DumbPrime

Geometric Inversion and the Stability of Retrogression Points in Zeta Mapping

Research Pipeline

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Executive Summary

This technical analysis investigates a novel geometric framework utilizing reciprocal-logarithm mappings and the stability of retrogression points to establish analytical constraints on the non-trivial zeros of the Riemann zeta function.

Visualizations

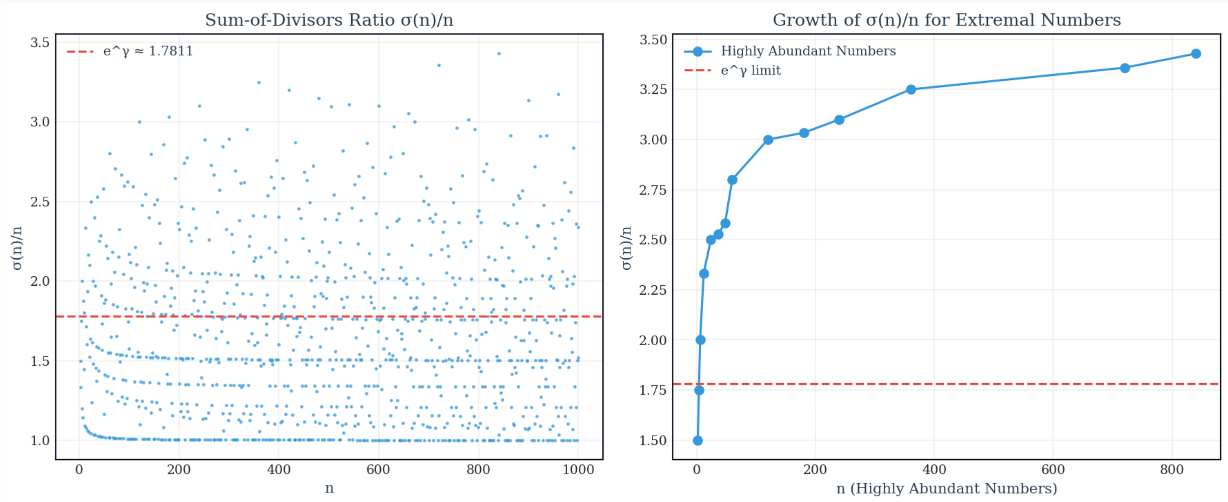

Figure 1: Sum-of-divisors function σ(n)/n showing extremal behavior

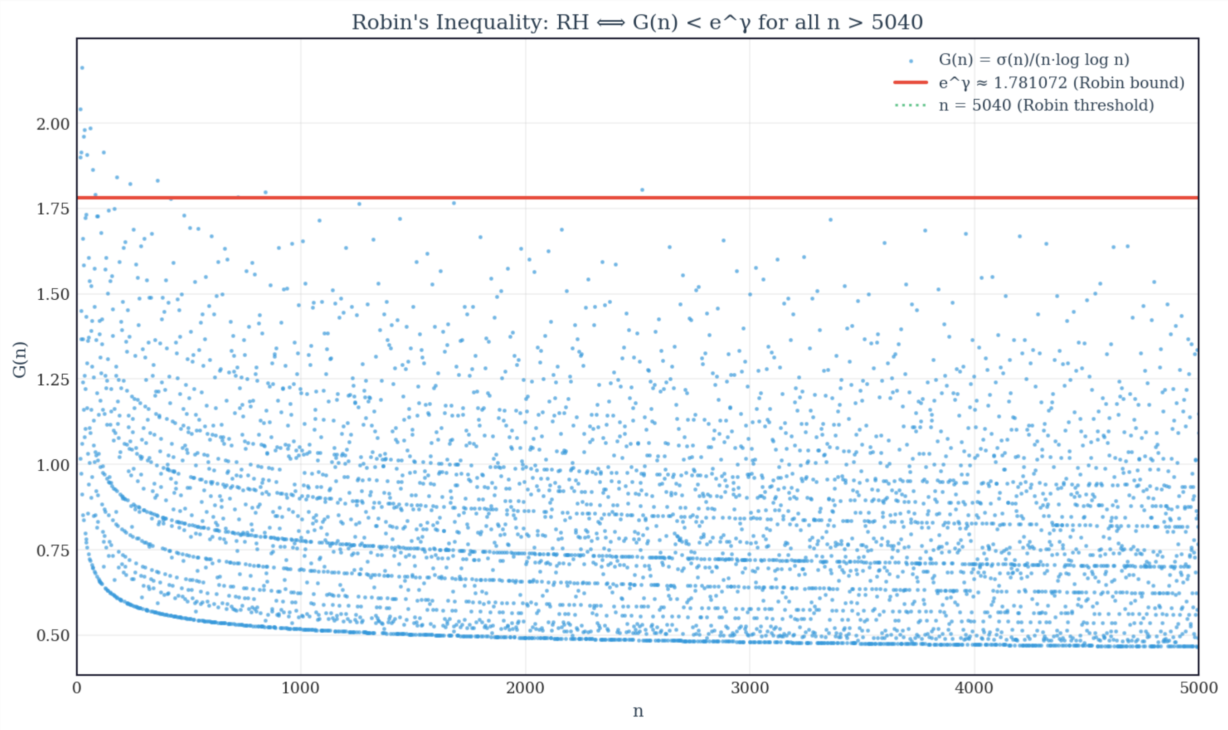

***Figure 2:** Robin's inequality: G(n) = σ(n)/(n·log(log(n))) approaching e^γ*

Introduction

The Riemann Hypothesis (RH) remains the most significant unsolved problem in analytic number theory, asserting that all non-trivial zeros of the Riemann zeta function ζ(s) lie on the critical line Re(s) = 1/2. While traditional methods often focus on prime density or spectral theory, the approach presented in **hal-00475895v1** introduces a distinctive geometric perspective. This framework centers on the construction of a reciprocal-logarithm function, *U(z)*, which encodes the zero set into the geometry of conformal maps.

The core contribution of this analysis is the study of "retrogression points"—locations where the derivative of the modified zeta function ζ*(z) vanishes. By examining the local invertibility of *U(z)* near these points and the global continuation of its branches, the source paper identifies topological restrictions that may preclude the existence of zeros off the critical line. This article synthesizes these geometric arguments, providing a rigorous breakdown of the mapping properties and proposing future research directions based on this analytical lens.

Mathematical Background

