Quantitative Finance

Lectures in Quantitative Finance

Spring Term 2022

3. Risk Measures

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Overview of the presentation

Literature

- Alexander C., Sheedy E., John C., *The Professional Risk Manager Handbook: A Comprehensive Guide to Current Theory and Best Practices*, PRMIA Publications, 2005.

- PRM Handbook, Professional Risk Manager Handbook Series, *Volume II: Mathematical Foundations of Risk Measurement*, 2015 Edition, PRMIA Publications, 2015.

Literature

- Artzner, Ph., Delbaen, F., Eber, J.-M., Heath, D., Coherent measures of risk, *Mathematical Finance* 9, 203-228, 1999.
- Farkas, W., Koch-Medina, P., Munari, C., Capital requirements with defaultable securities, *Insurance: Mathematics and Economics*, 55, 58-67, 2014.
- Föllmer, H., Schied, A., *Stochastic finance: an introduction in discrete time*, De Gruyter Studies in Mathematics 27, 2011.
- Kalkbrenner, M., Lotter, H., Overbeck, L., Sensible and efficient capital allocation for credit portfolios, *RISK*, 19-24, 2004.
- Leippold, M., Trojani, F., Vanini, P., Equilibrium impact of Value-at-Risk regulation, *Journal of Economic Dynamics and Control* 30, 1277-1313, 2006.
- McNeil, A.J., Frey, R., Embrechts, P., *Quantitative Risk Management: Concepts, Techniques, Tools*, Princeton University Press, 2005.

Introduction: What is risk?

What is risk?

- The Concise Oxford English Dictionary:
  "*hazard, a chance of bad consequences, loss or exposure to mischance*".

- A. McNeil, R. Frey, P. Embrechts, *Quantitative risk management: concepts, techniques and tools - revised version*, 2015, p.3:
  "*any event or action that may adversely affect an organization's ability to achieve its objectives and execute its strategies*".

- No single one-sentence definition captures all aspects of risk.

- **Market risk:** Risk of loss in a financial position due to changes in the underlying components/market prices (stocks, etc.).
- **Credit risk:** Risk of a counterparty failing to meet its obligations (loans/bonds etc.).
- **Operational risk (OpRisk):** Risk of loss resulting from inadequate or failed internal processes, people and systems or from external events (fraud, earthquakes, etc.).
- **Liquidity risk:** Market liquidity risk is the risk stemming from the lack of marketability of an investment that cannot be bought or sold quickly enough to prevent a loss. Funding liquidity risk refers to the ease with which institutions can raise funding.
- **Underwriting risk**
- **Model risk**

⋮

What is risk?

- *Financial firms* are not passive/defensive towards risk. They actively take *risks* because they seek a *return*.
- *Risk management* thus belongs to the *core competence* of banks and insurance companies.
- What does *managing risks* involve?
  - Determining enough *buffer capital* to absorb losses: i) for *regulatory* purposes (to satisfy regulators), and ii) for *economic capital* purposes (to allocate capital efficiently from the firm's perspective).
  - Making sure *portfolios* are well *diversified*.
  - Optimizing portfolios according to *risk-return* considerations (for example, via derivatives to hedge exposures to risks, or securitization, i.e., repackaging risks and selling them to investors).

What is risk?

Approaches to risk measurement:

- *Notional-amount approach*: The risk of a portfolio is defined as the (weighted) sum of the notional values of the individual securities.
- *Factor sensitivity measures*: Give the change in portfolio value for a given predetermined change in one of the underlying risk factors.
- *Scenario-based risk measures*: One considers a number of future scenarios and measures the maximum loss of the portfolio under these scenarios.
- *Risk measures based on loss distributions*: Statistical quantities describing characteristics of the loss distribution of the portfolio.

Risk measures are used for the following purposes:

- ▷ Determination of *risk capital*: A risk measure gives the amount of capital needed as a buffer against (unexpected) future losses to satisfy a regulator.
- ▷ *Management tool*: Risk measures are used in internal limit systems.
- ▷ *Insurance premia*: Risk measures are used in determining an insurance premia $\Rightarrow$ insurance risk $=$ reserve risk $+$ premium risk, where reserve risk is associated with historical years and premium risk is associated with future years.

Our *interpretation*:

- ▷ A *risk measure* gives the *amount of capital* that needs to be added to a position with loss L, so that the position becomes *acceptable* to an (internal/external) regulator.

Financial risk and regulators: The solvency problem

- *Liability holders* of a financial institution are concerned that the institution may become *insolvent*, i.e., may fail to honour its future obligations.
- This is the case if the institution's *financial position* will be negative in some *future* state of the economy.
- By *financial position* we mean:

$$\text{financial position} = \text{assets} - \text{liabilities}.$$

- How to act in order to *reduce* the *likelihood* of *insolvency*?

- ▷ We consider a one-period economy with dates $t = 0$ and $t = T$.

