Quantitative Finance

Lectures in Quantitative Finance

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7. Stochastic Calculus

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Introduction

Goal of chapter. We will introduce the theory of elementary *stochastic calculus*, in particular *Itô's lemma* and *stochastic differential equations*, with the application to the *Black-Scholes model*.

A *random* nature of financial markets make stochastic calculus extremely important in the mathematical modeling of financial processes.

We will discuss the basic theory behind

1. Brownian motions,
2. stochastic integration,
3. stochastic differential equations,
4. Itô's lemma, and
5. application to the Black-Scholes model.

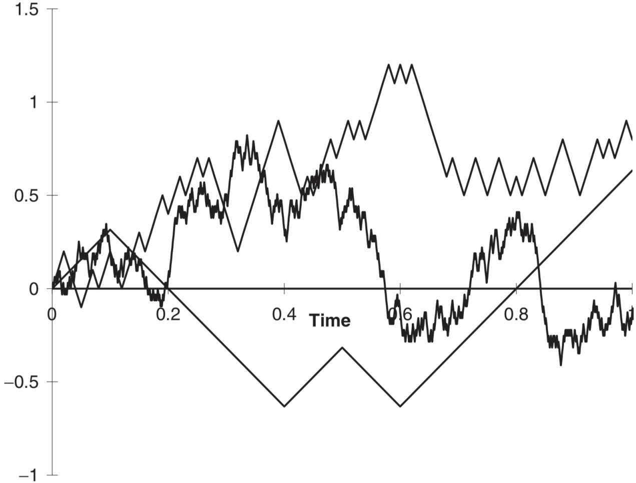

Figure 1: Brownian motion can be seen as a limit of (appropriately scaled) series of coin-tossing experiments.

- Discrete time stochastic processes presented in the previous chapters (*binomial trees*) can be seen as an *approximation* of the continuous time stochastic processes from this chapter.
- Equivalently, continuous time models can be obtained as a *limit* of the discrete time models, where the number of periods n goes to infinity (and hence $\Delta t \to 0$).
- In practice one does not observe asset prices following continuous time processes.
- Stochastic calculus equips us with the mathematical tools which are not available in discrete time (notably *Itô's lemma*).

Brownian motion

Since we work with stochastic processes, we have to equip the usual *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ with a *filtration* $(\mathcal{F}_t)_{t \geq 0}$. Recall:

Definition. A filtration $\mathbb{F}$ is an increasing sequence of σ-algebras on a measurable space. That is, given a measurable space $(\Omega, \mathcal{F})$, a filtration is a sequence of σ-algebras $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{F}$, such that $t_1 \leq t_2$ implies $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

A *filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, is a probability space equipped with the filtration $(\mathcal{F}_t)_{t \geq 0}$ of its σ-algebra $\mathcal{F}$.

Let $T > 0$ and suppose that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0 \le t \le T)})$ is given.

Definition. A one-dimensional standard *Brownian motion* is a stochastic process $(W_t)_{0 \le t \le T}$ with the following properties:

1. $W_0 = 0$,
2. $t \mapsto W_t(\omega)$ is a continuous function $\mathbb{P}$-a.s.,
3. $(W_t)_{0 \le t \le T}$ is adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$,
4. the increments $W_t - W_s$ are independent and normally distributed with variance $t - s$ and zero mean for any $0 \le s \le t$.

▷ *Intuitively*, the change of Brownian motion on the interval $(t, t + \Delta t)$ can be perceived as $\Delta W_t := W_{t+\Delta t} - W_t = \epsilon\sqrt{\Delta t}$, where $\epsilon \sim \mathcal{N}(0, 1)$.

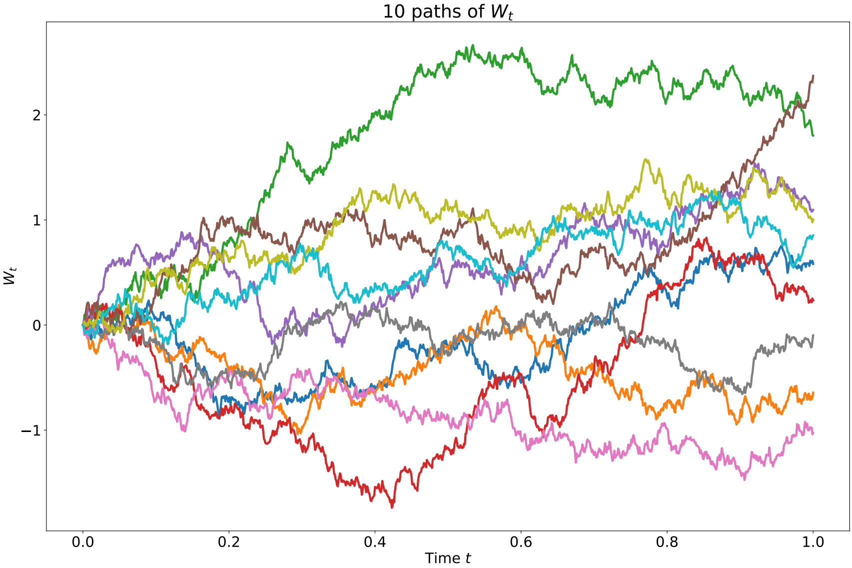

Figure 2: 10 paths of a one-dimensional standard Brownian motion.

