Insurance: Mathematics and Economics 52 (2013) 243–254

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Pricing catastrophe risk bonds: A mixed approximation method

Zong-Gang Ma *, Chao-Qun Ma

School of Business Administration, Hunan University, Changsha 410082, PR China

ARTICLE INFO

Article history:
Received June 2012
Received in revised form November 2012
Accepted 31 December 2012

Keywords:
Catastrophe risk bonds
Stochastic interest rates
Compound nonhomogeneous Poisson process
PCS loss
Mixed approximation method

ABSTRACT

This paper presents a contingent claim model similar to the one described by Lee and Yu (2002) for pricing catastrophe risk bonds. First, we derive a bond pricing formula in a stochastic interest rates environment with the losses following a compound nonhomogeneous Poisson process. Furthermore, we estimate and calibrate the parameters of the pricing model using the catastrophe loss data provided by Property Claim Services (PCS) from 1985 to 2010. As no closed-form solution can be obtained, we propose a mixed approximation method to find the numerical solution for the price of catastrophe risk bonds. Finally, numerical experiments demonstrate how financial risks and catastrophic risks affect the prices of catastrophe bonds.

1. Introduction

Insured property losses from natural catastrophic risk events, such as earthquakes, hurricanes, storms, floods, tornados or man-made catastrophes, are extremely large, when compared with other types of property and casualty losses. In general, catastrophe risks are of low-loss frequency and high-loss severity. Traditionally, insurance companies hedge and transfer catastrophe risk by purchasing reinsurance contracts. However, such a reinsurance contract could be less cost-effective to the reinsurance company and may pose a severe financial stress to the reinsurance company due to the unpredictable nature of large catastrophic losses. As a result, over the last twenty years it has become increasing difficult to find a reinsurance company to cover the catastrophic losses at a reasonable cost. In order to expand the capacity of reinsurance industry, securitization of accumulated catastrophic losses in financial markets has become a timely and desirable alternative to the traditional reinsurance norm (D'Arcy and France, 1992).

Catastrophe risk bonds (CAT bonds) are one of the most important insurance-linked financial securities. The losses caused by large catastrophes could lead to a significant amount of payment for the capital market investors. The first successful CAT bond was \$85 million issued by Hanover Re in 1994 (Laster, 2001). Another CAT bond was issued by a nonfinancial firm, in 1999, which covered the earthquake losses in Tokyo for the company Oriental Land (Cummins, 2008). A \$3.4 billion risk capital was issued through 18 transactions in 2009 and the catastrophe bond market was shown to be an increasingly attractive and worthwhile supplement to the sponsor risk transfer programs (Klein, 2010).

Although there have been a number of successful issuances of the CAT bonds in recent years, few academic studies have been conducted for the pricing of CAT bonds. Cox and Pedersen (2000) evaluated catastrophe risk bonds using a representative agent technique and developed a framework of pricing CAT bonds in the incomplete market setting. Lee and Yu (2002) developed a contingent claim model that incorporated stochastic interest rates and generic loss processes with considerations of other factors, such as moral hazard, basis risk, and default risk. Lee and Yu (2007) presented a contingent-claim framework for valuing reinsurance contract that can increase the value of a reinsurance contract and reduce its default risk by issuing the CAT bonds. Egami and Young (2008) developed a method for pricing structured CAT bonds based on utility indifference pricing. Unger (2010) proposed a formulation and discretization strategy for CAT bonds model by utilizing a numerical PDE approach.

Apart from the above-mentioned studies, there also exist other articles about the pricing of the CAT bonds. Baryshnikov et al. (2001) developed an arbitrage-free solution to the pricing of the CAT bonds under the condition of continuous trading, they used compound doubly stochastic Poisson process to incorporate various characteristics of the catastrophe process. Burnecki and Kukla (2003) corrected and applied the results of Baryshnikov et al. (2001) to calculate no-arbitrage prices of a zero-coupon

* Corresponding author. Tel.: +86 13667355716.
E-mail addresses: mazonggang2006@yahoo.com.cn (Z.-G. Ma), cqma1998@126.com (C.-Q. Ma).

0167-6687/\$ – see front matter © 2013 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2012.12.007

and coupon CAT bonds, and derived a pricing formula under the compound doubly stochastic Poisson model framework. Härdle and Cabrera (2010) applied the results of Burnecki and Kukla (2003) to examine the calibration of real parametric CAT bonds for earthquakes sponsored by the Mexican government. Also under an arbitrage-free framework, Vaugirard (2003a,b) adopted the jump-diffusion model of Merton (1976) to develop the first valuation model of insurance-linked securities that deal with catastrophic events and interest rate randomness. Fernández-Durán and Gergorio-Domínguez (2005) presented a methodology for the pricing of CAT bonds by considering the fact that the issuance of the CAT bond is done by the government and its main interest is to have additional funds to relieve the affected victims. Burnecki (2005) evaluated CAT bonds using a compound nonhomogeneous Poisson model with left truncated loss distribution. Jarrow (2010) developed a simple closed form solution for valuing CAT bonds, while the formula is consistent with any arbitrage-free model for the evolution of the LIBOR term structure of interest rates.

As the occurrence of catastrophe is largely unpredictable, valuing CAT bonds is very difficult. But a study of the pricing bond model plays a key role in the prevention and mitigation of natural disasters. Unfortunately, most prior studies did not take into account diverse factors that affect bond prices. In this paper, we consider a variety of factors that affect bond prices, such as loss severity distribution, claim arrival intensity, threshold level and interest rate uncertainty. Consequently, we derive a simple pricing formula for CAT bonds in a stochastic interest rates environment and show that the loss process follows a compound nonhomogeneous Poisson process.

The remainder of the paper is organized as follows. Section 2 briefly describes a CAT bond. Section 3 derives the pricing model of the CAT bonds. Section 4 conducts parameters calibration of the pricing model, and Section 5 is devoted to numerical analysis. Finally, Section 6 summarizes the paper and gives the conclusion. For ease of exposition, most proofs are presented in Appendix.

2. Brief description of a CAT bond

The simple structure of a CAT bond is presented in Fig. 1. It involves a sponsor (e.g. insurer, reinsurer, or government), who seeks to transfer the risk to investors who accept the risk for higher expected returns. The transfer of the risk to the capital market is achieved by creating a special purpose vehicle (SPV) that provides coverage to the sponsor and issues bonds to investors. The sponsor pays a premium in exchange for a pre-specified coverage if a catastrophic event of a certain magnitude takes place and investors purchase a bond. The SPV collects the capital and invests the proceeds in safe and short-term securities (e.g. Treasure bonds), which are held in a trust account. The returns generated from this trust account are usually swapped for floating returns based on the London interbank offered rate (LIBOR) that are supplied by a highly rated swap counterparty. The reason for the swap is to immunize the sponsor and the investors from interest rate (mark-to-market) risk and default risk (Cummins, 2008).

If the covered event (also called trigger event) does not happen during the term of the CAT bond, investors receive their principal plus a compensation for the catastrophic risk exposure. However, if a catastrophic risk event occurs and triggers specified in the bond contract during the risk-exposure period, then the SPV compensates the sponsor according to the CAT bond contract. This results in a partial or full principal to the investors (Loubergé et al., 1999).

Obviously, defining the default-trigger event plays an important role in structuring CAT bonds. This catastrophic event should be measurable and easily understood. In general, there are three types of triggering variables: indemnity triggers, index triggers and hybrid triggers. If the trigger event is based on the level of actual monetary losses suffered by the sponsor, then it is called an indemnity trigger. This triggering type is subject to the highest degree of the moral hazard. This phenomenon appears when the sponsor no longer tries to limit its potential losses as the risk is transferred to the investors. Therefore, moral hazard occurs due to loss control efforts by the sponsor (Lee and Yu, 2002). Although suffering from moral hazard risk, indemnity triggers eliminate basis risk by offering indemnity against modeled perils (Harrington and Niehaus, 1999).

There are three broad types of indices that can be used as CAT bond triggers: industry loss indices, modeled loss indices, and parametric indices. With industry loss indices, the payoff on the bond is triggered when estimated industry-wide losses from a catastrophic event exceed a specified threshold level. A modeled-loss index is calculated using the model provided by one of the major catastrophe-modeling firms-Applied Insurance Research Worldwide, EQECAT, or Risk Management solutions. Lastly, with a parametric trigger, the bond payoff is triggered by specified physical measures of the catastrophic events such as wind speed or the location and magnitude of an earthquake (Cummins, 2008). Index triggers help the sponsor in avoiding detailed information disclosure to the competitors, so that they can minimize the problem of the moral hazard. However, index triggers are subject to basis risk as the sponsor's losses may differ from industry losses. Here, the basis risk differs from the mismatch between the index and the sponsor's losses. Therefore, hybrid triggers can be resolved between the moral hazard and the basis risk. For example, under both index- and parametric-based triggers, the sponsor is limited to no capability in over-statistic the losses (Cummins et al., 2004; Damnjanovic et al., 2010).

3. Valuation framework

3.1. Modeling assumptions

Let $(\Omega, \mathcal{F}, \mathcal{P})$ denote a probability space, where Ω is the set of states of the world, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and $\mathcal{P}$ is a probability measure on $\mathcal{F}$. Continuous trading interval $[0, T]$ for a fixed $T > 0$. An increasing filtration $\mathcal{F}_t \subset \mathcal{F}, t \in [0, T]$. Let $\{V_t : t \in [0, T]\}$ denote the CAT bond price process for all $T \in [0, T]$, which is modeled by many factors: type of region, kind of loss event, sort of insured property, and interest rates uncertainty, etc. Let $\{L_t : t \in [0, T]\}$ denote the aggregate loss process; $\{\mathcal{N}_t : t \in [0, T]\}$ is a (non)homogeneous Poisson process with an intensity parameter λ_t; $\{X_j : j \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. In addition, let $\{r_t : t \in [0, T]\}$ denote the spot interest rate process (or the force of interest). $\{\mathcal{W}_t : t \in [0, T]\}$ is a standard Brownian motion and accounts for the uncertainty of interest rates.

