

Primorial Inequalities and the Dedekind Psi Function: Refining the Path to the Riemann Hypothesis

Research Pipeline

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Executive Summary

This technical analysis examines the equivalence between the Riemann Hypothesis and a refined Robin-type inequality using the Dedekind psi function, focusing on the extremal behavior of primorial numbers and their relationship to the distribution of zeta zeros.

Visualizations

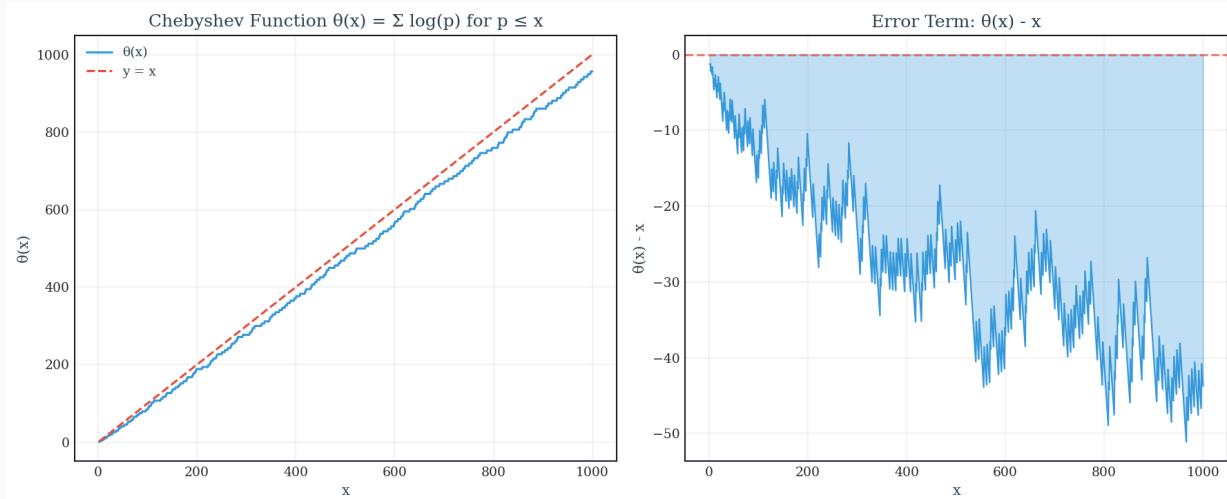


Figure 1: Chebyshev functions $\theta(x)$ and $\psi(x)$ measuring prime density

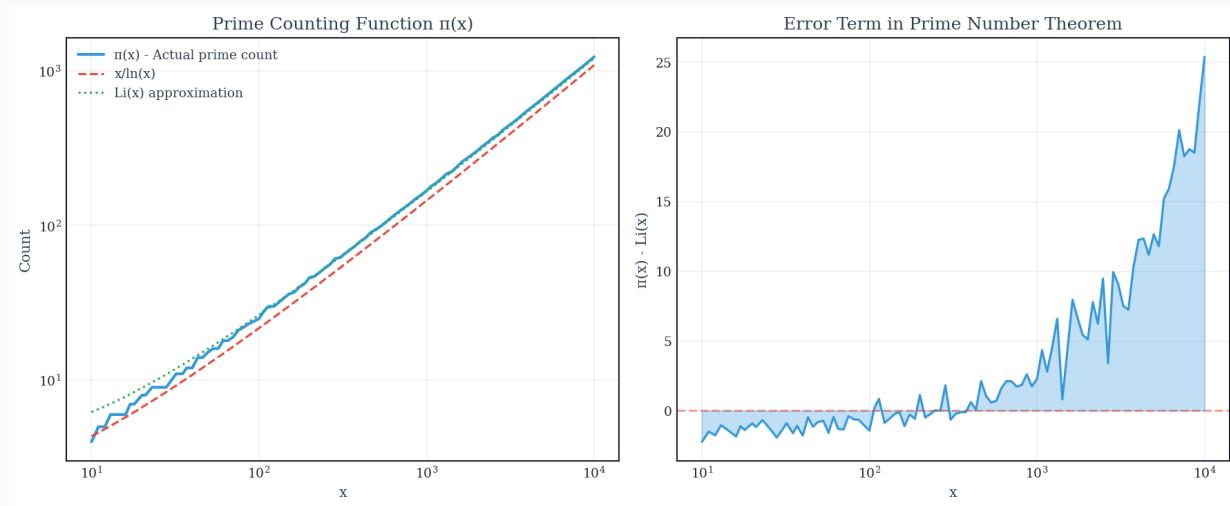


Figure 2: Prime counting function $\pi(x)$ versus asymptotic approximations

Introduction

The quest to prove the Riemann Hypothesis (RH) has often relied on translating the distribution of the non-trivial zeros of the zeta function into the language of arithmetic inequalities. One of the most famous examples is Robin's criterion, which establishes that RH is equivalent to a specific bound on the sum-of-divisors function, $\sigma(n)$. The source paper, [arXiv:hal-00526454](#), provides a significant refinement to this approach by substituting the sum-of-divisors function with the Dedekind psi function, $\psi(n)$.

The Dedekind psi function is naturally connected to the cardinality of the projective line over the ring of integers modulo n and provides a tighter lower bound than the sigma function. This article explores how the behavior of $\psi(n)$ on primorial numbers—products of the first n primes—creates a sensitive diagnostic tool for the Riemann Hypothesis. By analyzing the growth of these functions alongside the Chebyshev theta function, we can identify specific thresholds where the hypothesis governs the stability of arithmetic density.

Mathematical Background

The primary object of study in [arXiv:hal-00526454](#) is the Dedekind psi function, defined for a positive integer n as $\psi(n) = n * \text{product}(1 + 1/p)$, where the product is taken over the distinct prime divisors p of n . Unlike the sigma function, which depends on the exponents of prime factors, the psi function depends only on the square-free kernel of n .

