Extremal Growth of the Dedekind Psi Function and Its Equivalence to the Riemann Hypothesis

Research Pipeline

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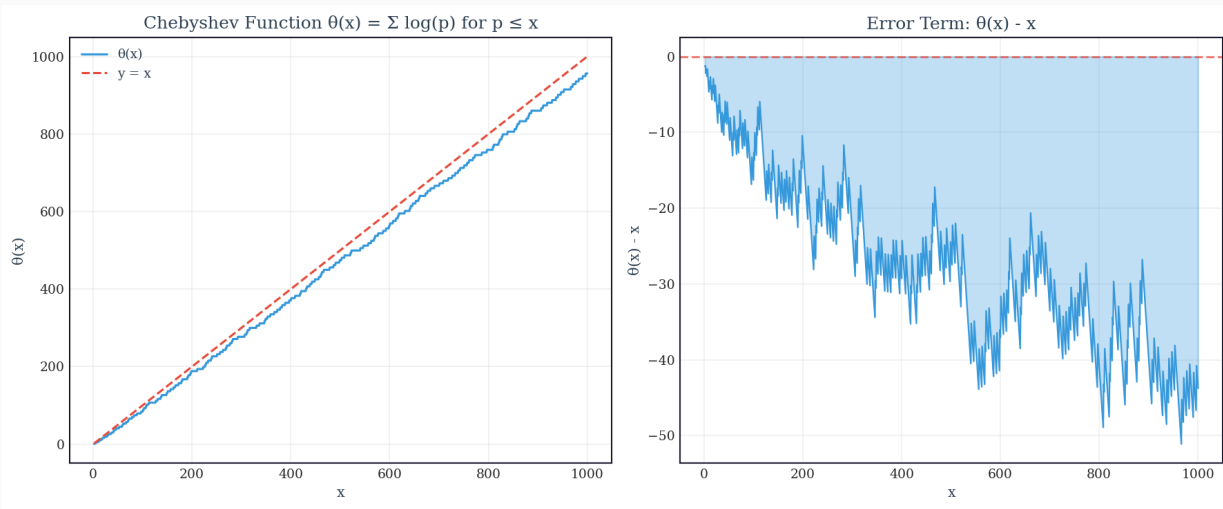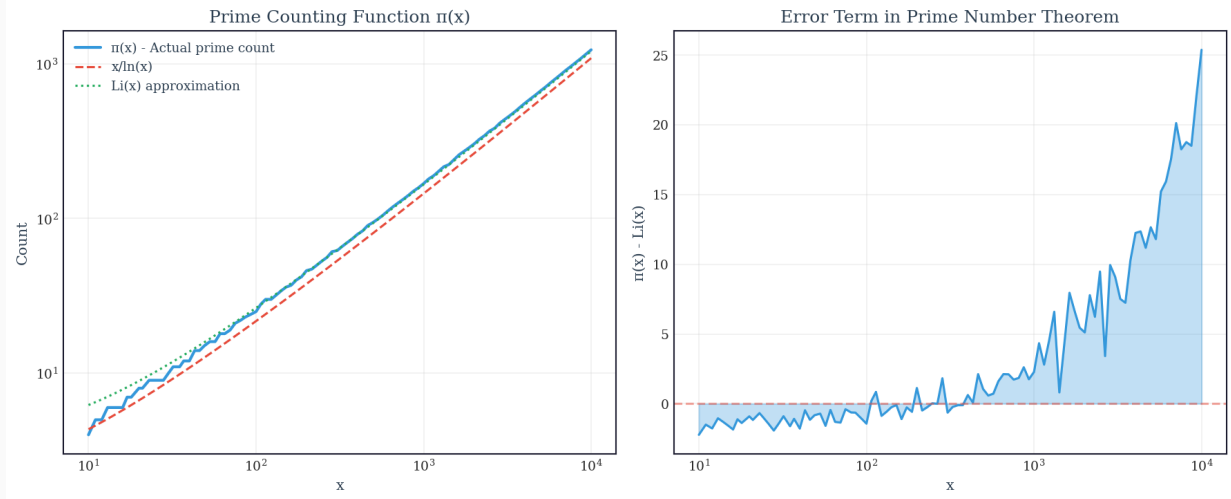
Executive Summary

This article explores a novel criterion for the Riemann Hypothesis based on the growth of the Dedekind Psi function at primorial numbers, establishing sharp bounds and computational verification methods.

