# Vieta's Formulas

Euclid Competition Prep

*Includes: core formulas, symmetric expressions, Newton's identities intro, problem set*

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## 1 The Core Idea

Vieta's formulas connect the **coefficients** of a polynomial to the **sums and products** of its roots — without needing to find the roots themselves. This is what makes them powerful for contest math.

If you know the roots are $r$ and $s$, you can expand:

$$(x-r)(x-s) = x^2 - (r+s)x + rs$$

Matching coefficients against $ax^2 + bx + c$ gives you Vieta's directly.

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## 2 Quadratic: $ax^2 + bx + c = 0$

Let $r, s$ be the two roots. Then:

$$\boxed{r+s = -\frac{b}{a}} \qquad \boxed{rs = \frac{c}{a}}$$

**Derivation:** Factor as $a(x-r)(x-s) = a[x^2 - (r+s)x + rs]$. Match with $ax^2 + bx + c$:

$$-a(r+s) = b \implies r+s = -\frac{b}{a}, \qquad a(rs) = c \implies rs = \frac{c}{a}$$

**Key Insight**

The sign on the sum is negative. $r+s = -b/a$, not $+b/a$. This is the most common error. Memorise it as: sum of roots $= -$(middle coefficient)/(leading coefficient).

### Example

For $2x^2 - 7x + 3 = 0$: $\quad r+s = \frac{7}{2}, \quad rs = \frac{3}{2}$.

Check: roots are $x = 3$ and $x = \frac{1}{2}$. Indeed $3 + \frac{1}{2} = \frac{7}{2}$ and $3 \cdot \frac{1}{2} = \frac{3}{2}$. ✓

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# 3 Cubic: $ax^3 + bx^2 + cx + d = 0$

Let $r, s, t$ be the three roots. Then:

$$\boxed{r + s + t = -\frac{b}{a}}$$

$$\boxed{rs + rt + st = \frac{c}{a}}$$

$$\boxed{rst = -\frac{d}{a}}$$

The pattern: alternating signs, elementary symmetric polynomials of the roots equal ratios of coefficients.

### General Pattern (degree $n$)

For $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with roots $r_1, \ldots, r_n$:

$$\sum_i r_i = -\frac{a_{n-1}}{a_n}, \qquad \sum_{i<j} r_i r_j = \frac{a_{n-2}}{a_n}, \qquad \prod_i r_i = (-1)^n \frac{a_0}{a_n}$$

You will not need beyond cubic on Euclid. Focus on quadratic cold, cubic comfortably.

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# 4 Symmetric Expressions — The Real Skill

Vieta's gives you $r + s$ and $rs$. Most Euclid problems ask for something derived from those. You need to express the target in terms of $r + s$ and $rs$.

### Essential Identities

$$r^2 + s^2 = (r+s)^2 - 2rs \tag{1}$$

$$r^3 + s^3 = (r+s)^3 - 3rs(r+s) \tag{2}$$

$$(r-s)^2 = (r+s)^2 - 4rs \tag{3}$$

$$r^2 + s^2 + t^2 = (r+s+t)^2 - 2(rs+rt+st) \tag{4}$$

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} \tag{5}$$

**Key Insight**

Every symmetric expression in the roots can be reduced to elementary symmetric polynomials (i.e. the Vieta quantities). The identities above cover 90% of Euclid cases. Memorise (1), (3), and (5) — they appear constantly.

### Worked Example

**Question:** $r$ and $s$ are roots of $x^2 - 5x + 3 = 0$. Find $r^2 + s^2$.

By Vieta's: $r + s = 5$, $rs = 3$.

$$r^2 + s^2 = (r+s)^2 - 2rs = 25 - 6 = 19$$

**Question:** Same equation. Find $\frac{1}{r} + \frac{1}{s}$.

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = \frac{5}{3}$$

**Question:** Same equation. Find $r^3 + s^3$.

$$r^3 + s^3 = (r+s)^3 - 3rs(r+s) = 125 - 45 = 80$$

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## 5 Constructing Polynomials from Roots

The reverse direction: given desired root properties, build the polynomial.

