Quantitative Finance

Lectures in Quantitative Finance
Spring Term 2022
2. Bond fundamentals

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Overview of the lecture

Bonds: Definition and Examples

Bonds: Definition

- A *bond* is an instrument of indebtedness, under which the *issuer* (debtor) owes the *holder* (creditor) a debt, and is obliged – depending on the terms – to pay them interest (i.e., the *coupon*) as well as to repay the *principal* at the *maturity*.
- *Interest* is usually payable at *fixed intervals* (semiannual, annual, sometimes monthly).
- Generally, the *ownership* of the bond can be transferred in the *secondary market* $\Longrightarrow$ *high liquidity* of the bond market.
- A bond is a form of *loan* $\Longrightarrow$ bonds provide the borrower with external funds to finance *long-term investments* or – in the case of government bonds – to finance *current expenditure*.

In practice, depending on the nature of the issues, one distinguishes between different types of bonds:

- **Government or Treasury Bonds**: issued by governments, primarily to finance the shortfall between public revenues and expenditures and to pay off earlier debts.
- **Municipal Bonds**: issued by municipalities, e.g. cities and towns to raise the capital needed for various infrastructure works such as roads, bridges, sewer systems, and so on.
- **Mortgage Bonds**: issued by special agencies who use the proceeds to purchase real estate loans extended by commercial banks.
- **Corporate Bonds**: issued by large corporations to finance the purchase of property, plant and equipment.

- Among all assets, the simplest (i.e., most basic) to study are fixed-coupon bonds as their cash-flows are predetermined.
- The **valuation of bonds** requires a good understanding of concepts such as *compound interest*, *discounting*, *present value*, and *yield*.
- For **hedging** and **risk management** of bond portfolios (risk) sensitivities such as *duration* and *convexity* are important.
- Bonds and stocks are both *securities*, but the major difference between the two is that (capital) stockholders have an equity stake in a company (i.e. they are owners), whereas *bondholders* have a *creditor* stake in the company (i.e. they are lenders).
- Being a creditor, *bondholders* have *priority* over stockholders. This means they will be repaid in advance of stockholders.

Bonds: Default

- For bonds issued by some companies and countries, there is a non-negligible *risk* that the issuer will *default* and the holder of the bond will not receive the promised payments.
- The price of the bond has to reflect that $\Longrightarrow$ *credit risk*.
- *Credit quality* tells investors how likely the borrower is going to default.
- *High-yield bonds* are bonds that are rated below investment grade by the credit rating agencies. As these bonds are *riskier* than investment grade bonds, investors expect to earn a higher yield.
- The bonds issued by some *national governments* (often also called Treasury bonds) are sometimes treated as *risk-free* and not exposed to default risk. Risk-free bonds are thus the safest bonds, with the lowest interest rate.

Bonds: Trading

- The issue price at which investors buy the bonds when they are first issued will typically be approximately equal to the nominal amount.

- The *market price* of the bond *varies* over its life: it may trade at a *premium* (above par, usually because market interest rates have fallen since issue), or at a *discount* (price below par, if market rates have risen or there is a high probability of default on the bond).

- Bonds are bought and *traded* mostly by *institutions* such as central banks, sovereign wealth funds, pension funds, insurance companies, hedge funds, and banks.

Zero-Coupon Bonds

A *zero-coupon* bond promises no coupon payments, only the repayment of the principal at maturity.

▷ Consider an investor who wants a zero-coupon bond, which

- pays 100 CHF
- in 10 years, and
- has no default risk.

Since the payment occurs at a future data – here after 10 years – the value of this investment is surely less than an up-front payment of 100 CHF (assuming a positive interest rate environment).

▷ To *value* this payment one needs two ingredients:

- the prevailing *interest rate* per period
- and the *tenor*, denoted T, which gives the number of periods until maturity, expressed in years.

- The *present value* of a zero-coupon bond is:

$$P(y) = \frac{C_T}{(1+y)^T} ,$$

where C_T is the principal (or face value) and y is the discount rate.

- For instance, a payment of $C_T = 100$ CHF in 10 years discounted at 6% is (only) worth 55.84 CHF.

Note:

- The (market) value of zero-coupon bonds decreases with longer maturities;
- keeping T fixed, the value of a zero-coupon bond decreases as the interest rate (or yield) increases.

Zero-Coupon Bonds

- Analogously to the notion of *present value*, we can define the notion of *future value* (FV) for an initial investment of amount F:

$$FV = F \cdot (1 + y)^T \ .$$

- For example, an investment now worth $F = 100$ CHF growing at 6% per year will have a future value of 179.08 CHF in 10 years.

- The internal rate of return of a bond, or annual growth rate, is called the *yield*, or *yield-to-maturity (YTM)*.
- Yields are usually easier to deal with than CHF values.
- Rates of return are directly comparable across assets (when expressed in percentage terms and on an annual basis).
- The yield y of a bond is the solution to the (non-linear) equation:

$$P = P(y),$$

where "P" is the (market) price of the bond and $P(\cdot)$ is the price of the bond as a function of the yield y; recall that in case of a zero-coupon bond

$$P(y) = \frac{C_T}{(1+y)^T}.$$

- The yield of bonds with the same characteristics but with different maturities can differ strongly. Hence, the yield (usually) depends on the maturity of the bond.
- The *yield curve* is the set of yields as a function of maturity.
- Under "normal" circumstances, the yield curve is upward sloping. That means the longer you lock in your money, the higher your return.

Important: state the method used for compounding:

- *Annual compounding* (usually the norm):

$$P(y) = \frac{C_T}{(1+y)^T} .$$

- *Semi-annual compounding* (e.g. used in the U.S. Treasury bond market): interest rate y_s is derived from:

$$P(y_s) = \frac{C_T}{\left(1+\frac{y_s}{2}\right)^{2T}} ,$$

where $2T$ is the number of periods, T is the number of years.

- *Continuous compounding* (used ubiquitously in the quantitative finance literature): interest rate y_c is derived from:

$$P(y_c) = \frac{C_T}{\exp(y_c T)} = e^{-y_c T} C_T .$$

Example. Consider our example of the zero-coupon bond, which pays 100 CHF in 10 years, once again. Recall that the present value of the bond is approximately equal to 55.8395 CHF.

Now we can compute the 3 yields as follows:

- Annual compounding:

$$P(y) = \frac{C_T}{(1+y)^{10}} \;\Rightarrow\; y = 6\%.$$

- Semi-annual compounding:

$$P(y) = \frac{C_T}{(1+y_s/2)^{20}} \;\Rightarrow\; (1+y_s/2)^2 = 1+y \;\Rightarrow\; y_s = 5.91\%.$$

- Continuous compounding:

$$P(y) = \frac{C_T}{\exp(y_c T)} \;\Rightarrow\; \exp(y_c) = 1+y \;\Rightarrow\; y_c = 5.83\%.$$

Note: higher (compounding) frequency results in a lower equivalent yield.

Exercise. Assume a semi-annual compounded rate of 8% (per annum). Then the equivalent annual compounded rate is

(1) 9.20%,

(2) 8.16%,

(3) 7.45%,

(4) 8.00%.

Exercise. Assume a semi-annual compounded rate of 8% (per annum). Then the equivalent annual compounded rate is

(1) 9.20%,

(2) 8.16%,

(3) 7.45%,

(4) 8.00%.

Solution: The correct answer is: (2). This is derived from

$$(1 + \frac{y_s}{2})^2 = (1 + y)$$

or, equivalently, $(1 + \frac{0.08}{2})^2 = (1 + y)$, which gives $y = 8.16\%$.

