

# Math 39100 K (33191)

## - Homework Solutions - Post 02

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Chapter 3, Section 3.3 - BD 10; BDM 7 :  $y'' + 2y' + 2y = 0$   
with characteristic equation  $r^2 + 2r + 2 = 0$ . Roots:  
 $r = [-2 \pm \sqrt{4 - 8}]/2 = -1 \pm i$ .

$$y = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t.$$

Contrast with  $y'' + 2y' - 2y = 0$  with characteristic equation  
 $r^2 + 2r - 2 = 0$ . Roots:  $r = [-2 \pm \sqrt{4 + 8}]/2 = -1 \pm \sqrt{3}$ .

$$y = C_1 e^{(-1+\sqrt{3})t} + C_2 e^{(-1-\sqrt{3})t}.$$

BD 17; BDM 12 :  $y'' + 4y = 0$  with characteristic equation  $r^2 + 4 = 0$ . Roots:  $r = \pm 2i$ .

$$y = C_1 \cos(2t) + C_2 \sin(2t),$$
$$y' = -2C_1 \sin(2t) + 2C_2 \cos(2t).$$

$y(0) = 0$  and  $y'(0) = 1$ . So  $0 = C_1(1) + C_2(0)$  and  $1 = -2C_1(0) + 2C_2(1)$ ..

$C_1 = 0$ ,  $C_2 = \frac{1}{2}$  and so

$$y = \sin(2t)/2$$

Example 3.4/25 :  $t^2y'' + 3ty' + y = 0, t > 1$  with  $y_1(t) = t^{-1}$ .

$$\begin{aligned} 1 \times y_2 &= ut^{-1} \\ 3t \times y_2' &= -ut^{-2} + u't^{-1} \\ t^2 \times y_2'' &= 2ut^{-3} - 2u't^{-2} + u''t^{-1} \\ 0 &= 0 + u' + u''t \end{aligned}$$

Let  $v = u'$  and  $v' = u''$ .  $t \frac{dv}{dt} = -v$ .

$$\ln v = -\ln t, \quad \text{and so} \quad \frac{du}{dt} = v = t^{-1}.$$

Hence,  $u = \int t^{-1} dt = \ln t$ .

$$y_2 = ut^{-1} = t^{-1} \ln t.$$

Example 3.5/9 :  $2y'' + 3y' + y = t^2 + 3 \sin t$ .

1. Homogeneous Equation  $2y'' + 3y' + y = 0$  has characteristic equation  $2r^2 + 3r + 1 = (2r + 1)(r + 1) = 0$ . So  $y_h = C_1 e^{-t/2} + C_2 e^{-t}$ .

2.  $t^2$  has associated root 0 and  $3 \sin t$  has associated root the conjugate pair  $\pm i$ . So our first guess for the test solution is  $Y = At^2 + Bt + C + D \cos t + E \sin t$ . Since none of the associated roots is a root of the characteristic equation, this is what we use for  $Y_p$ .

3. Substitute in the equation.

$$\begin{array}{rclclcl} 1 \times Y_p & = & At^2 & + Bt & + C & + D \cos t & + E \sin t, \\ 3 \times Y'_p & = & & 6At & + 3B & + 3E \cos t & - 3D \sin t, \\ 2 \times Y''_p & = & & & +4A & - 2D \cos t & - 2E \sin t. \end{array}$$

$$\text{So } t^2 + 3 \sin t = At^2 + (B + 6A)t + (C + 3B + 4A) \\ + (-D + 3E) \cos t + (-E - 3D) \sin t.$$

$$A = 1.$$

$$B + 6A = 0, \text{ and so } B = -6.$$

$$C + 3B + 4A = 0, \text{ and so } C = 14.$$

$$-D + 3E = 0 \text{ and so } D = 3E.$$

$$-E - 3D = 3 \text{ and so } -10E = 3.$$

$$\text{Thus, } E = -3/10, \quad D = -9/10.$$

$$y_g = C_1 e^{-t/2} + C_2 e^{-t} + t^2 - 6t + 14 - (9/10) \cos t - (3/10) \sin t.$$

Example 3.5/17 :

$$y'' - 2y' + y = te^t + 4, \quad y(0) = 1, y'(0) = 1.$$

1. Homogeneous Equation  $y'' - 2y' + y = 0$  has characteristic equation  $r^2 - 2r + 1 = (r - 1)^2 = 0$ . So  $y_h = C_1e^t + C_2te^t$ .

