Asset Management: Advanced Investments

Bonds / Interest Rates

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Overview

Overview

Definitions and Notation: Bond Prices and Yields

- We will be working with **discount / zero-coupon bonds** which promise to pay one currency unit at maturity.
- We denote the price at time t of an n-year discount bond by $P_t^{(n)}$. The corresponding log bond price is $p_t^{(n)}$.
- Note that **coupon bonds** are just **portfolios of zero-coupon bonds**. Letting CF_n denote the coupon bond's cash flow in n periods, one has:

$$\text{Coupon bond price} = CF_1 \cdot P^{(1)} + CF_2 \cdot P^{(2)} + \ldots \tag{1}$$

- The (gross) **yield** on a discount bond with n years to maturity, $Y_t^{(n)}$, solves:

$$P_t^{(n)} = \frac{1}{\left(Y_t^{(n)}\right)^n} \quad \rightarrow \quad Y_t^{(n)} = \left(\frac{1}{P_t^{(n)}}\right)^{\frac{1}{n}} . \tag{2}$$

- The **continuously compounded yield** (or log yield) is:

$$y_t^{(n)} = -\frac{1}{n} p_t^{(n)} = \ln\left(Y_t^{(n)}\right) . \tag{3}$$

Definitions and Notation: Forward Rates

- The **forward rate** is the rate at which you can contract today to lend or borrow in the future. We denote the discrete and continuously compounded forward rates for a loan from time $t+n-1$ to time $t+n$ by $F_t^{(n)}$ and $f_t^{(n)}$, respectively.

- No arbitrage implies that

$$F_t^{(n)} = \frac{P_t^{(n-1)}}{P_t^{(n)}} \ , \tag{4}$$

$$f_t^{(n)} = p_t^{(n-1)} - p_t^{(n)} \ . \tag{5}$$

- Bond prices can be expressed as their discounted present value using forward rates:

$$P_t^{(n)} = \left(\prod_{j=1}^{n} F_t^j \right)^{-1} \ , \tag{6}$$

$$p_t^{(n)} = -\sum_{j=1}^{n} f_t^j \ . \tag{7}$$

Definitions and Notation: Bond Returns

- The realized discrete and continuously compounded **returns from holding an n-period bond** from time t to time $t+1$ are:

$$R_{t+1}^{(n)} = \frac{P_{t+1}^{(n-1)}}{P_t^{(n)}} \ , \tag{8}$$

$$r_{t+1}^{(n)} = p_{t+1}^{(n-1)} - p_t^{(n)} \ . \tag{9}$$

- Note that the **return from holding a one-period bond** is just its yield, i.e.

$$R_{t+1}^{(1)} = Y_t^{(1)} \ , \tag{10}$$

$$r_{t+1}^{(1)} = y_t^{(1)} \ . \tag{11}$$

Definitions and Notation: Excess Returns

- We denote **excess log returns** from holding an n-period bond by

$$rx_{t+1}^{(n)} = r_{t+1}^{(n)} - y_t^{(1)} \ . \qquad (12)$$

- Note that the excess log return can be decomposed into the **initial yield spread** and the **change in the bond's log yield** from period t to period $t+1$ scaled by its (remaining) duration:

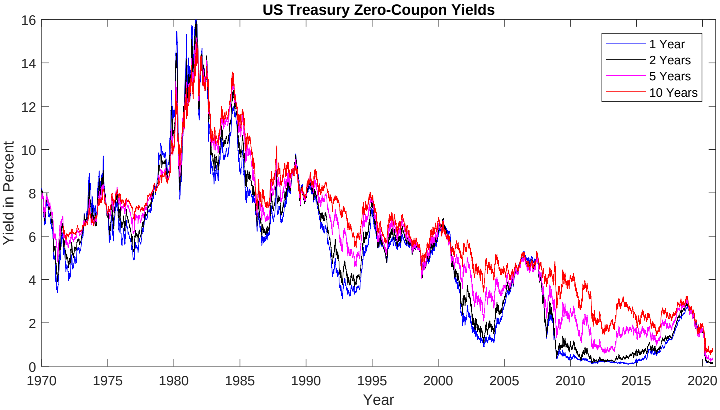
$$\begin{aligned} rx_{t+1}^{(n)} &= r_{t+1}^{(n)} - y_t^{(1)} \\ &= p_{t+1}^{(n-1)} - p_t^{(n)} - y_t^{(1)} \\ &= -(n-1)y_{t+1}^{(n-1)} + ny_t^{(n)} - y_t^{(1)} \\ &= \underbrace{\left(y_t^{(n)} - y_t^{(1)}\right)}_{\text{Initial yield spread}} - \underbrace{(n-1)}_{\text{Duration}} \underbrace{\left(y_{t+1}^{(n-1)} - y_t^{(n)}\right)}_{\text{Yield change}} \ . \end{aligned} \qquad (13)$$

Overview

Three Basic Questions

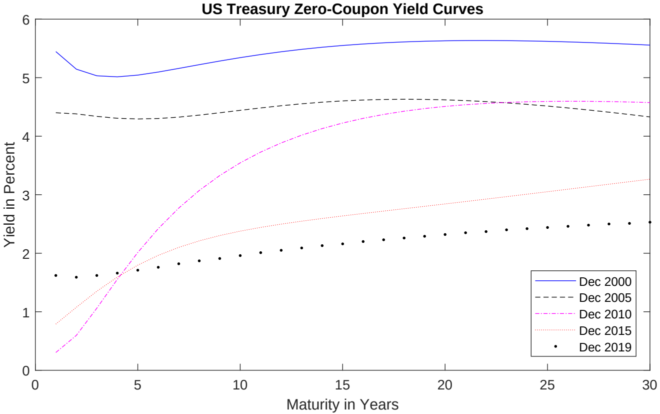
- Yields are just a convenient way to describe zero bond prices. We will be concerned with the evolution of yields over time, i.e. the expected value of future yields.
- If we can describe the evolution of yields, we can also model bond returns for portfolio construction and risk management.
- The literature has been concerned with three basic questions:
  1. Why do interest rates move?
  2. Why do yields differ across maturities?
  3. Why does the yield spread vary over time?

Why Do Interest Rates Move?



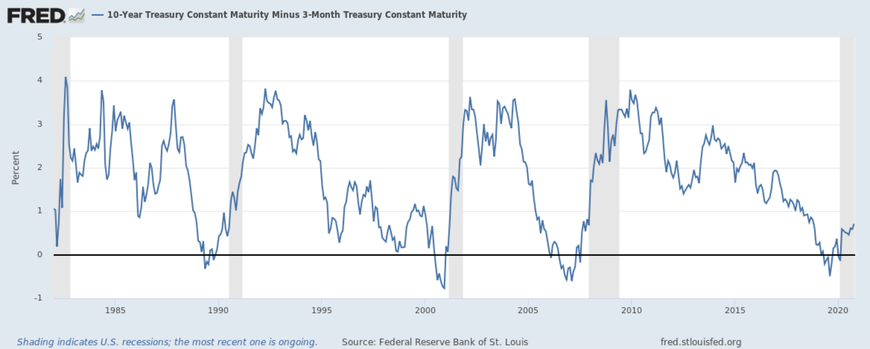
- Yields are characterized by a dominant **level factor** shifting yields of all maturities up and down.
- The short rates are set by the central bank taking macroeconomic factors such as unemployment and expected inflation into account.

