Quantitative Finance

Lectures in Quantitative Finance

Spring Term 2022

5. Discrete Time Models

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In this chapter we will have a closer look at multiperiod discrete-time models.

We will

- define all necessary concepts for *risk-neutral pricing* and *replication*,
- focus on the *binomial model*, and
- discuss *Black-Scholes' formula* in a discrete time setting.

Model setup

- *Trading periods*: $t \in \{0, \ldots, T\}$ for some $0 < T < \infty$.
- Market consisting of $d + 1$ *assets*:
  - asset 0 is consider as a *riskless bond*,
  - assets $1, \ldots d$ are *risky assets*.
- The *price at time t of the riskless bond* is a function of t and given by $S_t^0 = (1 + r)^t$, where $r > -1$ denotes the risk-free interest rate.
- The *price of asset $k \in \{1, \ldots, d\}$ at time t* is modelled by a non-negative random variable S_t^k, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Let $\mathbb{F} := (\mathcal{F}_t)_{t=0,\ldots,T}$ be a *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$. This means all $\mathcal{F}_t$ are sub-sigma-algebras of $\mathcal{F}$ and $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $s \leq t$. We call the space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a *filtered probability space*.
- For our model we choose $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F}$.
- We assume that for every $t \in \{0, \ldots, T\}$ the random vector $\overline{S}_t = (S_t^0, S_t)^\intercal = (S_t^0, S_t^1, \ldots, S_t^d)^\intercal$ is $\mathcal{F}_t$-measurable.

Interpretation.

- The σ-algebra $\mathcal{F}_t$ represents all the information available up to time t. It is natural to assume $\mathcal{F}_s \subseteq \mathcal{F}_t$ for any $s \leq t$, since there is no loss of information over time.
- We assume that S_t is $\mathcal{F}_t$-measurable. This means that the prices at time t of all risky assets are based only on information up to time t.

Adaptedness and predictability

We call an indexed family of random variables a *stochastic process*.

Definition. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space.

- A stochastic process $Z = (Z_t)_{t=0,\dots,T}$ is called $\mathbb{F}$*-adapted* if Z_t is $\mathcal{F}_t$-measurable for every $t = 0, \dots, T$.
- A stochastic process $Y = (Y_t)_{t=1,\dots,T}$ is called $\mathbb{F}$*-predictable* if Y_t is $\mathcal{F}_{t-1}$-measurable for every $t = 1, \dots, T$.

In our model, the *price process* $\overline{S} = \left(\overline{S}_t\right)_{t=0,\dots,T}$ is $\mathbb{F}$-adapted.

Trading strategies

Definition. An $\mathbb{R}^{d+1}$-valued process $\overline{\xi} = (\xi^0, \xi)^\intercal = (\xi^0, \xi^1, \ldots, \xi^d)^\intercal$ is called a *trading strategy* if it is $\mathbb{F}$-predictable.

In other words, ξ_t^k is $\mathcal{F}_{t-1}$-measurable for every $t \in \{1, \ldots, T\}$, $k \in \{0, \ldots, d\}$.

Interpretation.

- ξ_t^k represents the number of shares of asset k held during the t^{th} trading period between times $t-1$ and t.
- $\xi_t^k S_{t-1}^k$ denotes the amount invested in the k^{th} asset at time $t-1$, while $\xi_t^k S_t^k$ is the resulting value at time t.
- Predictability of the strategy represents the fact that *any investment must be allocated at the beginning of each trading period*, without anticipating future prices.

Definition. A trading strategy $\overline{\xi} \in \mathbb{R}^{d+1}$ is called *self-financing* if

$$\overline{\xi}_t \cdot \overline{S}_t = \overline{\xi}_{t+1} \cdot \overline{S}_t \quad \text{for every } t = 1, \dots, T-1.$$

This means that for every $t \in \{1, \dots, T-1\}$

$$\sum_{k=0}^{d} \xi_t^k S_t^k = \sum_{k=0}^{d} \xi_{t+1}^k S_t^k \text{ or rewritten } \sum_{k=0}^{d} (\xi_{t+1}^k - \xi_t^k) S_t^k = 0.$$

Interpretation.

- The portfolio of a self-financing strategy is rearranged in such a way that its *present value is preserved*.
- Any change in the portfolio value is due to *price fluctuations* of the assets and not to some external factors.

Consider the case of $d = 1$, i.e. we have only one risky asset.

The self financing condition can thus be written at $t = 1$ as:

$$\overline{\xi_1} \cdot \overline{S}_1 = \overline{\xi_2} \cdot \overline{S}_1,$$

which is equivalent to:

$$\xi_1^0 S_1^0 + \xi_1^1 S_1^1 = \xi_2^0 S_1^0 + \xi_2^1 S_1^1.$$

$\xi_1^0 S_1^0 + \xi_1^1 S_1^1$ represent the value of the trading strategy (portfolio) just before trading while $\xi_2^0 S_1^0 + \xi_2^1 S_1^1$ represent the value of the trading strategy (portfolio) immediately after trading.

Intuitively, with no exogenous infusion or withdrawal of money, the purchase of a new asset must be financed by the sale of an old one.

Discounted price process

To compare prices at different trading times, we have to consider discounted values.

Using the riskless asset as a *numéraire*, we define the *discounted price process* $\overline{X} = (\overline{X}_t)_{0 \le t \le T} = (X_t^0, X_t)^{\mathsf{T}}_{0 \le t \le T}$ by

$$X_t^0 = \frac{S_t^0}{S_t^0} \equiv 1\,, \quad \text{and}$$

$$X_t^k = \frac{S_t^k}{S_t^0} = \frac{S_t^k}{(1+r)^t} \quad \text{for } k \in \{1, \dots d\}\,,$$

for all $t \in \{0, \dots, T\}$.

Value process: definition

The *(discounted) value process* $V = (V_t)_{t \in \{0,\dots,T\}}$ of a trading strategy $\overline{\xi}$ is given by

$$V_0 = \overline{\xi}_1 \cdot \overline{X}_0 \quad \text{and} \quad V_t = \overline{\xi}_t \cdot \overline{X}_t \text{ for } t = 1, \dots, T.$$

V_t represents the *portfolio value* at the end of the t^{th} trading period.

Consider the case $d = 1$ and $T = 3$. The (discounted) value process can thus be written as:

$$
\begin{aligned}
V_0 &= \xi_1^0 X_0^0 + \xi_1^1 X_0^1, \\
V_1 &= \xi_1^0 X_1^0 + \xi_1^1 X_1^1, \\
V_2 &= \xi_2^0 X_2^0 + \xi_2^1 X_2^1, \\
V_3 &= \xi_3^0 X_3^0 + \xi_3^1 X_3^1.
\end{aligned}
$$

Arbitrage Opportunities

Arbitrage opportunity: definition

A self-financing trading strategy $\overline{\xi}$ is called an *arbitrage opportunity* if the corresponding value process V satisfies

$$V_0 \leq 0, \quad V_T \geq 0 \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[V_T > 0] > 0.$$

The market model is *arbitrage-free* if no such arbitrage opportunity exists.

The market is arbitrage-free if and only if there are no arbitrage opportunities for each single trading period.

Definition. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Then a stochastic process $M = (M_t)_{t=0,\dots,T}$ is called a *$(\mathbb{P}, \mathbb{F})$-martingale* if

- M is $\mathbb{F}$-adapted,
- $\mathbb{E}_{\mathbb{P}}[|M_t|] < \infty$ for every $t \in \{0, \dots, T\}$,
- $\mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_s] = M_s$, for all $0 \le s \le t \le T$.

Interpretation. The *best prediction* given the information up until now is exactly the current value. This is our definition of a *'fair game'*.

Note: Since we usually work with a given filtration $\mathbb{F}$, most of the time we will drop $\mathbb{F}$ and only say '$\mathbb{P}$-martingale'. If there is no ambiguity, we can also drop $\mathbb{P}$ and just say 'martingale'. However, be aware that the martingale property depends on the probability measure and the filtration!