2.  $te^t$  has associated root 1 and 4 has associated root 0. So our first guess for the test solution is  $Y = Ate^t + Be^t + C$ . Since 1 is a root for the homogeneous, we must multiply the block  $Ate^t + Be^t$  first by  $t$  and then by another  $t$  since 1 is a repeated root. So  $Y_p = At^3e^t + Bt^2e^t + C$ .

3. Substitute in the equation. Leave columns for the  $te^t$  and  $e^t$  terms.

$$\begin{aligned} 1 \times Y_p &= At^3e^t + Bt^2e^t + C, \\ -2 \times Y_p' &= -2At^3e^t + (-6A - 2B)t^2e^t - 4Bte^t, \\ 1 \times Y_p'' &= At^3e^t + (6A + B)t^2e^t + (6A + 4B)te^t + 2Be^t. \end{aligned}$$

So  $te^t + 4 = 0 + 0 + 6Ate^t + 2Be^t + C$ , Notice that the first two columns on the right add up to 0.

$A = 1/6, B = 0, C = 4$ , So that

$$y_g = C_1e^t + C_2te^t + (1/6)t^3e^t + 4,$$
$$y'_g = (C_1 + C_2)e^t + C_2te^t + (1/2)t^2e^t + (1/6)t^3e^t.$$

$$1 = y(0) = C_1 + 4, \quad 1 = y'(0) = C_1 + C_2.$$

$$C_1 = -3, \quad C_2 = 4.$$

$$y = -3e^t + 4te^t + (1/6)t^3e^t + 4.$$

For 19-23, just solve the homogeneous equation and get the test function  $Y(t)$ . Don't try to solve for the coefficients.

Example 3.5/21 :  $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$ . The characteristic equation of the homogeneous equation is  $r^2 + 3r = 0$  with roots  $r = 0, -3$ . So  $y_h = C_1 + C_2e^{-3t}$ .

For  $2t^4$  the associated root is 0

For  $t^2e^{-3t}$  the associated root is  $-3$

For  $\sin 3t$  the associated root(s) are  $\pm 3i$ .

First version of test function is  $\bar{Y}(t) =$

$$(At^4 + Bt^3 + Ct^2 + Dt + E) + (Ft^2 + Gt + H)e^{-3t} + (I \cos 3t + J \sin 3t).$$

Because 0 and  $-3$  are roots from the homogeneous equation, each of those blocks must be multiplied by  $t$  to get  $Y(t) =$

$$t(At^4 + Bt^3 + Ct^2 + Dt + E) + t(Ft^2 + Gt + H)e^{-3t} + (I \cos 3t + J \sin 3t).$$

Example 3.5/22 :  $y'' + y = t(1 + \sin t) = t + t \sin t$ .

Characteristic equation is  $r^2 + 1 = 0$  with roots  $r = \pm i$ .

$$y_h = C_1 \cos t + C_2 \sin t.$$

For  $t$  the associated root is 0.

For  $t \sin t$  the associated root is  $\pm i$ .

First version of test function is

$$\bar{Y}(t) = (At + B) + ((Ct + D) \cos t + (Et + F) \sin t).$$

Because  $\pm i$  are roots from the homogeneous equation, the corresponding block must be multiplied by  $t$  to get

$$Y(t) = (At + B) + t((Ct + D) \cos t + (Et + F) \sin t).$$

Example 3.5/25 :  $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t \sin 2t$ .

Characteristic equation is  $r^2 - 4r + 4 = (r - 2)^2 = 0$  with roots  $r = 2, 2$ .  $y_h = C_1 e^{2t} + C_2 t e^{2t}$ .

For  $2t^2$  the associated root is 0.