Why Do Yields Differ Across Maturities?



- The **yield curve** is a plot of yields of zero-coupon bonds as a function of their maturity. It is also called the **term structure of interest rates**.
- The level and shape of the yield curve vary over time depending on economic conditions.

Why Does the Yield Spread Vary Over Time?



- The **yield spread** (also called the term spread or term premium) represents the **slope of the yield curve**.
- During recessions (highlighted in grey in the figure) the yield curve steepens, while during expansions it flattens or sometimes even inverts.
- Yield curve inversions tend to occur **right before recessions**.

Overview

The Expectations Hypothesis

- The **expectations hypothesis** (EH) is the classic theory for understanding the shape of the yield curve.
- The expectations hypothesis can be stated in three mathematically equivalent forms:
  1. The *n*-**period yield** is the **average of expected future one-period yields**:

$$y_t^{(n)} = \frac{1}{n}\mathbb{E}_t\left[y_t^{(1)} + y_{t+1}^{(1)} + \ldots + y_{t+n-1}^{(1)}\right] + rpy^{(n)} \tag{14}$$

  2. The **forward rate** equals the **expected future spot rate**:

$$f_t^{(n)} = \mathbb{E}_t\left[y_{t+n-1}^{(1)}\right] + rpf^{(n)} \tag{15}$$

  3. The **expected returns** from holding bonds of **all maturities** are **identical**:

$$\mathbb{E}_t\left[r_{t+1}^{(n)}\right] = y_t^{(1)} + rph^{(n)} \tag{16}$$

- Although we have included risk premia *rpy*, *rpf* and *rph* in the above equations, these are zero in the purest form of the expectations hypothesis.

Risk Premia

- The above expressions say that alternative ways of getting money from one period to another must have the same expected value (plus a risk premium).
- The inclusion of a risk premium can be justified because one way is riskier than the other:
  1. From the expression $y_t^{(n)} = \frac{1}{n}\mathbb{E}_t\left[y_t^{(1)} + y_{t+1}^{(1)} + \ldots + y_{t+n-1}^{(1)}\right] + rpy^{(n)}$, rpy is the expected return from holding a n-period bond to maturity, financed by rolling over short-term bonds.
  2. From the expression $f_t^{(n)} = \mathbb{E}_t\left[y_{t+n-1}^{(1)}\right] + rpf^{(n)}$, the forward premium rpf is the expected return from planning to borrow for a year in the future spot market and agreeing today to lend in the forward market.
  3. From the expression $\mathbb{E}_t\left[r_{t+1}^{(n)}\right] = y_t^{(1)} + rph^{(n)}$, rph is the one-period excess return from holding a long-term bond for one period, financed by issuing a one-period bond. This is typically the dependent variable in return predictability regressions.
- If one allows for arbitrary time-varying risk premia, there is nothing left to test. Thus, tests of the expectations hypothesis involve testing restrictions on the risk premia (typically whether they are constant or not).

Overview

Risk Premia

- Empirically, average holding period returns of bonds of different maturities are quite similar, despite the increasing standard deviation of longer-maturity bond returns.

- For example, for the period 1961-2019, average continuously compounded one-year returns on zero-coupon US Treasury bonds of varying maturity are:

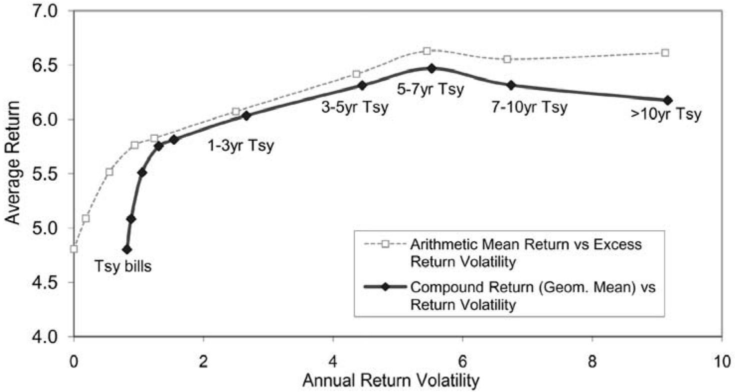
Zero-coupon bond returns

Bond maturity	Average holding period return	Standard deviation
1	5.00	3.24
2	5.40	3.73
3	5.77	4.46
4	6.08	5.28
5	6.33	6.13

Calculations using US Treasury rates from 1961 to 2019, annual data. Data from https://www.quandl.com/data/FED/SVENY.

Risk Premia

- The reward for extending the duration is highest at short maturities and decays at longer maturities.



Source: Ilmanen (2011)

Risk Premia

- The Sharpe ratios obtained by extending the duration exceed one at short maturities (if the one-month Treasury bill is used as the riskless rate) and decline monotonically from the shortest to the longest portfolios.

	1 mo	*0–3 mo*	*3–6 mo*	*6–9 mo*	*9–12 mo*	*1–3 yr*	*3–5 yr*	*5–7 yr*	*7–10 yr*	*10 yr+*
Arithmetic mean	4.81	5.09	5.52	5.76	5.83	6.07	6.42	6.63	6.55	6.61
Geometric mean	4.80	5.08	5.51	5.75	5.81	6.04	6.31	6.47	6.32	6.17
Return volatility	0.82	0.88	1.05	1.31	1.54	2.66	4.45	5.52	6.75	9.16
Average excess return	0.00	0.28	0.71	0.96	1.02	1.27	1.61	1.82	1.75	1.81
Excess return volatility	0.00	0.18	0.55	0.94	1.24	2.50	4.36	5.45	6.69	9.13
Sharpe ratio	NA	1.55	1.30	1.02	0.82	0.51	0.37	0.33	0.26	0.20
Average duration (approx.)	0.1	0.1	0.4	0.6	0.8	1.6	3.5	4.8	6	10

Summary statistics of returns on bond portfolios of different maturities.

Source: Ilmanen (2011)

Overview

Basic Idea

Under the expectations hypothesis, risk premia are **constant** and **unpredictable**.