- ▷ Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where
  - Ω represents the *set of all future scenarios* of the economy,
  - $\mathcal{F}$ is the σ-algebra, and
  - $\mathbb{P}$ is a *probability measure*.

- ▷ Denote the set of all random variables from Ω to $\mathbb{R}$ by $\mathcal{X}$.

- ▷ We model assets and liabilities at time $t = T$ as random variables

$$A : \Omega \to \mathbb{R} \quad \text{and} \quad L : \Omega \to \mathbb{R}\,.$$

- ▷ The *net financial position* is also a random variable

$$X := A - L \in \mathcal{X}\,.$$

- The value $X(\omega) = A(\omega) - L(\omega) \in \mathbb{R}$ represents the *capital position* at time $t = T$ in case the scenario ω will occur.
- Three cases:

$$X(\omega) = A(\omega) - L(\omega) \begin{cases} > 0 & \text{(gain)} \\ = 0 & \text{(neither gain nor loss)} \\ < 0 & \text{(loss)} \end{cases}$$

- If $X(\omega) \geq 0$ for every $\omega \in \Omega$, the company is always solvent.
- However, typically we have $\mathbb{P}[X < 0] > 0$.

The role of regulators

- To protect liability holders (among other reasons), financial institutions are subject to several *regulatory regimes*.
- *Main goal of regulation*: Ensure that financial institutions have enough capital to remain solvent.
- Financial institutions are required to hold *risk capital* as a buffer reserve against unexpected losses $\Rightarrow$ The key question is *how much* risk capital a financial institution should be required to hold to be deemed *adequately capitalised* by the regulator.
- Some regulatory frameworks:
  - Basel (now Basel III/IV): *Banking* system;
  - Solvency II: *Insurance* companies within EU;
  - Swiss Solvency Test: *Insurance* companies in CH.

The role of regulators

Basel Committee of Banking Supervision (BCBS): The Basel Committee does not have legal force but it formulates standards/best practices/guidelines, the *Basel Accords*, in the expectation that government authorities will take steps to implement them.

The second Basel Accord (Basel II, 2004) introduced a *three pillar concept*:

1. *Minimal capital charge*: Requirements for the calculation of the regulatory capital to ensure that a bank holds sufficient capital for its credit risk in the banking book, its market risk in the trading book, and operational risk.
2. *Supervisory review process*: Local regulators review the checks and balances put in place for capital adequacy assessments, ensure that banks have adequate regulatory capital, and perform stress tests of a bank's capital adequacy.
3. *Market discipline*: Banks are required to make their risk management processes more transparent.

The role of regulators

- Basel III does not replace Basel II, but rather *extends* it.
- The main intention is to *increase bank liquidity* and *decrease bank leverage*.
- Basel III was originally scheduled to be introduced between 2013 and 2015. Implementation was extended repeatedly to March 31st, 2019, and then January 1st, 2022.
- Basel IV is *anticipated*: i) would require more stringent capital requirements, ii) emphasizes simpler or standardized models in place of bank internal models, and iii) requires more detailed disclosure of reserves and other financial statistics.

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Parallel developments in *insurance regulation*:

- More fragmented, much *less international coordination* of efforts.
- Exception: *Solvency* I and II frameworks in the EU.
- Overseen by EIOPA (European Insurance and Occupational Pensions Authority). But implementation is a matter for *national regulators*.
- *Swiss Solvency Test* (SST) is specific to Switzerland (in force since January 2011).
  - Implements its own principles-based risk-capital regulation for insurers.
  - Similar to Solvency II, but differs in its treatment of different types of risk and puts more emphasis on the development of internal models.
  - The implementation of the SST is supervised by the *Swiss Financial Markets Supervisory Authority* (FINMA).

A *societal view*:

- Society relies on the *stability* of the banking and insurance system. The regulatory process, from which Basel II and Solvency II resulted, was motivated by the desire to prevent insolvency of individual institutions and thus *protect customers* (*microprudential* perspective).
- Since the 2007–2009 crisis, the reduction of *systemic risk* has become an important secondary focus (*macroprudential* perspective).
- The interests of society are served by enforcing the discipline of risk management in financial firms, through the use of *regulation*.
- Better *risk management* can reduce the risk of company failure and protect customers and policyholders. However, regulation must be designed with care and should not promote herding, procyclical behaviour, or other forms of endogenous risk that could result in a systemic crisis. Individual firms need to be allowed to fail on occasion.

Why manage financial risk?

A *shareholders' view*:

- The Modigliani–Miller theorem, which marks the beginning of modern *corporate finance* theory, states that, in an ideal world without taxes, bankruptcy costs and informational asymmetries, and with frictionless and arbitrage-free capital markets, the financial structure of a firm (thus its risk management decisions) is *irrelevant* for a *firm's value*.
- To find reasons for corporate Risk Management (RM), one has to "turn the Modigliani–Miller theorem upside down":
  - RM can *reduce taxes*.
  - RM can be beneficial, since a company may have a better *access* to *capital markets* than individual investors.
  - RM can increase the firm value in the presence of bankruptcy costs (liquidation costs or litigation costs), as it *reduces* the *likelihood of bankruptcy*.
  - RM can reduce the *impact* of costly *external financing*.