Given a random variable X with $\mathbb{E}[X] < \infty$ and a σ-algebra $\mathcal{A} \subset \mathcal{F}$, the conditional expectation $E[X|\mathcal{A}]$ is the almost surely unique random variable such that:

- $\mathbb{E}[X|\mathcal{A}]$ is $\mathcal{A}$-measurable,
- $\mathbb{E}[X\mathbb{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{A}]\mathbb{1}_B]$ for any $B \in \mathcal{A}$.

The second condition is thus equivalent to saying that the integrals over any element of $\mathcal{A}$ coincide.

When $\mathcal{A} = \mathcal{F}_0 = \{\emptyset, \Omega\}$, i.e. the trivial σ-algebra, then $\mathbb{E}[X|\mathcal{A}] = \mathbb{E}[X]$.

Conditional Expectation: Continued

We can also talk about the conditional expectation given another random variable: $\mathbb{E}[X|Y]$.

In this case, $\mathbb{E}[X|Y]$ is again a random variable that is only dependent on Y, i.e. $\mathbb{E}[X|Y] = g(Y)$ for some function g (the randomness is only inherited from Y).

If $\mathbb{E}[X|Y] = g(Y)$ then $\mathbb{E}[X|Y = y] = g(y)$. This can also be written formally in discrete time as:

$$
\begin{aligned}
\mathbb{E}(X \mid Y = y) &= \sum_x x\mathbb{P}(X = x \mid Y = y) \\
&= \sum_x x\frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}
\end{aligned}
$$

Example. Consider U and Y two independent random variables such that $U \sim \mathcal{U}[0,1]$ and $Y \sim \mathcal{U}[-1,1]$. With $X = U(Y+1)$, we have that:

$$\mathbb{E}[X|Y] = \mathbb{E}[U(Y+1)|Y] = (Y+1)\mathbb{E}[U] = \frac{1}{2}(Y+1).$$

Example. Let X and Y be two random variables such that:

$$Y = \begin{cases} 1 & \text{with probability } 1/3, \\ 2 & \text{with probability } 2/3, \end{cases}$$

and

$$X|Y = \begin{cases} Y & \text{with probability } 1/4, \\ 2Y & \text{with probability } 3/4, \end{cases}$$

Conditional expectation of X given $Y = y$ is a number which depends on y:

- If $Y = 1$ then $X|(Y = 1) = \begin{cases} 1 & \text{with probability } 1/4, \\ 2 & \text{with probability } 3/4, \end{cases}$
- If $Y = 2$ then $X|(Y = 2) = \begin{cases} 2 & \text{with probability } 1/4, \\ 4 & \text{with probability } 3/4, \end{cases}$

This gives:

$$\mathbb{E}[X|Y = 1] = 1 \times \frac{1}{4} + 2 \times \frac{3}{4} = \frac{7}{4},$$
$$\mathbb{E}[X|Y = 2] = 2 \times \frac{1}{4} + 4 \times \frac{3}{4} = \frac{14}{4}.$$

Therefore:

$$\mathbb{E}[X|Y=y]=\begin{cases}7/4 & \text{if } y=1,\\ 14/4 & \text{with if } y=2,\end{cases}$$

which shows that $\mathbb{E}[X|Y=y]$ is a number depending on y.

On the other hand, the above implies:

$$\mathbb{E}[X|Y]=\begin{cases}7/4 & \text{if } Y=1 \text{ (probability } 1/3),\\ 14/4 & \text{if } Y=2 \text{ (probability } 2/3),\end{cases}$$

which is equivalent to:

$$\mathbb{E}[X|Y]=\begin{cases}7/4 & \text{(probability } 1/3),\\ 14/4 & \text{(probability } 2/3).\end{cases}$$

Thus, $\mathbb{E}[X|Y]$ is a random variable with randomness inherited only from Y.

Example. (One-period binomial model) Consider a one-period binomial model with two assets, a risk-free and a risky asset.

- Let $\Omega = \{\omega_{\mathrm{u}}, \omega_{\mathrm{d}}\}$, where ω_{u} and ω_{d} represent the event of an *up* and *down* price movement of the risky asset, respectively.
- Denote the initial price of the *risk-free asset* by $B_0 = 1$ and the interest rate by $r > 0$, that means $B_1 = (1 + r)B_0$.
- Denote the initial price of the *risky asset* by $S_0 > 0$ and the future price by the random variable $S_1 : \Omega \to \mathbb{R}$.
- Let $d < 0 < u$ and let $p \in (0, 1)$ be the probability of an up movement,

$$p = \mathbb{P}[\{\omega_{\mathrm{u}}\}] = \mathbb{P}[S_1 = (1 + u)S_0] = 1 - \mathbb{P}[S_1 = (1 + d)S_0].$$

One period binomial model

The riskfree asset moves according to the figure below.

$$B_0 \xrightarrow{\quad 1 \quad} B_1 = (1+r)B_0$$

The 1 above the arrow indicates that this happens with certainty. So $\mathbb{P}[B = (1+r)B_0] = 1$. In contrast, S_1 is a random variable on Ω and can move up and down with probabilities p and $(1-p)$, respectively.

$$S_0 \xrightarrow{\quad p \quad} S_1(\omega_{\mathrm{u}}) = (1+u)S_0$$

$$S_0 \xrightarrow[\quad 1-p \quad]{} S_1(\omega_{\mathrm{d}}) = (1+d)S_0$$

Exercise. (Martingales in the one-period binomial model) Consider the model from above.

a) Let $r = 0$ and let the probability of an up-movement of the risky asset be $p = \frac{1}{3}$. For which of the following choices of S_0, u, and d is the discounted price process a martingale?
   - ▷ $S_0 = 100,\ u = 0.2,\ d = -0.1,$
   - ▷ $S_0 = 50,\ u = 0.2,\ d = -0.6,$
   - ▷ $S_0 = 150,\ u = 0.3,\ d = -0.15.$
   - ▷ $S_0 = 100,\ u = 0.3,\ d = 0.1.$

b) Pick the two which are not martingales under $\mathbb{P}$ defined by $p = \frac{1}{3}$. For each of them can you find another measure $\mathbb{P}^*$ defined by a different up-movement probability p^* such that $(S_t)_{t=0,1}$ is a $\mathbb{P}^*$-martingale? *Hint:* For the first one it is possible and you should get $p^* = \frac{3}{4}$. For the second it is not possible.

Definition. A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is called a *martingale measure* if the discounted price process $(X_t)_{t=0,\cdots,T}$ is a $\mathbb{P}^*$-martingale,

$$\mathbb{E}_{\mathbb{P}^*}[X_t] < \infty \quad \text{and}$$
$$\mathbb{E}_{\mathbb{P}^*}[X_t^k \mid \mathcal{F}_s] = X_s^k \ \text{ for } \ 0 \le s \le t \le T,\ k = 1, \dots, d.$$

Definition. We call two probability measures $\mathbb{P}$ and $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ *equivalent* if for any set $A \subseteq \mathcal{F}$ we have

$$\mathbb{P}[A] = 0 \ \text{ if and only if } \ \mathbb{P}^*[A] = 0\,.$$

In this case, we write $\mathbb{P} \sim \mathbb{P}^*$.

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, denote by $\mathcal{P}$ the set of all martingale measures, which are equivalent to $\mathbb{P}$, that is

$$\mathcal{P} = \{\mathbb{P}^* : \mathcal{F} \to [0,1] \mid \mathbb{P}^* \text{ is a martingale measure and } \mathbb{P}^* \sim \mathbb{P}\}.$$

Theorem. (Fundamental theorem of asset pricing, FTAP) A market model is free of arbitrage if and only if the set $\mathcal{P}$ of all equivalent martingale measures is not empty.