Analyses of predictability involve testing the various forms of the EH. Here is an overview before we look at the details:

Form 1: The *n*-period yield is the average of expected future one-period yields:

- This form of the EH implies that a **steep yield curve should predict an increase in future short rates**.
- This is tested by checking whether $\beta = 1$ in the regression:

$$\frac{1}{n}\sum_{i=0}^{n-1}\left(y_{t+i}^{(1)}\right) - y_t^{(1)} = \alpha + \beta\left(y_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+n-1}\ . \qquad (17)$$

- Empirically yield spreads do predict changes in short-term rates, and $\beta \approx 1$. The EH seems to hold.

Basic Idea

Form 2: The forward rate equals the expected future spot rate:

- This form of the EH implies that **forward rates should predict future short rates**.
- Equivalently, the spread between the forward 1-year rate and today's 1-year yield should predict the change in the 1-year yield over the next $n-1$ years.
- This is tested by checking whether $\beta = 1$ in the regression:

$$y_{t+n-1}^{(1)} - y_t^{(1)} = \alpha + \beta \left(f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+n-1} \ . \tag{18}$$

- Empirical findings:
  - At **short horizons**, the spread does **not** predict future yields but predicts future returns instead. The EH doesn't hold.
  - At **longer horizons**, the spread starts to predict changes in interest rates. The EH seems to work in the long-run.
- Thus a large spread implies that risk premia / expected excess returns on long-term bonds are high, so long-term bonds are a good investment.

Basic Idea

Form 3: The expected returns from holding bonds of all maturities are identical:

- This form of the EH implies that a **steep yield curve should predict an increase in future long rates** but not excess returns on bonds.
- This is tested by checking whether $b_1 = 1$ and $b_2 = 0$ in the regressions:

$$y_{t+1}^{(n-1)} - y_t^{(n)} = a_1 + b_1 \left(\frac{y_t^{(n)} - y_t^{(1)}}{n-1} \right) + \eta_{t+1} \ , \tag{19}$$

$$r_{t+1}^{(n)} - y_t^{(1)} = a_2 + b_2 \left(\frac{y_t^{(n)} - y_t^{(1)}}{n-1} \right) + \eta_{t+1} \ . \tag{20}$$

- Empirically yield spreads cannot predict changes in long-rates but predict high excess returns instead: $b_1 \neq 1$ and not significant, while b_2 is large and highly significant. The EH fails.
- Again, a steep yield curve implies that risk premia / expected excess returns on long-term bonds are high, so long-term bonds are a good investment.

Overview

Testing the EH Using Forward-Spot Spreads

- The starting point is **Form 2** of the EH (expression (15)), which states that forward rates should forecast future short rates:

$$f_t^{(n)} = \mathbb{E}_t \left[y_{t+n-1}^{(1)} \right] + rpf^{(n)} \ .$$

- This would imply a regression like

$$y_{t+n-1}^{(1)} = \alpha + \beta f_t^{(n)} + \varepsilon_{t+n-1} \ . \tag{21}$$

- The EH predicts that $\beta = 1$.

- Since interest rates are very persistent, it is better to use spreads instead:

$$y_{t+n-1}^{(1)} - y_t^{(1)} = \alpha + \beta (f_t^{(n)} - y_t^{(1)}) + \varepsilon_{t+n-1} \ . \tag{22}$$

- Under the EH, the spread between the forward 1-year rate and today's 1-year rate should predict the change in the 1-year yield over the next $n-1$ years.

Testing the EH Using Forward-Spot Spreads

- Note that in the special case $n = 2$, the forward-spot spread $f_t^{(2)} - y_t^{(1)}$ can be rewritten as

$$
\begin{aligned}
f_t^{(2)} - y_t^{(1)} &= p_t^{(1)} - p_t^{(2)} - y_t^{(1)} \\
&= \left(p_{t+1}^{(1)} - p_t^{(2)} + p_t^{(1)}\right) + \left(-p_{t+1}^{(1)} - y_t^{(1)}\right) \qquad (23) \\
&= \left(r_{t+1}^{(2)} - y_t^{(1)}\right) + \left(y_{t+1}^{(1)} - y_t^{(1)}\right) .
\end{aligned}
$$

- This is an identity which must hold **both ex ante and ex post**.

- Thus, a positive forward-spot spread *must* imply a **positive excess return on two-year bonds** or an **increase in the 1-year rate**.

- The forward-spot spread must predict either future excess returns or future yields. This is the same intuition as we had for price-dividend ratios, returns and dividend growth.

Fama-Bliss Regressions

- The decomposition of the forward-spot spread for $n = 2$,

$$f_t^{(2)} - y_t^{(1)} = \left(r_{t+1}^{(2)} - y_t^{(1)} \right) + \left(y_{t+1}^{(1)} - y_t^{(1)} \right)$$

suggests two complementary regressions, which were first run by Fama and Bliss (1987):

$$y_{t+1}^{(1)} - y_t^{(1)} = \alpha + \beta \left(f_t^{(2)} - y_t^{(1)} \right) + \varepsilon_{t+1} \ , \quad (24)$$

$$r_{t+1}^{(2)} - y_t^{(1)} = a + b \left(f_t^{(2)} - y_t^{(1)} \right) + \eta_{t+1} \ . \quad (25)$$

- The EH predicts that:
  - Forward rates forecast future short rates ($\beta = 1$).
  - Risk premia are constant and unpredictable ($b = 0$).
- Note that the decomposition identity implies $\beta + b = 1$.

Fama-Bliss Regressions

- For general n, the forward-spot spread $f_t^{(n)} - y_t^{(1)}$ can be decomposed as

$$\begin{aligned} f_t^{(n)} - y_t^{(1)} &= p_t^{(n-1)} - p_t^{(n)} - y_t^{(1)} \\ &= \left(p_{t+1}^{(n-1)} - p_t^{(n)} - y_t^{(1)}\right) + \left(-p_{t+1}^{(n-1)} + p_t^{(n-1)}\right) \\ &= \left(r_{t+1}^{(n)} - y_t^{(1)}\right) + (n-1)\left(y_{t+1}^{(n-1)} - y_t^{(n-1)}\right) . \end{aligned} \tag{26}$$

  Again, the spread must predict **future returns** or **changes in yields**.

- For general n, the Fama and Bliss (1987) regressions are:

$$y_{t+n-1}^{(1)} - y_t^{(1)} = \alpha + \beta\left(f_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+n-1} , \tag{27}$$

$$r_{t+1}^{(n)} - y_t^{(1)} = a + b\left(f_t^{(n)} - y_t^{(1)}\right) + \eta_{t+1} . \tag{28}$$

- Again, under the EH:
  - Forward rates forecast future short rates ($\beta = 1$).
  - Risk premia are constant and unpredictable ($b = 0$).

- Note that for $n > 2$, β and b need not sum to 1.

