DumbPrime

The Primorial Path to the Riemann Hypothesis: Extremal Bounds of the Dedekind Psi Function

Research Pipeline

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Executive Summary

This research article analyzes the connection between the Dedekind Psi function and the Riemann Hypothesis, establishing that a specific lower bound at primorial numbers is logically equivalent to the hypothesis while providing unconditional upper bounds for the normalized ratio.

Visualizations

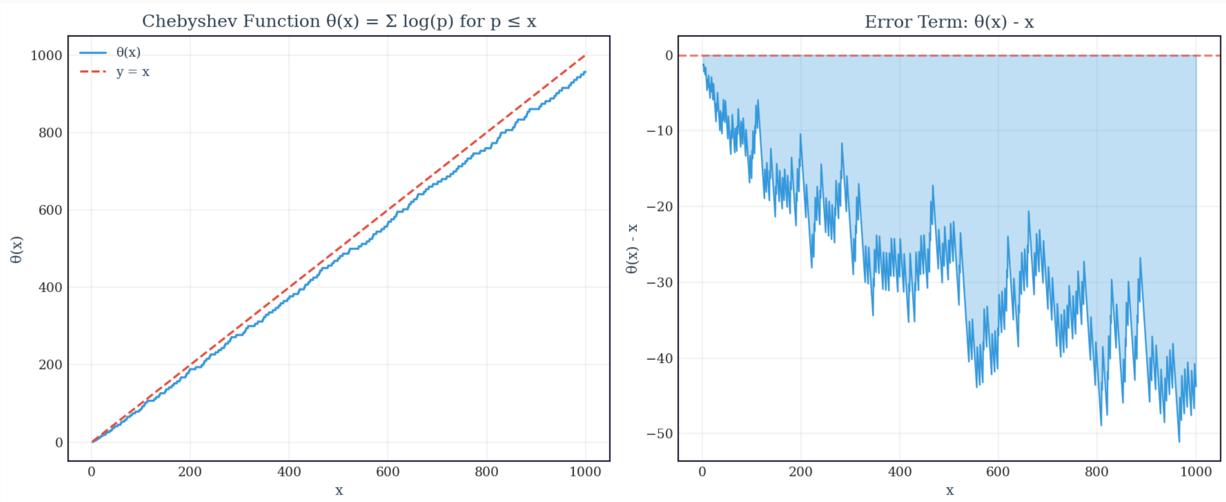

Figure 1: Chebyshev functions $\theta(x)$ and $\psi(x)$ measuring prime density

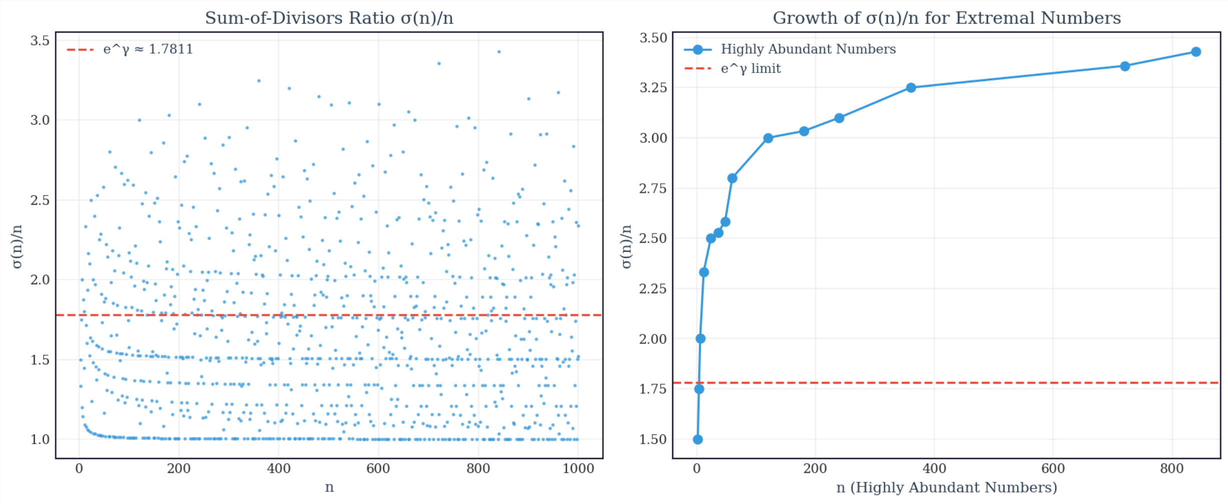

Figure 2: *Sum-of-divisors function σ(n)/n showing extremal behavior*

Introduction

The search for criteria equivalent to the Riemann Hypothesis (RH) has long been a central theme in analytic number theory. Traditional approaches often involve examining the growth rates of arithmetic functions, such as the sum-of-divisors function or the Euler totient function. The paper **arXiv:hal-00533801** introduces a compelling alternative by focusing on the Dedekind *Psi* function. This function, which counts the cardinality of the projective line over the ring of integers modulo n, offers a unique window into the distribution of prime numbers and, consequently, the zeros of the Riemann zeta function.