- *Martingale*: Given the information up to $s < t$ the conditional expectation of W_t is W_s, that is

$$\mathbb{E}(W_t \mid \mathcal{F}_s) = W_s.$$

- *Quadratic variation*: If we divide up the time interval $[0, t]$ into a partition P with $n + 1$ points t_i, then

$$\langle W \rangle_t = \lim_{n\to\infty} \sum_{t_i \in P} (W_{t_i} - W_{t_{i-1}})^2 \xrightarrow[a.s.]{} t. \tag{1}$$

- *Markov property*: The conditional distribution of W_t given information up until $s < t$ depends only on W_s. That is, for any event A and $s < t$ it holds that

$$\mathbb{P}(W_t \in A \mid \mathcal{F}_s) = \mathbb{P}(W_t \in A \mid W_s).$$

- *Normality*: Over finite time increments t_{i-1} to t_i, $W_{t_i} - W_{t_{i-1}}$ is normally distributed with mean zero and variance $t_i - t_{i-1}$.

- *Continuity*: The paths of Brownian motion are continuous, there are no discontinuities. Brownian motion is the continuous-time limit of the discrete-time random walk.

- *Non-differentiability*: Almost surely (with probability 1), the paths are not differentiable at any point. Therefore, *it is not possible* to define $\int_0^t f(s)dW_s$ as $\int_0^t f(s)\frac{dW_s}{ds}\,ds$, i.e.

$$\int_0^t f(s)dW \neq \int_0^t f(s)\frac{dW_s}{ds}\,ds.$$

Stochastic integration

In order to construct a *stochastic integral* let us first define the notion of a simple stochastic process.

Definition. A process $(H_t)_{0\leq t\leq T}$ is a *simple process* if it can be written as

$$H_t(\omega) = \sum_{i=1}^{p} \phi_i(\omega)\mathbb{I}_{(t_{i-1},t_i]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_p = T$ and ϕ_i is $\mathcal{F}_{t_{i-1}}$-measurable.

Definition. A *stochastic integral* of a simple process $H = (H_t)_{0 \le t \le T}$ is a process $(\int_0^t H_s dW_s)_{0 \le t \le T}$ given by

$$\int_0^t H_s dW_s = \sum_{i=1}^p \phi_i (W_{t_i \wedge t} - W_{t_{i-1} \wedge t}),$$

where $a \wedge b = \min\{a, b\}$.

- Note that in particular $\int_0^t 1 \, dW_s = W_t$.
- Given a $(\mathcal{F}_t)_{0 \le t \le T}$-Brownian motion $(W_t)_{0 \le t \le T}$ and a $(\mathcal{F}_t)_{0 \le t \le T}$-adapted process $(H_t)_{0 \le t \le T}$, one is able to *define* the stochastic integral $(\int_0^t H_s dW_s)_{0 \le t \le T}$ as soon as $\int_0^T H_s^2 dW_s < \infty$.
- The *general* concept *extends* that for simple processes.

Stochastic integration: Example

To illustrate why the *direct use* of the definitions is *complicated*, let us compute the stochastic integral $\int_0^t W_s dW_s$ for a Brownian motion W_s.

Consider a partition P with $n+1$ points t_i. Because

$$2W_{t_i}(W_{t_{i+1}} - W_{t_i}) = W_{t_{i+1}}^2 - W_{t_i}^2 - (W_{t_{i+1}} - W_{t_i})^2,$$

we obtain by summing up that

$$\sum_{t_i \in P} W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{1}{2}(W_t^2 - W_0^2) - \frac{1}{2}\sum_{t_i \in P}(W_{t_{i+1}} - W_{t_i})^2.$$

If $n \to \infty$, then the sum on the right-hand side becomes the quadratic variation in (1) and converges $\mathbb{P}$-a.s. to t. We therefore expect to obtain

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t,$$

and we will see later from Itô's lemma that this is indeed correct.

- Note that we should expect the first term $\frac{1}{2}W_t^2$ from *classical calculus* (where we have $\int_0^x y\, dy = \frac{1}{2}x^2$).
- The second-order correction term $\frac{1}{2}t$ appears due to the *quadratic variation* of Brownian trajectories.
- Since the direct use of definitions is complicated, we will now present the basic tool for working with stochastic processes and especially stochastic integrals in continuous time $\Longrightarrow$ *Itô's lemma*.

Stochastic differential equations

Let us use the notion of stochastic integral and extend it to the concept of *stochastic differential equations*.

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space and $(W_t)_{0 \leq t \leq T}$ a $\mathbb{F}$-Brownian motion. An $\mathbb{R}$-valued process $(X_t)_{0 \leq t \leq T}$ is called an *Itô process* if it can be written as

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s, \tag{2}$$

where

- X_0 is $\mathcal{F}_0$-measurable,
- $(K_t)_{0 \leq t \leq T}$ and $(H_t)_{0 \leq t \leq T}$ are $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted processes,
- $\int_0^t |K_s| ds < \infty \quad \mathbb{P}$-a.s.,
- $\int_0^t H_s^2 ds < \infty \quad \mathbb{P}$-a.s.

- Consider an Itô process $X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$ from the previous slide. Its shorthanded version

$$dX_t = K_t dt + H_t dW_t,$$

is called a *stochastic differential equation*.

- Note that its precise meaning comes, however, from the technically more accurate equivalent stochastic integral. In practice, the shorthanded version is used almost everywhere.

- Think of dX_t as an *increment* of the process X_t.

- A Brownian motion $(W_t)_{0\leq t\leq T}$ is itself an Itô process (2), where $W_0 = 0, K_s = 0$ and $H_s = 1$.

Itô's lemma

The theorem below represents the main result of *stochastic calculus*.