Fama-Bliss Regressions

- Estimating these regressions using US Treasury rates from 1961 to 2019 with annual data yields:

	$y_{t+n-1}^{(1)} - y_t^{(1)} = \alpha + \beta \left(f_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+n-1}$			$r_{t+1}^{(n)} - y_t^{(1)} = a + b \left(f_t^{(n)} - y_t^{(1)}\right) + \eta_{t+1}$		
n	β	$t(\beta)$	R^2_{adj}	b	$t(b)$	R^2_{adj}
2	0.17	0.60	−0.01	0.83	2.82	0.10
3	0.62	1.59	0.06	0.97	2.69	0.10
4	0.87	3.03	0.15	1.14	2.76	0.10
5	0.81	4.11	0.15	1.29	2.83	0.10

Data from https://www.quandl.com/data/FED/SVENY.

- At **short horizons**, the spread does not predict future yields but predicts future **returns** instead. Excess returns move almost one-for-one with the spread.

- At **longer horizons**, the spread starts to predict changes in **interest rates**. So the EH seems to work in the long-run.

Fama-Bliss Regressions

- Risk premia are **not constant**, contrary to what the EH would predict. Rather, they vary over time and with the forward-spot spread.

  $\rightarrow$ In the **short run**, changes in the forward-spot spread translate to changes in the **risk premium**.

- The EH does a much better job at long horizons. There is "sluggish adjustment" of yields. Future yields eventually start rising. In the meantime, long-term bonds earn positive excess returns.

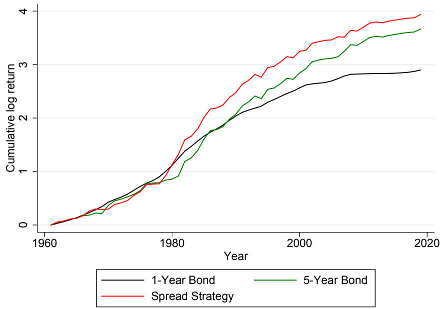
  $\rightarrow$ In the **long run**, changes in the forward-spot spread translate to changes in **yields**.

- The violation of the EH is often called the expectations hypothesis puzzle.

A Simple Forward-Spot Spread Strategy

To take advantage of return predictability, consider the following simple strategy:

- At the end of each year t, compute the forward-spot spread $f_t^{(5)} - y_t^{(1)}$.
- If $f_t^{(5)} - y_t^{(1)} > 0$, invest in the five-year bond.
- If $f_t^{(5)} - y_t^{(1)} < 0$, short the five-year bond and invest the capital and the short sale proceeds in the one-year bond.



Strategy	5-Y Bond	Spread
Mean Xs Return	1.41%	1.77%
Std Xs Return	5.79%	5.67%
Sharpe Ratio	0.24	0.31
Skewness	0.38	0.38
Kurtosis	2.92	2.82
Min	−10.46%	−8.44%
Max	16.77%	16.77%

Overview

Testing the EH Using Yield Spreads

- Recall **Form 1** of the EH (expression (14)):

$$y_t^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t\left(y_{t+i}^{(1)}\right) + rpy^{(n)} \ .$$

This expression can be rewritten as:

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t\left(y_{t+i}^{(1)}\right) - y_t^{(1)} + rpy^{(n)} = y_t^{(n)} - y_t^{(1)} \ . \tag{29}$$

- Similarly, rewriting **Form 3** (expression (16)) in excess log return form:

$$\mathbb{E}_t\left[r_{t+1}^{(n)}\right] - y_t^{(1)} = \mathbb{E}_t\left[rx_{t+1}^{(n)}\right] = rph^{(n)} \tag{30}$$

and inserting the expression for the excess log return (13) yields

$$\mathbb{E}_t\left[\left(y_t^{(n)} - y_t^{(1)}\right) - (n-1)\left(y_{t+1}^{(n-1)} - y_t^{(n)}\right)\right] = rph^{(n)} \tag{31}$$

or

$$\mathbb{E}_t\left[y_{t+1}^{(n-1)}\right] - y_t^{(n)} = \frac{y_t^{(n)} - y_t^{(1)}}{n-1} - \frac{1}{n-1} rph^{(n)} \ . \tag{32}$$

Campbell-Shiller Regressions

- This suggests two regressions to test the EH using yield spreads instead of the forward-spot spread (Campbell and Shiller (1991)):

$$\frac{1}{n}\sum_{i=0}^{n-1}\left(y_{t+i}^{(1)}\right) - y_t^{(1)} = \alpha + \beta\left(y_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+n-1}\ , \tag{33}$$

$$y_{t+1}^{(n-1)} - y_t^{(n)} = a + b\left(\frac{y_t^{(n)} - y_t^{(1)}}{n-1}\right) + \eta_{t+1}\ . \tag{34}$$

- The EH predicts that $\beta = 1$ and $b = 1$.
- Thus, if the EH holds, a steep yield curve should predict:
  - an increase in future short rates;
  - an increase in future long rates.

Predicting Short-Rate Changes

- Yield spreads do predict changes in short-rates. A **steep yield curve** implies an **increase in future short-term rates**.
- $\beta \approx 1$ and significant, the EH seems to hold.

	Bond Maturity τ								
Country	2	3	4	5	6	7	8	9	10
CA	0.834	0.832	0.896	0.976	1.000	1.043	1.087	1.128	1.144
	(0.17)	(0.12)	(0.09)	(0.08)	(0.07)	(0.07)	(0.07)	(0.07)	(0.08)
	[0.35]	[0.24]	[0.18]	[0.14]	[0.11]	[0.11]	[0.11]	[0.10]	[0.11]
$\bar{R}^2$	9.9	18.2	31.2	46.4	56.3	60.3	61.6	65.3	61.8
GM	0.479	0.526	0.677	1.298	1.401	1.453	1.504	1.530	
	(0.07)	(0.07)	(0.08)	(0.10)	(0.09)	(0.09)	(0.08)	(0.07)	
	[0.13]	[0.13]	[0.16]	[0.18]	[0.16]	[0.16]	[0.14]	[0.13]	
$\bar{R}^2$	19.3	21.6	28.1	46.0	57.2	62.6	69.6	78.3	
JP	0.612	0.690	0.426	0.371	0.443	0.622	0.942	1.467	
	(0.17)	(0.20)	(0.18)	(0.16)	(0.15)	(0.15)	(0.16)	(0.15)	
	[0.32]	[0.41]	[0.39]	[0.33]	[0.30]	[0.31]	[0.38]	[0.30]	
$\bar{R}^2$	5.3	5.1	2.2	2.3	4.4	9.0	17.8	41.7	
U.K.	0.505	0.637	0.755	0.872	0.981	1.041	1.125	1.162	1.250
	(0.11)	(0.10)	(0.10)	(0.09)	(0.08)	(0.08)	(0.07)	(0.07)	(0.07)
	[0.22]	[0.16]	[0.17]	[0.16]	[0.16]	[0.15]	[0.14]	[0.13]	[0.13]
$\bar{R}^2$	8.1	15.2	22.9	32.2	42.6	51.2	60.2	67.1	73.3
U.S.	0.485	0.715	0.786	1.143	1.414	1.387	1.369	1.299	1.114
	(0.14)	(0.12)	(0.14)	(0.13)	(0.11)	(0.09)	(0.07)	(0.06)	(0.04)
	[0.31]	[0.26]	[0.27]	[0.24]	[0.16]	[0.15]	[0.14]	[0.10]	[0.06]
$\bar{R}^2$	4.5	13.1	13.4	29.3	47.3	57.4	68.8	77.9	82.5

Source: Tang and Xia (2007); results correspond to regression model (33). OLS standard errors in parentheses, Newey-West standard errors in brackets.