For  $4te^{2t}$  the associated root is 2.

For  $t \sin 2t$  the associated root is  $\pm 2i$ .

First version of test function is  $\bar{Y}(t) =$

$$(At^2 + Bt + C) + (Dt + E)e^{2t} + ((Ft + G) \cos 2t + (Ht + I) \sin 2t).$$

Because 2 is a twice repeated root of the homogeneous equation, the corresponding block must be multiplied by  $t^2$ . The test function  $Y(t) =$

$$(At^2 + Bt + C) + t^2(Dt + E)e^{2t} + ((Ft + G) \cos 2t + (Ht + I) \sin 2t).$$

Example 4.2/11 :  $y''' - y'' - y' + y = 0$ . Characteristic equation is  $r^3 - r^2 - r + 1 = 0$ . Factor by grouping:  
 $r^3 - r^2 - r + 1 = r^2(r - 1) - (r - 1) = (r^2 - 1)(r - 1) = (r + 1)(r - 1)(r - 1)$  with roots  $-1, 1, 1$ .

$$y_g = C_1 e^{-t} + C_2 e^t + C_3 t e^t.$$

Example 4.2/13 :  $2y''' - 4y'' - 2y' + 4y = 0$ . Divide the characteristic equation by 2 and factor by grouping to get  $r^3 - 2r^2 - r + 2 = (r^2 - 1)(r - 2)$  with roots  $1, -1, 2$ .

Example 4.2/15 :  $y^{(6)} + y = 0$ . The characteristic equation  $r^6 = -1$  requires DeMoivre's Theorem. The modulus is  $1^{1/6} = 1$ . Each step is  $60^\circ$ , beginning with half-step  $30^\circ$ . The six roots are

$$(\cos 30) \pm i(\sin 30) = (\sqrt{3}/2) \pm i(1/2), (\cos 90) \pm i(\sin 90) = \pm i,$$

$$(\cos 150) \pm i(\sin 150) = (-\cos 30) \pm i(\sin 30) = (-\sqrt{3}/2) \pm i(1/2).$$

$$y_g = C_1 e^{t\sqrt{3}/2} \cos(t/2) + C_2 e^{t\sqrt{3}/2} \sin(t/2) + C_3 \cos(t) + C_4 \sin(t)$$

$$+ C_5 e^{-t\sqrt{3}/2} \cos(t/2) + C_6 e^{-t\sqrt{3}/2} \sin(t/2).$$

Example 4.2/21 :  $y^{(8)} + 8y^{(4)} + 16y = 0$ . The characteristic equation is  $r^8 + 8r^4 + 16 = (r^4 + 4)^2 = 0$ .  $r^4 = -4$  requires DeMoivre's Theorem. The modulus is  $4^{1/4} = \sqrt{2}$ . Each step is  $90^\circ$ , beginning with half-step  $45^\circ$ . The four roots are

$$\sqrt{2}((\cos 45) \pm \mathbf{i}(\sin 45)) = 1 \pm \mathbf{i},$$

$$\sqrt{2}((\cos 135) \pm \mathbf{i}(\sin 135)) = \sqrt{2}(-(\cos 45) \pm \mathbf{i}(\sin 45)) = -1 \pm \mathbf{i}.$$

Each pair of complex roots is repeated.

$$y_g = C_1 e^t \cos t + C_2 e^t \sin t + C_3 t e^t \cos t + C_4 t e^t \sin t + C_5 e^{-t} \cos t + C_6 e^{-t} \sin t + C_7 t e^{-t} \cos t + C_8 t e^{-t} \sin t$$

Example 4.3/13 :  $y''' - 2y'' + y' = t^3 + 2e^t$ . The characteristic equation for the homogeneous equation is  $r^3 - 2r^2 + r = r(r - 1)^2 = 0$  with roots 0, 1, 1.

$$y_h = C_1 + C_2e^t + C_3te^t.$$

For  $t^3$  the associated root is 0.

For  $2e^t$  the associated root is 1.