Theorem. (*Itô's lemma* or *Itô's formula*) Let $(X_t)_{0\leq t\leq T}$ be an Itô process, $X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$. Let $(t,x) \mapsto f(t,x)$ be a function which is twice differentiable w.r.t. x and once w.r.t. t such that these partial derivatives are continuous w.r.t. (t,x). Then

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)dX_s + \\ + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)d\langle X\rangle_s,$$

where by definition

$$\int_0^t \frac{\partial f}{\partial x}(s,X_s)dX_s =: \int_0^t \frac{\partial f}{\partial x}(s,X_s)K_s ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)H_s dW_s$$

and

$$\langle X\rangle_t := \int_0^t H_s^2 ds.$$

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Why is *Itô's lemma* so useful?

Take an Itô process (X_t), given by $dX_t = K_t dt + H_t dW_t$. It makes sense to consider another process $Y_t = f(t, X_t)$, which is a *function* of the *former*. The Itô's formula yields

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)d\langle X\rangle_t, \qquad (3)$$

where the differentials are formally computed according to the following rules

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0 \quad \text{and} \quad dW_t \cdot dW_t = dt.$$

Note that (3) is just a shorthanded version of Itô's lemma from the previous slide for $Y_t = f(t, X_t)$.

▷ *Financial relevance* of Itô's lemma: Think of X_t as some underlying financial asset and of $Y_t = f(t, X_t)$ as a new product obtained from the underlying by a possibly nonlinear transformation f.

▷ Then (3) shows us how the financial product reacts to changes in the underlying.

▷ The important message of *Itô's lemma* is then that when using stochastic models (for X_t), a simple linear approximation is not good enough, one must also account for the second order behavior of X_t.

Example.

Let $(X_t)_{0\le t\le T}$ be the Brownian motion $(W_t)_{0\le t\le T}$ and $f(t,x)=x^2$. Then

$$W_t = 0 + \int_0^t 0 ds + \int_0^t \underbrace{1}_{H_s} dW_s, \qquad \langle W\rangle_t = \int_0^t 1^2 ds = t,$$

and the differentials are

$$\frac{\partial f}{\partial t}(t,x) = 0, \qquad \frac{\partial f}{\partial x}(t,x) = 2x, \qquad \frac{\partial^2 f}{\partial x^2}(t,x) = 2.$$

By *Itô's formula* we have

$$f(t, W_t) = f(0, \underbrace{0}_{=W_0}) + \int_0^t \frac{\partial f}{\partial t}(s, W_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, W_s)dW_s + \\ + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, W_s)d\langle W\rangle_s,$$

with

$$W_t^2 = 0 + 0 + 2\int_0^t W_s\, dW_s + \frac{1}{2}\int_0^t 2\, dt = 2\int_0^t W_s\, dW_s + t,$$

or equivalently

$$W_t^2 - t = 2\int_0^t W_s\, dW_s,$$

which exactly ties up with the example on stochastic integration we have seen earlier.

Remark. Let $x : [0, \infty) \to \mathbb{R}$ be a function $t \mapsto x(t)$ and think of x as a typical *trajectory* $t \mapsto X_t(\omega)$ of X. The classical *chain rule* from analysis then says that if x is in C^1 (i.e. *continuously differentiable*) and $f : \mathbb{R} \to \mathbb{R}$ is in C^1, the *composition* $f \circ x : [0, \infty) \to \mathbb{R}$, $t \to f(x(t))$ is again in C^1 and its derivative is given by

$$\frac{d}{dt}(f \circ x)(t) = \frac{df}{dx}(x(t))\frac{dx}{dt}(t),$$

or more compactly

$$(f \circ x)^{\boldsymbol{\cdot}}(t) = f'\big(x(t)\big)\,\dot{x}(t),$$

where the dot $^{\boldsymbol{\cdot}}$ denotes the derivative with respect to t and the prime $'$ is the derivative with respect to x.

Remark. In formal *differential* notation, we can rewrite this as

$$d(f \circ x)(t) = f'\big(x(t)\big)\, dx(t), \tag{4}$$

or in *integral* form

$$f\big(x(t)\big) - f\big(x(0)\big) = \int_0^t f'\big(x(s)\big)\, dx(s).$$

In this last form, the chain rule can be extended to the case where f is in C^∞ (i.e. *smooth function*) and x is *continuous* and of *finite variation*.

Remark.

- The trajectories of Brownian motion are of *infinite variation* (and finite quadratic variation) and the classical chain rule does not apply.
- One can view Itô's formula as a purely analytical result which provides an *extension* of the *chain rule* for $f \circ x$ to functions x that have a nonzero quadratic variation.
- Comparing (3) to (4) shows that we have in comparison to the classical chain rule an *extra* second-order *term* coming from the *quadratic variation* of X.

Black-Scholes model

- The *Black-Scholes model* is the original breakthrough in field of *quantitative finance* that led to huge growth in the industry and the development of the subject.
- It is a mathematical model for the dynamics of a financial market containing *derivative* investment *instruments*.
- The model describes and explains the basic building blocks of derivatives theory (such as *delta hedging* and *no arbitrage*).
- Its derivation applies the mathematical tools presented during this lecture.
- These fundamental principles lead to the *Black–Scholes* derivatives pricing *equation*.

Assumptions: the market consists of at least one *risky asset* (usually called the stock) and one *riskless asset* (usually called the money market, cash, or bond).