Acceptable future positions

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- The first step is to discriminate between "good" and "bad" *financial positions* by introducing the concept of an *acceptance set*.

- A set $\mathcal{A}$ of random variables is called an *acceptance set* if

$$X \in \mathcal{A},\ Y \geq X \implies Y \in \mathcal{A}\,.$$

  This property is referred to as *monotonicity*.

- By $Y \geq X$ we mean $Y(\omega) \geq X(\omega)$ for all $\omega \in \Omega$.

- The acceptance set is *specified by the regulator*.

Testing for acceptability

▷ Let $\mathcal{A}$ be the *acceptance set* specified by the *regulator*.

▷ Testing whether a company is *adequately capitalized* reduces to establishing whether (or not) its financial position X belongs to $\mathcal{A}$.

▷ Two situations arise:

- If $X \in \mathcal{A}$, then the company is *not required* to hold risk capital.
- If $X \notin \mathcal{A}$, then the company is *forced* to hold risk capital. But the question is *how much*?

Example: Value-at-Risk acceptability

- Fix a level $\alpha \in (0,1)$, and define an *acceptance set* $\mathcal{A}_\alpha$ by

$$\mathcal{A}_{\mathrm{VaR}_\alpha} := \{X\,;\ \mathbb{P}(X < 0) \leq \alpha\}\,.$$

  Typically α is small, like $\alpha = 5\%$, $\alpha = 1\%$, $\alpha = 0.1\%$.

- Then $X \in \mathcal{A}_{\mathrm{VaR}_\alpha}$ is equivalent to X having a *default probability* capped by α.

Outlook:

- The corresponding *risk measure* $\rho_{\mathcal{A}_{\mathrm{VaR}_\alpha}}$ (more to come!) is called *Value-at-Risk*:

$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X + m < 0) \leq \alpha\}\,.$$

- Value-at-Risk is at the *core* of the Basel and Solvency regimes.

Quantile functions and quantiles

- Let Y be a real-valued random variable.
  The *quantile function* $Q_Y : (0,1) \to \mathbb{R}$ is the *generalized inverse* of the cdf F_Y of Y, which is defined as

$$Q_Y(p) = \inf\{y \in \mathbb{R} \mid F_Y(y) \geq p\}\,.$$

- Notice, if F_Y is *continuous and strictly increasing*, then it has an inverse function and it coincides with the quantile function, $F_Y^{-1} = Q_Y$.
- Now, for some probability level $\beta \in (0,1)$ the *β-quantile* $q_\beta(Y)$ is just the value $Q_Y(\beta)$.
- The $\frac{1}{2}$-quantile is also called the *median*.
- We will see quantiles again soon when we arrive at risk measures, in particular, *Value-at-Risk*.

- Recall: for some probability level $\beta \in (0,1)$, the *β-quantile* $q_\beta(Y)$ is simply the value $Q_Y(\beta)$.
- Note, this is the same as saying that for some probability level $\alpha \in (0,1)$, the (1-α)-*quantile* $q_{1-\alpha}(Y)$ is just the value $Q_Y(1-\alpha)$,

$$q_{1-\alpha}(Y) = Q_Y(1-\alpha) = \inf\{y \in \mathbb{R} \mid F_Y(y) \geq 1-\alpha\}.$$

Example: the (1-α)-*quantile* for $Y = -X$.

For some probability level $\alpha \in (0,1)$, the (1-α)-*quantile* for $Y = -X$ is just the value $Q_{-X}(1-\alpha)$,

$$\begin{aligned}
q_{1-\alpha}(-X) &= Q_{-X}(1-\alpha) \\
&= \inf\{y \in \mathbb{R} \mid F_{-X}(y) \geq 1-\alpha\} \\
&= \inf\{y \in \mathbb{R} \mid \mathbb{P}[-X \leq y] \geq 1-\alpha\} \\
&= \inf\{y \in \mathbb{R} \mid \mathbb{P}[X \geq -y] \geq 1-\alpha\} \\
&= \inf\{y \in \mathbb{R} \mid 1 - \mathbb{P}[X < -y] \geq 1-\alpha\} \\
&= \inf\{y \in \mathbb{R} \mid \mathbb{P}[X < -y] \leq \alpha\} \\
&= \inf\{-x \in \mathbb{R} \mid \mathbb{P}[X < x] \leq \alpha\} \\
&= \inf\{-x \in \mathbb{R} \mid \mathbb{P}[X - x < 0] \leq \alpha\} \\
&= \inf\{m \in \mathbb{R} \mid \mathbb{P}[X + m < 0] \leq \alpha\}.
\end{aligned}$$

Example. Consider $X \sim \mathcal{U}(a, b)$. Recall that

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b\,. \end{cases}$$

Hence, F_X is continuous and if we restrict x to (a, b), it is strictly increasing.