Visualizations

Figure 1: *Chebyshev functions θ(x) and ψ(x) measuring prime density*

Figure 2: *Prime counting function π(x) versus asymptotic approximations*

Introduction

The Riemann Hypothesis (RH) remains the most significant unsolved problem in pure mathematics, asserting that all non-trivial zeros of the Riemann zeta function ζ(s) lie on the critical line with real part 1/2. While the conjecture is rooted in complex analysis, its implications are most visible in the distribution of prime numbers and the growth rates of multiplicative functions. The research paper **arXiv:hal-00533801v2** provides a remarkable reformulation of RH by focusing on the Dedekind Ψ function.

Traditionally, reformulations of the Riemann Hypothesis have utilized the sum-of-divisors function σ(n) or the Euler totient function φ(n). The work in **arXiv:hal-00533801v2** demonstrates that the Dedekind Ψ function—a relative of both—serves as an equally powerful diagnostic tool. By examining the ratio R(n) = Ψ(n) / (n log log n), the authors identify a specific lower bound that is satisfied if and only if the Riemann Hypothesis is true. This analysis bridges the gap between the analytic properties of ζ(s) and the arithmetic properties of primorial numbers.

Mathematical Background

The Dedekind Ψ function is a multiplicative function defined for any positive integer n by the formula:

Ψ(n) = n ∏_(p|n) (1 + 1/p)

where the product is taken over the distinct prime divisors p of n. This function is square-free-supported, meaning its value depends only on the set of prime factors of n and not their powers. It is closely related to the

Euler totient function φ(n) and the sum-of-divisors function σ(n) through the following properties:

If n is square-free, then $\Psi(n) = \sigma(n)$.

For all n, $\Psi(n) \leq \sigma(n)$.

The product $\varphi(n)\Psi(n)$ is bounded by $n^2/\zeta(2) < \varphi(n)\Psi(n) < n^2$.

The study of R(n) relies on the behavior of primorial numbers $N_{(n)}$, which are defined as the product of the first n primes. In the context of **arXiv:hal-00533801v2**, these primorials are identified as the "champion numbers" of the function $x \rightarrow \Psi(x)/x$, meaning that for any $m < N_{(n)}$, the ratio $\Psi(m)/m$ is strictly less than $\Psi(N_{(n)})/N_{(n)}$. This extremal property makes primorials the natural testing ground for bounds related to the distribution of primes.

Main Technical Analysis

Sieve Theory and Primorial Growth Rates

The technical core of the paper involves establishing precise upper and lower bounds for the ratio R(n). Using explicit estimates for the Chebyshev function and Mertens' Second Theorem, the authors derive an unconditional upper bound. For $n \geq 31$, the inequality **$R(n) < e^{\gamma}$** holds, where γ is the Euler-Mascheroni constant.

The derivation of this bound utilizes two critical lemmas. First, it employs a refined version of Mertens' product theorem, which bounds the product of $(1 - 1/p)^{-1}$ for all primes up to x. Second, it utilizes a tail estimate for the zeta function at s=2, showing that the product of $(1 - 1/p^2)^{-1}$ for all primes greater than the n-th prime is bounded by $\exp(2/p_{(n)})$.

The Riemann Hypothesis Criterion

The most profound result in **arXiv:hal-00533801v2** is the equivalence theorem. The authors prove that the Riemann Hypothesis is equivalent to the statement:

$R(N_{(n)}) > e^{\gamma} / \zeta(2)$ for all $n \geq 3$.