One-period binomial model

Consider again the one-period binomial market with one risk-free and one risky asset.

$$B_0 = 1 \xrightarrow{\quad 1 \quad} B_1 = (1+r)B_0$$

$$S_0 \begin{cases} \xrightarrow{\quad p \quad} S_1(\omega_{\mathrm{u}}) = (1+u)S_0 \\ \xrightarrow[\quad 1-p \quad]{} S_1(\omega_{\mathrm{d}}) = (1+d)S_0 \end{cases}$$

Corollary. This market is arbitrage-free if, and only if, $d < r < u$.

Compare this result to the exercise above.

Proof. According to the FTAP, the no arbitrage condition is equivalent to the existence of an equivalent martingale measure $\mathbb{P}^*$. Hence, we need to find a measure $\mathbb{P}^* \sim \mathbb{P}$ such that $\mathbb{E}_{\mathbb{P}^*}\left[\frac{S_1}{B_1} \mid \mathcal{F}_0\right] = \mathbb{E}_{\mathbb{P}^*}\left[\frac{S_1}{B_1}\right] = S_0$.

In this one-period binomial model, this means we search for a measure $\mathbb{P}^*$ identified by $(p^*, 1-p^*)$ such that

$$p^* \frac{S_0(1+u)}{1+r} + (1-p^*)\frac{S_0(1+d)}{1+r} = S_0\,.$$

This is equivalent to

$$p^*(1+u) + (1-p^*)(1+d) = 1+r\,.$$

Solving for p^* we arrive at

$$p^* = \frac{r-d}{u-d} \, .$$

We want that $\mathbb{P}^* \sim \mathbb{P}$. Since $p > 0$ we must also require $p^* > 0$ (and of course the same for $(1-p^*)$). In other words, we require that

$$d < r < u \, .$$

This conclude the proof. $\square$

We also have a stronger version of arbitrage which is called free lunch.

Definition. A portfolio $\overline{\xi} \in \mathbb{R}^{d+1}$ is called a *free lunch* if $\overline{\xi} \cdot \overline{S}_0 < 0$ and $\overline{\xi} \cdot \overline{S}_1 \geq 0$ $\mathbb{P}$-a.s.. If no such strategy exists, we say that the market model satisfies the *no-free-lunch condition (NFL)*.

Theorem Consider a market model with $n \in \mathbb{N}$ possible future states described by $\Omega = \{\omega_1, \dots, \omega_n\}$ with $\mathbb{P}[\{\omega_k\}] > 0$ for all $k = 1, \dots, n$.

Then, the following are equivalent,

1. The market model satisfies the (NFL) condition.
2. There exists a martingale measure $\mathbb{P}^*$.

Notice: In contrast to the FTAP, here we have no statement about the equivalence of the risk-neutral measure $\mathbb{P}^*$ and the underlying measure $\mathbb{P}$.

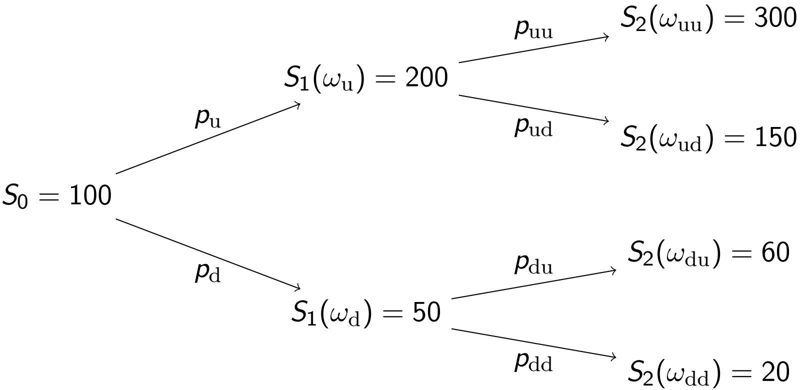
No free lunch in the binomial model

Exercise. Show that in the one-period binomial model with one riskless and one risky asset, there is no free lunch if, and only if, $d \leq r \leq u$.

Hint: Start as before in the corollary. Then argue that, since we do not require equivalence, we do not need strict inequalities.

Example. (Two-period Model) Consider a two-period model consisting of a riskless and a risky asset with dynamics

$$B_0 = 1 \xrightarrow{\quad 1 \quad} B_1 = (1+r)B_0 \xrightarrow{\quad 1 \quad} B_2 = (1+r)^2 B_0$$



Find the martingale measure $\mathbb{P}^*$, so that there is no arbitrage in both trading periods. For simplicity assume that $r = 0$, so that $B_t = 1$ for $t = 0, 1, 2$.

Hence, we want to a measure $\mathbb{P}^*$, such that

$$S_0 = \mathbb{E}_{\mathbb{P}^*}[S_1] \text{ and } S_1 = \mathbb{E}_{\mathbb{P}^*}[S_2 \mid \mathcal{F}_1].$$

In the binomial model, this translates to solving the following systems of equations (remember that all probabilities need to sum up to 1),

$$\begin{aligned}
100 = 200 p^*_{\mathrm{u}} + 50 p^*_{\mathrm{d}} \quad &\iff \quad p^*_{\mathrm{u}} = \frac{1}{3},\ p^*_{\mathrm{d}} = \frac{2}{3},\\
200 = 300 p^*_{\mathrm{uu}} + 150 p^*_{\mathrm{ud}} \quad &\iff \quad p^*_{\mathrm{uu}} = \frac{1}{3},\ p^*_{\mathrm{ud}} = \frac{2}{3},\\
50 = 60 p^*_{\mathrm{du}} + 20 p^*_{\mathrm{dd}} \quad &\iff \quad p^*_{\mathrm{du}} = \frac{3}{4},\ p^*_{\mathrm{dd}} = \frac{1}{4}.
\end{aligned}$$

Finally, the martingale measure is given by

$$\mathbb{P}^*[\{\omega_{\mathrm{uu}}\}] = p^*_{\mathrm{u}} p^*_{\mathrm{uu}} = \frac{1}{9}\,, \quad \mathbb{P}^*[\{\omega_{\mathrm{ud}}\}] = p^*_{\mathrm{u}} p^*_{\mathrm{ud}} = \frac{2}{9}\,,$$

$$\mathbb{P}^*[\{\omega_{\mathrm{du}}\}] = p^*_{\mathrm{d}} p^*_{\mathrm{du}} = \frac{1}{2}\,, \quad \mathbb{P}^*[\{\omega_{\mathrm{dd}}\}] = p^*_{\mathrm{d}} p^*_{\mathrm{dd}} = \frac{1}{6}\,.$$

Remark. Notice that in the two-period binomial model $\Omega = \{\omega_{\mathrm{uu}}, \omega_{\mathrm{ud}}, \omega_{\mathrm{du}}, \omega_{\mathrm{dd}}\}$. We use ω_{u} and ω_{d} to clarify at which of the two possible states we are at time $t = 1$. However, $\omega_{\mathrm{u}}, \omega_{\mathrm{d}}$ are not in Ω per se.

Contingent claims

Definition A European call option on the stock S is a contract which gives the holder of the option the right but not the obligation to buy the stock at a fixed time in the future (called exercise time or maturity) T for a fixed price, called the strike price K.

The payoff of a call option is given by $(S_T - K)^+ = max(0, S_T - K)$.

Definition A European put option on the stock S is a contract which gives the holder of the option the right but not the obligation to sell the stock at a fixed time in the future (called exercise time or maturity) T for a fixed price, called the strike price K.

The payoff of a call option is given by $(K - S_T)^+ = max(0, K - S_T)$.

Definition A contingent claim C_T with maturity T is said to be *attainable (or replicable)* if there exists a self-financing strategy $\overline{\xi}$ whose terminal portfolio value coincides with C_T, that is

$$C_T = \overline{\xi}_T \cdot \overline{S}_T \quad \mathbb{P}\text{-a.s.}$$

In this case, the trading strategy $\overline{\xi}$ is called *replicating strategy* for C_T.