Exercise. Assume a continuously compounded rate of 10% (per annum). Then the equivalent semi-annual compounded rate?

(1) 10.25%,

(2) 9.88%,

(3) 9.76%,

(4) 10.52%.

Exercise. Assume a continuously compounded rate of 10% (per annum). Then the equivalent semi-annual compounded rate?

(1) 10.25%,

(2) 9.88%,

(3) 9.76%,

(4) 10.52%.

Solution: The correct answer is: (1). This is derived from

$$(1+\frac{y_s}{2})^2 = \exp(y_c)$$

or, equivalently, $(1+\frac{y_s}{2})^2 = 1.1056$, which gives $y_s = 10.25\%$.

Coupon Bonds

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While zero coupon bonds are a very useful (theoretical) concept, the bonds usually issued and traded are coupon bearing bonds.

Note:

- A zero coupon bond is a special case of a coupon bond (with zero coupon);
- A coupon bond can be seen as a portfolio of zero coupon bonds.

Consider now the price (or present value) of a coupon bond with a general pattern of fixed cash-flows.

The *price-yield relationship* is defined as follows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \, .$$

Here we have adopted the following notations:

- C_t: the cash-flow (coupon or principal) in period t;
- t: the number of periods (e.g. half-years) to each payment;
- T: the number of periods to final maturity;
- y: the discounting yield.

- As indicated earlier, the typical cash-flow pattern for bonds traded in reality consists of regular coupon payments plus repayment of the principal (or face value) at the expiration.

- Specifically, if we denote c the coupon rate and F the face value, then the bond will generate the following stream of cash flows:

$$\begin{aligned} C_t &= cF & &\text{prior to expiration:} \quad 0 \leq t \leq T-1 \\ C_T &= cF + F & &\text{at expiration} \quad (t = T). \end{aligned}$$

Using this particular cash-flow pattern, we obtain (by the geometric series formula) a more compact formula for the price of a coupon bond:

$$\begin{aligned} P(y) &= \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_{T-1}}{(1+y)^{T-1}} + \frac{C_T}{(1+y)^T} \\ &= \frac{cF}{1+y} + \frac{cF}{(1+y)^2} + \cdots + \frac{cF}{(1+y)^{T-1}} + \frac{cF+F}{(1+y)^T} \\ &= cF \cdot \frac{\frac{1}{1+y} - \frac{1}{(1+y)^{T+1}}}{1 - \frac{1}{1+y}} + \frac{F}{(1+y)^T} \\ &= \frac{cF}{y}\left(1 - \frac{1}{(1+y)^T}\right) + \frac{F}{(1+y)^T}. \end{aligned}$$

Coupon Bonds: par bonds

Definition. A *par bond* (or a bond priced *at par*) is a bond for which the coupon rate matches the yield ($c = y$) (using the same compounding frequency).

Remark. The price of a par bond equals its face value, since

$$P(y) = \frac{yF}{y}\left(1 - \frac{1}{(1-y)^T}\right) + \frac{F}{(1+y)^T} = F\,.$$

Example. Consider a bond that pays 100 CHF in 10 years and has a 6% annual coupon.

(a) What is the market value of the bond if the yield is 6%?

(b) What is the market value of the bond if the yield falls to 5%?

Solution:

(a) The cash flows are $C_1 = 6, \ldots, C_9 = 6,\ C_{10} = 106$. Discounting at 6% gives values of 5.66,. . . ,3.55, and 59.19, which sum up to 100 CHF. Hence, the bond is selling at par.

(b) Alternatively, discounting at 5% leads to a price of 107.72 CHF. Hence, the bond is selling *above par*.

Exercise. Consider a 1-year fixed-rate bond currently priced at 102.9 CHF, with face value is CHF 100 and paying a 4% coupon each six months.

Then the yield of the bond is:

(a) 8% (b) 7% (c) 6% (d) 5% .

Solution: The correct answer is: (d).

The (two) cash flows are:

- CHF 4 after 6 months and
- CHF 104 after 12 months

Consequently, we need to find y_s such that

$$\frac{4}{1+\frac{y_s}{2}} + \frac{104}{\left(1+\frac{y_s}{2}\right)^2} = 102.9$$

Solving, we find $y_s = 0.0499\ldots \approx 5\%$.

Another special case of a general coupon bond is the so-called *perpetual bond*, or *consol*.

Definition. A perpetual bond is a regular coupons paying bond with $C_t = cF$, but with *infinite maturity*.

Example. The price of a consol is given by

$$P(y) = \frac{c}{y} F.$$

Solution: To calculate the price of a consol, we use the same approach. However, the sum becomes a series and we get:

$$
\begin{aligned}
P(y) &= \frac{cF}{1+y} + \frac{cF}{(1+y)^2} + \frac{cF}{(1+y)^3} + \cdots \\
&= cF \left[\frac{1}{1+y} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots \right] \\
&= cF \frac{1}{1+y} \left[1 + \frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \cdots \right] \\
&= cF \frac{1}{1+y} \left[\frac{1}{1-(1/(1+y))} \right] \\
&= cF \frac{1}{1+y} \frac{1+y}{y} \\
&= \frac{c}{y} F.
\end{aligned}
$$

Taylor series

- To gain information on the price changes of a bond as effects of changes in the risk factors, in our case the yield, is very important for *Hedging* and *Risk management*

- We focus on the the yield as main risk factor and analyse the following question:
  What happens with the price of a bond when the yield moves form an initial value y_0 to a new value $y_1 = y_0 + \Delta y$ (assuming Δy is relatively "small")?

Price approximation for bonds

- We start with the price-yield relation $P = P(y)$, and consider an initial price of the bond $P_0 = P(y_0)$ and a new price $P_1 = P(y_1)$.
- For a small change $\Delta y = y_1 - y_0$ we could approximate P_1 through a *Taylor series*:

$$
\begin{aligned}
P(y_1) = P(y_0 + \Delta y) &= P(y_0) + P'(y_0)\Delta y + \frac{1}{2}P''(y_0)(\Delta y)^2 + \dots \\
&= \sum_{n=0}^{\infty} \frac{P^{(n)}(y_0)}{n!}(\Delta y)^n .
\end{aligned}
$$

- This is a series ("infinite sum") with increasing exponents of Δy.

But how exactly does a Taylor series work?

Let us consider a real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ and a fixed point $x_0 \in \mathbb{R}$. The following series is called *Taylor series of f at x_0* and is defined as

$$T_{f,x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots$$

We immediately notice that for the existence of the series the function f has to be infinitely differentiable at x_0. If the function is infinitely differentiable on its domain, we call it *smooth*.

Often the Taylor series is a good approximation of the function in a neighbourhood of x_0, that means at points x which are close to x_0.

Be careful! 'Often' in the previous sentence is not for rhetoric purposes as we will see in an example below.

In practice, for example with computers, we are often limited in handling infinities. So we simply truncate the series at a certain degree.

So we call

$$\sum_{n=0}^{1} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + f'(x_0)(x - x_0)$$

a *Taylor polynomial of degree one*,

$$\sum_{n=0}^{2} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

a *Taylor polynomial of degree two* and so on.