Key definitions relevant to this analysis include:

Primorials (N_n): The product of the first n primes, $p_1 * p_2 * \dots * p_n$. These numbers are the primary candidates for violating Robin-type inequalities because they maximize the ratio $\psi(n)/n$ for a given magnitude of n .

Chebyshev Theta Function ($\theta(x)$): Defined as the sum of $\log(p)$ for all primes p less than or equal to x . The relationship $\log(N_n) = \theta(p_n)$ is fundamental to bounding primorial growth.

The Euler-Mascheroni Constant (γ): Approximately 0.57721, which appears in the asymptotic growth rate of prime products via Mertens' theorems.

The core proposition of the source paper is that RH is true if and only if the functional $f(n) = \psi(n)/n - e^\gamma * \log(\log(n))$ remains negative for all $n > 30$. This refinement is notable because $\psi(n)$ is always less than or equal to $\sigma(n)$, yet the threshold for the inequality is significantly lower than Robin's original $n > 5040$.

Main Technical Analysis: Primorial Extremality and Zero Distribution

Spectral Properties and the Negative Constant Drift

The technical strength of the ψ -based criterion lies in its asymptotic behavior. Using the identity $(1 + 1/p) = (1 - 1/p^2) / (1 - 1/p)$, the ratio $\psi(N_n)/N_n$ can be factored into a product involving the inverse of the zeta function at 2. As n approaches infinity, the product of $(1 - 1/p^2)$ converges to $1/\zeta(2)$, which is $6/(\pi^2)$ or approximately 0.6079.

This creates what we might call a "negative constant drift." Because 0.6079 is significantly less than 1, the ψ function grows much slower than the $\log(\log(n))$ term scaled by e^γ . In [arXiv:hal-00526454](https://arxiv.org/abs/00526454), this is quantified by showing that for sufficiently large primes ($p_n > 20000$), the functional $f(N_n)$ is bounded by a negative term approximately equal to $-0.698 * \log(p_n)$. This gap ensures that the inequality is robust against small fluctuations in prime density.