Predicting Long-Rate Changes

However, yield spreads **cannot predict changes in long rates**. $b \neq 1$ and not significant, the EH fails.

	Bond Maturity τ								
Country	2	3	4	5	6	7	8	9	10
CA	0.669	0.384	0.180	0.081	0.081	−0.027	−0.316	−0.725	−1.140
	(0.33)	(0.32)	(0.31)	(0.33)	(0.36)	(0.40)	(0.45)	(0.51)	(0.56)
	[0.70]	[0.69]	[0.69]	[0.73]	[0.80]	[0.90]	[1.00]	[1.11]	[1.21]
$\bar{R}^2$	1.4	0.2	−0.3	−0.4	−0.4	−0.5	−0.2	0.5	1.4
GM	−0.041	−0.391	−0.481	−0.425	−0.435	−0.543	−0.679	−0.871	
	(0.13)	(0.16)	(0.17)	(0.19)	(0.22)	(0.25)	(0.29)	(0.33)	
	[0.25]	[0.28]	[0.33]	[0.36]	[0.40]	[0.47]	[0.56]	[0.65]	
$\bar{R}^2$	−0.4	2.4	3.0	1.7	1.4	1.7	2.0	2.6	
JP	0.225	−0.662	−1.717	−2.268	−2.451	−2.362	−2.177	−1.982	
	(0.33)	(0.49)	(0.50)	(0.49)	(0.50)	(0.52)	(0.54)	(0.55)	
	[0.64]	[1.07]	[1.19]	[1.20]	[1.21]	[1.25]	[1.30]	[1.32]	
$\bar{R}^2$	−0.3	0.4	4.7	8.5	9.5	8.1	6.5	5.2	
U.K.	0.009	−0.033	−0.041	−0.053	−0.086	−0.150	−0.248	−0.380	−0.545
	(0.22)	(0.25)	(0.29)	(0.32)	(0.35)	(0.37)	(0.40)	(0.42)	(0.44)
	[0.43]	[0.49]	[0.56]	[0.64]	[0.72]	[0.80]	[0.87]	[0.94]	[1.01]
$\bar{R}^2$	−0.4	−0.4	−0.4	−0.4	−0.4	−0.4	−0.3	−0.1	0.2
U.S.	−0.031	−0.260	−0.414	−0.406	−0.235	−0.222	−0.278	−0.238	−0.364
	(0.28)	(0.29)	(0.32)	(0.38)	(0.47)	(0.54)	(0.56)	(0.52)	(0.46)
	[0.62]	[0.64]	[0.70]	[0.82]	[0.96]	[1.02]	[1.05]	[1.07]	[0.97]
$\bar{R}^2$	−0.4	−0.1	0.3	0.1	−0.3	−0.4	−0.3	−0.4	−0.2

Source: Tang and Xia (2007); results correspond to regression model (34). OLS standard errors in parentheses, Newey-West standard errors in brackets.

Predicting Excess Returns

- The above tests of the EH are based on predictability in yield changes. An alternative is to consider **whether the yield spread predicts excess bond returns**.

- Recall from expression (13) that one-period excess returns $rx_{t+1}^{(n)} = r_{t+1}^{(n)} - y_t^{(1)}$ can be decomposed into the yield spread and yield change. Rewrite this expression as

$$r_{t+1}^{(n)} - y_t^{(1)} + (n-1)\left(y_{t+1}^{(n-1)} - y_t^{(n)}\right) = \left(y_t^{(n)} - y_t^{(1)}\right) \quad . \tag{35}$$

- Hence, yield spreads must predict **excess returns** or **changes in yields**.

- We test this by estimating

$$r_{t+1}^{(n)} - y_t^{(1)} = a + b\left(\frac{y_t^{(n)} - y_t^{(1)}}{n-1}\right) + \eta_{t+1} \quad . \tag{36}$$

- The EH implies constant and unpredictable risk premia, i.e. predicts $b = 0$.

Predicting One-Year Excess Returns

- Estimating this regression using US Treasury rates from 1961 to 2019 with annual data yields:

Maturity n	b	$t(b)$	R^2_{adj}
2	1.65	2.82	0.10
3	3.75	2.82	0.11
4	6.33	3.10	0.12
5	9.33	3.50	0.13

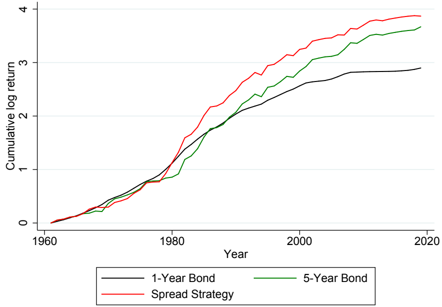
Data from https://www.quandl.com/data/FED/SVENY.

- A **steep yield curve** predicts **high excess returns**. b is large and highly significant, the EH fails.
- Hence, risk premia / expected excess returns are high when the price of long bonds is low. Put differently, a steep yield curve implies that long-term bonds are cheap and short-term bonds are expensive.
- As was the case for stocks, a low (relative) price predicts high future returns.

A Simple Yield Spread Strategy

To take advantage of this effect, consider the following simple strategy:

- At the end of each year t, compute the yield spread $y_t^{(5)} - y_t^{(1)}$.
- If $y_t^{(5)} - y_t^{(1)} > 0$, invest in the five-year bond.
- If $y_t^{(5)} - y_t^{(1)} < 0$, short the five-year bond and invest the capital and the short sale proceeds in the one-year bond.