The first guess for the test function  $Y_p^1$  is thus

$$Y_p^1 = (At^3 + Bt^2 + Ct + D) + (Ee^t).$$

Because 0 is a root for the homogeneous equation, we must multiply the first block by  $t$ .

Because 1 is a repeated root for the homogeneous equation, we must multiply the second block by  $t^2$ .

$$Y_p = t(At^3 + Bt^2 + Ct + D) + t^2(Ee^t).$$

Example 4.3/15 :  $y^{(4)} - 2y'' + y = e^t + \sin t$ . The characteristic equation for the homogeneous equation is  $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = 0$  with roots  $1, 1, -1, -1$ .

$$y_h = C_1 e^t + C_2 t e^t + C_3 e^{-t} + C_4 t e^{-t}.$$

For  $e^t$  the associated root is 1.

For  $\sin t$  the associated root is the complex pair  $\pm i$ .

The first guess for the test function  $Y_p^1$  is thus

$$Y_p^1 = (Ae^t) + (B \cos(t) + C \sin(t)).$$

Because 1 is a repeated root for the homogeneous equation, we must multiply the first block by  $t^2$ .

$$Y_p = t^2(Ae^t) + (B \cos(t) + C \sin(t)).$$

Example 4.3/16 :  $y^{(4)} + 4y'' = \sin 2t + te^t + 4$ . The characteristic equation for the homogeneous equation is  $r^4 + 4r^2 = r^2(r^2 + 4)$  with roots  $0, 0, \pm 2i$ .

$$y_h = C_1 + C_2 t + C_3 \cos(2t) + C_4 \sin(2t).$$

For  $\sin 2t$  the associated root is the complex pair  $\pm 2i$ .

For  $te^t$  the associated root is 1.

For 4 the associated root is 0.

The first guess for the test function  $Y_p^1$  is thus

$$Y_p^1 = (A \cos(2t) + B \sin(2t)) + (Ct + D)e^t + (F).$$

Because  $\pm i$  are roots for the homogeneous equation, we must multiply the first block by  $t$ .

Because 0 is a repeated root for the homogeneous equation, we must multiply the third block by  $t^2$ .

$$Y_p = t(A \cos(2t) + B \sin(2t)) + (Ct + D)e^t + t^2(F).$$

Example 4.3/18 :  $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t} \sin t$ .  
The characteristic equation for the homogeneous equation is  $r^4 + 2r^3 + 2r^2 = r^2(r^2 + 2r + 2) = 0$ . For the quadratic we use the Quadratic Formula, to get roots  $0, 0, -1 \pm \mathbf{i}$ .

$$y_h = C_1 + C_2 t + C_3 e^{-t} \cos(t) + C_4 e^{-t} \sin(t).$$

For  $3e^t$  the associated root is 1.

For  $2te^{-t}$  the associated root is  $-1$ .

For  $e^{-t} \sin t$  the associated root is  $-1 \pm \mathbf{i}$ .

The first guess for the test function  $Y_p^1$  is thus

$$Y_p^1 = (Ae^t) + (Bt + C)e^{-t} + (De^{-t} \cos(t) + Ee^{-t} \sin(2t)).$$

Because  $-1 \pm \mathbf{i}$  are roots for the homogeneous equation, we must multiply the third block by  $t$ .

$$Y_p = (Ae^t) + (Bt + C)e^{-t} + t(De^{-t} \cos(t) + Ee^{-t} \sin(2t)).$$

Example 3.6/3 :  $y'' + 2y' + y = 3e^{-t}$ . The roots of the characteristic equation for the homogeneous equation  $r^2 + 2r + 1 = (r + 1)^2 = 0$  are  $-1, -1$  and so  $y_h = C_1e^{-t} + C_2te^{-t}$ .

### Undetermined Coefficients:

The associated root for  $3e^{-t}$  is  $-1$ . Our first guess for test function  $Y_p^1 = Ae^{-t}$ .

