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Spectral Boundaries of Goldbach Generating Functions and Zeta Zero Distribution

Research Pipeline

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Executive Summary

This article investigates the meromorphic continuation of Goldbach generating functions as detailed in arXiv hal-00459918, demonstrating how the accumulation of Riemann zeta zeros on the critical line creates a natural boundary at Re(s) = 1 and provides a unique spectral signature.

Introduction

The relationship between the distribution of prime numbers and the zeros of the Riemann zeta function is one of the most profound connections in mathematics. While the Riemann Hypothesis (RH) suggests that all non-trivial zeros lie on the critical line Re(s) = 1/2, the research paper **arXiv hal-00459918** explores this through the lens of additive number theory. By examining the meromorphic continuation of the Goldbach generating function, researchers have identified a "spectral wall" where the accumulation of zeta zeros creates a natural boundary for analytic continuation.

The Goldbach conjecture, which posits that every even integer greater than two is the sum of two primes, is traditionally studied via the Hardy-Littlewood circle method. However, by shifting the focus toward the Dirichlet series representation Φ_(r)(s), we can observe how the analytical properties of prime representations are dictated by the fine structure of the zeta function's zeros. This analysis demonstrates that the boundary behavior of these functions at Re(s) = 1 provides an innovative analytical signature of the critical line.

Mathematical Background

The central objects of this study are the generating functions Φ*(r)(s), which encode the number of ways an* *integer can be expressed as the sum of r primes, weighted by the von Mangoldt function Λ(n). The binary Goldbach generating function is defined as the Dirichlet series Φ*(2)(s) = Σ G_(2)(n) n^(-s), where G_(2)(n) counts the prime pairs summing to n.

As detailed in **arXiv hal-00459918**, the analytic continuation of Φ*(r)(s) is intrinsically linked to the function M(s) = -ζ'(s)/ζ(s). A key structural result is the explicit representation of Φ*(2)(s) in terms of the non-trivial zeros ρ of the zeta function:

> **Explicit Formula:** Φ_(2)(s) = M(s-1)/(s-1) - Σ [(Γ(s-ρ)Γ(ρ))/Γ(s)] M(s-ρ) - M(s) log 2π + R(s).
>
> **Singularity Structure:** A more refined expansion reveals that the singular behavior is dominated by a double sum over zero pairs: - (1/Γ(s)) Σ [Γ(s+1-ρ)Γ(ρ)] / [(s - ρ - ρ′)ρ′].

This formula acts as a bridge: any statement about the location of zeros ρ is immediately reflected in the location of the candidate poles of the generating function.

Main Technical Analysis

The Natural Boundary at Re(s) = 1

The most significant technical insight involves the denominators (s - ρ - ρ′) in the double sum. These terms suggest that Φ_(2)(s) possesses potential poles at points where s equals the sum of two zeta zeros. If the Riemann Hypothesis is true, every non-trivial zero ρ has the form 1/2 + iγ. Consequently, the sum of any two zeros ρ + ρ′ results in a complex number with a real part exactly equal to 1 (1/2 + 1/2 = 1).

Under RH, the line Re(s) = 1 becomes a dense collection of singularities. Because the imaginary parts of the zeros are believed to be linearly independent, the values γ + γ′ populate the line Re(s) = 1 densely. This transformation of the line into a **natural boundary** prevents any further meromorphic continuation into the half-plane Re(s) < 1. This provides a definitive analytic signature: the existence of this boundary is a direct consequence of the zeros being perfectly aligned on the critical line.

Spectral Damping and Integral Estimates

To ensure the validity of these series, **arXiv hal-00459918** provides rigorous bounds on the growth of the coefficients using circle method estimates. Specifically, for the minor arcs, the paper establishes that the integral of the remainder terms is bounded by x^(r+2-k/2+ε). This "spectral damping" ensures that the contribution from zeros far from the point of interest decays exponentially due to the Gamma function weights, allowing for local dominance of specific zero pairs in the analysis.

Higher-Order Generalizations

For r > 2, the higher-order functions Φ_(r)(s) exhibit a recursive relationship. They can be expressed as linear combinations of shifted zeta functions and lower-order Φ functions. This implies that the analytic obstructions found in the binary case propagate to all higher-order Goldbach generating functions, shifted by a factor of (r-2).

Novel Research Pathways

1. Zero Correlation Statistics

The singularities of Φ_(2)(s) are located at s = ρ + ρ′. Research could investigate the "spectral density" of these pole clusters. If the imaginary parts γ follow the GUE (Gaussian Unitary Ensemble) distribution, the distribution of the sums γ + γ′ should follow predictable statistical patterns, providing a new method to test the GUE hypothesis through additive prime sums.

2. The Quasi-RH Equivalence

One could pursue the contrapositive strategy: if Φ_(2)(s) is shown to be meromorphic in any region crossing Re(s) = 1, it would imply that no sums of zeros ρ + ρ′ exist in that region. Establishing an equivalence between the "thickness" of the natural boundary and the distance of zeros from the critical line could lead to new proofs of zero-free regions.

Computational Implementation

The following Wolfram Language code visualizes the density of the potential poles on the natural boundary Re(s) = 1 by calculating the sums of the imaginary parts of the first 100 non-trivial zeta zeros.

```
Module[{zeros, sums, plotData, limit = 100},
  zeros = Table[Im[ZetaZero[n]], {n, 1, limit}];
  sums = Flatten[Table[zeros[[i]] + zeros[[j]], {i, 1, limit}, {j, i, limit}]];
  plotData = Histogram[sums, {1}, "Probability",
    PlotRange -> {{0, 500}, All},
    ChartStyle -> EdgeForm[Thin],
    PlotLabel -> "Density of Singularities on the Natural Boundary Re(s) = 1",
    AxesLabel -> {"Im(s)", "Relative Density"},
    LabelStyle -> {FontFamily -> "Arial", FontSize -> 12}];
  Print["Total potential singularities calculated: ", Length[sums]];
  plotData
]
```

Conclusions

The investigation into Goldbach generating functions reveals that the additive properties of primes are deeply embedded in the analytic structure of the Riemann zeta function. The discovery that Re(s) = 1 acts as a natural boundary under the Riemann Hypothesis provides a powerful new perspective on the significance of the critical line. By viewing the zeros of the zeta function as a spectral set, we can interpret the Goldbach problem as a study of the additive energy of that spectrum.

References

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