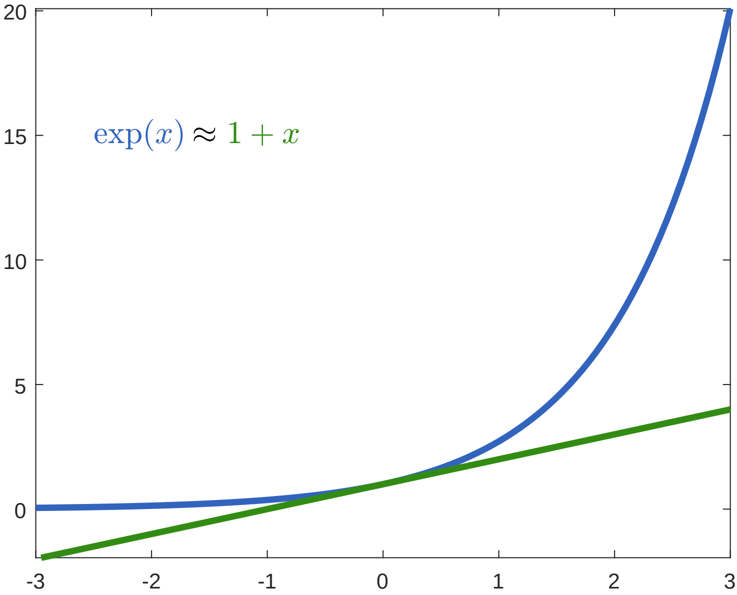
Example Consider $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \exp(x)$. We know that for the exponential function we have $f'(x) = f(x)$. We conclude that f is smooth and we can write down the Taylor series,

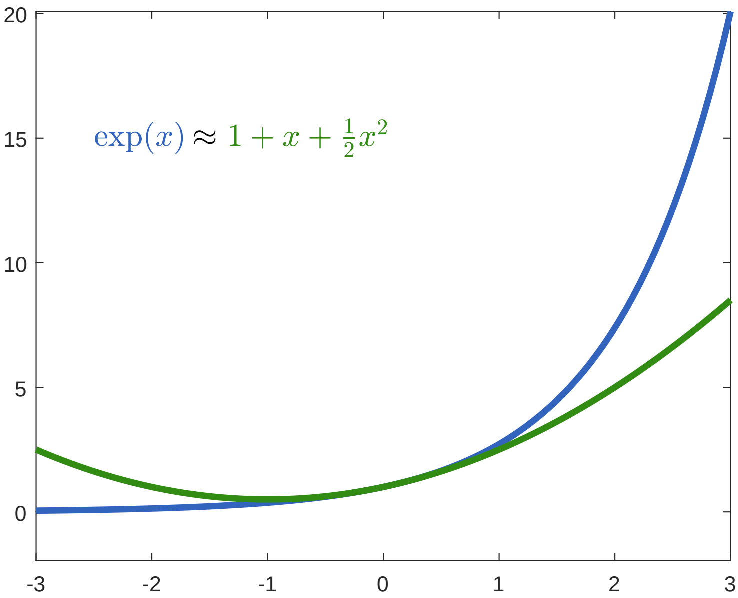
$$T_{f,x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=0}^{\infty} \frac{\exp(x_0)}{n!}(x - x_0)^n .$$

Let us look at the polynomials up to degree five at $x_0 = 0$.

Exercise. What is the converging neighbourhood in this example?
Hint: Do you recognise the resulting series, what is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

Taylor series of the exponential function





$\exp(x) \approx 1 + x + \frac{1}{2}x^2$
20
15
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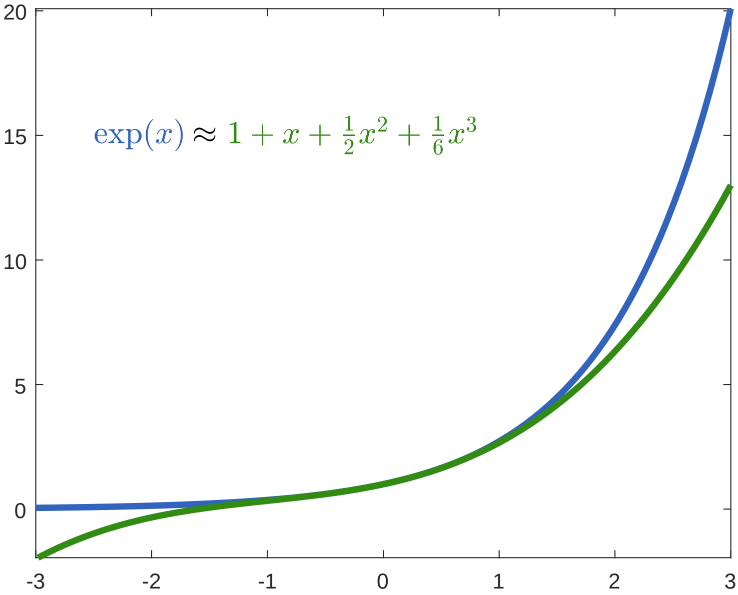
0
1
2
3


20
15
10
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-2
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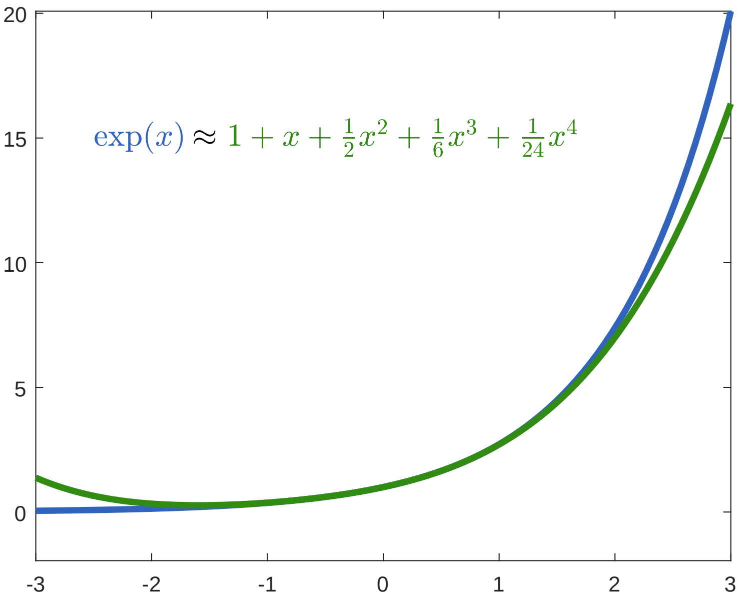
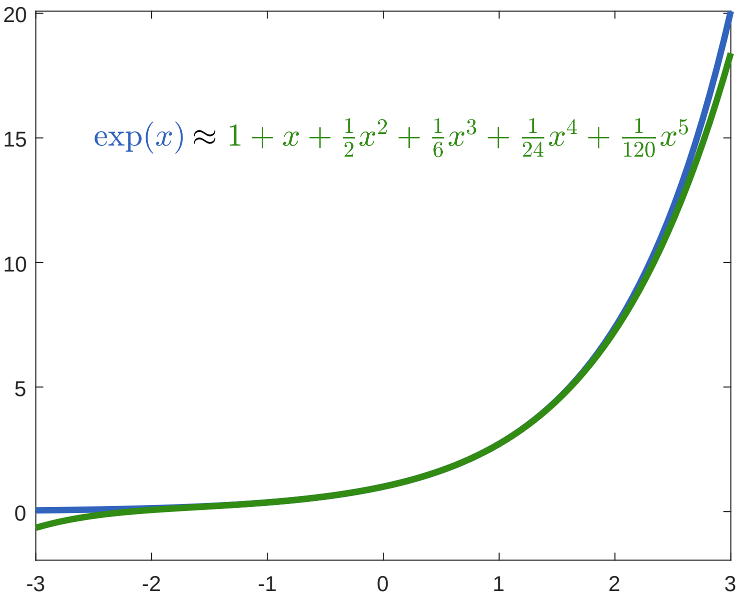
3
$\exp(x) \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

Taylor series of the exponential function



Taylor series of the exponential function



Example.* Let us look at another simple example. Consider $f : (-1, \infty) \to \mathbb{R}$ with $f(x) = \log(x+1)$.

