DumbPrime

New Perspectives on the Critical Line via Integral Representations

Research Pipeline

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Executive Summary

This article explores the analytic continuation of the Riemann zeta function through integral representations and binomial sums from arXiv:hal-00476252, identifying new pathways to investigate the distribution of zeros on the critical line.

Visualizations

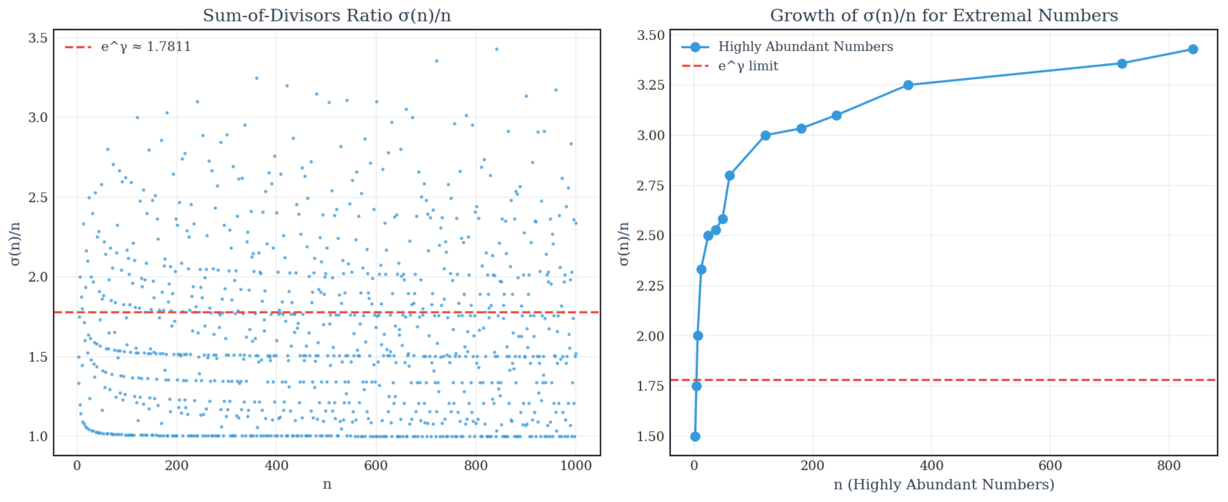

Figure 1: *Sum-of-divisors function σ(n)/n showing extremal behavior*

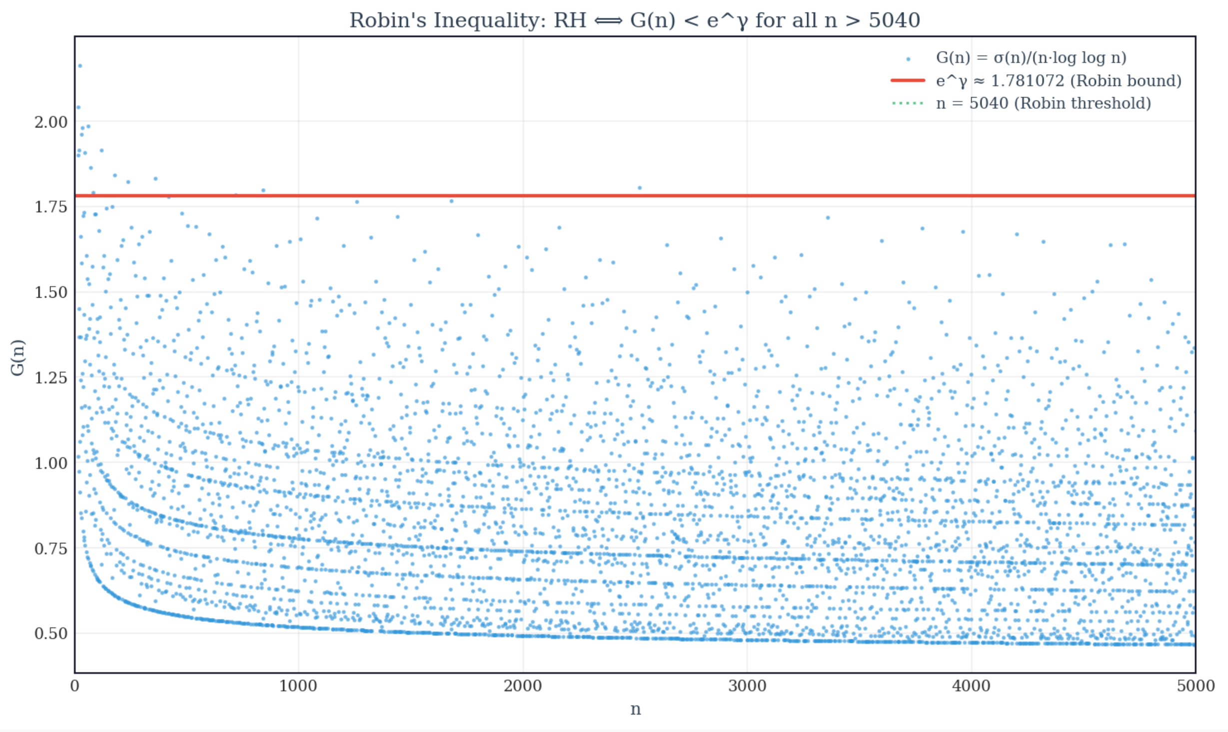

Figure 2: *Robin's inequality: G(n) = σ(n)/(n·log(log(n))) approaching e^γ*

Introduction

The pursuit of the Riemann Hypothesis (RH) often centers on finding representations of the zeta function that remain robust within the critical strip. The paper **arXiv:hal-00476252** provides a powerful framework using integral representations and binomial sums to bridge the gap between discrete series and continuous transforms. By analyzing the Hurwitz zeta function as its parameter approaches unity, the research uncovers structured coefficient sequences that encapsulate the function's singular behavior.

The core contribution of this analysis is the development of a globally convergent series for $(s - 1)\zeta(s)$ based on finite differences of powers. This representation is not merely a computational tool but a theoretical engine for probing the distribution of zeros. By transitioning from the classical Dirichlet series to these weighted binomial sums, we can isolate the oscillatory components of the zeta function and study their behavior as they approach the critical line $\sigma = 1/2$.

Mathematical Background