Thus, the quantile function is just the inverse of F_X, that is, for $p \in (0, 1)$, we have

$$Q_X(p) = F_X^{-1}(p) = (b - a)p + a\,.$$

Value-at-Risk and Expected Shortfall

Let X be a profit and loss distribution (assume that the loss is negative and the profit is positive).

Definition. The Value-at-Risk (VaR) at level $\alpha \in (0, 1)$ is:

the smallest number y such that

the probability that $Y := -X$ does not exceed y is at least $1 - \alpha$.

Mathematically, $VaR_\alpha(X)$ is the $(1 - \alpha)$-quantile of $Y := -X$, i.e.,

$$\begin{aligned} VaR_\alpha(X) &= F_Y^{-1}(1-\alpha) \\ &= \inf\{-x \in \mathbb{R} \mid \mathbb{P}[X - x < 0] \leq \alpha\} \\ &= \inf\{m \in \mathbb{R} \mid \mathbb{P}[X + m < 0] \leq \alpha\}. \end{aligned}$$

- How to *compute* $\mathrm{VaR}_\alpha(X)$ for some *fixed level* $\alpha \in (0,1)$ when X is a *discrete random variable*, for example

$$X = \begin{cases} 8 & [95\%] \\ -3 & [1\%] \\ 4 & [4\%] \end{cases} \quad ?$$

- Let $x_1 > x_2 > ... > x_n$ and consider a *general discrete* random variable X. Reorder its outcomes in such a way that

$$X = \begin{cases} x_1 & [p_1] \\ x_2 & [p_2] \\ \cdots & \cdots \\ x_{n-1} & [p_{n-1}] \\ x_n & [p_n] \end{cases}$$

For instance, in the above case we would have:

$$X = \begin{cases} 8 & [95\%] \\ 4 & [4\%] \\ -3 & [1\%] \end{cases}$$

Algorithm:

Start with line n:

- If $p_n > \alpha$ then STOP and $\mathrm{VaR}_\alpha(X) = -x_n$.
- Else, proceed to line $n-1$:
  - If $p_n + p_{n-1} > \alpha$, then STOP and $\mathrm{VaR}_\alpha(X) = -x_{n-1}$.
  - Else, proceed to line $n-2$:
  - If $p_n + p_{n-1} + p_{n-2} > \alpha$ then STOP and $\mathrm{VaR}_\alpha(X) = -x_{n-2}$.

Iterate until the algorithm stops.

Summary:

- Look at the *outcomes* defining X, starting from the *lowest*.
- As soon as you find a probability (sum of probabilities) which is *strictly greater* than α, take the *corresponding value* of X, with the *opposite sign*.

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Example.

▷ Consider again

$$X = \begin{cases} 8 & [p_1 = 95\%] \\ 4 & [p_2 = 4\%] \\ -3 & [p_3 = 1\%] \end{cases}$$

▷ Based on the previous *algorithm* we obtain the following results:

- If $\alpha = 0.5\%$, then $p_3 > \alpha$ and $\mathrm{VaR}_\alpha(X) = 3$.
- If $\alpha = 3\%$, then $p_3 < \alpha$ and $p_3 + p_2 > \alpha$, hence $\mathrm{VaR}_\alpha(X) = -4$.
- If $\alpha = 4\%$, then $p_3 < \alpha$ and $p_3 + p_2 > \alpha$, hence $\mathrm{VaR}_\alpha(X) = -4$.
- If $\alpha = 5\%$, then $p_3 < \alpha$, $p_3 + p_2 = \alpha$ and $p_3 + p_2 + p_1 > \alpha$, hence $\mathrm{VaR}_\alpha(X) = -8$.
- If $\alpha = 8\%$, then $p_3 < \alpha$, $p_3 + p_2 < \alpha$ and $p_3 + p_2 + p_1 > \alpha$, hence $\mathrm{VaR}_\alpha(X) = -8$.

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be a *normal random variable* with mean μ and variance σ^2. For $Z \sim \mathcal{N}(0,1)$, set $\Phi(z) := \mathbb{P}(Z \leq z)$. Then

$$\mathrm{VaR}_\alpha(X) = -\mu - \sigma\,\Phi^{-1}(\alpha)\,.$$

- **Proof.** Since $X \sim \mathcal{N}(\mu, \sigma^2)$, we have $Z := \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. Hence, we obtain

$$\begin{aligned}
\mathrm{VaR}_\alpha(X) &= \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X + m < 0) \leq \alpha\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \mathbb{P}\left(Z < \frac{-m-\mu}{\sigma}\right) \leq \alpha\right\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \mathbb{P}\left(Z \leq \frac{-m-\mu}{\sigma}\right) \leq \alpha\right\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \Phi\left(\frac{-m-\mu}{\sigma}\right) \leq \alpha\right\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \frac{-m-\mu}{\sigma} \leq \Phi^{-1}(\alpha)\right\} \\
&= \inf\{m \in \mathbb{R}\,;\ m \geq -\mu - \sigma\,\Phi^{-1}(\alpha)\} \\
&= -\mu - \sigma\,\Phi^{-1}(\alpha)\,.
\end{aligned}$$

