

MATHEMATICS

TRIGONOMETRY

High School Notes Series

Trigonometry is the study of the relationships between the angles and sides of triangles. It uses functions like sine, cosine, and tangent to analyze and model periodic patterns, waves, and various spatial relationships.

Fundamental Concepts

Types of Angles

Positive Angle An angle measured counterclockwise from the positive x-axis.

Negative Angle An angle measured clockwise from the positive x-axis.

Zero Angle An angle of 0° , representing no rotation, with its terminal arm on the positive x-axis at $(1,0)$.

Quadrantal Angle An angle where the terminal side lies on an axis (e.g., $0^\circ, 90^\circ, 180^\circ, 270^\circ$).

Coterminal Angles Angles that share the same terminal arm. Their relationship is given by $\beta = \alpha + 360k^\circ$, where k is an integer.

Key Vocabulary

Initial Arm The arm of an angle in standard position that lies on the positive x-axis.

Terminal Arm The arm that rotates from the initial arm to form the angle.

Principal Angle (θ) The counterclockwise angle from the initial arm to the terminal arm, where $0 \leq \theta < 360^\circ$.

Related Acute Angle (β) The acute angle formed between the terminal arm and the x-axis.

Identity An equation that is true for all possible values of its variables.

Ambiguous Case A situation in the Sine Law where two sides and a non-contained angle can form zero, one, or two possible triangles.

Trigonometric Ratios

Right-Triangle Trigonometry (SOH-CAH-TOA)

For a right triangle with hypotenuse h , opposite side o , and adjacent side a :

$$\begin{array}{ll} \sin \theta = \frac{o}{h} & \csc \theta = \frac{1}{\sin \theta} = \frac{h}{o} \\ \cos \theta = \frac{a}{h} & \sec \theta = \frac{1}{\cos \theta} = \frac{h}{a} \\ \tan \theta = \frac{o}{a} & \cot \theta = \frac{1}{\tan \theta} = \frac{a}{o} \end{array}$$

Note: $\csc x$ is the reciprocal of $\sin x$, not its inverse function, $\sin^{-1} x$ (or arcsin).

Trigonometry in the Cartesian Plane

For a point (x, y) at a distance $r = \sqrt{x^2 + y^2}$ from the origin:

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

Radians

To convert from degrees to radians, use the formula:

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180}$$

Signs of Ratios by Quadrant (CAST Rule)

The CAST rule helps us remember which trigonometric functions are positive in each quadrant.

Quadrant II (S) Sine is positive	Quadrant I (A) All are positive
Quadrant III (T) Tangent is positive	Quadrant IV (C) Cosine is positive

Special Angles

The exact trigonometric values for special angles are essential for quick calculations.

Degrees	0°	30°	45°	60°	90°	180°	270°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.	0	undef.

Trigonometric Identities

Pythagorean Identities

$$\begin{aligned}\sin^2 \theta + \cos^2 \theta &= 1 \\ 1 + \tan^2 \theta &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta\end{aligned}$$

Angle Addition and Subtraction Formulas

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\ \tan(\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}\end{aligned}$$

Double Angle Formulas

$$\begin{aligned}\sin(2\alpha) &= 2 \sin \alpha \cos \alpha \\ \cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha \\ \tan(2\alpha) &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha}\end{aligned}$$

Triangle Laws

Sine Law

For any triangle with sides a, b, c and opposite angles A, B, C :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Definition 1 (Ambiguous Case of Sine Law). The ambiguous case occurs in a Side-Side-Angle (SSA) scenario. Given sides a, b and angle A (opposite side a), the number of possible triangles depends on the relationship between a, b , and the height $h = b \sin A$.

- **If angle A is acute ($A < 90^\circ$):**
 - If $a < h$: No triangle exists.
 - If $a = h$: One right-angled triangle exists.
 - If $h < a < b$: Two distinct triangles exist.
 - If $a \geq b$: One triangle exists.
- **If angle A is obtuse ($A \geq 90^\circ$):**
 - If $a \leq b$: No triangle exists.
 - If $a > b$: One triangle exists.

Cosine Law

For any triangle, the Cosine Law relates the lengths of the sides to the cosine of one of its angles.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

It can be rearranged to find an angle if all three side lengths are known:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Sinusoidal Functions

The general forms for sinusoidal functions are $f(x) = a \sin(k(x - \phi)) + c$ and $f(x) = a \cos(k(x - \phi)) + c$. The parameters a , k , ϕ , c transform the basic sine or cosine graph.

Amplitude (a) The value $|a|$ is the amplitude, representing half the vertical distance between the maximum and minimum values. If $a < 0$, the function is reflected across its horizontal axis.

Period (k) The parameter k determines the period (length of one cycle), calculated as $P = \frac{2\pi}{|k|}$ (radians) or $P = \frac{360^\circ}{|k|}$ (degrees).

Phase Shift (ϕ) This is the horizontal shift. A positive ϕ shifts the graph right, and a negative ϕ shifts it left.

Vertical Shift (c) This parameter shifts the graph vertically and sets the axis of the curve at the line $y = c$.

The **Range** of the function is determined by the vertical shift and amplitude, and is given by the interval $[c - |a|, c + |a|]$.

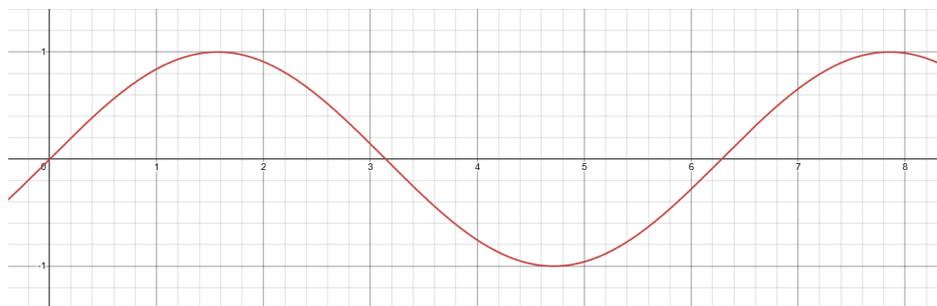


Figure 1: Graph of a sinusoidal function showing amplitude and period.