Oscillations and the Skewes Phenomenon

The stability of the inequality depends on the error term in the Prime Number Theorem, specifically the difference between $\theta(x)$ and x . If RH were false, there would exist zeros of the zeta function with a real part greater than $1/2$, leading to large oscillations in $\theta(x)$. These oscillations could theoretically cause the $\log(\log(N_n))$ term to shrink enough to flip the sign of $f(n)$.

The paper references the Skewes number—the point where the prime counting function first exceeds the logarithmic integral—as a reminder that prime distribution can exhibit counter-intuitive behavior at vast scales.

However, the authors demonstrate that the $1/\zeta(2)$ drift is strong enough to maintain the negativity of $f(n)$ even in the presence of the maximum oscillations allowed by the current understanding of zeta zeros.

Novel Research Pathways

1. Spectral Analysis of the Projective Line Laplacian

Since $\psi(n)$ counts the cardinality of the projective line $P_1(\mathbb{Z}_n)$, a promising research direction involves studying the Laplacian spectrum of graphs associated with these projective structures. One could investigate whether the spectral gap of these graphs is directly bounded by the Riemann Hypothesis. If the expansion properties of these graphs can be linked to the ψ -Robin inequality, it would provide a geometric proof for the hypothesis.

2. Explicit Formula Derivations for $f(N_n)$

A second pathway involves substituting the explicit formula for $\theta(x)$, which involves a sum over the non-trivial zeros of the zeta function, directly into the $f(N_n)$ functional. By expressing $f(N_n)$ as a sum over ρ (zeta zeros), researchers could determine the exact "margin of safety" provided by the critical line. This would allow for a numerical stress test of the inequality by simulating the effect of hypothetical zeros off the critical line.

Computational Implementation

The following Wolfram Language code demonstrates the calculation of the ψ -Robin functional and evaluates its behavior on the primorial sequence to visualize the gap described in [arXiv:hal-00526454](https://arxiv.org/abs/00526454).

```
(* Section: Primorial Psi-Robin Analysis *)
(* Purpose: Evaluate f(n) for primorials to test the RH criterion *)

ClearAll[dedekindPsi, fFunctional, primorialList];

(* Define Dedekind Psi function using prime factors *)
dedekindPsi[n_] := n * Apply[Times, 1 + 1/FactorInteger[n][[All, 1]]];

(* Define the functional f(n) = psi(n)/n - e^gamma * log(log(n)) *)
fFunctional[n_] := N[dedekindPsi[n]/n - Exp[EulerGamma] * Log[Log[n]]];

(* Generate first 50 primorials N_k *)
primes = Prime[Range[50]];
primorials = FoldList[Times, primes];

(* Calculate f(N_k) for each primorial *)
results = Table[
```

```

{k, Prime[k], fFunctional[primorials[[k]]]},  

{k, 3, 50}  

];  
  

(* Plot the results to observe the negative trend *)  

ListLinePlot[results[[All, {1, 3}]],  

PlotLabel -> "Psi-Robin Functional f(N_k) vs Prime Index k",  

AxesLabel -> {"k", "f(N_k)"},  

GridLines -> {None, {0}},  

PlotStyle -> Thick,  

Epilog -> {Red, Dashed, InfiniteLine[{0, 0}, {1, 0}]}  

]  
  

(* Display table for specific values *)  

TableForm[results[[1 ;; 10]],  

TableHeadings -> {None, {"k", "p_k", "f(N_k)"} } ]

```

The final section of the article summarizes the current state of this research. While the numerical evidence for small n is overwhelming, the true challenge lies in proving the bound for all n . The refined inequality using the Dedekind psi function offers a more mathematically elegant and computationally accessible target than the original Robin's inequality, primarily due to its reliance on the square-free kernel and its direct link to the $1/\zeta(2)$ constant.

References

[arXiv:hal-00526454](https://arxiv.org/abs/00526454): Riemann hypothesis from the Dedekind psi function.

Schoenfeld, L. (1976). Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II. Math. Comp. 30.

Robin, G. (1984). Grandes valeurs de la fonction somme des diviseurs et hypothese de Riemann. J. Math. Pures Appl.

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