Because  $-1$  is a root repeated twice, we must multiply by  $t^2$ . So  $Y_p = At^2e^{-t}$ .

$$1 \times Y_p = At^2e^{-t},$$

$$2 \times Y_p' = -2At^2e^{-t} + 4Ate^{-t},$$

$$1 \times Y_p'' = At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t}.$$

So  $3e^{-t} = 2Ae^{-t}$ . So  $A = 3/2$  and  $y_g = C_1e^{-t} + C_2te^{-t} + (3/2)t^2e^{-t}$ .

## Variation of Parameters:

We look for  $y_p = u_1 e^{-t} + u_2 t e^{-t}$ . We have the linear equations:

$$\begin{aligned}u_1'(e^{-t}) + u_2'(te^{-t}) &= 0 \\u_1'(-e^{-t}) + u_2'(e^{-t} - te^{-t}) &= 3e^{-t}.\end{aligned}$$

Wronskian is  $e^{-2t}$  so

$$u_1' = \begin{vmatrix} 0 & te^{-t} \\ 3e^{-t} & (e^{-t} - te^{-t}) \end{vmatrix} / e^{-2t},$$

$$u_2' = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & (3e^{-t}) \end{vmatrix} / e^{-2t}.$$

$$u_1' = -3t, u_2' = 3. \quad u_1 = -(3/2)t^2, u_2 = 3t.$$

$$Y_p = -(3/2)t^2 e^{-t} + (3t)te^{-t} = (3/2)t^2 e^{-t}.$$

Example 3.6/5 :  $y'' + y = \tan(t)$ . The roots of the characteristic equation for the homogeneous equation  $r^2 + 1 = 0$  are  $\pm i$  and so  $y_h = C_1 \cos(t) + C_2 \sin(t)$ .

We look for  $y_p = u_1 \cos(t) + u_2 \sin(t)$ . We have the linear equations:

$$\begin{aligned}u_1'(\cos(t)) + u_2'(\sin(t)) &= 0 \\u_1'(-\sin(t)) + u_2'(\cos(t)) &= \tan(t).\end{aligned}$$

Wronskian is 1.

$$u_1' = -\sin(t) \tan(t) = -\sin^2(t)/\cos(t) = (\cos^2(t) - 1)/\cos(t) = \cos(t) - \sec(t),$$

$$u_2' = \cos(t) \tan(t) = \sin(t).$$

$$u_1 = \sin(t) - \ln |\sec(t) + \tan(t)|, u_2 = -\cos(t),$$

$$y_p = -\sin(t) \ln |\sec(t) + \tan(t)|.$$

Example 3.6/10 :  $y'' - 2y' + y = e^t/(1 + t^2)$ . The roots of the characteristic equation for the homogeneous equation  $r^2 - 2r + 1 = 0$  are 1, 1 and so  $y_h = C_1 e^t + C_2 t e^t$ .

$$y_p = u_1 e^t + u_2 t e^t$$

$$u_1'(e^t) + u_2'(te^t) = 0$$

$$u_1'(e^t) + u_2'(e^t + te^t) = e^t/(1 + t^2).$$

Wronskian is  $e^{2t}$ .

$$u_1' = -t/(1 + t^2), u_2' = 1/(1 + t^2).$$

$$u_1 = -\frac{1}{2} \ln(1 + t^2), u_2 = \arctan(t).$$

$$y_p = -\frac{1}{2} e^t \ln(1 + t^2) + t e^t \arctan(t).$$

Example 3.6/14 :  $t^2y'' - t(t+2)y + (t+2)y = 2t^3$ . Divide by  $t^2$  to get

$$y'' - t^{-1}(t+2)y + t^{-2}(t+2)y = 2t.$$

Given  $y_1 = t$ ,  $Y_2 = te^t$ . (Check that these are solutions).

$$y_p = u_1t + u_2te^t.$$

$$u_1'(t) + u_2'(te^t) = 0$$

$$u_1'(1) + u_2'(e^t + te^t) = 2t.$$

Wronskian is  $t^2e^t$ .

$$u_1' = -2t^2e^t/t^2e^t = -2, u_2' = 2t^2/t^2e^t = 2e^{-t}.$$

$$u_1 = -2t, u_2 = -2e^{-t}.$$

$$y_p = -2t^2 - 2t.$$

Example 3.7/6 : Units are centimeters, grams and seconds.