3.2. Valuation theory

In an arbitrage-free opportunities financial market, the value of the contingent claim $\{\mathcal{C}_T : T > t\}$ at time t can be expressed as

$$V_t = E_t^Q(D(t, T)C_T | \mathcal{F}_t), \tag{1}$$

where E_t^Q denotes expectation under an equivalent martingale measure (often called the risk-neutral pricing measure), given $\mathcal{F}_t$. $D(t, T) = \exp(-\int_t^T r(s)ds)$ is a stochastic discount factor. This expression is very general, and it can be stated that in the absence of an arbitrage opportunities financial market, there exists a stochastic process $D(t, T)$ that prices the contingent claim $\mathcal{C}_T$.

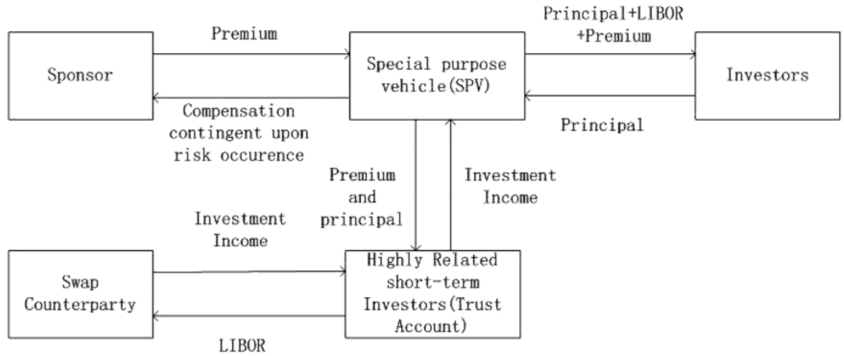

Fig. 1. Structure of CAT bonds.

3.3. *Interest rate process*

In this paper, the spot interest rate is assumed to follow the mean-reverting square-root process (Cox et al., 1985), which can be expressed as:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{2}$$

where $k > 0$ is the mean-reverting force measurement, $\theta > 0$ is the long-run mean of the interest rate, $\sigma > 0$ is the volatility parameter for the interest rate, and $\{W_t : t \in [0, T]\}$ is a standard Wiener process initialized at zero. And $2k\theta \geq \sigma^2$ guarantees that zero is an inaccessible boundary for spot interest rates. This setting can avoid possible negative interest rates r_t in Vasicek's model (Vasicek, 1977).

In the Cox–Ingersoll–Ross model (CIR model, for more details, please refer to Cox et al., 1985), the dynamics of bond prices are given by the stochastic differential equation:

$$\begin{aligned} dV(t,T) &= r_t[1 - \lambda_r B(t,T)]V(t,T)dt \\ &\quad - B(t,T)V(t,T)\sigma\sqrt{r_t}dW_t \end{aligned} \tag{3}$$

where

$$\begin{aligned} B(t,T) &= \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + k + \lambda_r)(e^{\gamma(T-t)} - 1) + 2\gamma}, \\ \gamma &= \sqrt{(k+\lambda_r)^2 + 2\sigma^2} \end{aligned} \tag{4}$$

and λ_r is a constant which determines the market price of risk.

Then, the market price of risk can be expressed as:

$$\begin{aligned} \Lambda_t &= \frac{\mu_t(V) - r_t V(t,T)}{\sigma_t(V)} \\ &= \frac{r_t[1 - \lambda_r B(t,T)]V(t,T) - r_t V(t,T)}{-B(t,T)\sigma\sqrt{r_t}} = \frac{\lambda_r\sqrt{r_t}}{\sigma}. \end{aligned} \tag{5}$$

For derivative pricing, it is a standard practice to use the device of a risk-neutral pricing measure. First we define a new process W_t^* by

$$\begin{aligned} W_t^* &= W_t - \int_0^t g_s ds = W_t + \int_0^t \Lambda_s ds \\ &= W_t + \int_0^t \frac{\lambda_r\sqrt{r_s}}{\sigma} ds, \end{aligned} \tag{6}$$

where g_s is the kernel in the transformation. The so-called "market price of risk", Λ_s is defined to be the negative of the Girsanov kernel, g_s (for more details, please refer to Duffie, 2001).

By Girsanov's theorem (ØKsendal, 2003), we get the following results:

(1) $\{W_t^* : 0 \leq t \leq T\}$ is a standard Brownian motion under the risk-neutral measure Q change for $P|_{F_t}$ defined by the Radon–Nikodym derivative:

$$\begin{aligned} &M_t = \exp\left\{-\int_0^t \frac{\lambda_r\sqrt{r_s}}{\sigma} dW_s - \frac{1}{2}\int_0^t \frac{\lambda_r^2 r_s}{\sigma^2} ds\right\}, \\ &\texttt{if } E\left[\exp\left\{\frac{1}{2}\int_0^T \frac{\lambda_r^2 r_s}{\sigma^2} ds\right\}\right] < \infty^{1}; \end{aligned} \tag{7}$$

(2)

$$\sigma\sqrt{r_t}\left(\frac{\lambda_r\sqrt{r_t}}{\sigma}\right) = k(\theta - r_t) - \alpha, \tag{8}$$

where α is a process.

By rearranging (8), we derive

$$\begin{aligned} \alpha &= k(\theta - r_t) - \lambda_r\sqrt{r_s} \\ &= k\theta - (k + \lambda_r)r_t \\ &= (k+\lambda_r)\theta - (k+\lambda_r)r_t - (k+\lambda_r)\left(\frac{\lambda_r\theta}{k+\lambda_r}\right) \\ &= (k+\lambda_r)\left(\frac{k\theta}{k+\lambda_r} - r_t\right). \end{aligned} \tag{9}$$

We now determine the interest rate process under the risk-neutral pricing measure Q, which can be expressed as:

$$dr_t = k^*(\theta^* - r_t)dt + \sigma\sqrt{r_t}dW_t^*, \tag{10}$$

where k^*, θ^*, and W_t^* are respectively defined as

$$\begin{aligned} k^* &= k + \lambda_r, \\ \theta^* &= \frac{k\theta}{k+\lambda_r}, \\ dW_t^* &= dW_t + \frac{\lambda_r\sqrt{r_t}}{\sigma}dt. \end{aligned} \tag{11}$$

$k^* > 0$ is the mean-reverting force measurement, $\theta^* > 0$ is the long-run mean of the interest rate, $\sigma > 0$ is the volatility parameter for the interest rates. The term λ_r is a constant market risk parameter, W_t^* is a standard Wiener process under risk-neutralized measure Q, and $2k^*\theta^* \geq \sigma^2$. This is a common transformation, and for the related reference, please refer to Shirakawa (2002), Lee and Yu (2002), Duan and Yu (2005), Hördahl and Vestin (2005), etc.

3.4. *Aggregate claims process*

The aggregate claims have a compound distribution with two main components: one characterizing the frequency (or incidence)

[1] This is called "Novikov's condition", which is sufficient to guarantee that the process $\{M_t\}_{t\leq T}$ is a martingale (with respect to F_t and P). See ØKsendal (2003, 159–168).

of catastrophic events and another describing the severity (or size or amount) of gain or loss resulting from the occurrence of a catastrophic event (Klugman et al., 2008; Tse, 2009). In this paper, we follow the approach of Aase (1999) and assume that the process of accumulated insured property losses follows a compound Poisson process. There are some assumptions about the aggregate loss process:

(1) There exists a Poisson point process N_t ($t \in [0, T]$) describing the flow of potentially catastrophic events of a given type in the region specified in the bond contract. The intensity of this Poisson point process is assumed to be a predictable bounded process λ_t. We denote the instants of these potentially catastrophic events as $0 \le t_1 \le \cdots \le t_j \le \cdots \le T$.
(2) τ is the time of the threshold event when the aggregate claims L_t exceed the threshold level D, i.e., $\tau = \inf\{t : L_t \ge D\}$.
(3) The insured losses incurred by each event in the flow $\{t_j\}_{j=1,\dots}$ are independent and identically distributed random variables $\{X_j\}_{j=1,\dots}$ with the distribution function $F(x) = P\{X_j < x\}$.

Therefore, a left-continuous and predictable aggregate loss process is defined as:

$$L_t = \sum_{t_j<\tau} X_j = \sum_{j=1}^{N_t} X_j, \tag{12}$$

where N_t and X_j are assumed to be independent processes.

3.5. Pricing model for the CAT bonds

The catastrophe loss process has jumps, so the markets are incomplete, and there is no unique pricing measure. That is, the methodology of replicating portfolios is not applicable. In this paper, we follow Merton (1976) and assume that the overall economy is only marginally influenced by localized catastrophe such as earthquakes, hurricanes, floods, etc., and the catastrophe losses pertain to idiosyncratic shocks to the capital markets. So the catastrophic risk associated with jumps can be diversified away, i.e., the catastrophic shocks will represent “nonsystematic” and have a zero risk premium. For a detailed discussion of this point, please refer to Cummins and Geman (1995), Cox and Pedersen (2000), Lee and Yu (2002) and Vaugirard (2003a,b). Furthermore, we follow Cox and Pedersen (2000), and assume that the aggregate loss process still retains their original distributional characteristics after changing from the physical probability measure P to the risk-neutral measure Q; under the risk-neutral pricing measure Q those events that depend only on financial variables are independent of those events that depend on catastrophic risk variables (for a detailed discussion of this point, please refer to Cox and Pedersen, 2000).