Assumptions on the *assets*:

- The rate of return on the riskless asset is constant, known and the same for all maturities (*risk-free interest rate*).
- The instantaneous *log return* of *stock* price follows an Itô process with *constant drift* and *volatility* (returns on the underlying stock are *normally distributed*).
- The derivative is European and can only be exercised at expiration.
- There are *no dividends* on the underlying during the life of the derivative.

Assumptions on the *market*:

- There are *no arbitrage* opportunities (i.e. there is no way to make a riskless profit).
- It is possible to *borrow* and *lend* any amount (*perfect divisibility*) of cash at the *riskless rate*.
- It is possible to buy and sell any amount (perfect divisibility) of the stock, including *short selling* (markets are perfectly *liquid*).
- The above transactions do not incur any fees, taxes or *transaction costs* (i.e. *frictionless market*).
- *Delta hedging* (as well as trading) is performed *continuously*.

Definition.

A stochastic process S_t is said to follow a *geometric Brownian motion* (GBM) if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = x_0,$$

where W_t is a Brownian motion, and $\mu \in \mathbb{R}$, $\sigma > 0$, and $x_0 > 0$ are constants.

- Note that the *instantaneous return* of the *underlying* following an *Itô process* $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$ implies a *GBM* for the *stock price*.
- For a *small* Δt (short time period), the percentage change in the stock price (return) can be *approximated* by $\frac{\Delta S}{S} \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t))$.
- *Next step*: solve the geometric Brownian motion stochastic differential equation.

Black–Scholes model

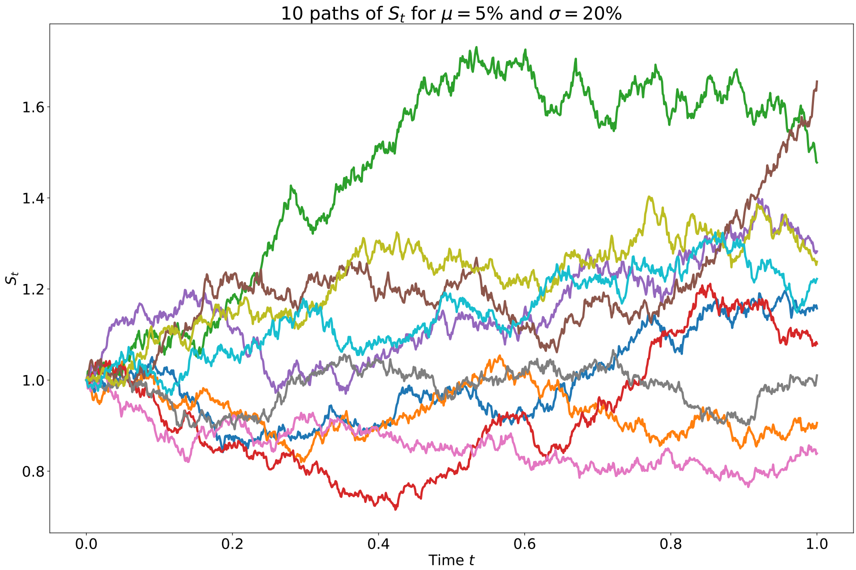

Figure 3: 10 paths of geometric Brownian motion with $\mu = 0.05$ and $\sigma = 0.2$.

Example.

Find the solution of

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = x_0.$$

Reformulation: We look for an adapted process $(S_t)_{0 \le t \le T}$ such that the integrals $\int_0^t S_s ds$ and $\int_0^t S_s dW_s$ exist and at any time t we have

$$S_t = x_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s \quad \mathbb{P} - \text{a.s.}$$

Formal computation: We write $Y_t = \log S_t$ where $(S_t)_{0 \le t \le T}$ is a solution of the initial equation. Note that (S_t) is an Itô process with

$$K_s = \mu S_s, \quad H_s = \sigma S_s.$$

Assuming $S_t > 0$, we apply Itô's lemma formally to $f(x) = \log x$ and obtain $\partial f/\partial t = 0$, $\partial f/\partial x = 1/x$, $\partial^2 f/\partial x^2 = -1/x^2$. The quadratic variation is given by

$$\langle S \rangle_t = \int_0^t H_s^2 ds = \int_0^t \sigma^2 S_s^2 \, ds.$$

Therefore, $d\langle S \rangle_t = \sigma^2 S_t^2 \, dt$ and we have

$$\log(S_t) = \log(S_o) + \int_0^t \frac{1}{S_s} dS_s + \frac{1}{2} \int_0^t -\frac{1}{S_s^2} \underbrace{\sigma^2 S_s^2 \, ds}_{d\langle S \rangle_s},$$

which can be rearranged as

$$Y_t = Y_0 + \int_0^t (\mu ds + \sigma dW_s) - \frac{1}{2}\sigma^2 t.$$

Solving the integral yields

$$Y_t = Y_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t,$$

where one can use the definition of Y_t and get

$$\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t.$$

Now we can apply the exponential function and obtain

$$S_t = S_0 \cdot \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t). \tag{5}$$

Conjecture: $S_t = x_0 \cdot \exp((\mu - \sigma^2/2)t + \sigma W_t)$ is a solution of the initial equation.

This has to be carefully proved.

Theorem.

Consider a real number μ, a strictly positive $\sigma > 0$ and $T > 0$, and a Brownian motion $(W_t)_{0 \leq t \leq T}$. Then, there exists a unique Itô process $(S_t)_{0 \leq t \leq T}$, which for any $t \leq T$ satisfies

$$S_t = x_0 + \int_0^t (\mu S_s ds + \sigma S_s dW_s) .$$

This process is given by

$$S_t = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) .$$

Proof.