- By the very definition, the VaR at the confidence level α does *not give* any *information* about the *severity* of *losses* which occur with a *probability* of *less* than $1 - \alpha$.
- Discussing VaR from the point of view of *coherence* some further problems appear: no accounting for *diversification*.
- Based on *historical data*, one cannot make accurate statements about the future *probability distribution*.
- Simon Johnson (MIT): "VaR misses everything that matters when it matters".

- Lord Turner, Chairman of the British Financial Services Authority (FSA) in The Turner Review: A regulatory response to the global banking crisis, March 2009, chapter I.4: "...misplaced reliance on sophisticated maths".

- Paper "An academic response to Basel II" by Jon Danielsson, Paul Embrechts, and others (LSE 2001).

- Coherent risk measures: P. Artzner, F. Delbaen, and others (1997, 1999).

Despite the criticism, the VaR approach is *still used*, mainly due to the Basel requirements (Basel Committee for Banking Supervision).

$\rightarrow$ *Questionable point* of the *international regulation*!

- "The proposed regulations fail to consider the fact that risk is endogenous, Value-at-Risk can destabilize an economy and induce crashes when they would not otherwise occur."
- "Statistical models used for forecasting risk have been proven to give inconsistent and biased forecasts, notably under-estimating the joint downside risk of different assets. The Basel Committee has chosen poor quality measures of risk when better risk measures are available."
- "Heavy reliance on credit agencies for the standard approach to credit risk is misguided as they have been shown to provide conflicting and inconsistent forecasts of individual clients' creditworthiness. They are unregulated and the quality of their risk estimates is largely unobservable."

Value-at-Risk: Not convex

▷ You are a *bank* and you give a *loan* of 100 CHF.

- The loan *interest rate* is $r = 2\%$.
- The *default probability* of the counterparty is $p = 0.8\%$.
- The VaR-level is $\alpha = 1\%$.

▷ The corresponding *position* at *maturity* is

$$X = \begin{cases} 100r = 100(1+r) - 100 & [\,1 - p = 99.2\%\,] \\ -100 & [\,p = 0.8\%\,] \end{cases}$$

▷ Recall the definition of Value-at-Risk

$$\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X + m < 0) \le \alpha\}\,.$$

▷ How to *compute* $\mathrm{VaR}_\alpha(X)$?

- Recall $\alpha = 1\%$ and

$$X = \begin{cases} 100r & [\,1 - p = 99.2\%\,] \\ -100 & [\,p = 0.8\%\,] \end{cases}$$

- It holds that $\mathrm{VaR}_\alpha(X) = -100r$.

- **Proof.** We have

$$\mathbb{P}(X + m < 0) = \begin{cases} 1 & \text{if } m < -100r \\ p = 0.8\% & \text{if } -100r \le m < 100 \\ 0 & \text{if } m \ge 100 \end{cases}$$

  Hence, $\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X + m < 0) \le \alpha\} = -100r$.

- Look at the *outcomes* defining X, starting from the *lowest*. As soon as you find a *probability* which is *strictly greater* than α, take the corresponding value of X, with the opposite sign.

- Next, assume you wish to *diversify*, and you give two 50 CHF loans.
  - The *default probability* of each counterparty is $p = 0.8\%$.
  - Defaults are *independent*.

- Take $Y, Z \sim X$ with Y, Z *independent*. The new position at *maturity* is

$$\frac{1}{2}Y + \frac{1}{2}Z = \begin{cases} 100r & [(1-p)^2 = 98.4064\%] \\ 50r - 50 & [2p(1-p) = 1.5872\%] \\ -100 & [p^2 = 0.0064\%] \end{cases}$$

- Given $\alpha = 1\%$, what is $\mathrm{VaR}_\alpha(\frac{1}{2}Y + \frac{1}{2}Z)$?

- Using the usual method, we obtain $\mathrm{VaR}_\alpha(\frac{1}{2}Y + \frac{1}{2}Z) = -(50r - 50)$.

- Finally, we show that Value-at-Risk is *not convex*.