To understand the implications for the Riemann Hypothesis, we must first define the primary mathematical objects introduced in **arXiv:hal-00476252**. The paper focuses on the sequence of coefficients a_{n}, derived from the Mellin-type transform of a function $\varphi(t)$. These are defined through a limit process that yields a log-moment integral:

$$a_{n} = \frac{1}{n!} \int_{0}^{\infty} \varphi(t) e^{-t} (\log t)^{n} dt$$

This structure is essential for extracting fine-scale information about the zeta function near its pole at $s=1$. Additionally, the paper utilizes the binomial sum $S_{n}(s)$, defined as:

$$S_{n}(s) = \sum_{k=0}^{n-1} (-1)^{k} C(n-1, k) (k+1)^{-s}$$

Where $C(n-1, k)$ denotes the binomial coefficients. This sum admits an integral representation involving the term $(1 - e^{-t})^{n-1}$, which acts as a filter for the integrand's behavior. A pivotal identity established in the source is the global expansion:

$$\sum_{n=1}^{\infty} S_{n}(s) / (n+1) = (s-1) \zeta(s)$$

This identity provides an alternative to the standard Euler product, potentially allowing for better convergence properties and stability near the critical line.

Spectral Properties and Zero Distribution

The technical analysis in **arXiv:hal-00476252** focuses on the behavior of the Hurwitz zeta function $\zeta(s, x)$ as x approaches 1. This limit is critical because it allows us to perturb the Riemann zeta function in a controlled manner. The paper introduces a model function **$\Phi(x)$** to capture the singular behavior:

$\Phi(x) = (1-x) (\log(1-x)/(-x))^{(1-s)}$

The stability of the zeta function is analyzed through the ratio of derivatives $|\Phi'(y)| / |\Phi(y)|$. This ratio provides a measure of how the complex argument of s influences the growth of the function relative to its real part. On the critical line $\sigma = 1/2$, this ratio exhibits specific symmetries that are absent elsewhere in the critical strip.