Strategy	5-Y Bond	Spread
Mean Xs Return	1.41%	1.90%
Std Xs Return	5.79%	5.62%
Sharpe Ratio	0.24	0.34
Skewness	0.38	0.42
Kurtosis	2.92	2.78
Min	−10.46%	−8.44%
Max	16.77%	16.77%

Overview

Single-Factor Model: Basic Idea

- We now investigate whether a **common factor** drives the expected returns on bonds of **different maturities**.
- Cochrane and Piazzesi (2005) forecast annual excess log returns rx on $2-5$ year bonds using five forward rates:

$$rx_{t+1}^{(n)} = \beta_0^{(n)} + \beta_1^{(n)} y_t^{(1)} + \beta_2^{(n)} f_t^{(2)} + \beta_3^{(n)} f_t^{(3)} + \beta_4^{(n)} f_t^{(4)} + \beta_5^{(n)} f_t^{(5)} + \varepsilon_{t+1} \ .$$

- Rather than using a single forward-spot spread or yield spread, they use **information from the entire yield curve**.
- The key findings are:
  1. This model generates substantially **higher R^2s** than Fama-Bliss and Campbell-Shiller regressions.
  2. The regression coefficients form a tent shape for bonds of **all maturities**.
  3. A **single factor** constructed as a linear combination of the forward rates predicts returns across **all maturities**.
  4. The single factor is **countercyclical** and also forecasts **stock returns**.

Cochrane-Piazzesi Regressions

Regression estimates for

$$rx_{t+1}^{(n)} = \beta_0^{(n)} + \beta_1^{(n)} y_t^{(1)} + \beta_2^{(n)} f_t^{(2)} + \beta_3^{(n)} f_t^{(3)} + \beta_4^{(n)} f_t^{(4)} + \beta_5^{(n)} f_t^{(5)} + \varepsilon_{t+1} \ .$$

Maturity n	const.	$y^{(1)}$	$f^{(2)}$	$f^{(3)}$	$f^{(4)}$	$f^{(5)}$	R^2	Level R^2	$\chi^2(5)$
2	−1.62	−0.98	0.59	1.21	0.29	−0.89	0.32	0.36	121.8
Large T	(0.69)	(0.17)	(0.40)	(0.29)	(0.22)	(0.17)			⟨0.00⟩
Small T	(0.86)	(0.30)	(0.50)	(0.40)	(0.30)	(0.28)	[0.19, 0.53]		32.9
EH							[0.00, 0.17]		⟨0.00⟩
3	−2.67	−1.78	0.53	3.07	0.38	−1.86	0.34	0.36	113.8
Large T	(1.27)	(0.30)	(0.67)	(0.47)	(0.41)	(0.30)			⟨0.00⟩
Small T	(1.53)	(0.53)	(0.88)	(0.71)	(0.53)	(0.50)	[0.21, 0.55]		38.6
EH							[0.00, 0.17]		⟨0.00⟩
4	−3.80	−2.57	0.87	3.61	1.28	−2.73	0.37	0.39	115.7
Large T	(1.73)	(0.44)	(0.87)	(0.59)	(0.55)	(0.40)			⟨0.00⟩
Small T	(2.03)	(0.71)	(1.18)	(0.94)	(0.71)	(0.68)	[0.24, 0.57]		46.0
EH							[0.00, 0.17]		⟨0.00⟩
5	−4.89	−3.21	1.24	4.11	1.25	−2.83	0.35	0.36	88.2
Large T	(2.16)	(0.55)	(1.03)	(0.67)	(0.65)	(0.49)			⟨0.00⟩
Small T	(2.49)	(0.88)	(1.46)	(1.16)	(0.88)	(0.85)	[0.21, 0.55]		39.2
EH							[0.00, 0.17]		⟨0.00⟩

Source: Cochrane and Piazzesi (2005). Large T denotes asymptotic standard errors, small T small-sample standard errors.

R^2s Are Higher than in Fama-Bliss Return Regressions

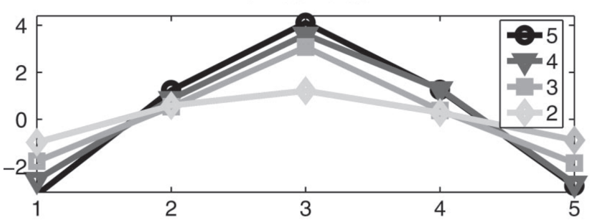
Estimates for Fama-Bliss excess return regressions

$$rx_{t+1}^{(n)} = \alpha + \beta(f_t^{(n)} - y_t^{(1)}) + \varepsilon_{t+1} \ .$$

Maturity n	α	β	R^2	$\chi^2(1)$	p-value
2	0.07	0.99	0.16		
Large T	(0.30)	(0.26)		18.4	⟨0.00⟩
Small T	(0.16)	(0.33)	[0.01, 0.33]	9.1	⟨0.00⟩
EH			[0.00, 0.12]		⟨0.01⟩
3	−0.13	1.35	0.17		
Large T	(0.54)	(0.35)		19.2	⟨0.00⟩
Small T	(0.32)	(0.41)	[0.01, 0.34]	10.8	⟨0.00⟩
EH			[0.00, 0.14]		⟨0.01⟩
4	−0.40	1.61	0.18		
Large T	(0.75)	(0.45)		16.4	⟨0.00⟩
Small T	(0.48)	(0.48)	[0.01, 0.34]	11.2	⟨0.00⟩
EH			[0.00, 0.14]		⟨0.01⟩
5	−0.09	1.27	0.09		
Large T	(1.04)	(0.58)		5.7	⟨0.02⟩
Small T	(0.64)	(0.64)	[0.00, 0.24]	4.0	⟨0.04⟩
EH			[0.00, 0.14]		⟨0.13⟩

Source: Cochrane and Piazzesi (2005).

Plot of Slope Coefficients by Maturity



Source: Cochrane and Piazzesi (2005). The x-axis gives the maturity of the forward rate used as predictive variable. The legend gives the maturity of the bond whose excess return is forecast.

- The **same function** of forward rates forecasts excess returns at **all maturities**.
- This suggests that one can predict returns on all bonds using a **single factor**.

Single-Factor Model: Factor Construction

- Regress average bond returns on the forward rates

$$\frac{1}{4}\sum_{n=2}^{5}\left(r_{t+1}^{(n)} - y_t^{(1)}\right) = \gamma_0 + \gamma_1 y_t^{(1)} + \gamma_2 f_t^{(2)} + \gamma_3 f_t^{(3)} + \gamma_4 f_t^{(4)} + \gamma_5 f_t^{(5)} + \varepsilon_{t+1}$$
$$= \boldsymbol{\gamma}'\mathbf{f}_t + \varepsilon_{t+1} \ , \qquad (37)$$

where $\boldsymbol{\gamma} = [\gamma_0\ \gamma_1\ \gamma_2\ \gamma_3\ \gamma_4\ \gamma_5]'$ and $\mathbf{f}_t = [1\ y_t^{(1)}\ f_t^{(2)}\ f_t^{(3)}\ f_t^{(4)}\ f_t^{(5)}]'$.