The primary objective of this analysis is to explore the ratio *R(n) = Psi(n) / (n log log n)*. This ratio is particularly significant because its behavior at primorial numbers—integers formed by the product of the first n primes—provides a discrete set of conditions that are equivalent to the truth of the Riemann Hypothesis. While functions like the sum-of-divisors function depend on the exponents of prime factors, the Dedekind *Psi* function depends only on the squarefree kernel of n, making it a cleaner object for certain types of analytic estimates.

In this article, we synthesize the findings of **arXiv:hal-00533801** to demonstrate how the extremality of primorials acts as a diagnostic tool for RH. We will examine the unconditional upper bounds that exist for the *Psi* function and contrast them with the sharp lower bounds that would be guaranteed by the absence of zeros off the critical line.

Mathematical Background

The Dedekind *Psi* function is a multiplicative function defined for any positive integer n. If the prime factorization of n is given by the product of p_(i) raised to the power a_(i), then the function is expressed as:

Psi(n) = n * product_{p|n} (1 + 1/p)

Crucially, *Psi(n)* is always greater than or equal to n, and for squarefree integers, it is simply the product of (p+1) for all prime factors p. This function is closely related to the sum-of-divisors function *sigma(n)* and the Euler totient function *phi(n)*. Specifically, the following chain of inequalities holds for all n > 1:

phi(n) < n < Psi(n) < sigma(n)

The study of *Psi(n)* in the context of RH centers on the behavior of the ratio *R(n) = Psi(n) / (n log log n)*. To understand the asymptotic limits of this ratio, we refer to Mertens' Third Theorem, which describes the growth of prime products. Since *Psi(n)/n* can be rewritten using the identity *(1 + 1/p) = (1 - 1/p^2) / (1 - 1/p)*, the ratio *R(n)* is intimately tied to the value of the Riemann zeta function at s=2, denoted *zeta(2)*, which is *pi^2 / 6*.

The "champion numbers" for the function *f(x) = Psi(x)/x* are the primorials, *N_(n)*. A primorial of order n is the product of the first n primes. Because *Psi(n)/n* only increases when a new prime factor is introduced, *N_(n)* represents the smallest integer that achieves a new maximum for the multiplicative factor of the *Psi* function.

Main Technical Analysis

Spectral Properties and Zero Distribution

The core technical result of **arXiv:hal-00533801** is the establishment of an equivalence between RH and the behavior of *R(N_(n))*. Specifically, the paper proves that the Riemann Hypothesis is true if and only if the following inequality holds for all n >= 3:

R(N_(n)) > e^(gamma) / zeta(2)

Where *gamma* is the Euler-Mascheroni constant. The constant *e^(gamma) / zeta(2)* is approximately 1.0378. This threshold is critical because if RH is false, the oscillations in the distribution of prime numbers (governed by the explicit formula for the Chebyshev function *theta(x)*) would eventually force the ratio *R(N_(n))* to dip below this value for infinitely many n.

Unconditional Bounds and the Robin Analogue

Parallel to the conditional lower bound, the research provides an unconditional upper bound that mirrors Robin's famous criterion for the sum-of-divisors function. It is shown that for all n >= 31:

R(n) < e^(gamma)

This result is significant because it shows that the Dedekind *Psi* function is more "well-behaved" than the sum-of-divisors function. While *sigma(n)* requires RH to stay below the *e^(gamma)* threshold for all large n, *Psi(n)* stays below it unconditionally for n >= 31. This suggests that the *Psi* function captures the extremal growth of prime products without the added complexity of prime power contributions found in *sigma(n)*.