C_T is attainable if and only if its corresponding discounted claim $H_T = \frac{C_T}{(1+r)^T}$ can be written as

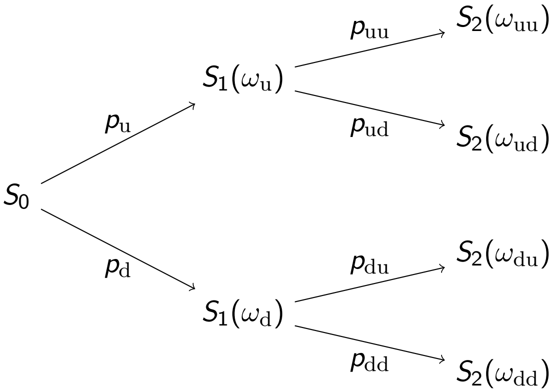
$$H_T = \overline{\xi}_T \cdot \overline{X}_T = V_T = V_0 + \sum_{t=1}^{T} \xi_t \cdot (X_t - X_{t-1}),$$

for a self-financing trading strategy $\overline{\xi}$ with value process V.

In this case, we say that the discounted claim H_T is attainable with the replicating strategy $\overline{\xi}$.

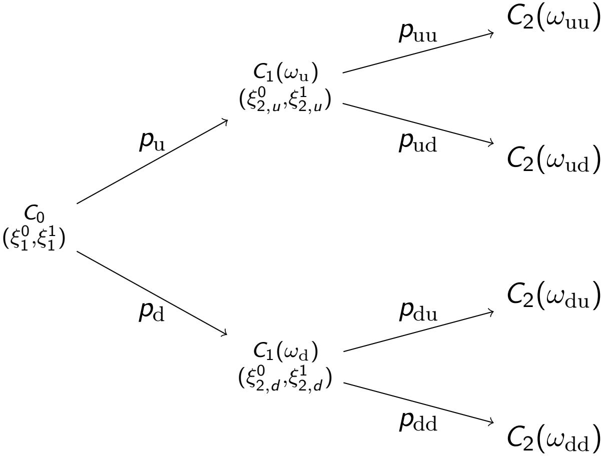
Example. (Two-period Model) Consider a two-period model consisting of a riskless and a risky asset with dynamics

$$B_0 = 1 \xrightarrow{\quad 1 \quad} B_1 = (1+r)B_0 \xrightarrow{\quad 1 \quad} B_2 = (1+r)^2 B_0$$



Attainable payoffs: Two period model exampl

An option C would then follow the following process:



with ξ denoting the replicating strategy of C.

The replicating strategy $(\overline{\xi_1}, \overline{\xi_2})$ must satisfy the following:

1. The self-financing property $\overline{\xi_1} \cdot \overline{S}_1 = \overline{\xi_2} \cdot \overline{S}_1$:

$$\xi_1^0 \cdot B_1 + \xi_1^1 \cdot S_1(\omega_u) = \xi_2^0 \cdot B_1 + \xi_2^1 \cdot S_1(\omega_u),$$
$$\xi_1^0 \cdot B_1 + \xi_1^1 \cdot S_1(\omega_d) = \xi_2^0 \cdot B_1 + \xi_2^1 \cdot S_1(\omega_d),$$

2. The replicating property $C_2 = \overline{\xi_2} \cdot \overline{S_2}$:

$$\xi_{2,u}^0 \cdot B_2 + \xi_{2,u}^1 \cdot S_2(\omega_{uu}) = C_2(\omega_{uu}),$$
$$\xi_{2,u}^0 \cdot B_2 + \xi_{2,u}^1 \cdot S_2(\omega_{ud}) = C_2(\omega_{ud}),$$
$$\xi_{2,d}^0 \cdot B_2 + \xi_{2,d}^1 \cdot S_2(\omega_{du}) = C_2(\omega_{du}),$$
$$\xi_{2,d}^0 \cdot B_2 + \xi_{2,d}^1 \cdot S_2(\omega_{dd}) = C_2(\omega_{dd}).$$

Theorem. Let H_T be a discounted, attainable contingent claim. Then H_T is integrable with respect to any equivalent martingale measure, that means

$$\mathbb{E}_{\mathbb{P}^*}[H_T] < \infty \quad \text{for all } \mathbb{P}^* \in \mathcal{P}.$$

Moreover, for every $\mathbb{P}^* \in \mathcal{P}$ the value process associated with the replicating strategy of H_T can be written as

$$V_t = \mathbb{E}_{\mathbb{P}^*}\left[H_T \mid \mathcal{F}_t\right] \quad \mathbb{P}\text{-a.s.} \quad \text{for } t = 0, \dots, T.$$

Hence, V is a non-negative $\mathbb{P}^*$-martingale for every equivalent martingale measure $\mathbb{P}^* \in \mathcal{P}$.

Arbitrage-free prices

The goal is to price a discounted contingent claim H_T without introducing arbitrage in the market. If H_T is attainable, then the discounted initial investment needed for replicating H_T,

$$\overline{\xi}_1 \cdot \overline{X}_0 = V_0 = \mathbb{E}_{\mathbb{P}^*}[H_T] \quad \text{for } \mathbb{P}^* \in \mathcal{P}\,,$$

can be interpreted as the unique discounted arbitrage-free price for H_T.

If H_T is not attainable, than there is no unique arbitrage-free price for H_T.

Let H_T be a discounted contingent claim. Then a real number $\pi(H_T) \geq 0$ is an *arbitrage-free price* for H_T if there exists an adapted stochastic process X^{d+1} such that

$$
\begin{aligned}
X_0^{d+1} &= \pi(H_T), \\
X_t^{d+1} &\geq 0 \quad \text{for } t = 1, \ldots, T-1, \\
X_T^{d+1} &= H_T,
\end{aligned}
$$

and such that the enlarged market model with price process $\left(X^0, \ldots, X^d, X^{d+1}\right)$ is arbitrage-free.

A priori, the claim H_T could have more than one arbitrage-free price.

Denote by $\Pi(H_T)$ the set of all arbitrage-free prices for H_T,

$$\Pi(H_T) = \{\pi(H_T) \in \mathbb{R} \mid \pi(H_T) \text{ is an arbitrage-free price for } H_T\}.$$

Theorem. Let H_T be a discounted contingent claim. Then, the set $\Pi(H_T)$ is non-empty and given by

$$\Pi(H_T) = \{\mathbb{E}_{\mathbb{P}^*}[H_T] \mid \mathbb{P}^* \in \mathcal{P} \text{ such that } \mathbb{E}_{\mathbb{P}^*}[H_T] < \infty\}.$$

The lower and upper bounds of $\Pi(H_T)$ are

$$\pi^H_{\min} = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}_{\mathbb{P}^*}[H_T] \quad \text{and} \quad \pi^H_{\max} = \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}_{\mathbb{P}^*}[H_T].$$

Theorem. Consider an arbitrage-free primary market model and a discounted contingent claim H_T such that $H_T \geq 0$. Then the following assertions hold.

1. If H_T is attainable, then $\Pi(H_T)$ consists of the unique element V_0, where V denotes the value process corresponding to the replicating strategy of H_T.
2. If H_T is not attainable, then the set of arbitrage-free price is an open interval of the form $\Pi(H_T) = (\pi^H_{\min}, \pi^H_{\max})$.

Under the assumption of no arbitrage opportunities, two portfolios with same payoff at time T, must have the same value at any prior time.

If at some point, one portfolio is cheaper than the other, then one could go long the cheaper portfolio and short the more expensive portfolio. At time T, the overall portfolio will have value zero. Therefore, the profit we made is riskless which violates the assumption of no arbitrage.

The Put-Call Parity is an important relationship between the price of a put and call option on the same underlying with the same maturity and strike price.

Consider two portfolios:

- P_1: Long K bonds and one call, short one put,
  $V_t(P_1) = KB_t + C_t - P_t$,
- P_2: Long one share, $V_t(P_2) = S_t$,

where the bonds have maturity T and the call and the put have both strike K and maturity T.