Consider $g(t,x) = x_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma x)$ and $S_t = g(t, W_t)$. Then

$$\frac{\partial g}{\partial t} = x_0\,(\mu - \frac{\sigma^2}{2})\ \exp((\mu - \frac{\sigma^2}{2})t + \sigma x),$$
$$\frac{\partial g}{\partial x} = x_0\,\sigma\ \exp((\mu - \frac{\sigma^2}{2})t + \sigma x),$$
$$\frac{\partial^2 g}{\partial x^2} = x_0\,\sigma^2\ \exp((\mu - \frac{\sigma^2}{2})t + \sigma x).$$

Since $\langle W\rangle_t = t$, Itô's formula is applicable and

$$S_t = x_0 + \int_0^t S_s(\mu - \frac{\sigma^2}{2})ds + \int_0^t S_s \sigma dW_s + \frac{1}{2}\int_0^t S_s \sigma^2 ds$$
$$= x_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s.$$

- In the derivation above we showed that the Black-Scholes model yields

$$\log S_T - \log S_0 \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T).$$

- This implies that the *log return* of the underlying from time 0 to maturity T follows a *normal distribution* with

$$\log \frac{S_T}{S_0} \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T),$$

and the *stock price* follows a *log-normal distribution* with

$$\log S_T \sim \mathcal{N}(\log S_0 + (\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T).$$

- The next goal is to derive the *Black-Scholes differential equation*.
- This equation must be satisfied by the price of *any* European type *derivative* dependent on a non-dividend paying stock (under the Black-Scholes market model).
- We will use *Itô's lemma* and *no-arbitrage principle* in determining the Black-Scholes equation.
- These arguments are similar to the no-arbitrage arguments we used to value stock options using *binomial* (or multinomial) *trees*.

- Consider a *derivative* with price at time t given by $V(t, S_t)$.
- The idea is to *construct* a *self-financing portfolio* Π_t of one long derivative position and a short position in some quantity θ_t of the underlying

$$\Pi_t = V(t, S_t) - \theta_t S_t, \tag{6}$$

and chose θ_t such that the portfolio becomes *riskless*.

- The *underlying* follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

- The self-financing condition implies the change in the portfolio value is

$$d\Pi_t = dV(t, S_t) - \theta dS_t. \tag{7}$$

- Using *Itô's lemma* we obtain

$$dV(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial S}(t, S_t)dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t)dt. \tag{8}$$

- Inserting (8) into (7) yields the portfolio change of

$$d\Pi_t = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial S}(t, S_t)dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t)dt - \theta_t dS_t.$$

- Our goal is to obtain a portfolio Π_t which is riskless, i.e. the *portfolio* should have *no randomness*. This is achieved when

$$(\frac{\partial V}{\partial S}(t, S_t) - \theta_t)dS_t = 0.$$

▷ The *randomness* is reduced to *zero* if we chose

$$\theta_t = \frac{\partial V}{\partial S}(t, S_t) := \Delta_V. \tag{9}$$

▷ Any *reduction* in *randomness* is generally termed *hedging*.

▷ The perfect elimination of risk, by exploiting correlation between two instruments (in this case a derivative and its underlying), is generally called *delta hedging*.

▷ Delta hedging is an example of a *dynamic hedging strategy* (the perfect hedge must be *continuously rebalanced*).

- After choosing the quantity θ as suggested above, we hold a portfolio whose value changes by the amount

$$d\Pi_t = \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t) \right) dt. \tag{10}$$

- This change is completely *riskless*.
- If we have a completely risk-free change $d\Pi_t$ in the portfolio value Π_t then it must be the same as the growth we would get if we put the equivalent amount of cash in a *risk-free interest*-bearing account

$$d\Pi_t = r\Pi_t dt. \tag{11}$$

- This is an example of the *no-arbitrage* principle.

- Substituting (6), (9) and (10) into (11) we find that

$$\left(\frac{\partial V}{\partial t}(t,S_t)+\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}(t,S_t)\right)dt = r\left(V(t,S_t)-\frac{\partial V}{\partial S}(t,S_t)S_t\right)dt.$$

- After rearranging we get

$$\frac{\partial V}{\partial t}(t,S_t)+\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}(t,S_t)+rS_t\frac{\partial V}{\partial S}(t,S_t)-rV(t,S_t)=0. \tag{12}$$

- This is the *Black–Scholes partial differential equation* (PDE).

- Notice that the Black–Scholes equation does *not depend* on μ.
- Why? Since we can *perfectly hedge* the option with the underlying we should *not* be *rewarded* for taking unnecessary risk (only the risk-free rate of return is in the equation).
- We can use the same Black–Scholes argument to *replicate* an option just by buying and selling the underlying asset.
- This leads to the idea of a *complete market*.
- In a complete market an option can be replicated with the underlying, thus making options *redundant* (many things conspire to make markets *incomplete*, such as *transaction costs*).

- The Black-Scholes equation (12) knows nothing about what kind of option we are valuing, whether it is a call or a put, nor what is the strike and the expiry.
- Hence, one must prescribe $V(T, S_T)$, the pay-off. For example, if we have a *call option* with the strike price K, then we know that

$$V(T, S_T) := C(T, S_T) = max(S_T - K, 0),$$

and similarly for the *put option*

$$V(T, S_T) := P(T, S_T) = max(K - S_T, 0).$$

- Consider a call option $C(t, S_t)$. From the continuous time model formula we obtain that if the asset price is ever zero, then S_t remains zero for all the time and hence the pay-off will be zero at expiry:

$$C(t, 0) = 0 \quad \text{for all} \quad 0 \leq t \leq T\,.$$

- These two conditions are so-called *boundary conditions*.