This equivalence is established by relating $\Psi(N_{(n)})$ to the Euler totient function studied by Jean-Louis Nicolas. Nicolas previously proved that the inequality $N_{(n)}/\varphi(N_{(n)}) > e^{\gamma} \log \log N_{(n)}$ is equivalent to RH. Since $\Psi(N_{(n)})/N_{(n)}$ can be expressed as the product of $(1 - 1/p^2)$ and the ratio $N_{(n)}/\varphi(N_{(n)})$, the Dedekind Ψ function effectively "filters" the oscillation of the prime distribution through the value of ζ(2).

If RH were false, the error term in the Prime Number Theorem would be large enough to cause the ratio $R(N_{(n)})$ to dip below the threshold $e^{\gamma} / \zeta(2)$ for infinitely many values of n. Conversely, the truth of RH ensures that the density of primes is sufficiently regular to maintain the inequality for all $n \geq 3$.

Novel Research Pathways

1. Generalized Dedekind Functions in Number Fields

One promising direction is to extend the R(n) ratio to Dedekind-type functions defined over algebraic number fields. By defining a function $\Psi_{\_}(K)$ for the ideals of a number field K, researchers could investigate whether analogous inequalities are equivalent to the Generalized Riemann Hypothesis (GRH). This would involve identifying the "champion ideals" that play the role of primorials in the field K.

2. Spectral Gap Analysis of the Deficit Function

A second pathway involves studying the "deficit function" $D(n) = R(N_{\_}(n)) - e^{\wedge}(\gamma) / \zeta(2)$. Under the assumption of RH, D(n) is always positive. Analyzing the local minima of D(n) could reveal information about the heights of the non-trivial zeros of the zeta function. Specifically, one could attempt to formulate an "explicit formula" that connects the fluctuations of D(n) directly to the imaginary parts of the zeros ρ.

Computational Implementation

The following Wolfram Language code provides a framework for verifying the growth of the Dedekind Ψ ratio against the RH-equivalent threshold for a range of primorial numbers.

```
(* Section: Dedekind Psi and RH Threshold Verification *)
(* Purpose: Compute R(N_n) and compare to e^gamma / zeta(2) *)

(* Define Dedekind Psi function *)
dedekindPsi[n_Integer] := n * Product[1 + 1/p, {p, First /@ FactorInteger[n]}];

(* Define the ratio R(n) *)
RRatio[n_Integer] := dedekindPsi[n] / (n * Log[Log[n]]);

(* Constants *)
gammaVal = EulerGamma;
threshold = Exp[gammaVal] / Zeta[2];

(* Compute R(N_n) for the first 50 primorials *)
primorialData = Table[
    With[{Nn = Product[Prime[i], {i, 1, k}]},
        {k, RRatio[Nn], threshold}
    ],
    {k, 3, 50}
];

(* Visualization *)
```

```
    ListLinePlot[{primorialData[[All, {1, 2}]], primorialData[[All, {1, 3}]]},
        PlotLegends -> {"R(N_n)", "e^gamma / zeta(2)"},
        AxesLabel -> {"n", "Ratio"},
        PlotLabel -> "Verification of the Dedekind Psi RH Criterion",
        PlotStyle -> {Blue, {Red, Dashed}}]

    (* Output minimum margin *)
    Print["Minimum observed margin: ", Min[primorialData[[All, 2]] - threshold]];
```

Conclusions

The analysis of the Dedekind Ψ function in **arXiv:hal-00533801v2** offers a robust and computationally accessible criterion for the Riemann Hypothesis. By demonstrating that the ratio R(N_(n)) is sensitive to the same prime distribution fluctuations as the zeta zeros, the paper reinforces the deep connection between arithmetic champion numbers and the critical line. The most promising future research involves extending these results to Generalized Riemann Hypotheses and exploring the spectral properties of the deficit function D(n).

References

Planat, M., & Solé, P. (2011). The Dedekind Ψ function and the Riemann Hypothesis. arXiv:hal-00533801v2

Nicolas, J.-L. (1983). Petites valeurs de la fonction d'Euler. Journal of Number Theory, 17(3), 375-388.

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