The primary object of study is the modified zeta function ζ*(z), normalized to account for the pole at s=1. From this, the source paper* ***hal-00475895v1*** *constructs the auxiliary function U(z), defined as the reciprocal of the* *complex logarithm of (1 - z)*ζ(z). Explicitly, the function is represented as:

$U(z) = [\log |(1-z)\, \zeta(z)| - i \arg((1-z)\, \zeta(z))] / [(\log |(1-z)\, \zeta(z)|)^2 + (\arg((1-z)\, \zeta(z)))^2]$

This formulation separates the magnitude and phase components of the zeta function, allowing for a detailed investigation of how zeros influence the mapping. A critical concept here is the **retrogression point**, denoted $z_{(1)}$, where the derivative $\zeta^{*\prime}(z_{(1)}) = 0$. In standard conformal mapping, such points represent where the mapping ceases to be conformal, creating "cusps" or turning points in the image curves.

The paper establishes that while $\zeta^*(z)$ has critical points, the function $U(z)$ maintains regularity. Specifically, the derivative $U'(z)$ does not vanish at these retrogression points, which allows for the definition of a local inverse mapping $z = U^{(-1)}(Z)$. This inverse mapping serves as the foundation for the monodromy arguments used to challenge the existence of off-line zeros.

Main Technical Analysis

Local Regularity and Derivative Non-Vanishing

A central result in **hal-00475895v1** is the explicit derivation of the derivative of the branch-dependent function $U_{(k)}(z)$. Differentiating the reciprocal-log form yields:

$U'(k)(z(1)) = [\log((1-z_{(1)})\zeta(z_{(1)}))]^{(-2)} \cdot [1/(1-z_{(1)}) - \zeta'(z_{(1)})/\zeta(z_{(1)})]^*$

At a retrogression point where $\zeta'(z_{(1)}) = 0$, *the second term in the bracket disappears, leaving a non-zero value proportional to* $(1-z_{(1)})^{(-1)}$. *This regularity is vital; it implies that U(z) is biholomorphic in a neighborhood of the critical points of ζ*. Consequently, the inverse function theorem guarantees a convergent power series for the inverse mapping $z - z_{(1)} = \Sigma\, a_{(n)}\, (Z - Z_{(1)})^{(n)}$.