At time T, one can observe that $K + (S_T - K)^+ - (K - S_T)^+ = S_T$. This implies that $V_T(P_1) = V_T(P_2)$.

From the law of one price we get that the value of the portfolios must be the same at any point in time:

$$KB_t + C_t - P_t = S_t$$

Example. Consider again the one-period binomial model with the following dynamics

$$B_0 \xrightarrow{\quad 1 \quad} B_1 = (1+r)B_0$$

$$S_0 \begin{cases} \xrightarrow{\quad p \quad} S_1(\omega_{\mathrm{u}}) = (1+u)S_0 \\ \xrightarrow{\quad 1-p \quad} S_1(\omega_{\mathrm{d}}) = (1+d)S_0 \end{cases}$$

Recall that the market is arbitrage-free if, and only if, $d < r < u$.

Recall that the *unique* equivalent martingale measure $\mathbb{P}^*$ is given by

$$\mathbb{P}^*(\{\omega_{\mathrm{u}}\}) = p^* = \frac{r-d}{u-d} \quad \text{and} \quad \mathbb{P}^*(\{\omega_{\mathrm{d}}\}) = 1 - p^* = \frac{u-r}{u-d}\,.$$

Consider a European call option on this stock with strike price $K > 0$. So the payoff if given by $C_1^{\mathrm{call}} = (S_1 - K)^+$.

We want to find an arbitrage-free price of this claim, $\pi(C_1^{\mathrm{call}})$, so we want to *price this call option*.

We know by the previous theorem that arbitrage-free prices are given by the expectation of the discounted claim. Since we have a unique equivalent martingale measure $\mathbb{P}^*$, we get the unique price

$$\begin{aligned}
\pi(C_1^{\mathrm{call}}) &= \mathbb{E}_{\mathbb{P}^*}\left[\frac{C_1^{\mathrm{call}}}{1+r}\right] \\
&= p^* \frac{C_1^{\mathrm{call}}(\omega_{\mathrm{u}})}{1+r} + (1-p^*)\frac{C_1^{\mathrm{call}}(\omega_{\mathrm{d}})}{1+r} \\
&= \frac{1}{1+r}\left(\frac{r-d}{u-d}((1+u)S_0-K)^+ + \frac{u-r}{u-d}((1+d)S_0-K)^+\right).
\end{aligned}$$

This is the price of a European call option on that particular asset. So we conclude that the European call option must have an attainable payoff. Later, we will indeed derive its replicating strategy.

To round up this example, let us plug in some values. Let $S_0 = 100$, $S_1(\omega_{\mathrm{u}}) = 120$ and $S_1(\omega_{\mathrm{d}}) = 80$, and assume the strike price is $K = 110$.

Hence, $u = 0.2$ and $d = -0.2$. Suppose the interest rate is $r = 0.05$. We check that indeed $d < r < u$, so that the market is arbitrage-free.

Finally, we can calculate the price of the call option as

$$\begin{aligned}\pi(C_1^{\mathrm{call}}) &= \frac{1}{1.05}\left(\frac{0.05-(-0.2)}{0.2-(-0.2)}(120-110)^+ + \frac{0.2-0.05}{0.2-(-0.2)}(80-110)^+\right)\\ &= \frac{1}{1.05}\left(\frac{5}{8}10+0\right) \approx 5.95\,[\mathrm{CHF}]\,.\end{aligned}$$

Exercise. Using the same procedure, calculate the price of a put option on the same asset with the same strike price K. You should get $\pi(C_{\mathrm{put}}) \approx 10.71\ \mathrm{CHF}$. Check your result with the put-call parity!

Complete Markets

A (multiperiod) arbitrage-free market model is called *complete* if *every* contingent claim is attainable.

From the previous theorem, it follows that in a complete market model every contingent claim has a unique arbitrage-free price.

Theorem. An arbitrage-free market model is complete if and only if there exists just one equivalent martingale measure, that means $|\mathcal{P}| = 1$.

Replicating strategies: binomial model

Example. Let us once again consider the one-period binomial model with the following dynamics

$$B_0 \xrightarrow{\quad 1 \quad} B_1 = (1+r)B_0$$

$$S_0 \xrightarrow{\quad p \quad} S_1(\omega_{\mathrm{u}}) = (1+u)S_0$$

$$S_0 \xrightarrow{\quad 1-p \quad} S_1(\omega_{\mathrm{d}}) = (1+d)S_0$$

So now we know that this market is arbitrage-free *and complete* if, and only if, $d < r < u$.

As promised, we will now derive a replicating strategy.

Let $C_1^{\mathrm{call}} = (S_1 - K)^+$ be the payoff of the call option on the underlying risky asset with strike price $K > 0$.

To replicate C_1^{call}, we need to determine a replicating strategy $\overline{\xi} = (\xi^0, \xi^1)^\intercal$ satisfying

$$(S_1 - K)^+ = \xi^0 B_1 + \xi^1 S_1 = \xi^0(1+r) + \xi^1 S_1 \,.$$

Since we only have two states of the world, ω_1, ω_2, we get a system of two equations,

$$(S_1(\omega_{\mathrm{u}}) - K)^+ = \xi^0(1+r) + \xi^1 S_1(\omega_{\mathrm{u}}) \,,$$
$$(S_1(\omega_{\mathrm{d}}) - K)^+ = \xi^0(1+r) + \xi^1 S_1(\omega_{\mathrm{d}}) \,,$$

or, when plugging in the model specifications,

$$(S_0(1+u) - K)^+ = \xi^0(1+r) + \xi^1 S_0(1+u) \,,$$
$$(S_0(1+d) - K)^+ = \xi^0(1+r) + \xi^1 S_0(1+d) \,.$$

Again in words: we search for numbers ξ^0, ξ^1. They represent how much we have invested in the risk-free asset and in the risky asset, respectively. We want to choose these numbers such that our portfolio *replicates* the payoff profile of the call option.

Solving the above system, we get the following replicating strategy

$$\xi^0 = \frac{(1+u)(S_0(1+d)-K)^+ - (1+d)(S_0(1+u)-K)^+}{(1+r)(u-d)},$$

$$\xi^1 = \frac{(S_0(1+u)-K)^+ - (S_0(1+d)-K)^+}{S_0(u-d)}.$$

Notice that $\xi^0 \leq 0$.

This means, in order to replicate a call option, we have to *borrow money* at the risk-free rate.

The unique arbitrage-free price of C_{call} is then given by

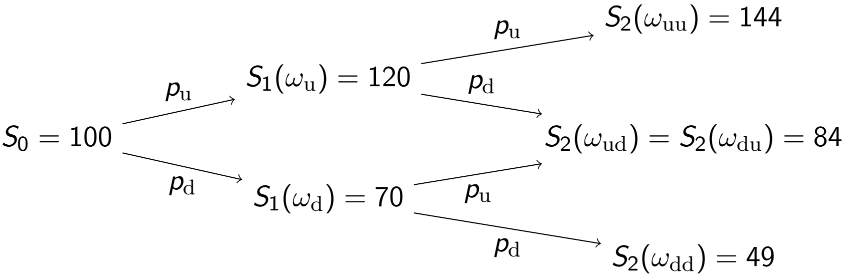
$$\pi(C_1^{\mathrm{call}}) = \begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix} \cdot \begin{pmatrix} B_0 \\ S_0 \end{pmatrix} = \xi^0 + \xi^1 S_0$$

$$= \frac{1}{1+r}\left(\frac{r-d}{u-d}((1+u)S_0 - K)^+ + \frac{u-r}{u-d}((1+d)S_0 - K)^+\right).$$

Notice that this formula indeed coincides with the one from the attainable payoffs section, which we derived via $\pi(C_1^{\mathrm{call}}) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C_1^{\mathrm{call}}}{1+r}\right]$.