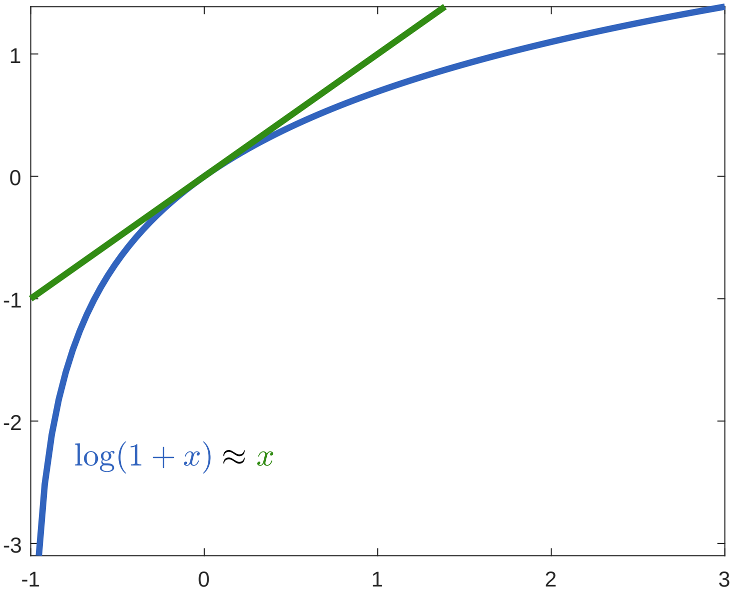
The first derivative is given by $f'(x) = \frac{1}{x+1}$, and the second as $f''(x) = -\frac{1}{(x+1)^2}$. In general, the n-th derivative, $n \geq 1$, is given by

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(x+1)^n} \quad \text{(exercise!)} .$$

So the coefficients of the Taylor series for $n \geq 1$ are given by

$$\frac{f^{(n)}(x_0)}{n!} = \frac{(-1)^{n-1}}{n(x_0+1)^n} .$$

So for $x_0 = 0$ we get $0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}$, and so on.


1
0
-1
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-3
-1
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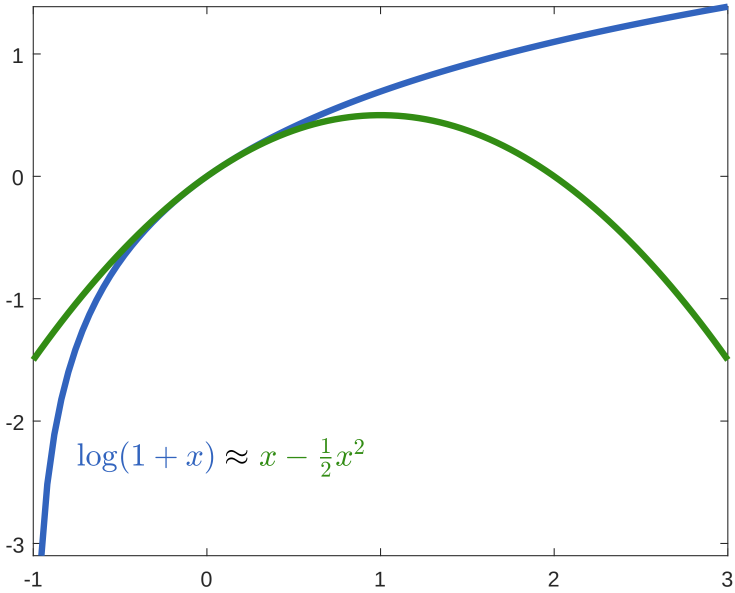
1
2
3
$\log(1+x) \approx x$


1
0
-1
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-3
-1
0
1
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3
$\log(1+x) \approx x - \frac{1}{2}x^2$

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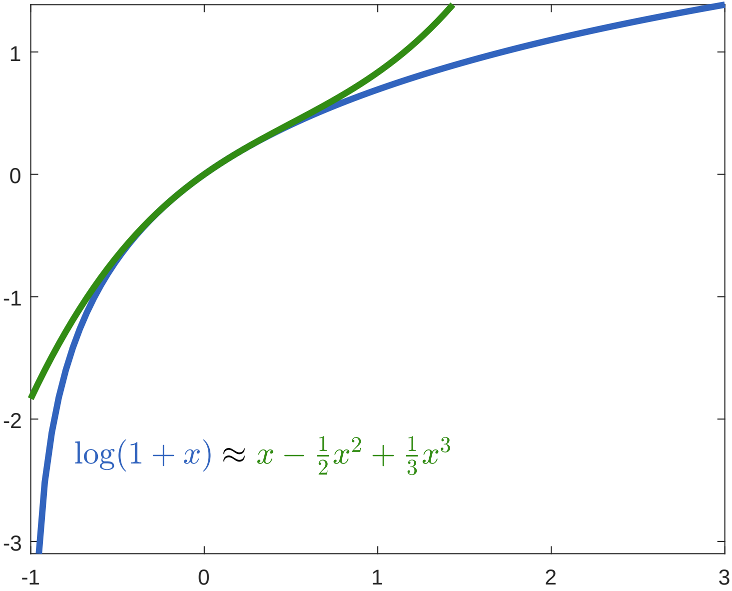

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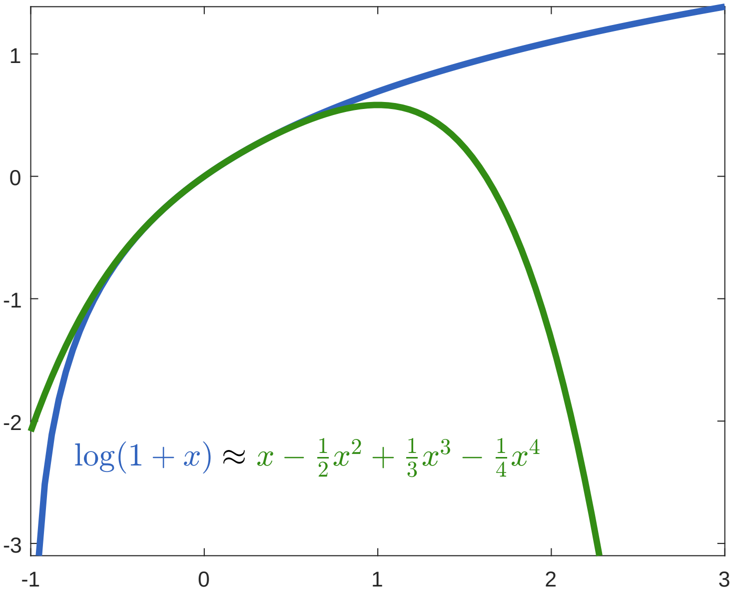
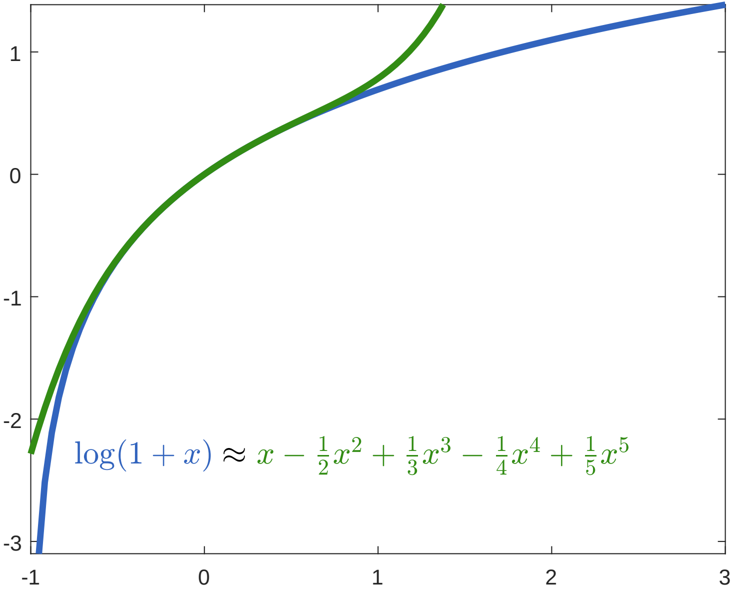



1
0
-1
-2
-3
-1
0
1
2
3
$\log(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$

It is important to understand the following points:

- If f is smooth, then we can write down the Taylor series, since f is infinitely differentiable at any point in its domain.