First, we consider a zero-coupon CAT bond with the payoffs ($P_{\text{CAT}}(T)$) of a certain amount $PV = Z$ at maturity time T. The structure of payoffs is given by

$$P_{\text{CAT}}(T) = \begin{cases} Z & \texttt{if } L_T \le D \\ pZ & \texttt{if } L_T > D, \end{cases} \tag{13}$$

where D is the threshold value in the CAT bond provisions; L_T is the aggregate claims at maturity T; p denotes the fraction of the principal needed to be paid to bondholder in the case where the aggregate claims are above the trigger at time T.

The other is the payment of the principal (face value) at maturity time T and coupon payments C which stop immediately at time T when $\tau \le T$. That is to say, if $L_\tau > D$ ($\tau \le T$), then its principal is protected and only the interest will be forfeited. The structure of payoffs is given by

$$P'_{\text{CAT}}(T) = \begin{cases} Z + C & \texttt{if } L_T \le D \\ Z & \texttt{if } L_T > D, \end{cases} \tag{14}$$

where D is the threshold value in the CAT bond provisions; L_T is the aggregate claims at maturity T.

To summarize, we obtain the following results.

The zero-coupon CAT bond prices at time t associated with the threshold D, the catastrophic flow N_t, and a loss distribution function $F(x)$, paying principal Z at time t to maturity T under the risk-neutralized pricing measure Q, is given by

$$\begin{aligned} V_t &= E^Q\left(e^{-\int_t^T r_s ds} P_{\text{CAT}}(T) \mid \mathcal{F}_t\right) \\ &= B_{\text{CIR}}(t,T) Z \Bigg[p + (1-p) \\ &\quad \times \sum_{n=0}^{\infty} e^{-\lambda_t (T-t)} \frac{(\lambda_t (T-t))^n}{n!} F^{*n}(D) \Bigg] \end{aligned} \tag{15}$$

where

$$F^{*n}(D) = \Pr(X_1 + X_2 + \cdots + X_n \le D) \tag{16}$$

denote the n-th convolution of F, and

$$\begin{aligned} &B_{\text{CIR}}(t,T) = A(t,T) e^{-B(t,T)r}, \\ &A(t,T) = \left[\frac{2\gamma e^{(k^*+\gamma)(T-t)/2}}{(k^*+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2k^*\theta^*}{\sigma^2}}, \\ &B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+k^*)(e^{\gamma(T-t)}-1)+2\gamma}, \\ &\gamma = \sqrt{(k^*)^2 + 2\sigma^2} \end{aligned} \tag{17}$$

Proof. See Appendix A.

The principal protected coupon CAT bond price associated with the threshold D, the catastrophic flow N_s, and a loss distribution function $F(x)$, paying the principal Z at time to maturity T and coupon payments C_s with a fixed spread c over LIBOR which cease at threshold time T when $\tau' \le T$ under the risk-neutralized pricing measure Q, is given by

$$\begin{aligned} V_t &= E^Q\left(e^{-\int_t^T r_s ds} P'_{\text{CAT}}(T) \mid \mathcal{F}_t\right) \\ &= B_{\text{CIR}}(t,T) \left[Z + C \sum_{n=0}^{\infty} e^{-\lambda_t (T-t)} \frac{(\lambda_t (T-t))^n}{n!} F^{*n}(D) \right] \end{aligned} \tag{18}$$

where

$$F^{*n}(D) = \Pr(X_1 + X_2 + \cdots + X_n \le D) \tag{19}$$

denote the n-th convolution of F, and

$$\begin{aligned} &B_{\text{CIR}}(t,T) = A(t,T) e^{-B(t,T)r}, \\ &A(t,T) = \left[\frac{2\gamma e^{(k^*+\gamma)(T-t)/2}}{(k^*+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2k^*\theta^*}{\sigma^2}}, \\ &B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+k^*)(e^{\gamma(T-t)}-1)+2\gamma}, \\ &\gamma = \sqrt{(k^*)^2 + 2\sigma^2}. \quad \square \end{aligned} \tag{20}$$

Proof. See Appendix B. □

3.6. A mixed approximation method

Under the assumption that the insured losses incurred by each catastrophic event are independent and identically distributed random variables $X_j \geq 0$ with the distribution function $F(x) = P\{X_j < x\}$ and claims arrival following a nonhomogeneous Poisson process with the intensity λ_t, the exact distribution $F(x, t)$ of the aggregate loss $L_t = \sum_{j=1}^{N_t} X_j$ in the interval $[0, t]$ is given by

$$\begin{aligned} F(x,t) &= \sum_{n=0}^{\infty} \exp\{-\lambda_t t\} \frac{(\lambda_t t)^n}{n!} F^{*n}(x), \quad x > 0 \\ &= \exp\{-\lambda_t t\}, \quad x = 0 \end{aligned} \tag{21}$$

where $F^{*n}(x)$ denotes the n-fold convolution F with itself, which is extremely difficult to compute because of its high-order convolutions, and only a few closed-form solutions are known.

Due to numerical difficulties, there are plenty of approximation methods of the exact cumulative distribution function $F(x, t)$, such as the Fast Fourier Transform, inversion method, recursive method, simulation method, approximation method, etc. In this paper, we employ an approximation method to evaluate the aggregate claims distribution. Therefore, we need to determine a good approximation for the aggregate claims distribution.

In actuarial science, the most well-known ones are the normal approximation and its improvements (the normal power ($N - P_2$) approximation and the Edgeworth approximation), the gamma approximation and the Esscher approximation. In general, the basic principle of the approximation is the density $f_L(l)$ (or distribution function $P(L \leq l)$) approximated by a function that uses the mean μ_L, variance σ_L^2, skewness $\kappa_{3L} = \frac{\tilde{\mu}_{3L}}{\sigma_L^3}$ (where $\tilde{\mu}_{3L}$ is the third moment about the mean μ_L, σ is the standard deviation) and even the excess kurtosis $\kappa_{4L} = \frac{\tilde{\mu}_{4L}}{\sigma_L^4} - 3$ (where $\tilde{\mu}_{4L}$ is the forth moment about the mean μ_L, σ_L is the standard deviation). Here, $\tilde{\mu}_{jL}$ denotes the jth central moment of L (Panjer and Willmot, 1992).

Chaubey et al. (1998) proposed a new approximation to the aggregate loss distribution based on the inverse-Gaussian (IG) distribution, and they used an IG-gamma mixture to approximate the true distribution accurately. Reijnen et al. (2005) compare five approximations ($N-P_2$, Edgeworth expansions, Gamma, IG and IG-gamma) of the Stop-loss premium by assuming different models for L_t and different underlying distributions. In their method, they proposed a rule of thumb: if $\kappa_{3X} \in [0, 5]$ and $\kappa_{4L} \in [0, 1.5]$ (where, κ_{3X} and κ_{4L} denote the skewness of the claim size X and the kurtosis of aggregate claims L, respectively), then it can use the gamma-IG approximation to evaluate aggregate loss distribution; otherwise, if $\kappa_{3X} \in (5, 15)$ or $\kappa_{4L} \in (1.5, 50)$, then the IG approximation is more accurate.

We propose a mixed approximation method to evaluate the aggregate claims distribution based on the value of the skewness κ_{3X} of the claim size X and the kurtosis κ_{4L} of the aggregate claims L. If κ_{3X} and κ_{4L} are consistent with the rule of thumb during the period of numerical calculation, we choose the IG-gamma method or the IG method to approximate the aggregate claims distribution.

Next we give an overview of the probability density functions of the approximations that we will use in this study.

3.6.1. The gamma approximation

This approximation is the most famous approximation in this field (Seal, 1977), with the probability density function (p.d.f.) given by

$$f_L(l) \approx f_{\text{gamma}}(l) = \frac{\beta^{\alpha}(l - x_0)^{\alpha-1} \exp(-\beta(l - x_0))}{\Gamma(\alpha)}, \tag{22}$$

where $\alpha = (\frac{2}{\kappa_{3L}})^2$, $\beta = \frac{2}{\kappa_{3L}\sigma_L}$, $x_0 = \mu_L - \frac{2\sigma_L}{\kappa_{3L}}$, $\Gamma(\cdot)$ denotes a usual gamma function.

3.6.2. The inverse Gaussian approximation

This approximation is quite new, and it is introduced by Chaubey et al. (1998), with the p.d.f. given by

$$f_L(l) \approx f_{\text{IG}}(l) = \frac{\alpha}{\sqrt{2\pi\beta(l - x_0)^3}} \exp\left(-\frac{(\alpha - \beta(l - x_0))^2}{2\beta(l - x_0)}\right), \tag{23}$$

where $\alpha = (\frac{3}{\kappa_{3L}})^2$, $\beta = \frac{3}{\kappa_{3L}\sigma_L}$, $x_0 = \mu_L - \frac{3\sigma_L}{\kappa_{3L}}$.

3.6.3. The gamma-IG approximation

The IG-gamma approximation is a combination of the IG approximation and the gamma approximation (Chaubey et al., 1998), with the p.d.f. given by

$$f_L(l) \approx f_{\text{gamma}-\text{IG}}(l) = \omega f_{\text{gamma}}(l) + (1 - \omega) f_{\text{IG}}(l), \tag{24}$$

where $\omega = \frac{\kappa_{4L} - \kappa_{4IG}}{\kappa_{4gamma} - \kappa_{4IG}} = \frac{\kappa_{4L} - \frac{5}{3}\kappa_{3L}^2}{\frac{3}{2}\kappa_{3L}^2 - \frac{5}{3}\kappa_{3L}^2} = \frac{10\kappa_{3L}^2 - 6\kappa_{4L}}{\kappa_{3L}^2}$, κ_{3L} and κ_{4L} denote the corresponding skewness and kurtosis of L, respectively.

4. Parameter calibration of the pricing model

In order to estimate and calibrate the parameters of the pricing model, we have to fit both the distribution function of the insured losses F and the claim arrival process N_t governing the flow of natural catastrophic events.

