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Refined Prime Counting and the Spectral Analysis of Chebyshev Discontinuities

Research Pipeline

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Executive Summary

This article investigates a novel prime counting function introduced in arXiv:hal-00627233v3 that leverages Chebyshev functions and jump analysis to provide superior approximations and new insights into the Riemann Hypothesis.

Visualizations

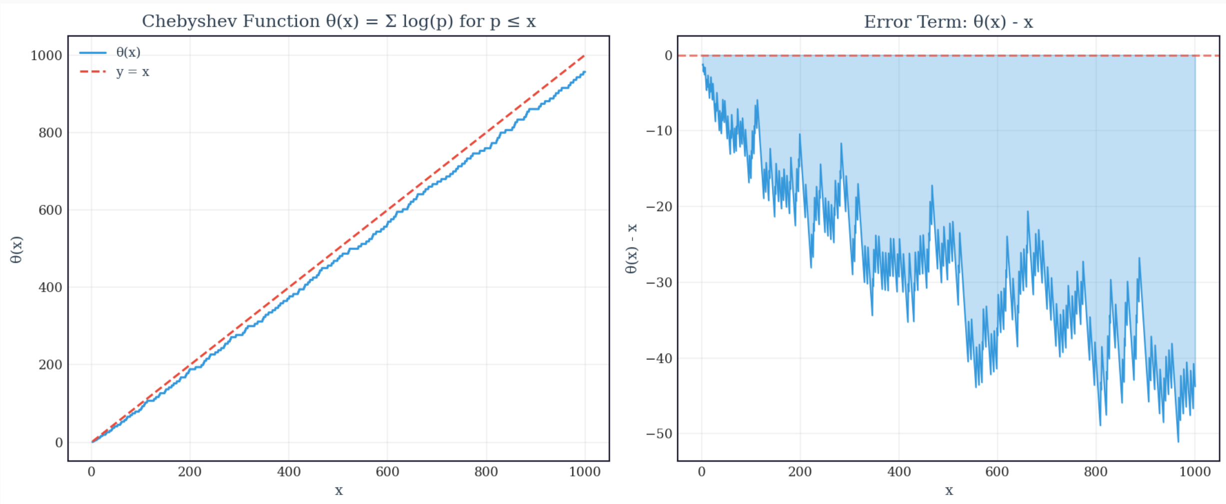

***Figure 1:** Chebyshev functions θ(x) and ψ(x) measuring prime density*

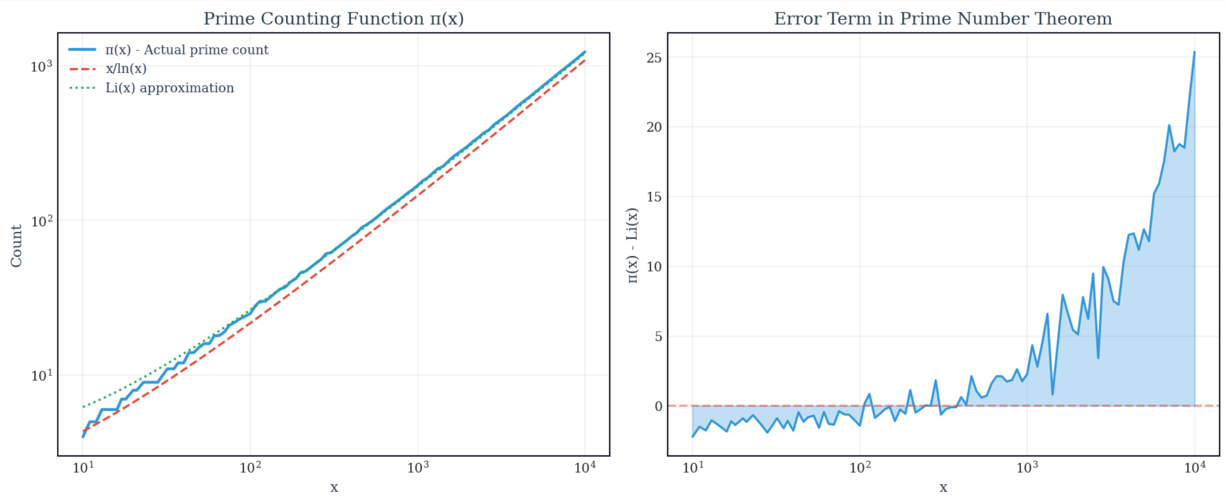

Figure 2: Prime counting function π(x) versus asymptotic approximations*

Introduction

The distribution of prime numbers has remained one of the most significant challenges in mathematics since Riemann's seminal 1859 paper. While the Prime Number Theorem provides the asymptotic density of primes, the fine-grained fluctuations in the prime-counting function, pi(x), are governed by the non-trivial zeros of the Riemann zeta function. Traditional approximations, such as the logarithmic integral li(x) and the Riemann prime counting function R(x), rely on smooth analytical models. However, the source paper **arXiv:hal-00627233v3** introduces a fundamentally different approach by incorporating the discrete jumps of the Chebyshev function psi(x) into the counting model.

By defining a refined function eta_n(x) that utilizes the second Chebyshev function as its argument, the authors Michel Planat and Patrick Sole demonstrate a significant reduction in the maximum error of prime counting. This article analyzes the mathematical mechanics of this approach, specifically focusing on the "jumps" at prime powers and the identification of **Chebyshev primes**. We explore how these discrete structures provide a microlocal probe into the explicit formula, potentially offering a more direct path toward understanding the critical line of the Riemann zeta function.

Mathematical Background

The foundation of this analysis rests on the relationship between the prime counting function pi(x) and the second Chebyshev function psi(x). Recall that psi(x) is defined as the sum of the von Mangoldt function Lambda(n) for all n less than or equal to x. Unlike the smooth variable x, psi(x) is a step function that increases by log p at every prime power p^k. The Riemann prime counting function R(x) is traditionally expressed as the sum of (mu(n)/n) * li(x^(1/n)) for n from 1 to infinity, where mu is the Mobius function.