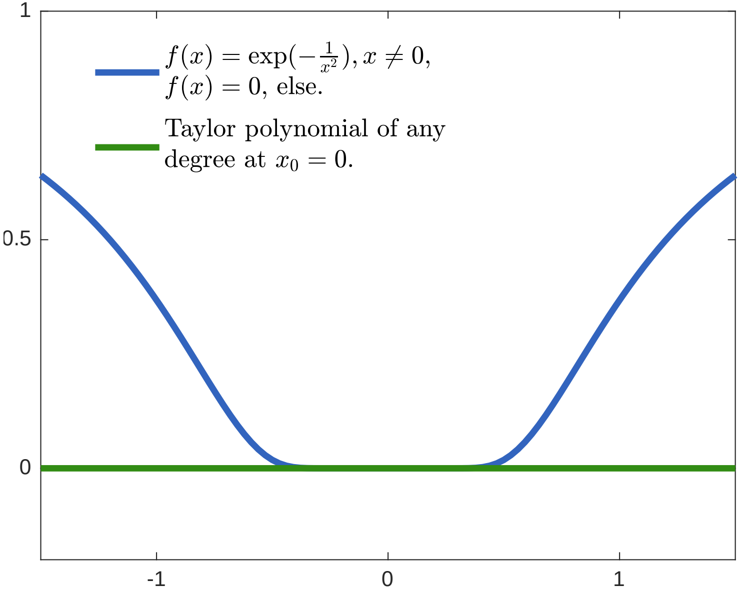
▷ However, it does not mean, the series has a limit at every point. That means, we cannot assign a unique real number to it. Consider for example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} e^{-\sqrt{2^n}} \cos(2^n x)$.

- And even if the limit exists, it may not be the same value as the function! Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0\,, \\ 0 & x = 0\,. \end{cases}$$

Indeed, one can show that f is infinitely differentiable at $x_0 = 0$ and all its derivatives are equal 0. Hence, the Taylor series is also equal 0. But f is not the zero function!

A function that is equal to its Taylor series in all neighbourhoods is called an *analytic function*.



$f(x) = \exp(-\frac{1}{x^2}), x \neq 0,$
$f(x) = 0$, else.
Taylor polynomial of any degree at $x_0 = 0$.
1
0.5
0
-1
0
1

Visualisation of Taylor series

Watch this! One of the nicest visualisations and explanations of the Taylor series is done by 3Blue1Brown, https://www.youtube.com/watch?v=3d6DsjIBzJ4.

Duration

Now that we understand what a Taylor series is, we move back to bonds. With respect to the first and second derivative we will now introduce two concepts from finance, duration and convexity[1].

For very small changes it is often enough to use only the first two terms of the Taylor series. Be careful though and remember the pitfalls!

[1] Beware that we do not talk about the actual convexity from mathematics. A somewhat misleading name!

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Definition.

- The *Dollar-Duration (DD)* is the negative of the first derivative

$$DD(y) = -P'(y).$$

- The *Modified Duration (D*)* is a normalized version of the Dollar-Duration:

$$D^*(y) = \frac{DD(y)}{P(y)} = -\frac{P'(y)}{P(y)}.$$

- The *Macaulay Duration (D_M)*, yet another normalized version:

$$D_M(y) = (1+y)D^*(y) = -(1+y)\frac{P'(y)}{P(y)}.$$

Duration: derivation

Let us derive the different notions of duration for our coupons paying bond:

- We start with the present value of the bond,

$$P(y) = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \ldots + \frac{C_T}{(1+y)^T} = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \, .$$

- Since differentiation is a linear operation, we can differentiate each term in the sum (!) separately. Hence, for each $t \in \{1, \ldots, T\}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{C_t}{(1+y)^t}\right) = -\frac{tC_t}{(1+y)^{t+1}} \, .$$

- Thus, the first derivative of the bond price with respect to the yield is

$$P'(y) = \frac{\mathrm{d}P}{\mathrm{d}y}(y) = -\sum_{t=1}^{T} \frac{tC_t}{(1+y)^{t+1}} = -\frac{1}{1+y}\sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}$$

$$= -\frac{1}{1+y}\left(\frac{C_1}{1+y} + \frac{2C_2}{(1+y)^2} + \ldots + \frac{TC_n}{(1+y)^T}\right) .$$

Hence, we get

- for the Dollar-Duration

$$DD(y) = -P'(y) = \sum_{t=1}^{T} \frac{t\, C_t}{(1+y)^{t+1}}\,,$$

- for the modified duration

$$D^*(y) = -\frac{P'(y)}{P(y)} = \frac{1}{P(y)} \sum_{t=1}^{T} \frac{t\, C_t}{(1+y)^{t+1}}\,,$$

- and for the Macaulay Duration

$$D_M(y) = -(1+y)\frac{P'(y)}{P(y)} = \frac{1}{P(y)} \sum_{t=1}^{T} \frac{t\, C_t}{(1+y)^{t}}\,.$$

- Notice that we can "rearrange" the modified duration as

$$D^*(y) = -\frac{P'(y)}{P(y)} = \frac{1}{(1+y)} \sum_{t=1}^{T} t \cdot \frac{C_t/(1+y)^t}{P(y)}$$

$$= \frac{1}{1+y} \left(\frac{C_1/(1+y)}{P(y)} + 2\frac{C_2/(1+y)^2}{P(y)} + \ldots + T\frac{C_T/(1+y)^T}{P(y)} \right)$$

- The expression in the bracket is a *weighted sum of times*. The times are the times t at which a cash payment is received.

- For each time the weight is $\frac{C_t/(1+y)^t}{P(y)}$. This is the ratio of the present value of the corresponding cash payment and the total present value.

- Notice also that the weights sum up to 1. Indeed,

$$\sum_{t=1}^{T} \frac{C_t/(1+y)^t}{P(y)} = \frac{1}{P(y)} \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} = \frac{P(y)}{P(y)} = 1\,.$$

  Hence, the expression can be interpreted as an average, the *average (or expected) time at which cash is received*.

- The concept of duration is closely related to the concept of a Taylor expansion. An approximation using duration corresponds to a first order Taylor expansion, that is a Taylor expansion which is cut after the first linear term.

- With this linear approximation the price change is usually underestimated.

- An approximation of the price change based on duration is pessimistic.

Example. Let us consider a three-year bond with a face value of 100 CHF, paying annual coupons of 4 CHF and trading on a yield-to-maturity of 5%. What is the modified duration?