TABLE 1—ESTIMATES OF THE SINGLE-FACTOR MODEL

A. Estimates of the return-forecasting factor, $\overline{rx}_{t+1} = \boldsymbol{\gamma}^{\top}\mathbf{f}_t + \bar{\varepsilon}_{t+1}$	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	R^2	$\chi^2(5)$
OLS estimates	−3.24	−2.14	0.81	3.00	0.80	−2.08	0.35	
			Asymptotic (Large T) distributions					
HH, 12 lags	(1.45)	(0.36)	(0.74)	(0.50)	(0.45)	(0.34)		811.3
NW, 18 lags	(1.31)	(0.34)	(0.69)	(0.55)	(0.46)	(0.41)		105.5
Simplified HH	(1.80)	(0.59)	(1.04)	(0.78)	(0.62)	(0.55)		42.4
No overlap	(1.83)	(0.84)	(1.69)	(1.69)	(1.21)	(1.06)		22.6
			Small-sample (Small T) distributions					
12 lag VAR	(1.72)	(0.60)	(1.00)	(0.80)	(0.60)	(0.58)	[0.22, 0.56]	40.2
Cointegrated VAR	(1.88)	(0.63)	(1.05)	(0.80)	(0.60)	(0.58)	[0.18, 0.51]	38.1
Exp. Hypo.							[0.00, 0.17]	

Source: Cochrane and Piazzesi (2005).

- Compute the factor as $\boldsymbol{\gamma}'\mathbf{f}_t$.

Single-Factor Model: Forecasting Performance

- The excess return on bonds of each maturity has some loading on this factor. To assess how much, estimate for each maturity $n = 2 - 5$ years

$$r_{t+1}^{(n)} - y_t^{(1)} = b_n \left(\gamma' \mathbf{f}_t\right) + \varepsilon_{t+1} \ . \tag{38}$$

B. Individual-bond regressions

	Restricted, $rx_{t+1}^{(n)} = b_n(\boldsymbol{\gamma}^\top \mathbf{f}_t) + \varepsilon_{t+1}^{(n)}$					Unrestricted, $rx_{t+1}^{(n)} = \boldsymbol{\beta}_n \mathbf{f}_t + \varepsilon_{t+1}^{(n)}$			
n	b_n	Large T	Small T	R^2	Small T	R^2	EH	Level R^2	$\chi^2(5)$
2	0.47	(0.03)	(0.02)	0.31	[0.18, 0.52]	0.32	[0, 0.17]	0.36	121.8
3	0.87	(0.02)	(0.02)	0.34	[0.21, 0.54]	0.34	[0, 0.17]	0.36	113.8
4	1.24	(0.01)	(0.02)	0.37	[0.24, 0.57]	0.37	[0, 0.17]	0.39	115.7
5	1.43	(0.04)	(0.03)	0.34	[0.21, 0.55]	0.35	[0, 0.17]	0.36	88.2

Notes: The 10-percent, 5-percent and 1-percent critical values for a $\chi^2(5)$ are 9.2, 11.1, and 15.1 respectively. All p-values are less than 0.005. Standard errors in parentheses "( )", 95-percent confidence intervals for R^2 in square brackets "[ ]". Monthly observations of annual returns, 1964–2003.

Source: Cochrane and Piazzesi (2005).

- The b_n coefficients say how much expected returns move with the factor. They **increase with maturity** – long maturities are scaled up versions of the factor.
- The R^2s are almost identical to those in the unrestricted regressions.

Single-Factor Model: Forecasting Performance

- Bond returns are **predictable**, so the **EH is rejected**. The single-factor model is a significant improvement on the Fama-Bliss regressions.
- The single factor **also predicts stock returns**, implying common changes in risk premia for stocks and bonds.

TABLE 3—FORECASTS OF EXCESS STOCK RETURNS

Right-hand variables		$\boldsymbol{\gamma}^\top \mathbf{f}$	(t-stat)	d/p	(t-stat)	$y^{(5)} - y^{(1)}$	(t-stat)	R^2
1	$\boldsymbol{\gamma}^\top \mathbf{f}$	1.73	(2.20)					0.07
2	D/p			3.30	(1.68)			0.05
3	Term spread					2.84	(1.14)	0.02
4	D/p and term			3.56	(1.80)	3.29	(1.48)	0.08
5	$\boldsymbol{\gamma}^\top \mathbf{f}$ and term	1.87	(2.38)			−0.58	(−0.20)	0.07
6	$\boldsymbol{\gamma}^\top \mathbf{f}$ and d/p	1.49	(2.17)	2.64	(1.39)			0.10
7	All $\mathbf{f}$							0.10
8	Moving average $\boldsymbol{\gamma}^\top \mathbf{f}$	2.11	(3.39)					0.12
9	MA $\boldsymbol{\gamma}^\top \mathbf{f}$, term, d/p	2.23	(3.86)	1.95	(1.02)	−1.41	(−0.63)	0.15

Notes: The left-hand variable is the one-year return on the value-weighted NYSE stock return, less the 1-year bond yield. Standard errors use the Hansen-Hodrick correction.

Source: Cochrane and Piazzesi (2005).

Single-Factor Model: Cyclicality

- Koijen et al. (2017) find that the CP factor helps price the **cross-section of stock returns**.
- The CP factor **increases during recessions**, similar to the term spread.

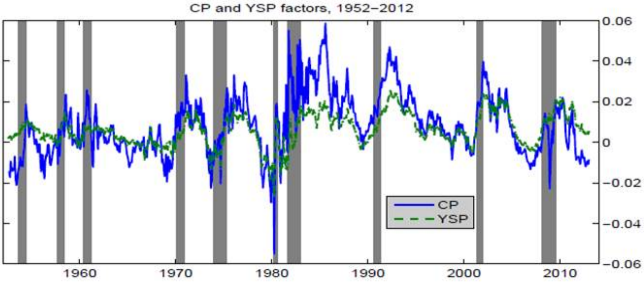


Figure A.2: *CP* factor and NBER recessions.

The figure plots the *CP* factor (solid line, against the right axis) and the NBER recessions (shaded areas). The sample is July 1952 until December 2011.

Source: Koijen et al. (2017).

Overview

Factor Model for Yields

- We now consider the ability of **multiple factors extracted from the term structure** to predict bond returns.
- Yields are subject to movements in the **level** and **slope**.
- We will model yields as functions of factors in order to capture their variation over time:

$$y_t^{(n)} = b_1^{(n)} F_1 + b_2^{(n)} F_2 + \dots \tag{39}$$

- These factors should be orthogonal to each other and maximize the explained variance of yields.
- This can be done using a **principal component analysis**. This amounts to doing an **eigenvalue decomposition of the yield covariance matrix**.

Principal Component Analysis

- Principal Component Analysis (PCA) is a statistical method used to reduce the dimensionality of a data set.
- The objective is to find a linear combination of the data that explains as much as possible of the observed variability of the data.
- Suppose you have demeaned data that is collected in a $(T \times N)$ matrix $\mathbf{X}$. Let $\boldsymbol{\Sigma}$ denote the covariance matrix of $\mathbf{X}$.
- Each principal component will be a $(T \times 1)$ vector and be orthogonal to the other principal components.