If you want a quadratic with roots $r$ and $s$:

$$x^2 - (r+s)x + rs = 0$$

**Example:** Find a quadratic whose roots are each 3 more than the roots of $x^2 - 4x + 1 = 0$.

Original roots: $r + s = 4$, $rs = 1$. New roots: $r + 3$ and $s + 3$.

$$\text{New sum} = (r+3) + (s+3) = r + s + 6 = 10$$

$$\text{New product} = (r+3)(s+3) = rs + 3(r+s) + 9 = 1 + 12 + 9 = 22$$

$$\text{New equation: } x^2 - 10x + 22 = 0$$

This type — "roots transformed by some operation" — appears on Euclid Q9 regularly.

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# 6 Discriminant Connection

For $ax^2 + bx + c = 0$:

$$(r - s)^2 = (r + s)^2 - 4rs = \frac{b^2 - 4ac}{a^2}$$

So the discriminant $\Delta = b^2 - 4ac$ tells you:

- $\Delta > 0$: two distinct real roots
- $\Delta = 0$: repeated root ($r = s$)
- $\Delta < 0$: complex roots (sum and product still real, still satisfy Vieta's)

**Warning**

Vieta's holds even when roots are complex. Do not assume roots are real unless the problem states it or $\Delta \geq 0$.

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# 7 Problem Set

Problems are ordered by difficulty. Do not look at the solution until you have a full answer or are genuinely stuck.

## Tier 1 — Direct Application

**P1.** The roots of $x^2 - 7x + 11 = 0$ are $r$ and $s$. Find:

(a) $r + s$ and $rs$
(b) $r^2 + s^2$
(c) $r^3 + s^3$
(d) $\frac{r}{s} + \frac{s}{r}$

**P2.** Find a quadratic with integer coefficients whose roots are $r + 2$ and $s + 2$, where $r, s$ are roots of $x^2 + 3x - 5 = 0$.

**P3.** The roots of $3x^2 - 12x + 7 = 0$ are $r$ and $s$. Without solving the equation, find $(r + 1)(s + 1)$.

## Tier 2 — One Step Further

**P4.** The roots of $x^2 - px + q = 0$ are $r$ and $s$. The roots of $x^2 - p'x + q' = 0$ are $r^2$ and $s^2$. Express $p'$ and $q'$ in terms of $p$ and $q$.

**P5.** For what value(s) of $k$ does $x^2 + kx + (k+3) = 0$ have a repeated root? Find that root.

**P6.** $r$ and $s$ are roots of $x^2 - bx + c = 0$. Given that $r^2 + s^2 = 11$ and $r^2s^2 = 16$, find all possible values of $b$ and $c$.

## Tier 3 — Euclid Level

**P7.** The three roots of $x^3 - 6x^2 + 11x - 6 = 0$ are $r, s, t$. Find $r^2 + s^2 + t^2$ and $r^3 + s^3 + t^3$.

**P8.** $r$ and $s$ are the roots of $x^2 - x - 1 = 0$. Prove that $r^n + s^n$ is always an integer for all positive integers $n$.
*Hint: find the recurrence that $r^n + s^n$ satisfies.*

**P9.** (Euclid-style) Let $r$ and $s$ be the roots of $x^2 - (k+1)x + k = 0$.

(a) Show that for any real $k$, the roots are always real.
(b) Find all values of $k$ for which $\frac{1}{r^2} + \frac{1}{s^2} = \frac{5}{4}$.

**P10.** (Hard) $r, s, t$ are roots of $x^3 + px + q = 0$ (note: no $x^2$ term).

(a) Write down $r + s + t$, $rs + rt + st$, and $rst$ in terms of $p$ and $q$.
(b) Find $r^2 + s^2 + t^2$ in terms of $p$.
(c) Find $r^3 + s^3 + t^3$ in terms of $p$ and $q$.