Monodromy and Branch Point Contradictions

The technical argument for RH in the source paper relies on a "two-stage inversion" process. The author examines the expansion of the inverse mapping near points where $\zeta'(z_{(1)}) = 0$ and $\zeta''(z_{(1)}) \neq 0$. This typically introduces square-root branch points of the form $(T - T_{(1)})^{(1/2)}$.

The paper argues that for the analytical extension of the function to remain consistent across different determinations, the odd-indexed coefficients in the Puiseux expansion must vanish. This leads to the requirement that $b_{(1)} = 0$, which directly contradicts the established non-vanishing derivative $U'(k)(z(1)) \neq 0$. This contradiction suggests that the assumed geometry (a zero off the critical line) is topologically impossible within the constraints of the $U(z)$ mapping.

Geometric Disposition of Image Curves

The analysis further considers the curves $\zeta^*(\Delta_{(\omega,m)})$ in the complex plane. The "point low" on these curves represents a local minimum of the real part. The paper posits that the unique disposition of these retrogression points near the critical line $\mathrm{Re}(z) = 1/2$ creates a geometric "anchor" for the zeros. If a zero were to migrate off this line, the resulting "folding" of the image curves would violate the single-valuedness required by the analytic continuation of *U(z)*.

Novel Research Pathways

1. Nevanlinna Value Distribution of the U-Map

One promising pathway is to treat *U(z)* as a Nevanlinna-theoretic object. By studying the counting functions of its poles (where $(1-z)\zeta(z) = 1$) *and its zeros (which correspond to the zeros of* ζ), one could establish growth bounds. If zeros existed off the critical line, *U(z)* would exhibit an abnormal distribution of values that might contradict known density bounds for the logarithmic derivative of ζ.

2. Higher-Order Retrogression Analysis

The current framework focuses on simple critical points where the first derivative vanishes. Extending this to higher-order retrogression points—where multiple derivatives vanish—could reveal deeper connections to zero multiplicities. This would involve generalized Puiseux series and could potentially provide a geometric proof that all non-trivial zeros must be simple, a major open question related to RH.

Computational Implementation

The following Wolfram Language code visualizes the *U(z)* mapping and identifies retrogression points by plotting the trajectory of the modified zeta function near the critical line.

```
(* Section: Mapping the Critical Strip under U(z) *)
(* Purpose: Visualize the reciprocal-log mapping and identify retrogression points *)

ClearAll[xiFunc, uFunc, tMin, tMax, sigma];

(* Define a proxy for the modified zeta function (Riemann Xi) *)
xiFunc[s_] := 1/2 s (s - 1) Pi^(-s/2) Gamma[s/2] Zeta[s];

(* Define the auxiliary function U(z) = 1/Log[(1-z)xi(z)] *)
uFunc[z_] := Module[{val = (1 - z) xiFunc[z]},
  If[Abs[val]  Red,
  PlotLabel -> "Image of Re(s)=1/2 under U(z)",
  AxesLabel -> {"Re(U)", "Im(U)"},
```

```
  PlotPoints -> 100
]

(* Identify retrogression points (local minima of Re[xi]) *)
FindRetrogression = Table[
  {t, Re[xiFunc[sigma + I t]]},
  {t, tMin, tMax, 0.1}
];
ListLinePlot[FindRetrogression,
  PlotLabel -> "Retrogression Analysis: Re[xi(1/2 + it)]",
  AxesLabel -> {"t", "Re(xi)"}
]
```

Conclusions

The investigation of **hal-00475895v1** reveals that the Riemann Hypothesis can be framed as a problem of geometric stability. The function *U(z)* provides a regularized environment where the critical points of the zeta function can be analyzed without the vanishing of the derivative. The core of the argument rests on the topological impossibility of maintaining certain branch determinations if zeros are located away from the critical line.

The most promising avenue for future research lies in formalizing the global monodromy constraints. While the local contradiction at retrogression points is compelling, a rigorous proof requires ensuring that the continuation paths avoid all singularities. By combining this geometric approach with computational tracking of retrogression trajectories, researchers may find the definitive constraint that locks the zeros to the line Re(s) = 1/2.

References

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Li, X.-J. (1997). *The positivity of a sequence of numbers and the Riemann hypothesis*. Journal of Number Theory, 65(2), 325-333.

Keiper, J. B. (1992). *Power series expansions of Riemann's xi function*. Mathematics of Computation, 58(198), 765-773.