Trigonometry

Pre-Calculus Notes

Covers: ratios, unit circle, identities, sine/cosine law, sinusoidal functions

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1 Key Vocabulary

- **Positive angle**: measured counterclockwise from the positive x-axis.
- **Negative angle**: measured clockwise from the positive x-axis.
- **Zero angle**: $0°$; terminal arm coincides with initial arm at $(1, 0)$.
- **Quadrantal angle**: terminal arm lies on an axis $(0°, 90°, 180°, 270°)$.
- **Initial arm**: fixed on the positive x-axis.
- **Terminal arm**: rotates from the initial arm; determines the angle.
- **Principal angle** θ: angle measured counterclockwise from initial to terminal arm.
- **Related acute angle** β: acute angle between terminal arm and the x-axis.
- **Identity**: an equation true for all values of the variable within stated restrictions.
- **Bearing**: clockwise angle from magnetic North to the object.
- **Ambiguous case**: when two sides and a non-included angle are given, more than one triangle may exist.

Angles sharing the same terminal arm form the set:

$$S = \{\beta \mid \beta = \alpha + 360\kappa,\ \kappa \in \mathbb{Z}\}$$

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2 Trigonometric Ratios

In a right triangle with opposite o, adjacent a, hypotenuse h (where $a^2 + o^2 = h^2$):

$$\sin\theta = \frac{o}{h}, \qquad \cos\theta = \frac{a}{h}, \qquad \tan\theta = \frac{o}{a}$$

Reciprocal ratios:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{h}{o}, \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{h}{a}, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

Warning

$\csc x \neq \sin^{-1} x$. The notation $\sin^{-1} x$ denotes the inverse function (arcsin), not the reciprocal. Calculators have no csc/sec/cot buttons — evaluate by computing $1 \div \sin\theta$ etc.

In the Cartesian Plane

For a point (x, y) at distance r from the origin, where $x^2 + y^2 = r^2$:

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}, \quad \csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}$$

On the **unit circle**, $r = 1$, so $\sin\theta = y$ and $\cos\theta = x$.

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3 Radians and Special Angles

Converting degrees to radians: $\text{rad} = \dfrac{\alpha}{180} \times \pi$

Special angles to memorise cold:

	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	N/A	0	N/A	0

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4 Signs by Quadrant (CAST Rule)

Quadrant I (0°–90°)

- $\sin\theta > 0$
- $\cos\theta > 0$
- $\tan\theta > 0$

Quadrant II (90°–180°)

- $\sin\theta > 0$
- $\cos\theta < 0$
- $\tan\theta < 0$

Quadrant III (180°–270°)

- $\sin\theta < 0$
- $\cos\theta < 0$
- $\tan\theta > 0$

Quadrant IV (270°–360°)

- $\sin\theta < 0$
- $\cos\theta > 0$
- $\tan\theta < 0$

Key Insight

CAST going counterclockwise from Q IV: **C**osine positive (Q IV), **A**ll positive (Q I), **S**ine positive (Q II), **T**angent positive (Q III).

Negative Angle Identities

$$\sin(-\theta) = -\sin\theta, \qquad \cos(-\theta) = \cos\theta, \qquad \tan(-\theta) = -\tan\theta$$

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5 Pythagorean Identities

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

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6 Sine and Cosine Laws

Sine law:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

Cosine law:

$$C = \sqrt{A^2 + B^2 - 2AB\cos c}$$

Ambiguous Case of the Sine Law

When angle a, and sides A and B are known, compute $h = B\sin a$.

If $\angle a$ is acute:

- $A < h$: no triangle.
- $A = h$: one right triangle.
- $A = B$ or $A > B$: one triangle.
- $h < A < B$: two triangles.

If $\angle a$ is obtuse:

- $A \leq B$: no triangle.
- $A > B$: one triangle.

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7 Compound Angle Identities

Sum and difference:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{1}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{2}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{4}$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{5}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{6}$$

Double angle (set $\alpha=\beta$):

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha, \qquad \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Sum-to-product identities:

$$\sin\alpha + \sin\beta = 2\sin\left(\tfrac{\alpha+\beta}{2}\right)\cos\left(\tfrac{\alpha-\beta}{2}\right) \tag{7}$$
$$\sin\alpha - \sin\beta = 2\cos\left(\tfrac{\alpha+\beta}{2}\right)\sin\left(\tfrac{\alpha-\beta}{2}\right) \tag{8}$$
$$\cos\alpha + \cos\beta = 2\cos\left(\tfrac{\alpha+\beta}{2}\right)\cos\left(\tfrac{\alpha-\beta}{2}\right) \tag{9}$$
$$\cos\alpha - \cos\beta = -2\sin\left(\tfrac{\alpha+\beta}{2}\right)\sin\left(\tfrac{\alpha-\beta}{2}\right) \tag{10}$$
$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \tag{11}$$
$$\tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \tag{12}$$

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8 Sinusoidal Functions

The general sinusoidal function:

$$\boxed{f(x) = a\cdot\sin\big(k(x-\phi)\big)+c} \qquad \text{or} \qquad \boxed{f(x) = a\cdot\cos\big(k(x-\phi)\big)+c}$$

a Amplitude.
$|a|>1$: vertical stretch by $|a|$. $0<|a|<1$: vertical compression. $a<0$: reflection across x-axis.

k	Period factor. $\lvert k \rvert > 1$: horizontal compression, period $= \frac{2\pi}{\lvert k \rvert}$ decreases. $0 < \lvert k \rvert < 1$: horizontal stretch, period increases. $k < 0$: reflection across y-axis.
ϕ	Phase shift. $\phi > 0$: shift right. $\phi < 0$: shift left.
c	Vertical shift. $c > 0$: shift up. $c < 0$: shift down.

Derived quantities:

- Period: $\dfrac{2\pi}{|k|}$
- Amplitude: $|a|$
- Range: $\{y \in \mathbb{R} \mid -|a| + c \leq y \leq |a| + c\}$
- Maximum (peak): $|a| + c$ Minimum (trough): $-|a| + c$
- Equation of axis: $y = c$

Base function $f(x) = \sin x$:

- Domain: $\{x \in \mathbb{R}\}$, Range: $[-1, 1]$, Period: 2π, Amplitude: 1
- Equivalent form: $\sin x = \cos(x - 90°)$

Key Insight

To find increasing/decreasing intervals for a transformed function: substitute $k(x - \phi)$ in place of x in the base intervals, then solve for x. Base increasing interval for sin: $\left[-\frac{\pi}{2} + 2\pi\kappa,\ \frac{\pi}{2} + 2\pi\kappa\right]$.