4.1. Data description

We take for our study the Property Claim Services (PCS) data covering losses resulting from natural catastrophic events in the US that occurred between 1985 and 2010,[2,3] and adjust for inflation using the Consumer Price Index (CPI) provided by the US Department of Labor.[4] That is, the losses are converted to 2010 dollars using the CPI. The Insurance Service Office's (ISO's) PCS unit is a US industry authority on insured property losses from catastrophes in the United States, Puerto Rico, and the US Virgin Islands. It tracks and estimates catastrophic property losses in the US on national, region, and state bases as well as for entire property and casualty industry. The PCS has maintained an insurance claim database and provides catastrophe indices (PCS indices) since 1949. Most catastrophe derivatives use a PCS index as a reference index. Moreover, loss transactions of a catastrophe-linked security are often based on a PCS index. Losses to investors/underwriters are calculated using a PCS as a proxy (Lin and Wang, 2009). Therefore, it is reasonable to use the PCS index losses from the entire property and casualty industry in the US as an estimation of the severity and frequency distributions for the pricing of the CAT bonds described in the previous section.

The adjusted PCS loss for catastrophic events during 1985–2010 is depicted in Fig. 2. Fig. 3 shows that the number of qualified catastrophic events in the US from 1985 to 2010. The twenty most costly insured CAT losses are shown in Table 1. In reality not all insurance losses over a certain time interval are recorded accurately. In the framework of catastrophe losses, PCS only records catastrophic events whose losses are over a predetermined threshold. The catastrophe loss threshold was changed from $1 million to $5 million in 1983, and increased again to $25 million in 1997.

2 http://www.iso.com/Products/Property-Claim-Services/Property-Claim-Services-PCS-info-on-losses-from-catastrophes.html.

3 Note: a set of 770 catastrophe loss data and 780 catastrophe events in US from 1985 to 2010.

4 ftp://ftp.bls.gov/pub/special.requests/cpi/cpiai.txt.

Table 1
The twenty most costly catastrophes in the United States.
Source: Property claims services, INC. (ISO), insurance information institute.

Event	Date	PCS loss ($ billions)	2010 dollars ($ billions)
Hurricane Katrina	25/08/05	41.10	45.89
Hurricane Andrew	24/08/92	15.50	24.09
Terrorist attacks in the US	11/09/01	18.78	23.12
Northridge earthquake and Winter Storm	17/01/94	13.30	19.57
Hurricane Ike	12/09/08	12.50	12.66
Hurricane Ivan and Jeanne	15/09/04	10.77	12.43
Hurricane Wilma	24/10/05	10.30	11.50
Hurricane Charley	13/08/04	7.48	8.63
Hurricane Hugo	17/09/89	4.20	7.38
Hurricane Rita	20/09/05	5.62	6.28
Hurricane Frances	03/09/04	4.59	5.30
Hurricane Georges	21/09/98	2.95	3.95
Wind and Thunderstorm Event	02/05/03	3.21	3.80
Tropical Storm Allison	05/06/01	2.50	3.08
Hurricane Opal	04/10/95	2.10	3.00
Wildland Fire Oakland Hills	20/10/91	1.70	2.72
Wind and Thunderstorm Event	06/04/01	2.20	2.71
Winter Storm	11/03/93	1.75	2.64
Hurricane Floyd	14/09/99	1.96	2.57
Wind and Thunderstorm Event	04/10/10	2.50	2.50

Note: Losses were adjusted to 2010 exposure and price level using the US consumer price index.

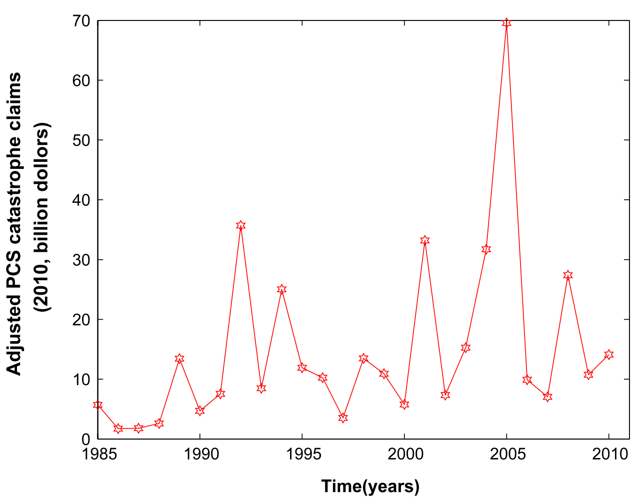

Fig. 2. The PCS catastrophe loss data in the United States from 1985 to 2010.

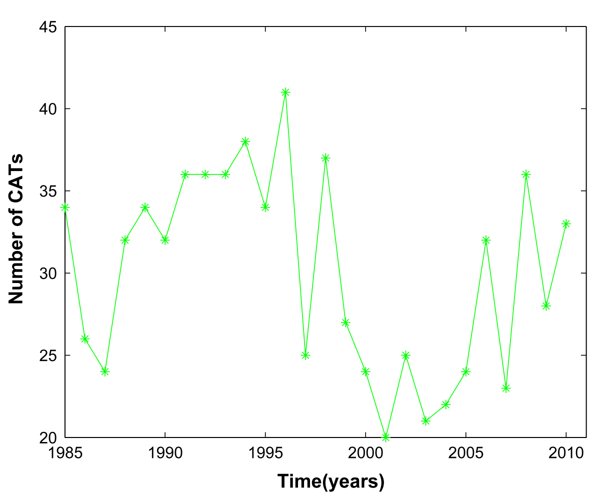

Fig. 3. Number of natural CATs in the United States from 1985 to 2010.

Because of confidentiality, the data cannot be explicitly provided. However, some features can be outlined as follows:

(1) the maximum value of the data is 90 times of the mean;
(2) there is 12.60% of the observations categorized as outliers if the $1.5 * IQR$ rule is used;
(3) the skewness and kurtosis for the data are 13.01 and 208.28; and
(4) 35.58% of the data is smaller than 1/500 of the maximum value, 92.34% of the data is smaller than 1/50 of the maximum value.

All points above suggest that the data is heavy-tailed.

4.2. Loss-severity distribution

In the actuarial literature, heavy-tailed data are commonly modeled by theoretical heavy-tailed distribution such as Log-normal, Weibull or Burr distributions, etc. So the following heavy-tailed distributions are often considered (within the domain $\Re_+$).

(1) Log-normal distribution, with the probability density function (p.d.f.) given by

$$f_X(x;\mu;\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), \quad x>0, \sigma>0, \mu\in\Re. \tag{25}$$

(2) Pareto distribution, with the p.d.f.,

$$f_X(x;\alpha;\beta) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}}, \quad x>\beta, \alpha>0, \beta>0. \tag{26}$$

(3) Inverse Gaussian distribution, with the p.d.f.,

$$f_X(x;\mu;\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \quad x>0, \mu>0, \lambda>0. \tag{27}$$

(4) Gamma distribution, with the p.d.f.,

$$f_X(x;\alpha;\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1}\exp(-x/\beta), \quad x>0, \alpha>0, \beta>0. \tag{28}$$

(5) Weibull distribution, with the p.d.f.,

$$f_X(x;\alpha;\beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\beta}\right)^\alpha\right), \quad x>0, \alpha>0, \beta>0. \tag{29}$$

(6) Generalized Extreme Value (GEV) Distribution, with the p.d.f.,

$$f_X(x;\mu;\sigma;\kappa) = \frac{1}{\sigma}\exp\left(-\left(1+k\left(\frac{x-\mu}{\sigma}\right)\right)^{-\frac{1}{k}}\right) \times \left(1+k\left(\frac{x-\mu}{\sigma}\right)\right)^{-1-\frac{1}{k}}, \quad k\neq 0;$$
$$= \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}-\exp\left(-\frac{x-\mu}{\sigma}\right)\right), \quad k=0 \tag{30}$$

where κ is the shape parameter, $\sigma > 0$ is the scale parameter, and μ is the location parameter of the distribution.

4.2.1. Non-parametric tests

Once the distribution is selected, we need to obtain parameter estimates. In general, parameters are found mainly by moment (for the Pareto distribution only) and maximum likelihood estimation.

As the choice of the distribution can influence the bond prices, we need to find a most suitable distribution function which fits the observed claims well. The next step is to evaluate the goodness-of-fit and the performance of the model selection criteria.

The goodness-of-fit problem is as follows: given a random sample $x_1, x_2, \ldots, x_n$, to test H_0; the sample comes from a population with a distribution function $F(x)$. A class of classical measures of fit is empirical distribution function (EDF) statistics. They are based on a comparison between the theoretical distribution $F(x)$ and the empirical distribution function $F_n(X)$.

In general, when $F(x)$ is continuous and completely specified, EDF statistics give more powerful tests of H_0 than the χ^2 test (Stephens, 1974). Moreover, the χ^2 goodness-of-fit test is of low power for small sample sizes (Levi and Partrat, 1991). Since empirical samples generally contain a rather small amount of extreme observations in our dataset of the catastrophe losses, it is questionable whether the χ^2 test would reveal a severe misfit of the tail. In this paper, we select the Kolmogorov–Smirnov (KS) test and the Anderson–Darling (AD) test to test of the appropriateness of the model selection. The K–S test has the advantage of making no assumption about the distribution data. Technically, it is non-parametric and distribution free. The AD test is a modification of the KS test and gives more weight to the tails than the KS test. In general, the theoretical distribution $F(x)$ with the smallest KS and AD values is determined to be the best fit to the empirical distribution $F_n(X)$ (Anderson and Darling, 1952; Law and Kelton, 2000; Wen and Yang, 2009).