- One also has a final condition

$$C(t, S_t) \approx S, \quad \text{for large } S_t.$$

- In order to obtain the derivative price $V(t, S_t)$, one needs to *solve* the *partial differential equation* (12) completed with the appropriate *boundary* and *final conditions* (which depend on the derivative type).
- Another way to obtain the derivative price is via the *martingale approach* (see below).
- One can as well use the *binomial* discrete time *model*, take the *limit* as the time step shrinks to zero and obtain the continuous-time Black–Scholes set up.
- In practice different *numerical methods* are often used (finite difference methods, Monte Carlo sampling, and others).

Remarks:

- ▷ The main idea is to set up a *riskless portfolio* consisting of a position in the *derivative* and a position in the *underlying stock*.
- ▷ In the *absence* of *arbitrage* opportunities, the *return* of such portfolio must be equal to the *risk-free rate*.
- ▷ The reason a riskless portfolio can be set up is that the stock price and the derivative are both affected by the *same* underlying *source* of *uncertainty*: stock price movements.
- ▷ In such portfolio, the gain/loss from the stock position always *offsets* the gain/loss from the derivative position so that the overall *value* of the portfolio at the end of the short period is *known* with *certainty*.
- ▷ This leads to the Black-Scholes partial differential equation.

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Example. Consider an at-the money call option

- on a stock worth $S_t = 100$,
- with a strike price $K = 100$,
- and maturity of six months.
- Assume the risk free rate is $r = 5\%$, and
- the stock has annual volatility $\sigma = 20\%$ and pays no dividend.

Compute the price of the call option under the Black-Scholes setting.

Solution.

- The present value factor $e^{-r(T-t)} = \exp(-5\% \cdot 6/12) = 0.9753$.
- The value of d_1:

$$d_1 = \frac{\log(S_t/(K \cdot e^{-r(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2} = 0.2475\,.$$

- $d_2 = d_1 - \sigma\sqrt{T-t} = 0.1061$.

Black-Scholes formula: Example

- Using standard normal tables (or the computer) we find $\Phi(d_1) = 0.5977$ and $\Phi(d_2) = 0.5422$. Note that both values are greater than 0.5 since d_1 and d_2 are positive.
- The *price* of the *call* is $C(t, S_t) = S_t \cdot \Phi(d_1) - K \cdot e^{-r(T-t)} \cdot \Phi(d_2) = 6.89$.
- The *price* of the *call option* can also be viewed as an equivalent position of $\Phi(d_1) = 59.77\%$ in the *stock* and some *borrowing*: $C(t, S_t) = 59.77 - 52.88 = 6.89$; thus, this is a *leveraged* position in the stock.
- The price of the put is $P(t, S_t) = 4.42$.
- Buying the call and selling the put costs $6.89 - 4.42 = 2.47$; this indeed equals to $S_t - K \cdot e^{-r(T-t)} = 100 - 97.53 = 2.47$ which confirms the *put-call parity*.

- Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,.$$

- Black-Scholes formula to price a call option:

$$C(t, S_t) = S_t \cdot \Phi(d_1) - K \cdot e^{-r(T-t)} \cdot \Phi(d_2)\,.$$

  - $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.
  - The number d_1 is given by

$$\begin{aligned} d_1 &= \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(S_t/(K \cdot e^{-r(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\,. \end{aligned}$$

  - The number d_2 is given by $d_2 = d_1 - \sigma\sqrt{T-t}$.

Black–Scholes model: Final remarks

- The two *classic references* are the paper (Black and Scholes 1973) by *Fischer Black* and *Myron Scholes* which derives the key equations and the paper (Merton, 1973) by *Robert Merton* which adds a rigorous mathematical analysis.

- Merton and Scholes were awarded the 1997 *Nobel Prize* in Economic Sciences for this work (Fischer Black died in 1995).

- Modern texts that give rigorous derivations of the Black-Scholes formula include Björk (1998), Duffie (2001), Karatzas & Shreve (1998), Oksendal (1998).

Conclusion

"In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant."

— Fisher Black

References

References

Books:

- Hull, John C., *Options, futures and other derivatives*, Sixth Edition, Prentice Hall, 2006, Chapters 13 and 14.
- Wilmott, Paul, *Paul Wilmott Introduces Quantitative Finance*, Second Edition, Wiley, 2007, Chapters 5 and 6.
- Björk, Tomas, *Arbitrage Theory in Continuous Time*, Third Edition, Oxford Finance Series, 2009, Chapters 4 and 5.

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Appendix: Black–Scholes formula: Martingale approach

Definition.

A *martingale* with respect to $\mathbb{P}$ and $(\mathcal{F}_t)_{t\geq 0}$ is a real-valued stochastic process $(M_t)_{t\geq 0}$ such that for every time t,

1. M_t is $\mathcal{F}_t$-adapted,
2. $M_t \in \mathcal{L}^1(\mathbb{P})$ for all t, i.e. $\mathbb{E}^{\mathbb{P}}[|M_t|] < \infty$,
3. For $s \leq t$:

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad \mathbb{P}\text{-almost surely.}$$

Intuition: Fair game.

Definition. Two measures $\mathbb{P}$ and $\mathbb{Q}$ under the same σ-algebra $\mathcal{F}$ are equivalent if and only if $\mathbb{P}(\omega) = 0 \iff \mathbb{Q}(\omega) = 0$.