Example. Consider the following two-period *reconnecting* binomial model consisting of a riskless and a risky asset with undiscounted dynamics given by

$$B_0 = 1 \xrightarrow{\quad 1 \quad} B_1 = 1.1 \xrightarrow{\quad 1 \quad} B_2 = 1.21$$



Here $u = 0.2$, $d = -0.3$, and $r = 0.1$.

First, we want to exclude arbitrage from the market. So we need to construct an equivalent martingale measure $\mathbb{P}^*$.

As before, we solve the following system of equations. Remember that all probabilities need to sum up to 1 and do not forget to discount (here $r \neq 0$).

Here, since u and d stay constant through both periods, the probabilities will be the same (hence the difference in notation compared to the first two-period binomial tree we saw!). Indeed,

$$
\begin{aligned}
1.1 \cdot 100 = 120 p_{\mathrm{u}}^* + 70 p_{\mathrm{d}}^* &\iff p_{\mathrm{u}}^* = \frac{4}{5},\ p_{\mathrm{d}}^* = \frac{1}{5}\,, \\
1.1 \cdot 120 = 144 p_{\mathrm{u}}^* + 84 p_{\mathrm{d}}^* &\iff p_{\mathrm{u}}^* = \frac{4}{5},\ p_{\mathrm{d}}^* = \frac{1}{5}\,, \\
1.1 \cdot 70 = 84 p_{\mathrm{u}}^* + 49 p_{\mathrm{d}}^* &\iff p_{\mathrm{u}}^* = \frac{4}{5},\ p_{\mathrm{d}}^* = \frac{1}{5}\,.
\end{aligned}
$$

Finally, the equivalent martingale measure $\mathbb{P}^*$ is defined by

$$\mathbb{P}^*[\{\omega_{\mathrm{uu}}\}] = p_{\mathrm{u}}^* p_{\mathrm{u}}^* = \frac{16}{25}\,, \quad \mathbb{P}^*[\{\omega_{\mathrm{ud}}\}] = p_{\mathrm{u}}^* p_{\mathrm{d}}^* = \frac{4}{25}\,,$$

$$\mathbb{P}^*[\{\omega_{\mathrm{du}}\}] = p_{\mathrm{d}}^* p_{\mathrm{u}}^* = \frac{4}{25}\,, \quad \mathbb{P}^*[\{\omega_{\mathrm{dd}}\}] = p_{\mathrm{d}}^* p_{\mathrm{d}}^* = \frac{1}{25}\,.$$

Since this is the unique equivalent martingale measure, the market model is arbitrage-free and complete.

Technical note: Notice that in our reconnecting tree, the events ω_{ud} and ω_{du} lead via different paths to the same result $S_2 = 84$. Depending on whether or not we want to know the path, we could have identified $S_2 = 84$ by just one event $\overline{\omega}$. However, the underlying probabilities stay the same, just the notation changes,

$$\mathbb{P}^*[S_2 = 84] = \mathbb{P}^*[\{\overline{\omega}\}] = \mathbb{P}^*[\{\omega_{\mathrm{ud}}\} \cup \{\omega_{\mathrm{du}}\}] = \frac{4}{25} + \frac{4}{25} = \frac{8}{25}\,.$$

Two-period binomial model

Example. Now we consider a European call option on this asset with strike price $K = 100$. Hence, the payoff is given by $C_2 = (S_2 - 100)^+$.

For simplicity we can draw the payoff diagram.

$$C_0 \xrightarrow{p_{\mathrm{u}}^*} C_1(\omega_{\mathrm{u}}) \xrightarrow{p_{\mathrm{u}}^*} C_2(\omega_{\mathrm{uu}}) = (144 - 100)^+ = 44$$

$$C_1(\omega_{\mathrm{u}}) \xrightarrow{p_{\mathrm{d}}^*} C_2(\omega_{\mathrm{ud}}) = C_2(\omega_{\mathrm{du}}) = (84 - 100)^+ = 0 \xleftarrow{p_{\mathrm{u}}^*} C_1(\omega_{\mathrm{d}})$$

$$C_0 \xrightarrow{p_{\mathrm{d}}^*} C_1(\omega_{\mathrm{d}}) \xrightarrow{p_{\mathrm{d}}^*} C_2(\omega_{\mathrm{dd}}) = (49 - 100)^+ = 0$$

Since the market is complete, the discounted claim $H_2 = \frac{C_2}{(1+r)^2}$ is attainable and its arbitrage-free price is equal to

$$\begin{aligned}\pi(H_2) &= \mathbb{E}_{\mathbb{P}^*}[H_2] = \sum_{\omega\in\Omega} H_2(\omega)\mathbb{P}^*[\{\omega\}] \\ &= H_2(\omega_{\mathrm{uu}})\mathbb{P}^*[\{\omega_{\mathrm{uu}}\}] + 2H_2(\omega_{\mathrm{ud}})\mathbb{P}^*[\{\omega_{\mathrm{ud}}\}] + H_2(\omega_{\mathrm{dd}})\mathbb{P}^*[\{\omega_{\mathrm{dd}}\}] \\ &= \frac{44}{1.21}\cdot\frac{16}{25} + 0 + 0 = \frac{400}{11}\cdot\frac{16}{25} = \frac{256}{11} = 23.\overline{27}\,.\end{aligned}$$

Exercise. Calculate the price of a European put option on the same asset with a strike price of $K = 120$. Notice, the strike is different, so you cannot use the put-call parity! If $K = 100$ the price would be ≈ 5.9174, this is too easy! You should get ≈ 11.8678.

T-period binomial model

Finally, we want to generalise the binomial model to T periods.

Consider a market model with one riskless and one risky asset in which trading is executed at time $t \in \{0, 1, \dots, T\}$.

- ▷ The riskless asset has price at time t given by $B_t = (1+r)^t$, where $r > -1$ denotes the risk-free interest rate.
- ▷ The risky asset has initial price $S_0 > 0$ and at time t its price is given by a non-negative random variable S_t defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which we will explicitly define later.
- ▷ Suppose that the *simple return* R_t of the t^{th} trading period can only take the values

$$R_t = \frac{S_t - S_{t-1}}{S_{t-1}} \in \{d, u\},$$

where, as before, $d < 0 < u$. Notice, this is the same idea as before. At each time, S_t can either move up, $S_t(1+u)$, or down, $S_t(1+d)$.

To formalise the model, it is useful to think of the binomial tree not as outcomes but as *paths*.

We notice that each path can be uniquely represented by the number of *up* and *down* movements when starting from S_0. In general,

$$S_t(\omega) = S_0 \cdot (1+u)^{j_t(\omega)}(1+d)^{t-j_t(\omega)},$$

where $j_t(\omega)$ is the number of up-moves in a total of t moves, when the event ω occurs.

So what are the events $\omega \in \Omega$?

Let $\Omega = \{-1, 1\}^T = \{\omega = (y_1, \dots, y_T) \mid \forall k : y_k \in \{-1, 1\}\}$ be the sample space.

For $\omega = (y_1, \dots, y_T) \in \Omega$ define the projection on the t^{th} coordinate of ω by $Y_t(\omega) = y_t$.

Using this notation, we can rewrite the simple return as

$$
\begin{aligned}
R_t(\omega) &= \frac{S_t(\omega) - S_{t-1}(\omega)}{S_{t-1}(\omega)} = (1+d)\frac{1 - Y_t(\omega)}{2} + (1+u)\frac{1 + Y_t(\omega)}{2} - 1 \\
&= \frac{1}{2}\big(d(1 - Y_t(\omega)) + u(1 + Y_t(\omega))\big) = \begin{cases} u, & \text{if } Y_t(\omega) = 1, \\ d, & \text{if } Y_t(\omega) = -1. \end{cases}
\end{aligned}
$$

The price process of the risky asset can be written as

$$S_t = S_0 \prod_{s=1}^{t} (1 + R_s).$$

The discounted price process is of the form

$$X_t = \frac{S_t}{B_t} = S_0 \prod_{s=1}^{t} \frac{1 + R_s}{1 + r} = \frac{S_0}{(1 + r)^t} \prod_{s=1}^{t} (1 + R_s).$$

The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is usually defined by

$$\mathcal{F}_t = \sigma(S_s \mid 0 \leq s \leq t) := \sigma\left(S_0, \ldots, S_t\right) = \sigma\left(X_0, \ldots, X_t\right)$$

for all $t = 0, \ldots, T$. These $\mathcal{F}_t$ are called *σ-algebras generated by the random variables $S_0, S_1, \ldots, S_t$*.