Model selection refers to the problem of using the data to select one model from the list of competing models. In essence, it involves the use of a model selection criteria to find the best fitting model to the data (Wasserman, 2000). In order to assess the performance of the above parametric distributions, we apply corrected Akaike's Information Criteria and Bayesian information criteria to rank these models, with the best approximating model being the one with the lowest *AICc* value and *BIC* value.

Kolmogorov–Smirnov statistic. Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from some distribution with cumulative function (CDF) $F(X)$. The empirical CDF is denoted by $F_n(x) = n^{-1} \times$ [number of observations $\leq x$]. The KS statistic (D) is based on the largest vertical distance between $F(x)$ and $F_n(x)$ for all values of x, i.e.,

$$D_n = \sup_x\{|F_n(x) - F(x)|\}. \tag{31}$$

The statistic D_n can be computed by calculating (Law and Kelton, 2000)

$$D_n^+ = \max_{i=1,2,\ldots,n}\left\{\frac{i}{n} - F(X_{(i)})\right\}, \quad D_n^- = \max_{i=1,2,\ldots,n}\left\{F(X_{(i)}) - \frac{i}{n}\right\} \tag{32}$$

where $X_{(i)}$ is the ith order statistic, and letting

$$D_n = \max\{D_n^+, D_n^-\}. \tag{33}$$

If the test statistic, D, is greater than the corresponding critical value C_α, we reject that the data follow the predetermined distribution.

Anderson–Darling (AD) statistic. The AD test is a form of minimum distance estimate, which assesses whether a sample comes from a predetermined distribution. The AD goodness of fit test is designed to detect the difference in the tails between the fitted distribution and the data. The AD statistic (A^2) is defined as (Anderson and Darling, 1952; Stephens, 1974):

$$A^2 = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\times[\ln F(X_i) + \ln(1 - F(X_{n-i+1}))] - n. \tag{34}$$

If the test statistic, A^2, is greater than the corresponding critical value C_α, we reject that the data follow the predetermined distribution.

Corrected Akaike's information criteria (AICc). Akaike (1973) found a simple relationship between the likelihood (L) and the number of parameters in the examined model (K):

$$AIC = -2\ln(L) + 2K \tag{35}$$

(where ln is the natural logarithm) to estimate the expected distance of a given model from truth. However, for small sample sizes (roughly approximated as being when $\frac{n}{k} < 40$ (where n is the sample size) in the most complex model), *AIC* might not be accurate. Then a second order bias correction for *AIC* was derived by Sugiura (1978) and Hurvich and Tsai (1989):

$$AICc = AIC + \frac{2K(K+1)}{n-K-1}. \tag{36}$$

As sample size (n) increases, *AICc* converges to *AIC*.

Bayesian information criteria (BIC). Another widely used information criteria is the *BIC*, which is derived within a Bayesian framework as an estimate of the Bayes factor for two competing models (Schwarz, 1978), formulated as

$$BIC = -2\ln(L) + K\ln(n) \tag{37}$$

where ln is the natural logarithm, n is the sample size, L is the value of the likelihood, K is the number of parameters in the examined model.

4.2.2. Empirical results

First, we calculated the parameters of loss distribution by moment (for the Pareto distribution only) and maximum likelihood estimation. The results of the parameter estimation are shown in Table 2. We apply the Kolmogorov–Smirnov test and Anderson–Darling statistic test to show that the Pareto distribution does not pass all tests at the 5% level, as shown in Table 3. Table 4 shows that the GEV distribution with parameters $k = 0.35431, \sigma = 6.5307$ and $\mu = 7.7158$ yields the best results, and the lognormal distribution with parameters $\mu = 2.3179, \sigma = 0.89666$ is the next best fit. So we will choose the GEV and the lognormal distribution for further analysis.

Table 2
Parameter estimates for the annual catastrophe loss amounts.

Distributions	Gen. Extreme Value	Inv. Gaussian	Gamma	Lognormal	Pareto	Weibull
Parameters	$k = 0.35431$ $\sigma = 6.5307$ $\mu = 7.7158$	$\mu = 14.96$ $\lambda = 15.221$	$\alpha = 1.0174$ $\beta = 14.704$	$\sigma = 0.89666$ $\mu = 2.3179$	$\alpha = 0.56579$ $\beta = 1.734$	$\alpha = 1.302$ $\beta = 14.135$

Table 3
Test statistics for different distributions.

Distribution	Gen. Extreme Value	Lognormal	Inv. Gaussian	Weibull	Gamma	Pareto
Test values (critical values for $\alpha = 0.05$)						
D_n (0.25907)	0.08784	0.09377	0.09401	0.10883	0.1306	0.29802
AD (2.5018)	0.2296	0.24384	0.38287	0.5989	0.62191	4.6711

Note: the critical values in parentheses can be easily found in the literature, as e.g. D'Agostino and Stephens (1986).

Table 4
The performance for different distributions.

Distribution						
Criteria	Gen. Extreme Value	Lognormal	Weibull	Gamma	Pareto	Inv. Gaussian
AICc	−**165.8349**	−163.3982	−154.6801	−142.3942	−97.2340	−15.6406
BIC	−**163.1516**	−161.8051	−153.0870	−140.8011	−95.6409	−14.0474

Note: The optimum values are in boldface.

4.3. Claim arrival process

A pure Poisson process has been widely used to describe the arrival rate of catastrophic events and is applied for pricing of catastrophe risk derivatives (Cummins and Geman, 1995; Lee and Yu, 2002, 2007; Burnecki and Kukla, 2003; Cox et al., 2004; Jaimungal and Wang, 2006; Lin and Wang, 2009; Härdle and Cabrera, 2010, etc.). In this paper, we assume that the number of claims observed till time t is a nonhomogeneous Poisson process (NHPP), denoted by N_t, with the intensity λ_t of the counting process varies with time t. In Fig. 3 we can see that the time series of the annual number of claims exhibit a cyclic and periodic trends. This suggests that calibrating a nonhomogeneous Poisson process with a cosine rate function would give an adapted model.

$$\begin{aligned}\lambda_t^1 &= a + b\sin^2(t+c) + \lambda_c \\ &= a + b\sin^2(t+c) + d\exp\left\{\cos\left(\frac{2\pi t}{\omega}\right)\right\}\end{aligned} \tag{38}$$

where $a > 0, d > 0, \omega > 0$, and λ_c is the intensity function discussed in Vere-Jones (1982). Obviously, λ_c is a cyclic function with period ω, the problem of estimating λ_c at a given $t \in [0, \infty)$ can be reduced to the problem of estimating λ_c at a given $t \in [0, \omega)$.

Applying the Nonlinear Least Squares procedure we estimate the parameters of the intensity function to the accumulated annual number of PCS losses. We conclude that $a = 27.4746, b = 2.1304, c = -0.3185, d = 1.1938$, and $\omega = 4.7938$. Burnecki et al. (2011a,b) propose an intensity function of the form $\lambda_t^2 = a + b \cdot 2\pi \cdot \sin\{2\pi(t-c)\}$. In the same way, using the Nonlinear Least Squares procedure we conclude that $a = 27.7986, b = 2.2628$, and $c = -0.0247$. We also consider homogeneous Poisson process with an annual intensity $\lambda_t^3 = 30$.

In order to compare these three Poisson intensities precisely, it is necessary to measure the errors based on the comparison of actual data with the values predicted by the Poisson intensities. In this paper, the performance of the Poisson intensities is evaluated using five goodness-of-fit measures: the mean absolute error (MAE), the root mean square error (RMSE), Theil's coefficient (U), Nash–Sutcliffe coefficient of efficiency (E) and the index of agreement (d) as shown in Eqs. (39)–(43).

$$\mathrm{MAE} = \frac{1}{N}\sum_{i=1}^{N}|O_i - P_i| \tag{39}$$

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(O_i - P_i)^2} \tag{40}$$

$$U = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}(O_i - P_i)^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}(O_i)^2} + \sqrt{\frac{1}{N}\sum_{i=1}^{N}(P_i)^2}} \tag{41}$$

$$E = 1.0 - \frac{\sum_{i=1}^{N}(O_i - P_i)^2}{\sum_{i=1}^{N}(O_i - \bar{O})^2} \tag{42}$$

$$\begin{aligned} d &= 1.0 - \frac{\sum_{i=1}^{N}(O_i - P_i)^2}{\sum_{i=1}^{N}(|P_i - \bar{O}| + |O_i - \bar{O}|)^2} \\ &= 1.0 - N\frac{\mathrm{MSE}}{\mathrm{PE}} \end{aligned} \tag{43}$$

where the overbar denotes the mean for the entire time period of the evaluation, the PE denotes the "potential error", O_i is the observed value at time i, P_i is the predicted value at time i and N is the total number of observations. The MAE, RMSE statistics have as the lower limit, the value of zero, which is the optimum value for them as it for U. Instead, higher values of the E and d indicate better agreement between the model and observations (Nash and Sutcliffe, 1970; Willmott, 1981, 1984; Legates and McCabe, 1999).

Table 5 gives a summary of five measurements. Obviously, λ_t^1 is superior to the two other intensity functions due to its larger E and d, and smaller MAE, RMSE and U.

Table 5
Summary of the statistical tests for the Poisson intensity.

Poisson intensity	Performance				
	MAE	RMSE	U	E	D
λ_t^1	**5.2379**	**5.83674**	**0.0963**	**0.0455**	**0.02879**
λ_t^2	5.4615	5.9743	0.0986	2.1538E−014	3.4436E−008
λ_t^3	5.4615	5.9743	0.1638	0	0

Note: the optimum value are in boldface.

Table 6
The kurtosis range of the aggregate claims L.