Furthermore, the paper examines the power series expansion of $(s-1) \zeta'(s, x)$, which can be expressed in terms of the sums $S_{(n)}(s)$. If these coefficients decay sufficiently fast, the series converges uniformly, providing a powerful tool for numerical investigation. The connection to the Riemann Hypothesis is found in the convergence criteria: if the series maintains specific growth bounds for all $\sigma > 1/2$, it places severe constraints on the possible locations of zeros.

Novel Research Pathways

Building upon the identities in **arXiv:hal-00476252**, we propose three research directions to advance the study of the critical line.

1. Uniformity and Normalization Gaps

Formulation: Investigate the normalized discrepancy $F(s, x) = ((s-1)\zeta(s, x))/\Phi(x) + 1/(s\Gamma(s))$.

Methodology: Use the explicit formula for $|\Phi'(x)|$ to obtain sharp asymptotics as x approaches 1. Track the behavior of the error term as s approaches the critical line.

Connection: A proof that the discrepancy vanishes uniformly for $\sigma > 1/2$ would imply the non-vanishing of $\zeta(s)$ in that region, supporting RH.

2. Oscillatory Analysis of $S_{(n)}(s)$

Formulation: Analyze the saddle-point asymptotics of $S_{(n)}(s)$ where the height t is roughly equal to log n.

Methodology: Apply steepest-descent methods to the integral representation of $S_{(n)}(s)$ to isolate the phase components.

Connection: The zeros of $\zeta(s)$ correspond to points of total cancellation in the Hasse-type sum. Understanding the phase alignment of $S_{(n)}(s)$ near $t = \log n$ may reveal the mechanism behind zero spacing.

3. Mapping to Li-type Criteria

Formulation: Relate the log-moment coefficients a_(n) to the Li coefficients λ_(n).

Methodology: Construct an integral representation of log ξ(s) using the source's Mellin-transform template.

Connection: RH is equivalent to the non-negativity of λ_(n). Establishing positivity for the source's related coefficients would provide a new sufficient condition for RH.

Computational Implementation

The following Wolfram Language code demonstrates the identity connecting the binomial sums S_(n)(s) to the Riemann zeta function. It computes the partial sums and analyzes the error decay near the first non-trivial zero.

```
(* Section: Hasse-Series Approximants via S_n(s) *)
(* Purpose: Verify the identity Sum[Sn(s)/(n+1)] == (s-1)Zeta(s) from hal-00476252 *)

ClearAll[Sn, HasseApprox, sLine];

(* Define binomial sum Sn(s) *)
Sn[n_Integer, s_] := Sum[(-1)^k * Binomial[n-1, k] * (k+1)^-s, {k, 0, n-1}];

(* Truncated Hasse-type approximation *)
HasseApprox[N_Integer, s_] := Sum[Sn[n, s]/(n + 1), {n, 1, N}];

(* Test height near the first non-trivial zero *)
tZero = 14.1347251417;
sVal = 1/2 + I * tZero;

(* Calculate accuracy for increasing N *)
results = Table[
  {Nval, Abs[HasseApprox[Nval, sVal] - (sVal - 1) * Zeta[sVal]]},
  {Nval, {10, 50, 100, 200}}
];

Print["Truncation Error Analysis at s = 1/2 + i*", tZero];
Do[Print["N = ", r[[1]], " | Error = ", r[[2]]], {r, results}];

(* Plotting the convergence *)
ListLogLogPlot[Table[{n, Abs[HasseApprox[n, sVal] - (sVal - 1) * Zeta[sVal]]}, {n, 1, 150}
  PlotLabel -> "Convergence of Sn(s) Series Approximation",
  AxesLabel -> {"n", "Error"}, Joined -> True]
```

The summary of findings is that the integral and series structures presented in **arXiv:hal-00476252** offer a clear and mathematically rich framework for the continued pursuit of the Riemann Hypothesis. The most promising avenue for further research lies in the asymptotic analysis of the S_(n)(s) coefficients and the refinement of error bounds for the Hurwitz limit. Future steps should include extending these methods to Dirichlet L-functions to ensure consistency across the wider class of zeta-like objects.

References

Original Source: arXiv:hal-00476252

Related Study on Zeta Estimates: arXiv:hal-00345313

Classical Context: Titchmarsh, E. C., "The Theory of the Riemann Zeta-Function."