$\mathcal{F}_0$ is the trivial sigma field, $\mathcal{F} := \mathcal{F}_T$ coincides with the power set of Ω. The random variables Y_t, R_t are $\mathcal{F}_t$-measurable for any trading time.

Fix any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ with $\mathbb{P}\left[\{\omega\}\right] > 0$ for all $\omega \in \Omega$.

The above model is called the *Cox-Ross-Rubinstein (CRR) model*.

Theorem. The Cox-Ross-Rubinstein model is arbitrage free if, and only if, $d < r < u$. In this case, the market is complete and so there exists a unique equivalent martingale measure $\mathbb{P}^*$.

In addition, the random variables $R_1, \dots, R_T$ are independent under $\mathbb{P}^*$ with joint distribution

$$\mathbb{P}^*[R_t = u] = \frac{r-d}{u-d} =: p^* ,$$

$$\mathbb{P}^*[R_t = d] = \frac{u-r}{u-d} =: 1 - p^* .$$

Since the binomial model is arbitrage-free and complete, any contingent claim C_T is attainable.

Thus, we can extend the model by defining the arbitrage-free undiscounted price process C for C_T as

$$C_t = (1+r)^t \, \mathbb{E}_{\mathbb{P}^*}\left[\frac{C_T}{(1+r)^T} \mid \mathcal{F}_t\right],$$

for $t = 0, \ldots, T$.

Theorem. (Black-Scholes' formula for the binomial model) Suppose we are in an arbitrage-free, complete binomial model. Then, the price at time $t \in \{0, \dots, T\}$ of an undiscounted call option payoff $C_T = (S_T - K)^+$ on the underlying risky asset with maturity T and strike price $K > 0$ is given by

$$C_t = \sum_{s=0}^{T-t} \binom{T-t}{s} (p^*)^s (1-p^*)^{T-t-s} \frac{(S_t(1+u)^s(1+d)^{T-t-s} - K)^+}{(1+r)^{T-t}}.$$

Exercise. Prove the theorem.

Examples and exercises

Exercise. Consider a one-period model with 2 assets. Let $\Omega = \{\omega_{\mathrm{u}}, \omega_{\mathrm{d}}\}$ and let $\mathbb{P}[\{\omega_{\mathrm{u}}\}] = 1 - \mathbb{P}[\{\omega_{\mathrm{d}}\}] > 0$.

Assume $r = 0$, $B_0 = 1$, $S_0 = 100$, $S_1(\omega_{\mathrm{u}}) = 120$, $S_1(\omega_{\mathrm{d}}) = 80$ and let $80 < K < 120$.

Consider a call and put option on the risky asset with payoffs $C_1^{\mathrm{call}} = (S_1 - K)^+$ and $C_1^{\mathrm{put}} = (K - S_1)^+$, respectively.

For which values of K do we have $\pi(C_1^{\mathrm{call}}) \leq \pi(C_1^{\mathrm{put}})$?

Solution.

i. First, we need to derive an equivalent risk-neutral measure. We already know that it will be unique, since $d < r < u$. So as before, we solve $p^*120 + (1 - p^*)80 = 100$. This gives the $p^* = \frac{1}{2} = 1 - p^*$.

ii. We also know that the binomial model is complete if $d < r < u$. So we can use p^* to price the contingent claims C_1^{call} and C_1^{put}. We get

$$\pi(C_1^{\mathrm{call}}) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C_1^{\mathrm{call}}}{1+r}\right] = \frac{1}{2}(120 - K)^+ + \frac{1}{2}(80 - K)^+ = 60 - \frac{K}{2}\,,$$

$$\pi(C_1^{\mathrm{put}}) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C_1^{\mathrm{put}}}{1+r}\right] = \frac{1}{2}(K - 120)^+ + \frac{1}{2}(K - 80)^+ = \frac{K}{2} - 40\,.$$

iii. Finally, $\pi(C_1^{\text{call}}) \leq \pi(C_1^{\text{put}})$ is equivalent to

$$60 - \frac{K}{2} \leq \frac{K}{2} - 40\,.$$

This implies $K \geq 100$. By assumption, $K < 120$ and thus, we get that for

$$100 \leq K < 120$$

the call option price is less than or equal to the put option price.

Exercise. Consider a two period binomial model with one risk-free and one risky asset. Let $u = 0.2$, $d = -0.2$, $r = 0.1$, $S_0 = 100$, $B_0 = 1$.

Compute the price at time $t = 0$ of a European put option on the risky asset with maturity 2 and strike price $K = 100$.

Solution. The payoff of the European put option is given by

$$C_2^{\text{put}} = (K - S_2)^+ .$$

Since $d < r < u$, the binomial model is arbitrage-free and complete. Thus, there exists a unique martingale measure $\mathbb{P}^*$ such that

$$\pi(C_2^{\text{put}}) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C_2^{\text{put}}}{(1+r)^2}\right].$$

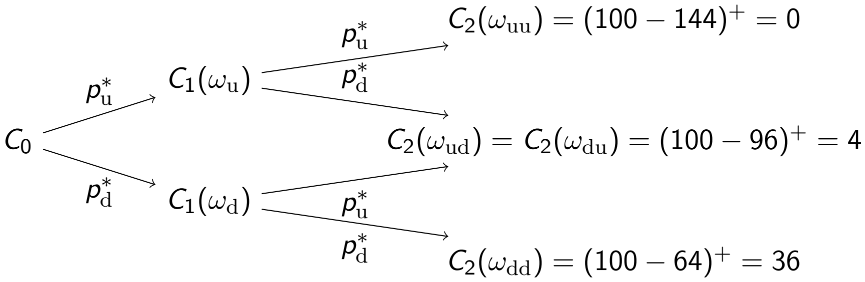
We know that in the binomial model the probability of an up and down movement in each period are given by

$$p^* = \frac{r-d}{u-d} = \frac{3}{4},$$
$$1-p^* = \frac{u-r}{u-d} = \frac{1}{4}.$$

Therefore,

$$\mathbb{P}^*[\{\omega_{\mathrm{uu}}\}] = \frac{9}{16} \quad \mathbb{P}^*[\{\omega_{\mathrm{ud}}\}] = \frac{3}{16}$$
$$\mathbb{P}^*[\{\omega_{\mathrm{du}}\}] = \frac{3}{16} \quad \mathbb{P}^*[\{\omega_{\mathrm{dd}}\}] = \frac{1}{16}.$$

So we have the following state of the payoff.



Exercise 2: Solution

Hence, the price of the put option is given by:

$$\pi(C_2^{\text{put}}) = \mathbb{E}_{\mathbb{P}^*}\left[\frac{C_2^{\text{put}}}{(1+r)^2}\right]$$

$$= \frac{1}{1.1^2}\cdot\left(0 + 2\cdot\frac{3}{16}\cdot 4 + \frac{1}{16}\cdot 36\right)$$

$$= \frac{15}{1.1^2\cdot 4} \approx 3.099\,.$$

Exercise. Consider a one-period market model with $d+1$ assets. We use the notation from above, $\overline{S}_0 = (S_0^0, S_0)^\intercal = (S_0^0, S_0^1, \dots, S_0^d)^\intercal$ is the initial price vector and $\overline{S}_1 = (S_1^0, S_1)^\intercal = (S_1^0, S_1^1, \dots, S_1^d)^\intercal$ is the future price random vector. As always, let $S_0^0 = 1$ and $S_1^0 = (1+r)$ be the prices of the riskless asset.