Distribution	Intensity ($t \in [\frac{1}{4}, 2.5]$)	Kurtosis range
Gen. Extreme Value (GEV)	λ_t^1	[0.6346, 6.0943]
	λ_t^2	[0.4943, 4.8447]
	λ_t^3	[0.6208, 6.2084]
Lognormal	λ_t^1	[0.3398, 3.2627]
	λ_t^2	[0.2646, 2.5937]
	λ_t^3	[0.3324, 3.3238]

5. Numerical analysis

In this section, we value the prices of CAT bond with face value $Z = 1$ USD and coupon $C \equiv 0.1$ USD for time $t = 0$ using the mixed approximation method. We will discuss various influences of the choice of the pricing model parameters. We consider the GEV distribution with parameters $\kappa = 0.35431$, $\sigma = 6.5307$ and $\mu = 7.7158$, and the lognormal distribution with parameters $\mu = 2.3179$, $\sigma = 0.89666$. We also consider homogeneous Poisson process with an annual intensity $\lambda_s^3 = 30$(HPP) and nonhomogeneous Poisson process with the intensity $\lambda_s^1 = 27.4746 + 2.1304\sin^2(s-0.3185)+1.1938\exp\{\cos(\frac{2\pi s}{4.7938})\}$(NHPP1) and the intensity $\lambda_s^2 = 27.7986 + 2.2628 \cdot 2\pi \cdot \sin\{2\pi(s + 0.0247)\}$ (NHHP2). For comparison purposes, we also consider an annual continuously-compound discount rate and $r = \ln(1+r_0)$ is a constant.

First, we assume the maturity time $T \in [\frac{1}{4}, 2.5]$ years and threshold level $D \in [374, 4488]$ ten million USD (quarterly-3∗annual average loss). Furthermore, we assume the term structure parameters within the ranges be used regularly in previous literature. The initial short-term interest rate r_0 and the long-run interest rate θ are both assumed to be 6% per annum. The mean-reverting force κ is set to be 0.2 and the volatility of the interest rate σ is 10%. The market risk parameter λ is −0.01. p is set to be 0.5 when the aggregate claims L_t exceed the threshold level D (or , $\tau \le T$).

Under the GEV distribution and the lognormal distribution, $\kappa_{3X}^1 = 3.7382$ and $\kappa_{3X}^2 = 1.9320$ denote the skewness of the claim size X respectively. The kurtosis of the aggregate claims L in the time $t \in [\frac{1}{4}, 2.5]$ are shown in Table 6. This obviously satisfies the rule of thumb. However, the kurtosis also increases as the time increases, we cannot use a single approximation method to evaluate aggregate claims distribution. Therefore, we use the gamma-IG approximation and the IG approximation to evaluate the compound distribution.

Now, we calculate the zero-coupon and coupon CAT bond prices using the approximating solution. Figs. 4 and 5 illustrate the zero-coupon and coupon CAT bond prices with respect to the threshold level and time to maturity under the GEV, NHPP1 and stochastic interest rates assumptions. Note that the CAT bond prices decrease as the time to maturity increases, and increase as the threshold level increases.

Fig. 6 shows that the price difference between stochastic interest rates and constant interest rate under the assumptions of the GEV loss distribution and the NHPP1. We observe that constant

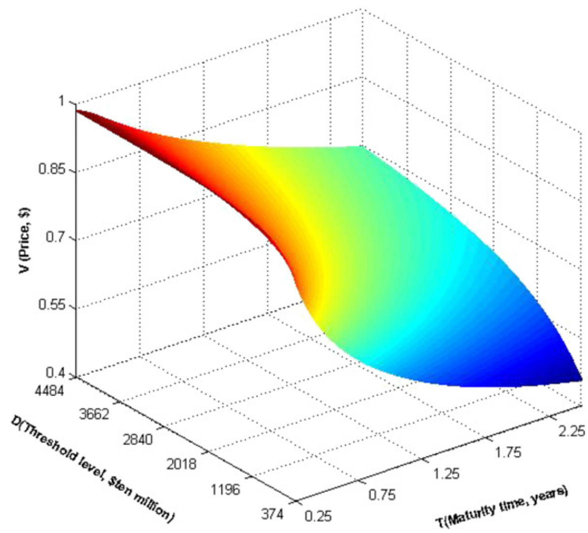

Fig. 4. The zero-coupon CAT bond prices under the GEV, the NHPP1 and stochastic interest rates assumptions.

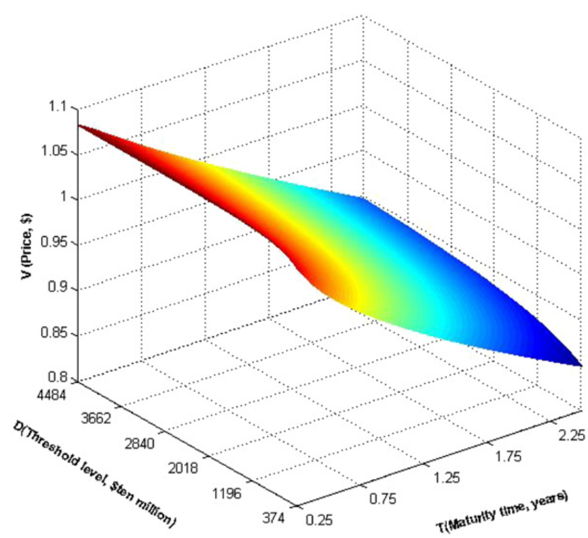

Fig. 5. The principal-protected CAT bond prices under the GEV, the NHPP1 and stochastic interest rates assumptions.

interest rate overestimates the bond prices change substantially from 0.25 to 2.5 years. The significant price differences indicate that the uncertainty of interest rates is an important factor and should be taken into account when pricing the CAT bonds.

In Fig. 7, we illustrate how the bond prices are affected by loss severity distribution. We can clearly observe that the difference between the zero-coupon CAT bond prices calculated in the GEV and the lognormal loss distribution conditions under the assumptions of the NHPP1 and stochastic interest rates. We also can observe that the differences of the bond prices vary from −3.43% to 7.97%. This difference is especially marked in the tail, so

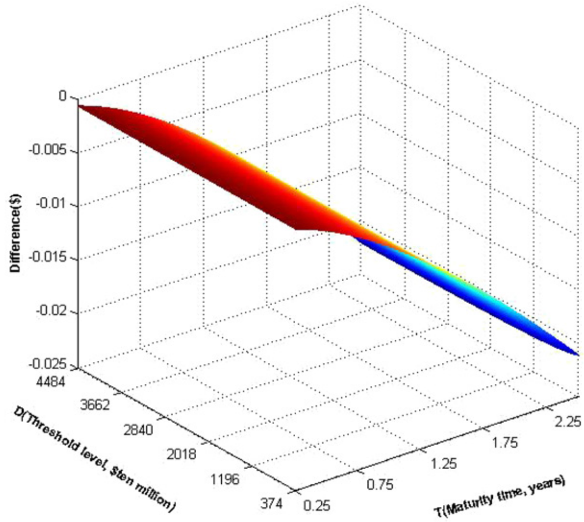

Fig. 6. The difference between zero-coupon CAT bond prices in stochastic interest rates and constant interest rate under the GEV and the NHPP1 assumptions.

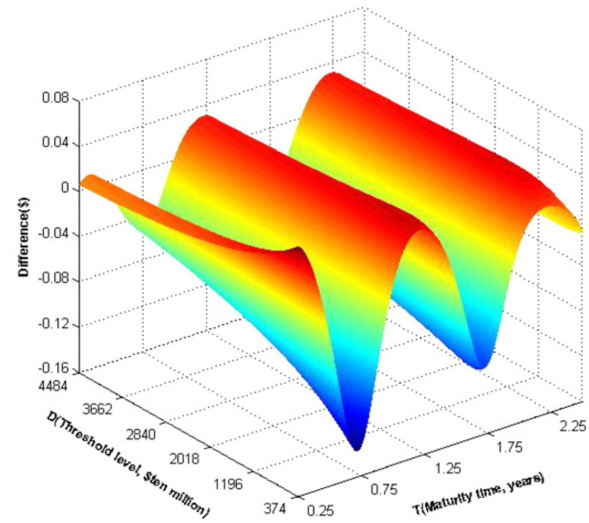

Fig. 8. The difference between zero-coupon CAT bond prices in the NHPP1 and the NHPP2 under the GEV and stochastic interest rates assumptions.

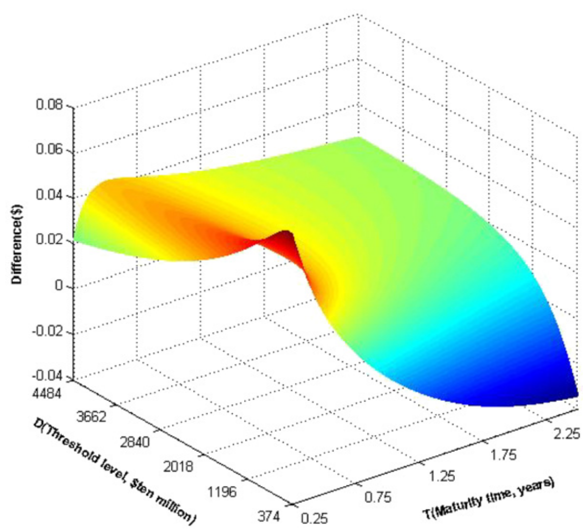

Fig. 7. The difference between zero-coupon CAT bond prices in the GEV and the lognormal distribution under the NHPP1 and stochastic interest rates assumptions.