Firstly, we need to calculate the price of the bond. Using the formula

$$P(y) = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3}$$

and inserting the values, we calculate the price of the bond as

$$P = \frac{4}{1.05} + \frac{4}{1.05^2} + \frac{104}{1.05^3} = 97.2768.$$

Secondly, we can calculate the modified duration using the formula derived above,

$$D^*(y) = \frac{1}{1+y}\left(\frac{C_1/(1+y)}{P(y)} + 2\frac{C_2/(1+y)^2}{P(y)} + 3\frac{C_3/(1+y)^3}{P(y)}\right).$$

After inserting the numbers, we get

$$\frac{1}{1.05}\left(\frac{4/1.05}{97.2768} + 2\frac{4/1.05^2}{97.2768} + 3\frac{104/1.05^3}{97.2768}\right) = 2.7470\,[\text{years}].$$

Note that if the cash flows arise at semi-annual intervals then the units of the modified duration will, correspondingly, be half-years. Thus, it will be necessary to divide by 2 to convert the modified duration into years.

Bond traders, portfolio managers and risk managers use modified duration through the following relationship. For the *price change for a basis point change in y*, ΔP_B, we have

$$\%\Delta P_B \approx \text{Modified Duration} \cdot 0.0001$$

This is a consequence of the approximation (for small differences ΔP and Δy)

$$\frac{\Delta P/P}{\Delta y/y} = \frac{\Delta P}{\Delta y}\frac{y}{P} = \frac{\frac{\Delta P}{\Delta y}}{P} y \approx -D^* \cdot y\,.$$

Modified duration and risk management

- In our example, the bond is selling at 97.2768 with a modified duration of 2.7470.
- Notation: BP usually stands for "basis point" and is the amount 0.01%.
- We can compute the percentage change for a one basis point (0.01%) change in yield as $2.7470 \cdot 0.0001 = 0.0002747$.
- Thus, the actual price change explained by a one basis point change in yield $= 0.0002747 \cdot 97.2768 = 0.026722$.
- This is known in the bond markets as the *dollar value of a basis point* (DVBP or *DV01*).

Convexity

To improve the approximation of the bond price sensitivity given by modified duration, one can consider the Taylor approximation up to the second derivative. This leads to the concept of *convexity*.[2]

- Again, we take the second derivative of each term in the sum separately. We get for all $t \in \{1, \ldots, T\}$,

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{C_t}{(1+y)^t}\right) = \frac{\mathrm{d}}{\mathrm{d}y}\left(-\frac{t\,C_t}{(1+y)^{t+1}}\right) = \frac{t(t+1)\,C_t}{(1+y)^{t+2}}.$$

- Hence, the second derivative of the bond price is given by

$$P''(y) = \frac{\mathrm{d}^2 P}{\mathrm{d}y^2}(y) = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} = \frac{1}{(1+y)^2}\sum_{t=1}^{T}\frac{t(t+1)C_t}{(1+y)^t}$$

$$= \frac{1}{(1+y)^2}\left(\frac{2C_1}{1+y} + \frac{6C_2}{(1+y)^2} + \ldots + \frac{T(T+1)C_n}{(1+y)^T}\right).$$

[2] *Do not confuse this with the actual mathematical meaning of convexity!*

Definition.

- *Dollar convexity (DC)* is defined as the second derivative of the bond price with respect to the yield, that means we set

$$DC(y) = \frac{\mathrm{d}^2 P}{\mathrm{d}y^2}(y) = P''(y).$$

- *Bond convexity (κ)* is defined to be the second derivative of the bond price with respect to the yield, divided by the bond price, that is

$$\kappa(y) = \frac{DC(y)}{P(y)} = \frac{P''(y)}{P(y)}.$$

Note that some practitioners refer to convexity as the second derivative divided by the present value and additionally multiplied by the factor $\frac{1}{2}$.

Bond convexity: example

We will illustrate the use of the second derivative in bond risk management by calculating the convexity of the three-year bond we looked at earlier.

Example. Consider again a three-year bond with a face value of 100 CHF, paying annual coupons of 4 CHF and trading on a yield-to-maturity of 5%.

To calculate the bond convexity, we need to calculate the second derivative of the bond price. Using the derivation above, we get for this bond,

$$P''(y) = \frac{1(1+1)C_1}{(1+y)^{1+2}} + \frac{2(2+1)C_2}{(1+y)^{2+2}} + \frac{3(3+1)C_3}{(1+y)^{3+2}}$$
$$= \frac{2C_1}{(1+y)^3} + \frac{6C_2}{(1+y)^4} + \frac{12C_3}{(1+y)^5}\,.$$

Inserting the numbers, we get

$$\frac{2\cdot 4}{(1.05)^3} + \frac{6\cdot 4}{(1.05)^4} + \frac{12\cdot 104}{(1.05)^5} = 1004.4962\,.$$

Given that the bond price is 97.2768, convexity per cash flow period is:

$$\kappa(y) = \frac{DC(y)}{P(y)} = \frac{P''(y)}{P(y)} = \frac{1004.4962}{97.2768} = 10.3262\,[\text{years}^2]$$

In this example, modified duration is measured in years so convexity is measured in years squared.

Caution: What do we adjust if we consider semi-annual yields?

Let $y = y_0 + \Delta y$, then we have the

- exact price-yield relation:

$$P(y) = P\,,$$

- (first order) approximation using duration:

$$P(y) \approx P(y_0) - D^* \cdot P(y_0) \cdot \Delta y\,,$$

- (second order) approximation using duration and convexity:

$$P(y) \approx P(y_0) - D^*(y_0) \cdot P(y_0) \cdot \Delta y + \frac{1}{2}\kappa(y_0) \cdot P(y_0) \cdot (\Delta y)^2\,.$$

Now let us investigate how well the different Taylor expansions approximate the change in price.

Example. Consider the three-year bond with a face value of 100 CHF, paying annual coupons of 4 CHF and trading on a yield-to-maturity of 5%. We calculated the price of this bond to be 97.2768 CHF.

- ▷ If the yield to maturity were to rise by 1% then the present value would fall from 97.2768 CHF to

$$P = \frac{4}{1.06} + \frac{4}{1.06^2} + \frac{104}{1.06^3} = 94.6540 \text{ [CHF]},$$

so a price change of $94.6540 - 97.2768 = -2.6228$ CHF.
This is the exact price difference.

- If we use *only the modified duration* (Taylor expansion of first degree), we approximate the change of price as

$$\begin{aligned} P(y) - P(y_0) &\approx -D^* \cdot P(y_0) \cdot \Delta y \\ &= -2.74703 \cdot 97.2768 \cdot 0.01 = -2.6722\,. \end{aligned}$$

Comparing this to the exact price change, this approximation overestimates the change of price by $2.6722 - 2.6228 = 0.0494$ [CHF].

- If we use the Taylor expansion up to the squared term, so if we use *modified duration and convexity*, we will get a more accurate figure.

  In this case, we get an additional term in the Tayor expansion,

  $$\frac{1}{2}\kappa(y_0)\cdot P(y_0)\cdot(\Delta y)^2 = \frac{1}{2}\cdot 10.3262\cdot 97.2768\cdot 0.01^2 = 0.0502\,.$$

  (Do not forget the factor 1/2 which arises from Taylor's expansion!)

  This improved approximation yields change of

  $$\begin{aligned} P(y) - P(y_0) &\approx -D^* \cdot P(y_0)\cdot \Delta y + \frac{1}{2}\kappa(y_0)\cdot P(y_0)\cdot(\Delta y)^2 \\ &= -2.6722 + 0.0502 = -2.622\,. \end{aligned}$$

  Relative to the exact value this is only an error of $2.6228 - 2.622 = 0.0008$ [CHF].