## 8 Solutions

**P1.**

(a) $r + s = 7$, $rs = 11$
(b) $r^2 + s^2 = (r + s)^2 - 2rs = 49 - 22 = 27$
(c) $r^3 + s^3 = (r + s)^3 - 3rs(r + s) = 343 - 231 = 112$
(d) $\frac{r}{s} + \frac{s}{r} = \frac{r^2 + s^2}{rs} = \frac{27}{11}$

**P2.** Original: $r + s = -3$, $rs = -5$. New roots $r + 2$, $s + 2$:

$$\text{sum} = r + s + 4 = 1, \quad \text{product} = rs + 2(r + s) + 4 = -5 - 6 + 4 = -7$$

Equation: $x^2 - x - 7 = 0$.

**P3.** $r + s = 4$, $rs = 7/3$.

$$(r + 1)(s + 1) = rs + (r + s) + 1 = \frac{7}{3} + 4 + 1 = \frac{22}{3}$$

**P4.** $p' = r^2 + s^2 = p^2 - 2q$. $\quad q' = r^2s^2 = (rs)^2 = q^2$.

**P5.** Repeated root $\Leftrightarrow \Delta = 0$: $k^2 - 4(k + 3) = 0 \Rightarrow k^2 - 4k - 12 = 0 \Rightarrow (k - 6)(k + 2) = 0$.
$k = 6$: root $= -k/2 = -3$. $\quad k = -2$: root $= 1$.

**P6.** $r^2 + s^2 = b^2 - 2c = 11$ and $r^2s^2 = c^2 = 16 \Rightarrow c = \pm 4$.

- $c = 4$: $b^2 = 19$, $b = \pm\sqrt{19}$
- $c = -4$: $b^2 = 3$, $b = \pm\sqrt{3}$

**P7.** $r + s + t = 6$, $rs + rt + st = 11$, $rst = 6$.

$$r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + rt + st) = 36 - 22 = 14$$

$$r^3 + s^3 + t^3 = (r + s + t)^3 - 3(r + s + t)(rs + rt + st) + 3rst = 216 - 198 + 18 = 36$$

**P8.** Let $f(n) = r^n + s^n$. From $x^2 = x + 1$: $r^n = r^{n-1} + r^{n-2}$ and same for $s$. So:

$$f(n) = f(n - 1) + f(n - 2)$$

Base cases: $f(1) = r + s = 1$ (integer), $f(2) = r^2 + s^2 = (r + s)^2 - 2rs = 1 + 2 = 3$ (integer). Since the recurrence is $f(n) = f(n - 1) + f(n - 2)$ and both base cases are integers, all $f(n)$ are integers by induction. ■

**P9.**

(a) $\Delta = (k + 1)^2 - 4k = k^2 + 2k + 1 - 4k = k^2 - 2k + 1 = (k - 1)^2 \geq 0$ for all real $k$. ✓

(b) By Vieta's: $r + s = k + 1$, $rs = k$.

$$\frac{1}{r^2} + \frac{1}{s^2} = \frac{r^2 + s^2}{(rs)^2} = \frac{(k+1)^2 - 2k}{k^2} = \frac{k^2+1}{k^2} = \frac{5}{4}$$

$$4(k^2 + 1) = 5k^2 \implies k^2 = 4 \implies k = \pm 2$$

Check $k = 0$ is excluded (division by zero). Both $k = 2$ and $k = -2$ are valid.

**P10.**

(a) No $x^2$ term means $b = 0$ in the general cubic. So $r + s + t = 0$, $\quad rs + rt + st = p$, $rst = -q$.

(b) $r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + rt + st) = 0 - 2p = -2p$.

(c) Use the identity $r^3 + s^3 + t^3 - 3rst = (r + s + t)(r^2 + s^2 + t^2 - rs - rt - st)$. Since $r + s + t = 0$, the right side is 0. So $r^3 + s^3 + t^3 = 3rst = -3q$.

*P8 and P9 are the Euclid-relevant template. If Q9 involves roots, it will look like one of these.*