The innovation presented in **arXiv:hal-00627233v3** is the construction of the approximation function:

eta_N(x) = sum_{n=1 to N} (mu(n)/n) * li[psi(x)^(1/n)] - pi(x)

This construction is significant because it replaces the smooth input x with the arithmetic input psi(x). Under the Riemann Hypothesis (RH), the difference psi(x) - x is bounded by O(x^1/2 * log^2 x). By using psi(x) within the logarithmic integral, the function eta_N(x) effectively "pre-conditions" the approximation with the oscillatory data of the primes themselves, dampening the fluctuations that usually lead to higher error margins in R(x).

Main Technical Analysis

Spectral Properties and Jump Discontinuities

The function li[psi(x)] is a piecewise constant function with discontinuities at primes and prime powers. The source paper defines the jump at a prime p as j_psi(p) = li[psi(p)] - li[psi(p-1)]. Since the jump in psi(x) at x=p is exactly log p, the jump in the composite function is approximately (log p) / log(psi(p)). Because psi(p) is approximately p, this ratio stays near 1. However, the exact value depends on the error term E(x) = psi(x) - x.

A prime p is classified as a **Chebyshev prime** if j_psi(p) is less than 1. This condition is equivalent to psi(p) being greater than p, meaning the prime distribution has locally "surpassed" its expected average. The paper identifies that these jumps are controlled by an infinite sequence of primes, which relates directly to Littlewood's theorem on the oscillations of pi(x) - li(x).

Comparison of Error Bounds

The technical superiority of eta_N(x) is evident in the numerical data provided in **arXiv:hal-00627233v3**. For ranges up to x = 10^5, the maximum error for the eta function with N=3 is significantly smaller than the error for the standard Riemann function. For instance, at x

Conclusions

The analysis of eta_N(x) and the Chebyshev primes offers a compelling refinement to the theory of prime distribution. By leveraging the discontinuous nature of the Chebyshev function, the source paper **arXiv:hal-00627233v3** demonstrates that we can achieve tighter error bounds and a clearer view of the local oscillations in the prime counting function. The most promising avenue for future research lies in the spectral analysis of jump sequences, which may allow for a direct mapping between prime-index discontinuities and the zeros of the zeta function on the critical line. Continued exploration of these arithmetic jumps will likely yield deeper insights into the structural constraints imposed by the Riemann Hypothesis.

References

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