Theorem. *First Fundamental Theorem of Asset Pricing (FTAP)*
No arbitrage $\iff$ There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted price process of every tradable asset is a $\mathbb{Q}$-martingale.

Remark. Such measure $\mathbb{Q}$ is called *Equivalent Martingale Measure* or *risk-neutral measure*.

Examples of martingales: $\{e^{-rt}S_t\}_{t\geq 0}$, $\{e^{-rt}C(t, S_t)\}_{t\geq 0}$.

- ▷ Next goal: find the *price* of the *call option* $C(t, S_t)$.

Consequence of the FTAP:

The value of any derivative can be calculated by discounting its final pay-off under the risk-neutral measure:

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT}C(T, S_T)|\mathcal{F}_t] = e^{-rt}C(t, S_t)$$

$$\iff$$

$$C(t, S_t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}C(T, S_T)|\mathcal{F}_t].$$

Remark.

- ▷ No mentioning of hedging,
- ▷ No need to know the investors' expectations/risk preferences.

The *call option price* can be written as:

$$
\begin{aligned}
C(t,S_t) &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}C(T,S_T)|\mathcal{F}_t] = \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K)^+|\mathcal{F}_t] = \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K)\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] = \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] = \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{Q}(S_T>K|\mathcal{F}_t),
\end{aligned}
\tag{13}
$$

where we used the pay-off of a call option

$$C(T,S_T) = \max(S_T-K,0) = (S_T-K)^+.$$

$\Longrightarrow$ Need to calculate $\mathbb{E}^{\mathbb{Q}}[S_T\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t]$ and $\mathbb{Q}(S_T>K|\mathcal{F}_t)$.

- The dynamics of S_t under $\mathbb{P}$:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t.$$

- What is the dynamics of S_t under $\mathbb{Q}$?

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu\, dt + \sigma\, dW_t = \\ &= r\, dt + \sigma\left(\frac{\mu - r}{\sigma} dt + dW_t\right) = \\ &= r\, dt + \sigma\, d\widetilde{W}_t.\end{aligned}$$

- If W_t is a Brownian motion under $\mathbb{P}$, then $\widetilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$ is a Brownian motion under $\mathbb{Q}$.

- Intuition: *change* of *probability measure* changes the probability of paths so that $\mathbb{E}^{\mathbb{Q}}[\frac{dS_t}{S_t}] = r\, dt \implies$ *Risk-neutral investor*.

Remarks:

- Remember equation (5): Given $s < t$, *Itô's formula* gives the expression of S_t under $\mathbb{Q}$:

$$S_t = S_s e^{(r-\frac{\sigma^2}{2})(t-s)+\sigma(\widetilde{W}_t - \widetilde{W}_s)}\Big|_{\mathcal{F}_s}, \quad \forall s, t \in [0, T].$$

- For any $s < t$ it holds that $\frac{\widetilde{W}_t - \widetilde{W}_s}{\sqrt{t-s}}\Big|_{\mathcal{F}_s}$ is a *standard normal* random variable.

- Denote the cdf of a standard normal distribution with $\Phi(\cdot)$. The *symmetry* of the distribution yields $1 - \Phi(x) = \Phi(-x)$ for any $x \in \mathbb{R}$.

- We will use these results in the derivation on the next slide.

$$
\begin{aligned}
\mathbb{Q}(S_T > K|\mathcal{F}_t) &= \mathbb{Q}\left(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)} > K \Big| \mathcal{F}_t\right) = \\
&= \mathbb{Q}\left(\widetilde{W}_T - \widetilde{W}_t > \frac{ln(\frac{K}{S_t}) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma} \Bigg| \mathcal{F}_t\right) = \\
&= 1 - \mathbb{Q}\left(\frac{\widetilde{W}_T - \widetilde{W}_t}{\sqrt{T-t}} \le \frac{ln(\frac{K}{S_t}) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \Bigg| \mathcal{F}_t\right) = \\
&= 1 - \Phi\left(\frac{ln(\frac{K}{S_t}) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) = \\
&= \Phi\left(\frac{ln(\frac{S_t}{K}) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) = \\
&= \Phi(d_2).
\end{aligned}
$$

- Recall:

$$C(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{Q}(S_T > K|\mathcal{F}_t).$$

- Above, we calculated $\mathbb{Q}(S_T > K|\mathcal{F}_t)$.

- *Remaining term* to calculate:

$$\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] = S_t e^{r(T-t)}\mathbb{E}^{\mathbb{Q}}[e^{-\frac{\sigma^2}{2}(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)}\mathbb{1}_{\{S_T>K\}}\big|\mathcal{F}_t].$$

- *Problem*: $e^{-\frac{\sigma^2}{2}(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)}$ and $\mathbb{1}_{\{S_T>K\}}$ are *not independent*.

Theorem. *Girsanov's Theorem*

Let W_t be a standard Brownian motion under a probability measure $\mathbb{P}$. We can define a new Brownian motion $W_t^{\mathbb{Q}}$ under a probability measure $\mathbb{Q}$ such that

$$W_t^{\mathbb{Q}} = W_t - \theta t,$$

where the *Radon-Nikodym derivative* (of the *change of measure* from $\mathbb{P}$ to $\mathbb{Q}$) is defined as

$$Z_t := \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\frac{\theta^2}{2}t+\theta W_t}.$$

$W_t^{\mathbb{Q}}$ is a standard (non-drifted) Brownian motion under $\mathbb{Q}$.

Finally, for every $\mathcal{F}_t$-measurable and integrable random variable X_t we have

$$\mathbb{E}^{\mathbb{P}}\left[\frac{Z_T}{Z_t} X_T | \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{F}_t].$$

Exercise: Show that Z_t is a $\mathbb{P}$-martingale.

