Exponential Functions

Pre-Calculus Notes

Covers: definitions, transformations, growth and decay, applications

1 Definitions and Key Vocabulary

All functions are relations where each value of the independent variable corresponds with exactly one value of the dependent variable.

Domain and Range of any exponential function $f(x) = b^x$:

Domain: $\{x \in \mathbb{R}\}$ Range: $\{y \in \mathbb{R} \mid y > 0\}$

- **Stretch**: a transformation that elongates a figure.
- **Compression**: a transformation that squeezes a figure.
- **Translation**: a transformation of n units in a specified direction.
- **Asymptote**: a line or curve that a figure approaches closely but never touches.

2 The Basic Exponential Function

The most basic exponential function is:

$$\boxed{f(x) = b^x}$$

Key properties:

- $f(x) \neq 0$ for any value of x, so a **horizontal asymptote** exists at $y = 0$.
- When $x = 0$, $y = 1$, so the **y-intercept is always 1** (for functions without a multiplier a).

Exponential growth or decay?

- $b > 1$: **exponential growth** (larger b means faster growth).
- $0 < b < 1$: **exponential decay** (smaller b means faster decay).

Key Insight

b must satisfy $b > 0$ and $b \neq 1$. A base of 1 gives the constant $f(x) = 1$. A negative base produces undefined outputs for non-integer exponents.

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3 Transformations

The general transformed exponential function is:

$$\boxed{f(x) = a \cdot b^{k(x-d)} + c}$$

a — $|a| > 1$: vertical stretch by factor $|a|$.
$0 < |a| < 1$: vertical compression by factor $|a|$.
$a < 0$: reflection across the x-axis (stretch/compression still applies).

b — The base. Must satisfy $b > 0$ and $b \neq 1$.

c — $c > 0$: translate c units **up**.
$c < 0$: translate $|c|$ units **down**.

d — $d > 0$: translate d units **right**.
$d < 0$: translate $|d|$ units **left**.

k — $|k| > 1$: horizontal compression by factor $\left|\frac{1}{k}\right|$.
$0 < |k| < 1$: horizontal stretch by factor $\left|\frac{1}{k}\right|$.
$k < 0$: reflection across the y-axis (stretch/compression still applies).

Warning

Only parameter c changes the asymptote. The horizontal asymptote shifts from $y = 0$ to $y = c$. Parameters a, k, and d do not affect it.

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4 Growth and Decay Applications

For applied problems, use $f(x) = a \cdot b^x$, where:

$f(x)$ — Final value.

a — Initial value.

b Growth/decay factor.
Growth: $b = 1 + r$ where r is the growth rate.
Decay: $b = 1 - r$ where r is the decay rate.

x Number of time periods.

Example. A population of 500 grows at 3% per year. After 10 years:

$$f(10) = 500 \times (1.03)^{10} \approx 672$$

Example. A car worth \$20,000 depreciates at 15% per year. After 5 years:

$$f(5) = 20{,}000 \times (0.85)^{5} \approx \$8{,}874$$

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5 Inverse: Logarithmic Functions

The inverse of $f(x) = b^x$ is $f^{-1}(x) = \log_b x$:

$$b^x = y \iff \log_b y = x$$

Domain and range swap on inversion:

- Exponential: domain $\mathbb{R}$, range $(0, \infty)$.
- Logarithm: domain $(0, \infty)$, range $\mathbb{R}$.

The horizontal asymptote $y = 0$ of the exponential becomes a **vertical asymptote** $x = 0$ for the logarithm.

Key Insight

$\log_b(y) = x$ asks: "b to the power of *what* gives y?" The answer is x. This is the single most useful way to think about any logarithm.