- Now we have, on one side

$$\mathrm{VaR}_\alpha\left(\frac{1}{2}Y + \frac{1}{2}Z\right) = 50 - 50r = 49\,,$$

while on the other side

$$\begin{aligned}\frac{1}{2}\mathrm{VaR}_\alpha(Y) + \frac{1}{2}\mathrm{VaR}_\alpha(Z) &= \frac{1}{2}\mathrm{VaR}_\alpha(X) + \frac{1}{2}\mathrm{VaR}_\alpha(X)\\ &= \mathrm{VaR}_\alpha(X)\\ &= -100r\\ &= -2\,.\end{aligned}$$

- Hence, we have

$$\mathrm{VaR}_\alpha\left(\frac{1}{2}Y + \frac{1}{2}Z\right) > \frac{1}{2}\mathrm{VaR}_\alpha(Y) + \frac{1}{2}\mathrm{VaR}_\alpha(Z)\,,$$

which shows that *Value-at-Risk* is *not* a *convex* risk measure.

▷ The *Expected Shortfall* (or Tail-Value-at-Risk) is defined as

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\, d\beta\,.$$

▷ Expected Shortfall is *implemented* in the Swiss Solvency Test. Under discussion about Basel III.

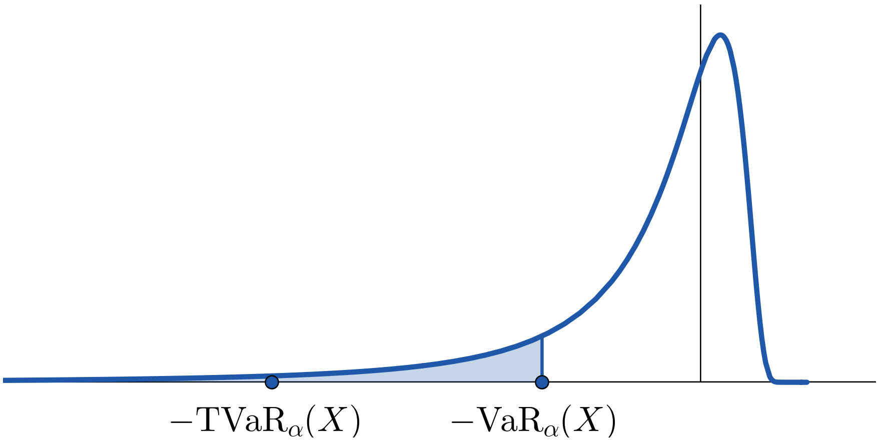

Figure 1: Visualization of Expected Shortfall for $X \sim \mathrm{GEV}(\mu, \sigma, \xi)$.

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then it holds

$$\mathrm{ES}_\alpha(X) = -\mu + \frac{\sigma}{\alpha}\,\Phi'(\Phi^{-1}(\alpha))\,.$$

- **Proof.**

  Since $\mathrm{VaR}_\beta(X) = -\mu - \sigma\,\Phi^{-1}(\beta)$, we have

$$\begin{aligned}
\mathrm{ES}_\alpha(X) &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\,d\beta \\
&= \frac{1}{\alpha}\int_0^\alpha \left(-\mu - \sigma\,\Phi^{-1}(\beta)\right) d\beta \\
&\overset{\beta=\Phi(z)}{=} -\mu - \frac{\sigma}{\alpha}\int_{-\infty}^{\Phi^{-1}(\alpha)} z\,\Phi'(z)\,dz \\
&\overset{\Phi'(z)=\frac{1}{\sqrt{2\pi}}\exp(-z^2/2)}{=} -\mu - \frac{\sigma}{\alpha}\left[-\Phi'(z)\right]_{-\infty}^{\Phi^{-1}(\alpha)} \\
&= -\mu + \frac{\sigma}{\alpha}\,\Phi'(\Phi^{-1}(\alpha))\,.
\end{aligned}$$

- VaR is a *frequency-based* risk measure.
- A position X is acceptable for VaR if, and only if,

$$\mathbb{P}(X < 0) \leq \alpha.$$

$\rightarrow$ No information about the *magnitude* of a *potential loss*!

- ES is a *severity-based* risk measure.
- A (continuous) position X is acceptable for ES if, and only if,

$$\mathbb{E}[X 1_{\{X \leq -\mathrm{VaR}_\alpha(X)\}}] \geq 0.$$

$\rightarrow$ Positions with a *heavy left tail* are likely to be *unacceptable*!

- There are less acceptable positions in the ES sense. Indeed,

$$\mathrm{ES}_\alpha(X) \geq \mathrm{VaR}_\alpha(X)\,.$$

In other words, ES defines *higher risk capital*.

Formalizing the capital adequacy problem

The core of this section aims to present the framework proposed in *Coherent Measures of Risk* published by P. Artzner, F. Delbaen, J.-M. Eber & D. Heath (1999).

As scientists we observe the real world and build mathematical models of it. This allows us to (approximately) describe the world and solve problems therein. One such problem evolves around capital adequacy:

- Liability holders and regulators of financial institutions are credit sensitive.
- They are concerned that the value of the institution's **assets** is insufficient to cover its **liabilities**.
- To address this concern financial institutions hold **risk capital**, which is meant to absorb unexpected losses.