1. Show that if there exists a free lunch, that is a strategy $\overline{\xi} \in \mathbb{R}^{d+1}$ such that $\overline{\xi} \cdot \overline{S}_0 < 0$ and $\overline{\xi} \cdot \overline{S}_1 \geq 0$ $\mathbb{P}$-a.s., then there exists an arbitrage opportunity.
2. Give an example to show that the reverse is not true.

Solution.

1. Suppose there exists a free-lunch, that means we find a $\overline{\xi} \in \mathbb{R}^{d+1}$ such that $\overline{\xi} \cdot \overline{S}_0 < 0$ and $\overline{\xi} \cdot \overline{S}_1 \geq 0$ $\mathbb{P}$-a.s..
   We want to show that this implies the existence of an arbitrage opportunity, that means there exists $\overline{\eta} \in \mathbb{R}^{d+1}$ such that:
   - $\overline{\eta} \cdot \overline{S}_0 \leq 0$,
   - $\overline{\eta} \cdot \overline{S}_1 \geq 0$ $\mathbb{P}$-a.s.,
   - $\mathbb{P}[\overline{\eta} \cdot \overline{S}_1 > 0] > 0$.

   By definition, a free-lunch implies a negative initial portfolio,

$$0 > \overline{\xi} \cdot \overline{S}_0 = \xi^0 + \xi \cdot S_1 =: -\delta\,,$$

   so that $\delta > 0$.

Now, we define $\eta^0 := \xi^0 + \delta$ and $\eta^k := \xi^k$ for $k = 1, \dots, d$. Then, for the new strategy η we get:

$$\overline{\eta} \cdot \overline{S}_0 = (\xi^0 + \delta) + \eta \cdot S_0 = 0\,.$$

Moreover,

$$\begin{aligned}\overline{\eta} \cdot \overline{S}_1 &= \eta^0(1+r) + \eta \cdot S_1 \\ &= \xi^0(1+r) + \delta(1+r) + \xi \cdot S_1 \\ &= \overline{\xi} \cdot \overline{S}_1 + \delta(1+r).\end{aligned}$$

By definition of free-lunch, we know that $\mathbb{P}[\overline{\xi} \cdot \overline{S}_1 \geq 0] = 1$. But since $\delta(1+r) > 0$, it follows that

$$\mathbb{P}[\overline{\eta} \cdot \overline{S}_1 > 0] = \mathbb{P}\left[\overline{\xi} \cdot \overline{S}_1 + \delta(1+r) > 0\right] = 1.$$

This proves that $\overline{\eta}$ is an arbitrage opportunity.

2. Consider a binomial model with 2 assets.

   Recall that no free lunch in this model is equivalent to $d \leq r \leq u$. Hence, choosing for example $d = r \leq u$ we get the no free lunch condition.

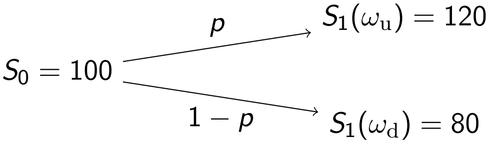
   However, no arbitrage is equivalent to $d < r < u$. And thus, with the above choice there must exist an arbitrage opportunity in the market.

   This example shows, that in general, the existence of an arbitrage opportunity does not imply the existence of free lunch.

A little extra

Example.

Now we want to use options to modify the risk of a position. Assume that the price movement of the risk asset is given by



Let $K = 110$.

If we invest in the risky asset, we have the following *simple return*

$$R_S = \frac{S_1 - S_0}{S_0}.$$

So, in our binomial model, we get

$$R_S(\omega_\mathrm{u}) = \frac{S_1(\omega_\mathrm{u}) - S_0}{S_0} = 0.2 = 20\%,$$
$$R_S(\omega_\mathrm{d}) = \frac{S_1(\omega_\mathrm{d}) - S_0}{S_0} = -0.2 = -20\%.$$

On the other hand, consider again $C_1^{\mathrm{call}} = (S_1 - 110)^+$, and suppose that $r = 0.05$.

We calculate the price of this option as $\pi(C_1^{\mathrm{call}}) \approx 5.95$.

Therefore, the simple return on the option is

$$R_C = \frac{(S_1 - K)^+ - \pi(C_1^{\mathrm{call}})}{\pi(C_1^{\mathrm{call}})}\,.$$

So this gives us

$$R_C(\omega_{\mathrm{u}}) \approx \frac{(120 - 110)^+ - 5.95}{5.95} \approx 68\%,$$
$$R_C(\omega_{\mathrm{d}}) \approx \frac{0 - 5.95}{5.95} = -100\%.$$

There is a dramatic increase of profit opportunities and (in this case) even more so in losses. This is called the *leverage effect* of options.

To reduce the risk of holding the asset, we can hold the portfolio consisting of one risky asset and one put option on that asset. The payoff profil of this portfolio is given by

$$\widetilde{C} = (K - S_1)^+ + S_1 \,.$$

Clearly, this "insurance" involves an additional cost.

Indeed, from the put-call parity, we can derive the price of the put option

$$\pi(C_1^{\mathrm{put}}) = \pi(C_1^{\mathrm{call}}) - S_0 + \frac{K}{1+r} \approx 10.71 \,.$$

So the portfolio of a risky asset and a put on that asset costs approximately $\pi(\widetilde{C}) = 100 + 10.71 = 110.71\ [\mathrm{CHF}]$ at time $t = 0$.

The simple return of this portfolio is

$$R_{\widetilde{C}} = \frac{(K - S_1)^+ + S_1 - \pi(\widetilde{C})}{\pi(\widetilde{C})},$$

which in the different cases is

$$R_{\tilde{C}}(\omega_{\mathrm{u}}) \approx \frac{(110 - 120)^+ + 120 - 110.71}{110.71} \approx 8.39\%$$

$$R_{\tilde{C}}(\omega_{\mathrm{d}}) \approx \frac{(110 - 80)^+ + 80 - 110.71}{110.71} \approx -0.64\%.$$

Hence, by holding this portfolio insurance we have reduce the risk of a loss, although the possibility of a big gain has decreased as well.

Exercise. Explain where the asymmetry of the reduction of profit and losses comes from.

Appendix

We have already seen simple European options.

In a multiperiod setup, we have many more types of options. In the following, we introduce the most important ones.

The outcome of an *Asian option* depends on the average price of the underlying asset S

$$S_{\mathrm{av}} = \frac{1}{|M|} \sum_{t \in M} S_t \,,$$

where $M \subset \{0, 1, \cdots, T\}$ is a subset of predetermined time periods.

The payoff of the *average price call option* with strike price K is given by

$$C^{\mathrm{av}}_{\mathrm{call}} = (S_{\mathrm{av}} - K)^+ \,,$$

and the payoff of the *average price put option* is

$$C^{\mathrm{av}}_{\mathrm{put}} = (K - S_{\mathrm{av}})^+ \,.$$

The payoff of a *barrier option* depends on whether or not the price of the underlying asset reaches a certain level (barrier) before maturity. Most commonly we have two types, the *knock-out* and *knock-in* options.

A *knock-out option* has zero payoff once the price of the underlying asset reaches the barrier $B \in \mathbb{R}$. For example, an *up-and-out call option* with strike price K has payoff

$$C^{\text{u\&o}}_{\text{call}} = \begin{cases} (S_T - K)^+ & \text{if } \max_{0 \le t \le T} S_t < B, \\ 0 & \text{otherwise.} \end{cases}$$

A *knock-in option* pays off only if the barrier B is reached. For example, a *down-and-in put option* with strike price K has payoff

$$C^{\text{d\&i}}_{\text{put}} = \begin{cases} (K - S_T)^+ & \text{if } \min_{0 \le t \le T} S_t \le B, \\ 0 & \text{otherwise.} \end{cases}$$

Acknowledgements

Many thanks to Alexander Smirnow for his support in preparing these slides and for challenging an earlier version of them!