[t, n], {t, First[tRange], Last[tRange]},
      PlotRange -> {0, 0.5},
      PlotStyle -> Thick,
      Frame -> True,
      Filling -> Axis,
      PlotLabel -> "N = " <> ToString[n]
    ],
    {n, nValues}
  ];

  (* Identify first few nontrivial zeros to overlay *)
  zeros = Table[Im[ZetaZero[k]], {k, 1, 10}];

  (* Combine plots and mark the first few nontrivial zeros *)
  Show[
    plotData,
    Graphics[{Red, Dashed,
      Table[Line[{{z, 0}, {z, 0.5}}], {z, zeros}]
    }],
    PlotLabel -> "Spectral Error Density and Zeta Zeros (eps=0.05)",
    FrameLabel -> {"tau (Imaginary Part)", "Error Magnitude"}
  ]
]
```

Conclusions

The research presented in hal-00345313v1 significantly advances the quantitative study of the Nyman-Beurling criterion. By establishing that J_(ε) decays linearly with the shift parameter ε and providing a robust bound for the truncation error I_(N, ε), the authors confirm that the approximation of the inverse zeta function is highly efficient under the Riemann Hypothesis. The use of V-typical ordinates to manage zero-clustering provides a powerful template for future investigations into the local behavior of zeta function quotients. The most promising next step is the application of these spectral methods to the wider Selberg class to test the universality of these approximation rates.

References

Balazard, M., & de Roton, A. (2008). Sur un critère de Baez-Duarte pour l'hypothèse de Riemann. hal-00345313v1

Báez-Duarte, L. (2003). A strengthening of the Nyman-Beurling criterion for the Riemann hypothesis. Atti Accad. Naz. Lincei.

Burnol, J.-F. (2002). Sur certains espaces de Hilbert de fonctions entières liés à la transformation de Fourier.