Capital adequacy

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- They are concerned that the value of the institution's **assets** is insufficient to cover its **liabilities**.
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Capital adequacy

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The capital adequacy problem:

- **How much risk capital** should a financial institution hold to be deemed **adequately capitalized** by the regulator?

Related regulatory frameworks:
Swiss Solvency Test, Solvency II, Basel III-IV.

The acceptance set

As mentioned, the first step is to discriminate between “good” and “bad” financial positions by introducing the concept of an acceptance set.

Definition. A set $\mathcal{A}$ of random variables is called an *acceptance set* if

- $\mathcal{A}$ is *non-trivial*, i.e. $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \subsetneq \mathcal{X}$,
- $\mathcal{A}$ is *monotone*, i.e. $X \in \mathcal{A},\ Y \geq X \implies Y \in \mathcal{A}$.

By $Y \geq X$ we mean either $Y(\omega) \geq X(\omega)$ for all $\omega \in \Omega$, or $\mathbb{P}$-almost surely.

The acceptance set is *specified by the regulator*.

Properties of acceptance sets

An acceptance set $\mathcal{A} \subset \mathcal{X}$ is called

- ▷ a *cone* if $X_T \in \mathcal{A} \implies \forall \lambda > 0 : \lambda X_T \in \mathcal{A}$,
- ▷ *convex* if $X_T, Y_T \in \mathcal{A} \implies \forall \lambda \in [0,1] : \lambda X_T + (1-\lambda) Y_T \in \mathcal{A}$,
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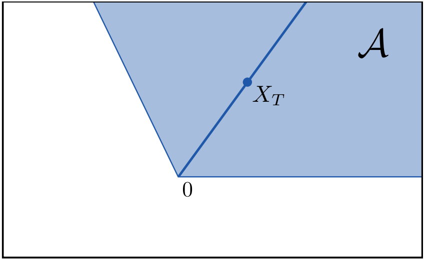

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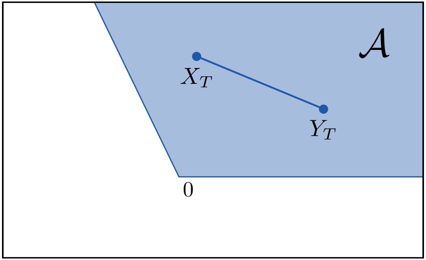

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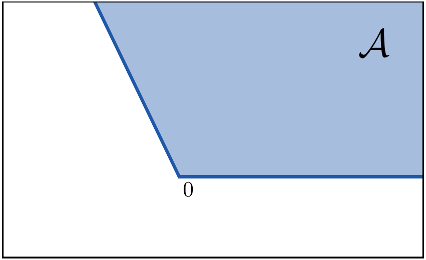

Testing for acceptability

Let $\mathcal{A}$ be the acceptance set specified by the regulator.

Testing whether a company is adequately capitalised or not reduces to establishing whether its financial position X belongs to $\mathcal{A}$ or not.

Two situations:

- if $X \in \mathcal{A}$, then the company is not required to hold additional risk capital,
- if $X \notin \mathcal{A}$, then the company is forced to hold risk capital...
  but how much?

Risk measures

Risk capital is determined by using appropriate *risk measures*.

Definition. The risk measure $\rho_{\mathcal{A}}$ associated to the acceptance set $\mathcal{A}$ is the map

$$\rho_{\mathcal{A}} : \mathcal{X} \to \mathbb{R} \quad \text{with} \quad X \mapsto \rho_{\mathcal{A}}(X)\,,$$

defined by

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}\,.$$

It gives a rule to compute risk capital according to the acceptance set $\mathcal{A}$.

- The quantity $\rho_{\mathcal{A}}(X) \in \mathbb{R}$ is an amount of capital, which we interpret as *risk capital*.

- More precisely, $\rho_{\mathcal{A}}(X)$ defines the minimal amount of capital that has to be added to X in order to transform X into an acceptable position. We can say that $\rho_{\mathcal{A}}(X)$ is the *cost* of making X *acceptable*.

- Risk measures give us a way to check if a financial position belongs to the acceptance set. If $\mathcal{A}$ is closed, then:

$$X \in \mathcal{A} \iff \rho_{\mathcal{A}}(X) \leq 0,$$
$$X \notin \mathcal{A} \iff \rho_{\mathcal{A}}(X) > 0.$$

Proposition. Let $\mathcal{A}$ be an acceptance set, and $\rho_{\mathcal{A}}$ the corresponding risk measure. Then $\rho_{\mathcal{A}}$ satisfies:

- ▷ *Monotonicity*: $X \le Y \implies \rho_{\mathcal{A}}(X) \ge \rho_{\mathcal{A}}(Y)$,
- ▷ *Cash-additivity*: $\rho_{\mathcal{A}}(X + c) = \rho_{\mathcal{A}}(X) - c$ for all $c \in \mathbb{R}$.