So  $g = 980\text{cm/s}^2$ .  $m = 100\text{gr}$ ,  $\Delta L = 5\text{cm}$ ,

From equilibrium, so  $y_0 = 0$ .

Downward velocity of  $10\text{cm/s}$ , so  $y'_0 = -10$ .

$w = mg = 98000$  (dynes).  $w = k\Delta L$ , so

$k = 98000/5 = 19600$ .

No damping, so  $c = 0$ . No external force.

$$100y'' + 19600y = 0, \quad \text{or} \quad y'' + 196y = 0,$$

Characteristic equation:  $r^2 + 196 = 0$  with roots  $r = \pm 14i$ . So

$$y = C_1 \cos 14t + C_2 \sin 14t$$
$$y' = 14C_2 \cos 14t - 14C_1 \sin 14t.$$

$$0 = y(0) = C_1, \quad -10 = y'(0) = 14C_2.$$

So the solution is  $y = -(5/7) \sin 14t$ .

Example 3.7/7 : Units are feet, pounds and seconds.

So  $g = 32\text{ft}/\text{s}^2$ .  $w = 3\text{lb}$ ,  $\Delta L = 3/12 = 1/4\text{ft}$ ,

Lifted up, so  $y_0 = 1/12\text{ft}$ .

Downward velocity of  $2\text{ft}/\text{s}$ , so  $y'_0 = -2$ .  $3w = mg = 32m$ , so  
 $m = 3/32$  (slugs).  $w = k\Delta L$ , so  $k = 3 \div (1/4) = 12$ .

No damping, so  $c = 0$ . No external force.

$$3/32y'' + 12y = 0, \text{ or } y'' + 128y = 0,$$

Characteristic equation:  $r^2 + 128 = 0$  with roots  $r = \pm 8\sqrt{2}i$ .  
So

$$y = C_1 \cos(8\sqrt{2}t) + C_2 \sin(8\sqrt{2}t)$$
$$y' = 8\sqrt{2}C_2 \cos(8\sqrt{2}t) - 8\sqrt{2}C_1 \sin(8\sqrt{2}t).$$

$$\frac{1}{12} = y(0) = C_1, \quad -2 = y'(0) = 8\sqrt{2}C_2.$$

So the solution is  $y = \frac{1}{12} \cos(8\sqrt{2}t) - \frac{1}{4\sqrt{2}} \sin(8\sqrt{2}t)$ .

$$A = \sqrt{\frac{1}{144} + \frac{1}{32}}, \quad \phi = \arctan\left(-\frac{3}{\sqrt{2}}\right), \quad y = A \cos(8\sqrt{2}t - \phi)$$

[This part I won't ask.]

Example 3.7/9 : Units are centimeters, grams and seconds with  $g = 980\text{cm}/\text{s}^2$ .

$$m = 20\text{gr}, \Delta L = 5\text{cm}, c = 400$$

Down  $2\text{cm}$ , so  $y_0 = -2$ . Released, so  $y'_0 = 0$ .

$$w = mg = 19600 \text{ (dynes)}. w = k\Delta L, \text{ so } k = 19600/5 = 3920.$$

No external force.

$$20y'' + 400y' + 3920y = 0, \quad \text{or} \quad y'' + 20y' + 196y = 0,$$

Characteristic equation:  $r^2 + 20r + 196 = 0$  with roots

$$r = \left(-\frac{\sqrt{20}}{2}\right) \pm \frac{\sqrt{384}}{2}\mathbf{i} = -10 \pm 4\sqrt{6}\mathbf{i}.$$

So

$$y = C_1 e^{-10t} \cos(4\sqrt{6}t) + C_2 e^{-10t} \sin(4\sqrt{6}t)$$
$$y' = (-10C_1 + 4\sqrt{6}C_2)e^{-10t} \cos(4\sqrt{6}t) +$$
$$(-10C_2 - 4\sqrt{6}C_1)e^{-10t} \sin(4\sqrt{6}t).$$

$$-2 = y(0) = C_1, \quad 0 = y'(0) = -10C_1 + 4\sqrt{6}C_2.$$

So the solution is

$$y = -2e^{-10t} \cos(4\sqrt{6}t) - \frac{5}{\sqrt{6}}e^{-10t} \sin(4\sqrt{6}t).$$

Example 3.8/10 : Units are feet, pounds and seconds.