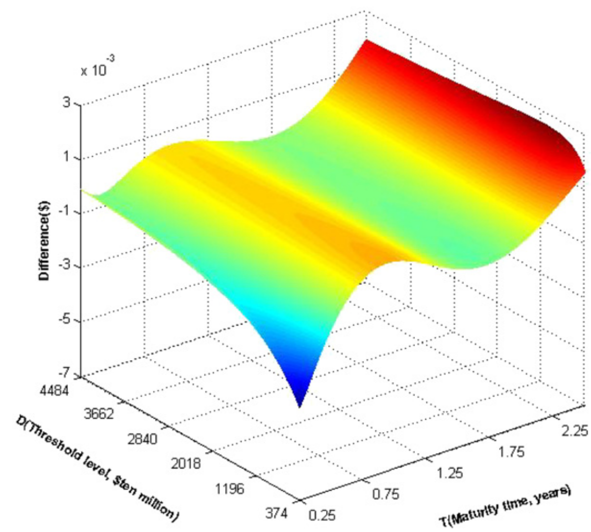

Fig. 9. The difference between zero-coupon CAT bond prices in the NHPP1 and the HPP under the GEV and stochastic interest rates assumptions.

the heavy-tailed distribution is much more suitable for catastrophe loss.

Figs. 8 and 9 show how the choice of the fitted claims arrival intensity influences the zero-coupon CAT bond prices. Fig. 8 illustrates the difference between the zero-coupon CAT bond prices calculated in the NHPP1 and NHPP2 conditions under the assumptions of the GEV loss distribution and stochastic interest rates. We clearly observe that the differences of the bond price significantly changes from −13.59% to 6.11%. Finally, Fig. 9 illustrates the differences between the zero-coupon CAT bond prices calculated in the NHPP1 and HPP conditions under the assumptions of the GEV loss distribution and stochastic interest rates. We can also observe that the differences of the bond prices vary from −0.36% to 0.25%.

6. Conclusions

This paper develops a simple contingent claim model to price catastrophe risk bonds under the risk-neutral pricing measure. Furthermore, we examine the calibration of the pricing model using PCS loss index and the annual number of natural catastrophic events in the US that occurred between 1985 and 2010.

Because the compound distribution is difficult to calculate, we adopt a mixed approximation method to approximate aggregate claims distribution. Based on the values of the skewness κ_{3l} of the claim size X and the kurtosis κ_{4L} aggregate claims L, we choose the gamma-IG or IG approximation method to get a more accurate approximation solution. Numerical experiments show that the mixed approximation method described in this paper is effective and feasible.

The numerical experiments show that interest rate uncertainty, loss severity, threshold level and claim arrival intensity have important implications for the CAT bonds pricing model. By comparing stochastic interest rates and constant interest rate, we observe that the constant interest derives up the bond prices substantially in the time $T \in [\frac{1}{4}, 2.5]$. As the threshold level increases, the bond price also significantly increases. However, as the time increases, the bond price flat decreases. This is because even though more coupons are received as the time increases, the probability of catastrophe risk also increases, i.e., partial coupons are offset by the catastrophic risk exposure. The numerical results further validate that the choices of the distribution and the intensity have a great impact on the bond prices.

Acknowledgments

The authors wish to extend our sincere appreciation to the managing editor Rob Kaas and two anonymous reviewers whose constructive suggestions helped to significantly improve the original version of the manuscript. We also wish to thank Gary and Tomasz from the Property Claim Services (PCS) Unit of the Insurance Service Office (ISO), for providing the data used in this study.

This work was supported by the National Science Fund for Distinguished Young Scholars of China under Grant No. 70825006 and the Program for Changjiang Scholars and Innovative Research Team in the University under Grant No. IRT0916 and the National Natural Science Innovation of China under Grant No. 71221001.

Appendix A

Proof. In order to derive formula (15), we first show some assumptions and main conclusions developed in the paper by Cox and Pedersen (2000).

Let $(\Omega, \mathcal{F}, \mathcal{P})$ define a probability space, where $\Omega := \Omega^1 \times \Omega^2$ is the set of states of the world, $\mathcal{F}$ is a σ-algebra of subsets of Ω, an increasing filtration $\mathcal{F}_k = \mathcal{F}_k^1 \times \mathcal{F}_k^2 \subset \mathcal{F}, k = 0, 1, \ldots, T$, and $\mathcal{P}(w) = \mathcal{P}_1(w^1)\mathcal{P}_2(w^2)$ is a probability measure on $\mathcal{F}$, $w = (w^1, w^2)$ with $w^1 \in \Omega^1$ as a generic state of the world describing the state of the financial market variables, $w^2 \in \Omega^2$ is a generic state of the world describing the state of the catastrophic risk variables. The financial market variables and the catastrophic risk variables are assumed to be modeled respectively on the filtered probability space $(\Omega^1, \mathcal{F}^1, \mathcal{P}_1)$ and $(\Omega^2, \mathcal{F}^2, \mathcal{P}_2)$.

In order to discuss random variables in their full model that depend only on financial variables or catastrophic risk variables, they define two new filtrations: $\mathcal{A}_t^1 := \mathcal{F}_t^1 \times \{\Phi, \Omega\}$ for $t \in [0, T]$ and $\mathcal{A}_t^2 := \{\Phi, \Omega\} \times \mathcal{F}_t^2$ for $t \in [0, T]$. They prove that the two σ-algebra $\mathcal{A}_T^1$ and $\mathcal{A}_T^2$ are independent under the probability measure $\mathcal{P}$:

$$\mathcal{P}(\alpha_1 \cap \alpha_2) = \mathcal{P}(\alpha_1)\mathcal{P}(\alpha_2)$$

where $\alpha_1 \in \mathcal{A}_T^1, \alpha_2 \in \mathcal{A}_T^2$ and $\alpha_1 = A_1 \times \Omega^2$ for some $A_1 \in \mathcal{F}_T^1$ and $\alpha_2 = \Omega^1 \times A_2$ for some $A_2 \in \mathcal{F}_T^2$.

A random variable X on $(\Omega, \mathcal{F}, \mathcal{P})$ is said to depend only on financial risk variables if it is measurable with respect to $\mathcal{F}_T^1$. Alternatively, a random variable X on $(\Omega, \mathcal{F}, \mathcal{P})$ is said to depend only on catastrophic risk variables if it is measurable with respect to $\mathcal{F}_T^2$.

A stochastic process Y is said to evolve through dependence only on financial risk variables if Y is adapted to $\mathcal{F}_t^1$. Alternatively, a stochastic process Y is said to evolve through dependence only on financial risk variables if Y is adapted to $\mathcal{F}_t^2$.

They also prove that under the assumption that aggregate consumption depends only on the financial variables, for any random variable X that depends only on catastrophic risk variables, $E^Q(X) = E^P(X)$, that is, the aggregate loss process still retains their original distributional distributions after changing from the physical probability measure P to the risk-neutral measure Q. And σ-algebra $\mathcal{A}_T^1$ and $\mathcal{A}_T^2$ are independent under Q (For a detailed proof, please refer to Cox and Pedersen, 2000).

From their conclusions, we can obtain that under the risk-neutral pricing measure Q those events that depend only on financial variables are independent of those events that depend on catastrophic risk variables.

Therefore, we can derive that the price of a CAT bond paying Z at maturity at time t is

$$\begin{aligned}
V_t &= E^Q\left(Ze^{-\int_t^T r_s ds}P_{\text{CAT}}(T)|\mathcal{F}_t\right) \\
&= E^Q\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right)E^Q(ZP_{\text{CAT}}(T)|\mathcal{F}_t) \\
&= B_{\text{CIR}}(t,T)E^P(ZP_{\text{CAT}}(T)|\mathcal{F}_t) \\
&= B_{\text{CIR}}(t,T)E^P(Z1\{L_T \le D\} + pZ1\{L_T > D\}|\mathcal{F}_t) \\
&= B_{\text{CIR}}(t,T)(ZP(L_T \le D) + pZP(L_T > D)) \\
&= B_{\text{CIR}}(t,T)Z(F(D) + p \times (1 - F(D))) \\
&= B_{\text{CIR}}(t,T)Z\left[p + (1-p)\sum_{n=0}^{\infty} e^{-\lambda_S T}\frac{(\lambda_S T)^n}{n!}F^n(D)\right] \qquad (44)
\end{aligned}$$

where

$$F^n(D) = \Pr(X_1 + X_2 + \cdots + X_n \le D) \qquad (45)$$

denote the n-th convolution of F, and the spot interest rate follows the square-root process of Cox et al. (1985), we have

$$\begin{aligned}
&B_{\text{CIR}}(t,T) = A(t,T)e^{-B(t,T)r}, \\
&A(t,T) = \left[\frac{2\gamma e^{(k^*+\gamma)\frac{T-t}{2}}}{(k^*+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2k^*\theta^*}{\sigma^2}}, \\
&B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+k^*)(e^{\gamma(T-t)}-1)+2\gamma}, \\
&\gamma = \sqrt{(k^*)^2 + 2\sigma^2}. \quad \square
\end{aligned} \qquad (46)$$

Appendix B

Proof. It is easy to verify that the price of a coupon CAT bond with coupon payments C_t to the threshold event τ' is

$$\begin{aligned}
V_t &= E^Q\left(Ze^{-\int_t^T r_s ds}P'_{\text{CAT}}(T)|\mathcal{F}_t\right) \\
&= E^Q\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right)E^Q(P'_{\text{CAT}}(T)|\mathcal{F}_t) \\
&= B_{\text{CIR}}(t,T)E^P(P'_{\text{CAT}}(T)|\mathcal{F}_t) \\
&= B_{\text{CIR}}(t,T)((Z+C)P(L_T \le D) + ZP(L_T > D)) \\
&= B_{\text{CIR}}(t,T)((Z+C)P(L_T \le D) + Z(1 - P(L_T \le D))) \\
&= B_{\text{CIR}}(t,T)(Z + CP(L_T \le D)) \\
&= B_{\text{CIR}}(t,T)\left[Z + C\sum_{n=0}^{\infty} e^{-\lambda_S T}\frac{(\lambda_S T)^n}{n!}F^n(D)\right] \qquad (47)
\end{aligned}$$

where

$$F^n(D) = \Pr(X_1 + X_2 + \cdots + X_n \le D) \qquad (48)$$

denote the n-th convolution of F, and the spot interest rate follows the square-root process of Cox et al. (1985), we have

$$\begin{aligned}
&B_{\text{CIR}}(t,T) = A(t,T)e^{-B(t,T)r}, \\
&A(t,T) = \left[\frac{2\gamma e^{(k^*+\gamma)\frac{T-t}{2}}}{(k^*+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2k^*\theta^*}{\sigma^2}}, \\
&B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+k^*)(e^{\gamma(T-t)}-1)+2\gamma}, \\
&\gamma = \sqrt{(k^*)^2 + 2\sigma^2}. \quad \square
\end{aligned} \qquad (49)$$

References

Aase, K., 1999. An equilibrium model of catastrophe insurance futures and spreads. Geneva Papers on Risk and Insurance Theory 24 (1), 69–96.