Proof. If $X \le Y$, then by the monotonicity of $\mathcal{A}$ we have

$$\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\} \subseteq \{m \in \mathbb{R} \mid Y + m \in \mathcal{A}\}$$

and monotonicity of $\rho_{\mathcal{A}}$ follows by taking the infimum on both sides.

For cash.additivity fix $c \in \mathbb{R}$. Then, using the substitution $k = c + m$ in the second equality, we obtain

$$\begin{aligned}\rho_{\mathcal{A}}(X + c) &= \inf\{m \in \mathbb{R} \mid X + c + m \in \mathcal{A}\}\\ &= \inf\{k - c \in \mathbb{R} \mid X + k \in \mathcal{A}\}\\ &= \inf\{k \in \mathbb{R} \mid X + k \in \mathcal{A}\} - c = \rho_{\mathcal{A}}(X) - c\,.\end{aligned}$$

Example. Fix a level $\alpha \in (0,1)$, and define the Value-at-Risk acceptance set by

$$\mathcal{A}_{\mathrm{VaR}_\alpha} := \{X \mid \mathbb{P}[X < 0] \le \alpha\}\,.$$

Typically, α is small, e.g. $\alpha = 5\%$, $\alpha = 1\%$, $\alpha = 0.1\%$.

Then, $X \in \mathcal{A}_{\mathrm{VaR}_\alpha}$ is equivalent to X having a *default probability* capped by α.

The corresponding risk measure $\rho_{\mathcal{A}_{\mathrm{VaR}_\alpha}}$ is the well-known *Value-at-Risk*:

$$\mathrm{VaR}_\alpha(X) := \rho_{\mathcal{A}_{\mathrm{VaR}_\alpha}}(X) = \inf\{m \in \mathbb{R} \,|\, \mathbb{P}(X + m < 0) \leq \alpha\}\,.$$

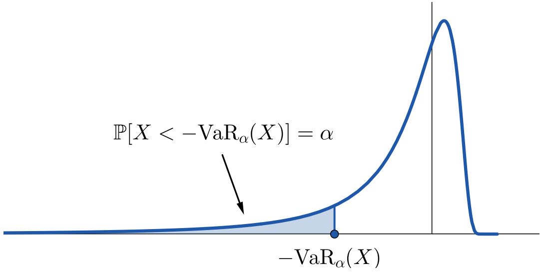

Figure 2: Visualisation of Value-at-Risk for $X \sim \mathrm{GEV}(\mu, \sigma, \xi)$.

Conclusion

S. Shreve, *Don't blame the quants*, 2008

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S. Shreve, *Don't blame the quants*, 2008

"When a bridge collapses, no one demands the abolition of civil engineering. [...]
If engineering is to blame the solution is *better – not less –* engineering.

Furthermore, it would be preposterous to replace the bridge with a slower, less efficient ferry rather than to rebuild the bridge and overcome the obstacle."

What to do now?

We need:

- Transparent *financial products* and robust procedures,
- Stronger buffers for *risk*,
- An intelligent *regulation* of financial markets on *international* level,
- Appropriate *stimulating* mechanisms.

This requires *more* quantitative analysis and mathematics, not less!

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... and especially a project of stronger international regulation!

More or less?

We need *more* quantitative analysis:

- Even a regulatory measure which seems very plausible at the beginning could be an open door for *arbitrage strategies*. Here, we need careful analysis supported by *mathematical models*.
- Even for incentive schemes mathematics is needed! In the economics literature it belongs to the so-called *principal-agent* problem and *mechanism design*.

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- Even a regulatory measure which seems very plausible at the beginning could be an open door for *arbitrage strategies*. Here, we need careful analysis supported by *mathematical models*.
- Even for incentive schemes mathematics is needed! In the economics literature it belongs to the so-called *principal-agent* problem and *mechanism design*.

We need *less* quantitative analysis:

- There is no "correct" mathematical model for the *financial markets*.
- Any model is somehow "naive" and can become *dangerous* when it replaces the reality.

Recent developments and concerns

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- *High frequency trading*
  Is it fair? Does it cause flash crashes?
- *Algorithmic trading*
  Does it increase or reduce liquidity? Does it make markets more volatile?
- *Commodities trading*
  Does it inflate food prices? Does it increase volatility?
- *Systemic risk*
  How can it be measured, managed, mitigated? How should regulators deal with it? How can systemically important financial institutions be identified?

- Main takeaways:
  - *Risk measure* defines the minimal amount of *capital* that has to be *added* to a net financial position of a company in order that such position becomes *acceptable*.
  - Value-at-Risk at the confidence level α is a frequency-based risk measure which does *not give* any information about the *severity* of *losses* which occur with a probability of less than $1 - \alpha$.
  - Expected Shortfall is a severity-based risk measure which depends also on the *heaviness* of the *left tail* of the position's distribution.
  - Rules will have to be both *tightened* and *better enforced* to prevent/mitigate future financial crises. However, all the reforms in the world will *never guarantee* total safety.

Thanks

Thank you very much!

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