$w = 8\text{lb}$ . So  $m = w/g = 8/32 = 1/4$ .  $\Delta L = 1/2$ . So

$k = w/\Delta L = 16$ .  $c = 0$ .

$y(0) = -1/4, y'(0) = 0$ .

$(1/4)y'' + 16y = 8 \sin(8t)$ , or  $y'' + 64y = 32 \sin(8t)$ .

The roots of the characteristic equation for the homogeneous equation  $r^2 + 64 = 0$  are  $\pm 8i$ . So the forcing occurs at resonance.

$Y_p = At \cos(8t) + Bt \sin(8t)$

$Y_p' = -8At \sin(8t) + 8Bt \cos(8t) + A \cos(8t) + B \sin(8t)$ .

$Y_p'' = -64At \cos(8t) - 64At \sin(8t) - 16A \sin(8t) + 16B \cos(8t)$ .

$32 \sin(8t) = Y_p'' + 64Y_p = -16A \sin(8t) + 16B \cos(8t)$ . So

$A = -2, B = 0$ .

$y = C_1 \cos(8t) + C_2 \sin(8t) - 2t \cos(8t)$ ,

$y' = -8C_1 \sin(8t) + 8C_2 \cos(8t) - 2 \cos(8t) + 16t \sin(8t)$ .

At  $t = 0$ :  $-1/4 = C_1, 0 = 8C_2 - 2$ . So

$$y = -(1/4) \cos(8t) + (1/4) \sin(8t) - 2t \cos(8t).$$

Example 5.2/9 :  $(1 + x^2)y'' - 4xy' + 6y = 0$

$$1y'' = \sum 1n(n-1)a_n x^{n-2} [k = n-2] = \sum (k+2)(k+1)a_{k+2} x^k.$$

$$+x^2y'' = \sum n(n-1)a_n x^n [k = n] = \sum k(k-1)a_k x^k.$$

$$-4xy' = \sum 4na_n x^n [k = n] = \sum -4ka_k x^k.$$

$$6y = \sum 6a_n x^n [k = n] = \sum 6a_k x^k.$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)} [-(k(k-1)+4k-6)a_k] = \frac{-k^2 + 5k - 6}{(k+2)(k+1)} a_k$$

$$= -\frac{(k-2)(k-3)}{(k+2)(k+1)} a_k.$$

$$k = 0 : a_2 = -3a_0,$$

$$k = 1 : a_3 = -\frac{1}{3}a_1.$$

$$k = 2 : a_4 = 0, \quad k = 3 : a_5 = 0. \text{ So } a_k = 0 \text{ for } k \geq 2.$$

Example 5.2/12,18:

$$(1-x)y'' + xy' + -y = 0, y(0) = -3, y'(0) = 2.$$

$$\begin{aligned} 1y'' &= \sum 1n(n-1)a_n x^{n-2} [k = n-2] = \sum (k+2)(k+1)a_{k+2} x^k \\ -xy'' &= \sum -n(n-1)a_n x^{n-1} [k = n-1] = \sum -(k+1)(k)a_{k+1} x^k. \\ +xy' &= \sum na_n x^n [k = n] = \sum ka_k x^k. \\ -y &= \sum -a_n x^n [k = n] = \sum -a_k x^k. \end{aligned}$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)} [(k+1)ka_{k+1} - (k-1)a_k], \quad a_0 = -3, a_1 = 2.$$

$$k = 0 : a_2 = a_0/2 = -3/2,$$

$$k = 1 : a_3 = a_2/3 = -1/2.$$

$$k = 2 : a_4 = [6a_3 - a_2]/12 = -3/4,$$

$$k = 3 : a_5 = [12a_4 - 2a_3]/20 = -2/5.$$