Akaike, H., 1973. Information theory and an extension of the maximum likelihood principle. In: Petrov, B.N., Csáki, F. (Eds.), Second International Symposium on Information Theory. Akadémiai Kiadó, Budapest, pp. 267–281.

Anderson, T.W., Darling, D.A., 1952. Asymptotic theory of certain ''goodness of fit'' criteria based on stochastic processes. The Annals of Mathematical Statistics 23 (2), 193–212.

Baryshnikov, Yu., Mayo, A., Taylor, D.R., 2001. Pricing of CAT Bonds. Working Paper.

Burnecki, K., 2005. Pricing catastrophe bonds in a compound non-homogeneous Poisson model with left-truncated loss distributions. Presentation, Wroclaw University of Technology.

Burnecki, K., Janczura, J., Weron, R., 2011a. Building loss models. In: Cizek, P., Härdle, W., Weron, R. (Eds.), Statistical Tools for Finance and Insurance, second ed. Springer, Berlin, pp. 293–328.

Burnecki, K., Kukla, G., 2003. Pricing of zero-coupon and coupon CAT bonds. Applicationes Mathematicae 30, 315–324.

Burnecki, K., Kukla, G., Taylor, D., 2011b. Pricing catastrophe bond. In: Cizek, P., Härdle, W., Weron, R. (Eds.), Statistical Tools for Finance and Insurance, second ed. Springer, Berlin, pp. 371–391.

Chaubey, Y.P., Garrido, J., Trudeau, S., 1998. On the computation of aggregate claims distributions: some new approximations. Insurance: Mathematics and Economics 23 (3), 215–230.

Cox, S.H., Fairchild, J.R., Pedersen, H.W., 2004. Valuation of structured risk management products. Insurance: Mathematics and Economics 34 (2), 259–272.

Cox, J., Ingersoll, J., Ross, S., 1985. A theory of the term structure of interest rates. Econometrica 53 (2), 385–407.

Cox, S.H., Pedersen, H.W., 2000. Catastrophe risk bonds. North American Actuarial Journal 4 (4), 56–82.

Cummins, J., 2008. CAT bonds and other risk-linked securities: state of the market and recent developments. Risk Management and Insurance Review 11 (1), 23–47.

Cummins, J., Geman, H., 1995. Pricing catastrophe insurance futures and call spreads: an arbitrage approach. Journal of Fixed Income 4, 46–57.

Cummins, J., Lalonde, D., Phillips, R., 2004. The basis risk of catastrophe-loss index securities. Journal of Financial Economics 71, 77–111.

Damnjanovic, I., Aslan, Z., Mander, J., 2010. Market-implied spread for earthquake CAT bonds: financial implications of engineering decisions. Risk Analysis 30 (12), 1753–1770.

D'Agostino, R.B., Stephens, M.A., 1986. Goodness-of-fit Techniques. Marcel Dekker, New York.

D'Arcy, S.P., France, V.G., 1992. Catastrophe futures: a better hedge for insurers. Journal of Risk and Insurance 59 (4), 575–600.

Duan, J.-C., Yu, M.-T., 2005. Fair insurance guaranty premia in the presence of risk-based capital regulations, stochastic interest rate and catastrophe risk. Journal of Banking & Finance 29 (10), 2435–2454.

Duffie, D., 2001. Dynamic Asset Pricing Theory, third ed. Princeton University Press, Princeton, N.J.

Egami, M., Young, V., 2008. Indifference prices of structured catastrophe(CAT) bonds. Insurance: Mathematics and Economics 42 (2), 771–778.

Fernández-Durán, J.J., Gergorio-Domínguez, M.M., 2005. Actuarial valuation of CAT bonds for natural disasters in Mexico. E1 Trimestre Económico 288 (4), 877–912 (in Spanish: valuación actuarial de bonos catastróficos para desastres naturales en México).

Härdle, W.K., Cabrera, B.L., 2010. Calibrating CAT bonds for Mexican earthquakes. Journal of Risk and Insurance 77 (3), 625–650.

Harrington, S., Niehaus, G., 1999. Basis risk with PCS catastrophe insurance derivative contracts. Journal of Risk and Insurance 66 (1), 49–82.

Hördahl, P., Vestin, D., 2005. Interpreting implied risk-neutral densities: the role of risk premia. Review of Finance 9, 97–137.

Hurvich, C.M., Tsai, C.-L., 1989. Regression and time series model selection in small samples. Biometrika 76 (2), 297–307.

Jaimungal, S., Wang, T., 2006. Catastrophe options with stochastic interest rates and compound Poisson losses. Insurance: Mathematics and Economics 38 (3), 469–483.

Jarrow, R., 2010. A simple robust model for CAT bond valuation. Finance Research Letters 7 (2), 72–79.

Klein, C., 2010. Reinsurance Market Review 2010. Guy Garpenter & Company. inc.

Klugman, S.A., Panjer, H.H, Willmot, G.E., 2008. Loss Models: From Data to Decisions, third ed.. Wiley, New York.

Laster, D. S., 2001. Capital market innovation in the insurance industry. Sigma No. 3, Zurich, Switzerland.

Law, A.M., Kelton, W.D., 2000. Simulation Modeling and Analysis, third ed. McGraw-Hill, New York.

Lee, J.-P., Yu, M.-T., 2002. Pricing default-risky CAT bonds with moral hazard and basis risk. Journal of Risk and Insurance 69 (1), 25–44.

Lee, J.-P., Yu, M.-T., 2007. Valuation of catastrophe reinsurance with catastrophe bonds. Insurance: Mathematics and Economics 41 (2), 264–278.

Legates, D.R., McCabe, G.J., 1999. Evaluating the use of ''goodness-of-fit'' measures in hydrologic and hydroclimatic model validation. Water Resources Research 35 (1), 233–241.

Levi, C., Partrat, C., 1991. Statistical analysis of natural events in the United States. Astin Bulletin 21 (2), 253–276.

Lin, S., Wang, T., 2009. Pricing perpetual American catastrophe put options: a penalty function approach. Insurance: Mathematics and Economics 44 (2), 287–295.

Loubergé, H., Kellezi, E., Gilli, M., 1999. Using catastrophe-linked securities to diversify insurance risk: a financial analysis of CAT bonds. Journal of Insurance Issues 22 (2), 126–146.

Merton, R.C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3 (1–2), 125–144.

Nash, J.E., Sutcliffe, J.V., 1970. River flow forecasting through conceptual models part I—a discussion of principles. Journal of Hydrology 10 (3), 282–290.

ØKsendal, B., 2003. Stochastic Differential Equations, sixth ed. Springer-Verlag, Berlin.

Panjer, H.H., Willmot, G.E., 1992. Insurance Risk Models. Society of Actuaries, Schaumburg, Ill.

Reijnen, R., Albers, W., Kallenberg, W.C.M., 2005. Approximations for stop-loss reinsurance premiums. Insurance: Mathematics and Economics 36 (3), 237–250.

Schwarz, G., 1978. Estimating the dimension of a model. The Annals of Statistics 6, 461–464.

Seal, H.L., 1977. Approximation to risk theory's $F(x, t)$ by means of the gamma distribution. Astin Bulletin 9, 281–289.

Shirakawa, H., 2002. Squared Bessel processes and their applications to the square root interest rate model. Asia-Pacific Financial Markets 9 (3–4), 169–190.

Stephens, M.A., 1974. EDF statistics for goodness of fit and some comparisons. Journal of the American Statistical Association 69 (347), 730–737.

Sugiura, N., 1978. Further analysts of the data by Akaike's information criterion and the finite corrections. Communications in Statistics—Theory and Methods 7 (1), 13–26.

Tse, Y.-K., 2009. Nonlife Actuarial Models: Theory, Models and Evaluation. Cambridge University Press, Cambridge.

Unger, A.J.A., 2010. Pricing index-based catastrophe bonds: part 1: formulation and discretization issues using a numerical PDE approach. Computers & Geosciences 36 (2), 139–149.

Vasicek, O.A., 1977. An equilibrium characterization of term structure. Journal of Financial Economics 5 (2), 177–188.

Vaugirard, V., 2003a. Valuing catastrophe bonds by Monte Carlo simulations. Applied Mathematical Finance 10 (1), 75–90.

Vaugirard, V., 2003b. Pricing catastrophe bonds by an arbitrage approach. The Quarterly Review of Economics and Finance 43 (1), 119–132.

Vere-Jones, D., 1982. On the estimation of frequency in point-process data. Journal of Applied Probability 19, 383–394.

Wasserman, L., 2000. Bayesian model selection and model averaging. Journal of Mathematical Psychology 44 (1), 92–107.

Wen, Fenghua, Yang, Xiaoguang, 2009. Skewness of return distribution and coefficient of risk premium. Journal of Systems Science and Complexity 22, 360–371.

Willmott, C.J., 1981. On the validation of models. Physical Geography 2 (2), 184–194.

Willmott, C.J., 1984. On the evaluation of model performance in physical geography. In: Gaile, G.L., Willmott, C.J. (Eds.), Spatial Statistics and Models. D. Reidel, Norwell, Mass, pp. 443–460.