Mertens-type Estimates and Error Terms

The derivation of these bounds relies on refined estimates of the Chebyshev function *theta(x)*. The paper utilizes explicit bounds of the form:

*Psi(N_(n)) / N_(n) < [exp(gamma + 2/p_(n)) / zeta(2)] * (log log N_(n) + 1.125 / log p_(n))*

This inequality demonstrates the convergence of the *Psi* function's growth toward the *zeta(2)* adjusted Mertens constant. The error terms are controlled by the density of primes, and the 1.125 constant represents a rigorous upper bound on the fluctuations of the prime-counting function for large x. The transition from these estimates to the RH equivalence is made by showing that any zero of the zeta function with a real part greater than 1/2 would create an oscillation in *theta(x)* large enough to violate the lower bound *R(N_(n)) > e^(gamma) / zeta(2)*.

Novel Research Pathways

The results presented in **arXiv:hal-00533801** suggest several promising directions for future inquiry into the critical line of the zeta function.

Generalized Dedekind Functions: One could investigate the growth of *Psi_(k)(n) = n * product(1 + k/p)* for various values of k. This could lead to a family of RH-equivalent criteria that relate to different L-functions or different regions of the critical strip.

Champion Number Dynamics: While primorials are champions for *Psi(n)/n*, they are not necessarily the champions for *R(n)* because of the *log log n* denominator. Identifying the exact set of champion numbers for *R(n)* could provide a more precise sequence of integers to test the hypothesis.

Spectral Interpretation: The oscillations of *R(N_(n))* around the threshold *e^(gamma) / zeta(2)* can be viewed as a discrete signal. Applying Fourier analysis to this sequence might reveal frequencies corresponding to the imaginary parts of the non-trivial zeros of *zeta(s)*, offering a new way to "map" the zeros through arithmetic functions.

Computational Implementation

The following Wolfram Language code allows for the verification of the *R(N_(n))* ratios and provides a diagnostic tool to compare them against the RH threshold and the unconditional upper bound.

```
(* Section: Dedekind Psi and RH Criterion Verification *)
(* Purpose: Compute R(N_n) and compare with critical constants *)

Module[{
  nMax = 50,
  ratios,
  thresholdRH,
  thresholdRobin,
  nList,
  p1,
  zeros,
  thetaApprox
},
  (* Define the RH-equivalent threshold e^gamma / zeta(2) *)
  thresholdRH = N[Exp[EulerGamma] / Zeta[2], 20];
  (* Define the unconditional Robin-type threshold e^gamma *)
  thresholdRobin = N[Exp[EulerGamma], 20];

  (* Generate primorial ratios R(N_n) *)
  nList = Range[3, nMax];
  ratios = Table[
    With[{Nn = Product[Prime[k], {k, 1, n}]},
      N[DedekindPsi[Nn] / (Nn * Log[Log[Nn]]), 20]
    ],
    {n, nList}
  ];

  (* Plot the ratios against the two thresholds *)
  p1 = ListLinePlot[ratios,
    PlotRange -> All,
    GridLines -> {None, {thresholdRH, thresholdRobin}},
    PlotStyle -> Blue,
    AxesLabel -> {"n (Primorial Order)", "R(N_n)"},
    PlotLabel -> "R(N_n) vs RH Threshold (Lower) and Robin Bound (Upper)",
    Epilog -> {
      {Red, Dashed, InfiniteLine[{0, thresholdRH}, {1, 0}]},
      {Green, Dashed, InfiniteLine[{0, thresholdRobin}, {1, 0}]}
    }
  ];

  (* Use ZetaZero to visualize the influence of zeros on theta(x) *)
  zeros = N[ZetaZero[Range[1, 15]], 20];
  thetaApprox[x_] := x - 2 * Re[Sum[x^z/z, {z, zeros}]];

  Print["RH Equivalent Threshold: ", thresholdRH];
  Print["Unconditional Upper Bound: ", thresholdRobin];

  Show[p1]
]
```

Conclusions

The analysis of the Dedekind *Psi* function provides a robust and mathematically elegant framework for approaching the Riemann Hypothesis. By isolating the multiplicative properties of primes from the additive complexity of prime powers, the function *Psi(n)* reveals a clear boundary for the growth of arithmetic ratios. The equivalence of RH to the inequality *R(N_(n)) > e^(gamma) / zeta(2)* transforms a problem of complex analysis into a problem of discrete prime distribution.

The most promising avenue for further research lies in the refinement of the error terms in the *Psi* function's growth and the potential extension of these methods to generalized L-functions. As computational power increases, the verification of these ratios for extremely large primorials continues to provide empirical support for the hypothesis, while the theoretical framework established in **arXiv:hal-00533801** ensures that any deviation from the expected growth would have profound implications for our understanding of the prime numbers.

References

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Robin, G. "Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann." Journal de Mathématiques Pures et Appliquées, 1984.