  This example illustrates that the second-order Taylor approximation can lead to better results than a first-order approximation.

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Example. Recall that a zero-coupon bond has only payment at maturity equal to the face value $C_T = F$,

$$P(y) = \frac{F}{(1+y)^T}.$$

The first derivative is

$$P'(y) = -T \cdot \frac{F}{(1+y)^{T+1}} = -\frac{T}{1+y} \cdot \frac{F}{(1+y)^T} = -\frac{T}{1+y} \cdot P(y).$$

Hence, the modified duration is $D^*(y) = T/(1+y)$ and the Macaulay duration is $D_M = T$. The second derivative is

$$P''(y) = T(T+1)\frac{F}{(1+y)^{T+2}} = \frac{(T+1)T}{(1+y)^2} \cdot P(y).$$

Hence, the convexity is given by $\kappa(y) = \frac{(T+1)T}{(1+y)^2}$.

Remark.

- Note the difference between the modified duration $D^*(y) = T/(1+y)$ and the Macaulay duration $D_M(y) \equiv T$.
- Duration is measured in periods, like T.
- Considering annual compounding, duration is measured in years, whereas with semi-annual compounding duration is in half-years and has to be divided by two for conversion to years.
- Dimension of convexity is expressed in periods squared.
- Considering semi-annual compounding, convexity is measured in half-years squared and has to be divided by four for conversion to years squared.

Summary. Using the duration-convexity terminology developed so far, we can rewrite the Taylor expansion for the change in the price of a bond, as follows:

$$\begin{aligned}\Delta P &:= P(y_0 + \Delta y) - P(y_0) \\ &= -D^*(y_0)P(y_0)\Delta y + \frac{1}{2}\kappa(y_0)P(y_0)(\Delta y)^2 + \dots ,\end{aligned}$$

where

- duration measures the first-order (linear) effect of changes in yield,
- convexity measures the second-order (quadratic) term.

Exercise. What is the price impact of a 10-BP increase in yield on a 10-year zero-coupon bond whose price, modified duration and convexity are $P = 100$ CHF, $D^* = 7$ and $\kappa = 50$, respectively.

(a) $= -0.705$ (b) $= -0.700$ (c) $= -0.698$ (d) $= -0.690$

Solution: The correct answer is: (c).

The initial price is $P(y_0) = 100$. The yield increase is 10-BP, which means $\Delta y = y_1 - y_0 = 10 \cdot 0.0001 = 0.001$.

The price impact is approximately

$$\Delta P = P(y_1) - P(y_0) \approx -D^*(y_0)P(y_0)\Delta y + \frac{1}{2}\kappa(y_0)P(y_0)(\Delta y)^2$$
$$= -7 \cdot 100 \cdot (0.001) + \frac{1}{2} \cdot 50 \cdot 100 \cdot (0.001)^2 = -0.6975\,.$$

Price approximation: Illustration

It is helpful to have a graphical representation of the duration-convexity approximation. The graph (on the next slide) compares the following three curves:

- The actual, exact price-yield relationship:

$$P(y) = P\,.$$

- The duration based estimate (first-order approximation):

$$P(y) = P(y_0) - D^*(y_0)P(y_0)\Delta y\,.$$

- The duration and convexity estimate (second-order approximation):

$$P(y) = P(y_0) - D^*(y_0)P(y_0)\Delta y + \frac{1}{2}\kappa(y_0)P(y_0)(\Delta y)^2.$$

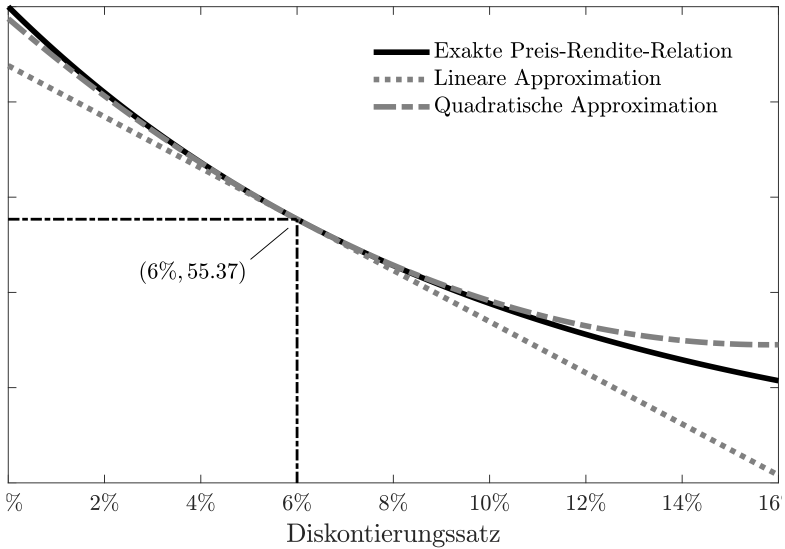

Figure 1: *Solid black*: The exact price-yield relationship. *Dotted grey*: Linear approximation. *Dash-dotted grey*: second order approximation.

Exercise.*

a) Consider the function $a : \mathbb{R} \to \mathbb{R}$ defined by $a(x) = \max(x, 0) = x_+$. Explain at which point x_0 and why the consideration of the Taylor expansion is problematic. Notice, the payoff of an option is of this form.

b) Derive the Taylor series of $b : (0, 1) \to \mathbb{R}$ given by $b(x) = \frac{1}{1-x}$ at the point $x_0 = 0$. What do you notice?

c) Consider the price of a zero-coupon bond, $P(y) = \frac{C_T}{(1+y)^T}$. Write down the Taylor polynomial up to the second derivative. In this expression, identify the modified duration D^* and the (bond) convexity κ. Argue how convexity got its name. Do you think it is an appropriate name? Compare also the example above.

Conclusions

Conclusions.

- For small movements in the yield the duration-based linear approximation provides a reasonable fit to the exact price. Including the convexity term, increases the range of yields over which the approximation remains reasonable.

- Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the point y_0.

- When the yield rises, the price drops but less than predicted by the tangent. If the yield falls, the price increases faster than in the duration model. In other words, the quadratic term is always beneficial.

Notes.

- In economic terms, duration is the average time to wait for each payment weighted by their present values.
- For the standard bonds considered so far, we have been able to compute duration and convexity analytically. However, in practice there exist bonds with more complicated features (such as mortgage-backed securities with an embedded prepayment option), for which it is not possible to compute duration and convexity in closed form.
- Instead, we need to resort to numerical methods, in particular, approximating the bond price sensitivities with finite differences.

Appendix: Examples

Example 1

Consider a zero-coupon bond with $C_T > 0$ CHF paid in T years.
Assuming that the annual compounding rate is $y = 10\%$ determine the

(a) the semi-annual compounding rate y_s and

(b) the continuous compounding rate y_c.

(a) For the semi-annual compounding rate we set

$$\frac{C_T}{(1+y)^T} = \frac{C_T}{\left(1+\frac{y_s}{2}\right)^{2T}}.$$

Cancelling C_T on both sides and "inverting" gives us

$$(1+y)^T = \left(1+\frac{y_s}{2}\right)^{2T}.$$

Hence, taking roots we get

$$(1+y) = \left(1+\frac{y_s}{2}\right)^2.$$

And finally we get

$$y_s = 2\sqrt{1+y} - 2.$$

Inserting $y = 10\%$ in the derived equation, we obtain

$$y_s = 2\sqrt{1.1} - 2 \approx 0.0976.$$

So the semi-annual yield is about 9.76%.