Idea: Define a new measure $\mathbb{Q}^*$ such that the *Radon-Nikodym derivative* is given by $\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} := Z_t$, where $\frac{Z_T}{Z_t} = e^{-\frac{\sigma^2}{2}(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)}$. Then we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] &= S_t e^{r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{-\frac{\sigma^2}{2}(T-t)+\sigma(\widetilde{W}_T-\widetilde{W}_t)}\mathbb{1}_{\{S_T>K\}}\Big|\mathcal{F}_t\right] = \\
&= S_t e^{r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\frac{Z_T}{Z_t}\mathbb{1}_{\{S_T>K\}}\Big|\mathcal{F}_t\right] = \\
&= S_t e^{r(T-t)}\mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{1}_{\{S_T>K\}}\Big|\mathcal{F}_t\right] = \\
&= S_t e^{r(T-t)}\mathbb{Q}^*(S_T > K|\mathcal{F}_t).
\end{aligned}$$

This was possible by setting $\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = Z_t = e^{-\frac{\sigma^2}{2}t+\sigma\widetilde{W}_t}$.

Next step: Calculate $\mathbb{Q}^*(S_T > K|\mathcal{F}_t)$.

- Given the definition of Z_t, *Girsanov theorem* tells us that $W_t^* = \widetilde{W}_t - \sigma t$ is a $\mathbb{Q}^*$-Brownian motion. Hence, the dynamics of S_t under $\mathbb{Q}^*$ are given by

$$\begin{aligned}\frac{dS_t}{S_t} &= r\,dt + \sigma\,d\widetilde{W}_t = \\ &= r\,dt + \sigma(W_t^* + \sigma\,dt) = \\ &= (r + \sigma^2)\,dt + \sigma\,dW_t^*.\end{aligned}$$

- *Itô's lemma* now gives

$$S_T = S_t e^{(r+\frac{\sigma^2}{2})(T-t)+\sigma(W_T^*-W_t^*)}\Big|_{\mathcal{F}_t}.$$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] &= S_t e^{r(T-t)}\mathbb{Q}^*(S_T > K|\mathcal{F}_t) = \\
&= S_t e^{r(T-t)}\mathbb{Q}^*\left(S_t e^{(r+\frac{\sigma^2}{2})(T-t)+\sigma(W_T^*-W_t^*)} > K\Big|\mathcal{F}_t\right) = \\
&= S_t e^{r(T-t)}\mathbb{Q}^*\left(\frac{W_T^*-W_t^*}{\sqrt{T-t}} > \frac{ln(\frac{K}{S_t})-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Big|\mathcal{F}_t\right) = \\
&= S_t e^{r(T-t)}\left(1-\mathbb{Q}^*\left(\frac{W_T^*-W_t^*}{\sqrt{T-t}} \le \frac{ln(\frac{K}{S_t})-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Big|\mathcal{F}_t\right)\right) = \\
&= S_t e^{r(T-t)}\left(1-\Phi\left(\frac{ln(\frac{K}{S_t})-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Big|\mathcal{F}_t\right)\right) = \\
&= S_t e^{r(T-t)}\Phi\left(\frac{ln(\frac{S_t}{K})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\Big|\mathcal{F}_t\right) = \\
&= S_t e^{r(T-t)}\Phi\left(d_1\right).
\end{aligned}
$$

- Now we just have to put everything together. As shown in equation (13), the call option price is given by

$$C(t,S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{Q}(S_T > K|\mathcal{F}_t).$$

- We calculated

$$\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] = S_t e^{r(T-t)}\Phi\left(d_1\right),$$

and

$$\mathbb{Q}(S_T > K|\mathcal{F}_t) = \Phi(d_2).$$

- This yields the *Black-Scholes formula* for *call options*:

$$C(t, S_t) = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2),$$

where

$$d_1 = \frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

- $\Phi(\cdot)$ denotes the cdf of a $\mathcal{N}(0,1)$ distributed random variable

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds.$$

- The *put option price* can be derived analogously (for example via the martingale approach) to the call price by imposing an appropriate *final pay-off*.

- Alternatively, we can derive it using the *put-call parity* relation

$$C(t, S_t) - P(t, S_t) = S_t - K \cdot e^{-r(T-t)} .$$

- The *put option price* is

$$P(t, S_t) = K \cdot e^{-r(T-t)} \cdot \Phi(-d_2) - S_t \cdot \Phi(-d_1) .$$

Setting: Derive the *Black-Scholes call option price* with the *martingale approach*.

Main steps:

1. Apply the *Fundamental Theorem of Asset Pricing* to write the call option as a discounted pay-off under $\mathbb{Q}$.
2. Decompose the price into
$$C(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T > K\}} | \mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{Q}(S_T > K | \mathcal{F}_t).$$
3. Solve for $\mathbb{Q}(S_T > K|\mathcal{F}_t)$ by writing the dynamics of S_t under $\mathbb{Q}$ and using the *normality* of the *Brownian motion*.
4. Using a *change of measure* from $\mathbb{Q}$ to $\mathbb{Q}^*$ write $\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T > K\}}|\mathcal{F}_t]$ as
$$\mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\{S_T > K\}}|\mathcal{F}_t] = S_t e^{r(T-t)}\mathbb{Q}^*(S_T > K|\mathcal{F}_t).$$
5. Solve for $\mathbb{Q}^*(S_T > K|\mathcal{F}_t)$ (same procedure as Step 3).