Principal Component Analysis

- The **first principal component**, denoted $\mathbf{y_1}$, is that linear combination of the original variables, $\mathbf{y_1} = \mathbf{a_1'X}$ whose sample variance $\mathbf{a_1'\Sigma a_1}$ is greatest. Since the variance of $\mathbf{y_1}$ could be increased without limit by increasing the elements of $\mathbf{a_1}$, one imposes the constraint that $\mathbf{a_1'a_1} = 1$. The optimization problem is

$$\max_{\mathbf{a_1}} \mathbf{a_1'\Sigma a_1} \quad \text{s.t.} \quad \mathbf{a_1'a_1} = 1 \ . \tag{40}$$

- The Lagrangian is

$$\mathcal{L} = \mathbf{a_1'\Sigma a_1} + \lambda_1(1 - \mathbf{a_1'a_1}) \ . \tag{41}$$

- The FOCs are

$$\frac{\partial \mathcal{L}}{\partial \mathbf{a_1}} = 2\mathbf{\Sigma a_1} - 2\lambda_1 \mathbf{a_1} = \mathbf{0} \quad \Rightarrow \quad \mathbf{\Sigma a_1} = \lambda_1 \mathbf{a_1} \ . \tag{42}$$

- Note that

$$\text{var}(\mathbf{y_1}) = \mathbf{a_1'\Sigma a_1} = \mathbf{a_1'}\lambda_1\mathbf{a_1} = \lambda_1\mathbf{a_1'a_1} = \lambda_1 \ . \tag{43}$$

  Thus, $\mathbf{a_1}$ is the eigenvector of $\mathbf{\Sigma}$ corresponding to its largest eigenvalue, λ_1.

- The **second principal component**, $\mathbf{y_2}$, is the linear combination of $\mathbf{X}$ that has the greatest variance subject to the conditions that $\mathbf{a}_2'\mathbf{a}_2 = 1$ and $\mathbf{y_2}$ be uncorrelated with $\mathbf{y_1}$. The optimization problem in this case is:

$$\max_{\mathbf{a}_2} \mathbf{a}_2'\boldsymbol{\Sigma}\mathbf{a}_2 \quad \text{s.t.} \quad \mathbf{a}_2'\mathbf{a}_2 = 1 \ , \quad \mathbf{a}_2'\mathbf{a}_1 = 0 \ . \tag{44}$$

- The Lagrangian is

$$\mathcal{L} = \mathbf{a}_2'\boldsymbol{\Sigma}\mathbf{a}_2 + \lambda_2(1 - \mathbf{a}_2'\mathbf{a}_2) - \mu\mathbf{a}_2'\mathbf{a}_1 \ . \tag{45}$$

- The FOCs are

$$\frac{\partial \mathcal{L}}{\partial \mathbf{a}_2} = 2\boldsymbol{\Sigma}\mathbf{a}_2 - 2\lambda_2\mathbf{a}_2 - \mu\mathbf{a}_1 = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\Sigma}\mathbf{a}_2 = \lambda_2\mathbf{a}_2 + \mu\mathbf{a}_1 \ . \tag{46}$$

- Note that

$$\text{var}(\mathbf{y_2}) = \mathbf{a}_2'\boldsymbol{\Sigma}\mathbf{a}_2 = \mathbf{a}_2'\lambda_2\mathbf{a}_2 + \mu\mathbf{a}_2'\mathbf{a}_1 = \lambda_2 \ . \tag{47}$$

  Thus, $\mathbf{a}_2$ is the eigenvector of $\boldsymbol{\Sigma}$ corresponding to its second largest eigenvalue, λ_2. (Remember that distinct eigenvectors are always orthogonal to each other.)

Principal Component Analysis

- Similarly,
  - the nth principal component is defined by the eigenvector associated with the nth largest eigenvalue of $\mathbf{\Sigma}$, λ_n; and
  - by choosing $\mathbf{a_n'a_n} = 1$, its variance is λ_n.

- Thus, the total variance of the N principal components will equal the variance of the original variables, i.e.

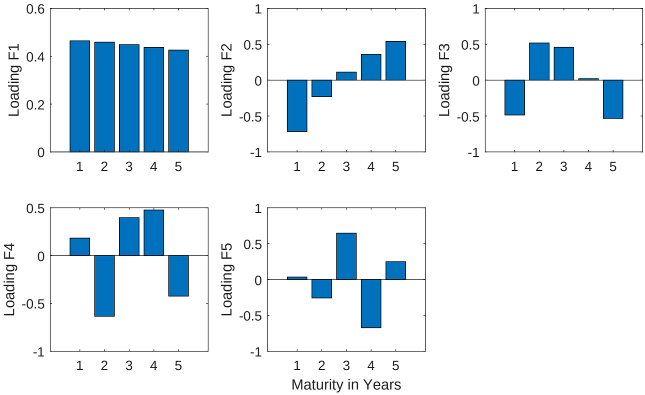
$$\sum_{n=1}^{N} \lambda_n = \text{tr}(\mathbf{\Sigma}) \tag{48}$$

and the nth principal component accounts for a proportion

$$z_n = \frac{\lambda_n}{\text{tr}(\mathbf{\Sigma})} \tag{49}$$

of the total variation in the original data.

Factor Loadings for US Yields, 1961-2020



- F_1 corresponds to the average yield level $\rightarrow$ **level factor**.
- F_2 captures changes in the slope of the yield curve $\rightarrow$ **slope factor**.
- F_3 captures changes in the curvature of the yield curve $\rightarrow$ **curvature factor**.

Predicting Bond Returns

	x_t	*level*	*slope*	*curve*	*factor4*	*factor5*	R^2
% of var(f)		97.7	2.2	0.10	0.01	0.00	
	0.24						0.46
	(9.00)						
		0.19					0.03
		(0.80)					
		0.21	-1.72				0.13
		(0.95)	(-2.51)				
		0.22	-1.74	4.79			0.25
		(1.11)	(-2.72)	(2.82)			
		0.20	-1.70	4.73	-0.28	-15.49	0.34
		(1.13)	(-2.60)	(2.80)	(-0.09)	(-2.87)	

Table 2. Regression coefficients, t-statistics and R^2 for forecasting the average (across maturity) excess return $\overline{rx}_{t+1}$ in GSW data, based on the return-forecast factor x_t and eigenvalue-decomposition factors of forward rates. The top row gives the fraction of variance explained, $100 \times \Lambda_i / \sum_{j=1}^{15} \Lambda_j$. Monthly observations of annual returns 1971-2006. Standard errors include a Hansen-Hodrick correction for serial correlation due to overlap.

Source: Cochrane and Piazzesi (2008).

- F_2, F_3, and F_5 are most important for predicting bond returns.
- The R^2 of the model with all 5 factors is similar to the explanatory power of the Cochrane-Piazzesi single-factor model.

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