(b) For the continuous compounding rate we set

$$\frac{C_T}{(1+y)^T} = \exp\left(-y_c T\right) C_T \,.$$

Again, cancelling C_T and inverting the fractions, we get

$$(1+y)^T = \exp\left(y_c T\right)$$

Finally, taking roots and applying the natural logarithm we get

$$(1+y) = \exp\left(y_c\right) \quad \text{and} \quad y_c = \ln\left(1+y\right) .$$

Example 1: Solution

As above, using $y = 10\%$, we get

$$y_c = 0.0953,$$

and so the continuous compounding rate is about 9.53%.

Remark: as expected we have the relation

$$y > y_s > y_c.$$

Consider a zero-coupon bond which pays C_T CHF in T years.

1. Assume that the semi-annual compounding rate is $y_s = 6\%$ and compute both the annual compounding rate y as well as the continuous compounding rate y_c.
2. Assume that the continuous compounding rate is $y_c = 7\%$ and compute both the annual compounding rate y as well as the semi-annual compounding rate y_s.

Consider a coupon bond with $T = 3$ years and a face value of 1000 CHF. We assume yearly coupon payments with $c = 2\%$ and a yield of 1%.

Using the Taylor expansion results, compute the effect of an increase of the yield with 1%.

Using second order expansion we know

$$P(y_1) \approx P(y_0) + P'(y_0)(y_1 - y_0) + \frac{1}{2}P''(y_0)(y_1 - y_0)^2.$$

So we first compute $P(y_0)$, $P'(y_0)$ and $P''(y_0)$.

It is easy to see that the cashflows are:

$$C_1 = C_2 = 0.02 \cdot 1000 = 20$$
$$C_3 = 1.02 \cdot 1000 = 1020.$$

The price is given by

$$P(y) = \sum_{k=1}^{3} \frac{C_k}{(1+y)^k}$$

Example 2: Solution

We differentiate to get

$$P'(y) = -\sum_{k=1}^{3} \frac{k \cdot C_k}{(1+y)^{k+1}} = -\left[\frac{20}{(1+y)^2} + \frac{40}{(1+y)^3} + \frac{3060}{(1+y)^4}\right],$$

and

$$P''(y) = \sum_{k=1}^{3} \frac{k(k+1)C_k}{(1+y)^{k+2}} = \frac{40}{(1+y)^3} + \frac{120}{(1+y)^4} + \frac{12240}{(1+y)^5}.$$

Now we insert $y_0 = 0.01$ and obtain

$$P(y_0) = \frac{20}{1.01} + \frac{20}{1.021} + \frac{1020}{1.0303} \approx 1029.4099\,,$$

$$P'(y_0) = -\left[\frac{20}{1.021} + \frac{40}{1.0303} + \frac{3060}{1.0406}\right] \approx -2999.0293\,,$$

$$P''(y_0) = \frac{40}{1.0303} + \frac{120}{1.0406} + \frac{12240}{1.0510} \approx 11800.0813\,.$$

Finally inserting these results in the Taylor expansion, we obtain with $y_1 = 0.02$,

$$\begin{aligned} P(y_1) &\approx P(y_0) + P'(y_0) \cdot 0.01 + \frac{1}{2} P''(y_0) \cdot 0.01^2 \\ &\approx 1029.4099 - 2999.0294 \cdot 0.01 + \frac{1}{2} 11800.0813 \cdot 0.01^2 \\ &\approx 1000.0096. \end{aligned}$$

Remark: This is a par bond since $y_1 =$ coupon rate. Consequently the effective price is 1000 CHF. Compared to the effective price of 1000 CHF, Taylor's approximation method delivers a quite good approximation!

Example 3

Using the concepts of duration and convexity, discuss the effect of a decrease of the yield with 5 BP on the price of a zero-coupon bond, with $T = 5$ years and $P_0 = 50$ CHF assuming $y_0 = 8\%$.

Compare these results with the effective change.

(a) Using **only** duration we have:

$$P(y_1) - P(y_0) \approx -D^*(y_0)P_0\Delta y.$$

Recall that we obtain $D^*(y_0)$ as before

$$D^*(y_0) = -\frac{P'(y_0)}{P(y_0)} = \frac{T}{1+y_0}.$$

So for $T = 5$ and $y = 0.08$ we get

$$D^*(y_0) = \frac{5}{1.08} \approx 4.6296.$$

So using only Duration, we obtain

$$\Delta P \approx -4.6296 \cdot 50 \cdot (-0.0005) = 0.11574074\ldots.$$

(b) Using duration **and** convexity the approximation result is

$$P(y_1) - P(y_0) \approx -D^*(y_0)P(y_0)\Delta y + \frac{1}{2}\kappa(y_0)P(y_0)\,(\Delta y)^2\,.$$

For κ we obtained:

$$\kappa(y_0) = \frac{P''(y_0)}{P(y_0)} = \frac{(T+1)\,T}{(1+y_0)^2}$$

Inserting $T = 5$ and $y = 0.08$ yields

$$\kappa(y_0) = \frac{6 \cdot 5}{(1.08)^2} \approx 25.7202\,.$$

Inserting this, we obtain for the approximation with duration **and** convexity

$$\begin{aligned} P(y_1) - P(y_0) &\approx 0.11574 + \frac{1}{2} \cdot 25.7202 \cdot 50 \cdot (-0.0005)^2 \\ &= 0.11590149\ldots\,. \end{aligned}$$

(*c*) Since we are considering a zero-coupon bond, the effective price $P(y_1)$ can be computed as

$$P(y_1) = P(y_0) \cdot \frac{(1+y_0)^T}{(1+y_1)^T}\,.$$

Choosing $T = 5$ und $y = 0.08$, on gets

$$P(y_1) = 50 \cdot \left(\frac{1.08}{1.0795}\right)^5 = 50.11590166\ldots\,.$$

Comparison:

Only duration	$\Delta P = 0.11574074\ldots$	$P_1 = 50.11574074\ldots$
duration & convexity	$\Delta P = 0.11590149\ldots$	$P_1 = 50.11590149\ldots$
effective value	$\Delta P = 0.11590166\ldots$	$P_1 = 50.11590166\ldots$

We see that when using duration and convexity we obtain a decent approximation of the effective value.

Appendix: Optional

- Choose a change in the yield, Δy, and reprice the bond under an up-move scenario $P_+ = P(y_0 + \Delta y)$ and a down-move scenario $P_- = P(y_0 - \Delta y)$.
- Then approximate the first-order derivative with a centered finite difference. From

$$D^*(y) = -\frac{1}{P(y)} \frac{\mathrm{d}P(y)}{\mathrm{d}y}$$

effective duration is estimated as:

$$D^*(y) \approx \frac{1}{P(y_0)} \cdot \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{2\Delta y} = -\frac{1}{P_0} \cdot \frac{P_+ - P_-}{2\Delta y}.$$

- Similarly, from

$$\kappa(y) = \frac{1}{P(y)} \frac{\mathrm{d}^2 P(y)}{\mathrm{d}y^2}$$

effective convexity is estimated as:

$$\begin{aligned}\kappa(y) &\approx \frac{1}{P(y_0)\Delta y} \cdot \left[\frac{P(y_0 - \Delta y) - P(y_0)}{\Delta y} - \frac{P(y_0) - P(y_0 + \Delta y)}{\Delta y} \right] \\ &= \frac{1}{P(y_0)} \cdot \frac{P_- - 2P(y_0) + P_+}{(\Delta y)^2} .\end{aligned}$$

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