Math 39100 K (33191) - Lectures 01

Ethan Akin
Email: eakin@ccny.cuny.edu

Spring, 2026

Contents

Introduction

- Officially, the book is Boyce,DiPrima and Meade *Elementary Differential Equations and Boundary Value Problems* (12^{th} Ed). However, it might be easier to find less expensive copies of either the 11^{th} edition or the 9^{th} or 10^{th} edition of Boyce and DiPrima *Elementary Differential Equations and Boundary Value Problems* . Make sure you get a version that has *...and Boundary Value Problems* in the title.

- Keep up with the homework.
- Ask questions.

- The course information sheet with the term's homework assignments is posted on my site:

  profakin.webflow.io

  Notice that this is a personal site. It is not part of the Math Dept website.

- I will be posting there a pdf of the slides I am using here. The first batch for the course is already up. I will post the remaining pieces as we get to them.
- The class will meet from 8am to 9:15 on Tuesdays and Thursdays in Marshak 02.
- Office: MR (Marshak) 325A

  Office Hours: Tuesday 11:00am-1:00pm,
  Thursday 11:00am-12:00.

  Email: eakin@ccny.cuny.edu

First Order Differential Equations - Introduction

Solving a differential equation means using given information about the derivative to find the original function.

Recall the three notations for the derivative of a function f with the equation $y = f(x)$ indicating that x is the *input* or *independent variable* and y is the *output* or *dependent variable*. We write the derivative as $y' = f'(x)$ or in Leibniz's notation $\frac{dy}{dx}$.

The Integral as Antiderivative

A general first order ode (= ordinary differential equation) is of the form $y' = F(x, y)$ or $\frac{dy}{dt} = F(t, y)$.

However, we will see equations written:
$p(t, y)\frac{dy}{dt} + q(t, y) = r(t, y)$. or, using Liebniz notation, we will see equations in differential form:

$$M(x, y)dx + N(x, y)dy = 0.$$

These are all algebraically equivalent. That is, we can rewrite any of these to get $y' = F(x, y)$ or $\frac{dy}{dt} = F(t, y)$.

You have seen the derivative y' written as a function of both x and y when you learned *implicit differentiation*.

The equation $xy = 1$ implicitly defines y equal to a function of x.

Of course we can solve explicitly to get $y = x^{-1}$ and differentiate to get $y' = -x^{-2}$.

We can also use implicit differentiation on the original expression to get $xy' + y = 0$ and so $y' = -y/x$.

Recall that integration was introduced as the *antiderivative*.
$y' = f(x)$ implies

Recall that integration was introduced as the *antiderivative*.
$y' = f(x)$ implies $y = \int f(x)dx + C$.

With different values of the integration constant C we obtain infinitely many parallel curves, each with the same shape determined by the derivative.

Linear Growth

$\frac{dy}{dt} = m$ (Absolute Growth Rate y' is a constant).
$dy = mdt$, and so

$$y = \int mdt = mt + C.$$

When $t = 0$, $y_0 = C$.

$$\text{So } \frac{dy}{dt} = m \text{ implies } y = mt + y_0.$$

Exponential Growth

$\frac{dy}{dt} = ky$ (Relative Growth Rate $\frac{y'}{y}$ is a constant).
$dy = kydt$, and so
$y = \int kydt = ky^2/2 + C$??? NO

$$\int \frac{dy}{y} = \int kdt$$

.

$$\ln|y| = kt + C,$$

$$y = Ce^{kt} \quad \text{different C.}$$

When $t = 0$, $y_0 = C$.

$$\text{So } \frac{dy}{dt} = ky \text{ implies } y = y_0 e^{kt}.$$

Let's do the last bit slowly. Recall $e^{A+B} = e^A e^B$.

$\ln|y| = kt + C$ implies $|y| = e^C \cdot e^{kt}$.

Recall that e^x is always positive and $|y_0| = e^C$.

So if $y_0 > 0$ then $y_0 = e^C$ and $y = y_0 e^{kt}$.

If $y_0 < 0$ then $-y_0 = |y_0| = e^C$ and so

$$y = -|y| = -e^C e^{kt} = y_0 e^{kt}.$$

If $y_0 = 0$ then $y = 0 = y_0 e^{kt}$ is the solution. Our previous

sloppy procedure works fine, but it is worth looking at a diagram.

The Fundamental Existence and Uniqueness Theorem, Section 2.4

An *initial value problem* (= an IVP) consists of a differential equation together with initial conditions.

In the first order case, this is a pair:

$$[ode] y' = F(x, y), \quad [ic] y(x_0) = y_0.$$

Solving the ode involves an integration which yields a single arbitrary constant. The solution with the arbitrary constant is the *general solution*. That is, there are infinitely many solutions obtained by making different choices of the constants. However, the constant is determined by the initial value. Theorem 2.4.2 in Section 2.4 says, as far as we are concerned:

Theorem An IVP has one and only one solution. That is, the solution exists and is unique.

Actually, some conditions are needed to make this work. It suffices that $F(x, y)$ have continuous partial derivatives. Example 3 of Section 2.4 illustrates what might otherwise happen.

$$IVP: \quad \frac{dy}{dt} = y^{1/3},\ y(0) = 0.$$

$$\int y^{-1/3} dy = \frac{3}{2} y^{2/3} = t - C.$$

Notice that $t - C$ has to be positive and so $t \geq C$ (which is why I used $-C$ instead of C). We obtain for any choice of C two solutions:

$$y = \begin{cases} 0 & t \leq C, \\ \pm[\frac{2}{3}(t - C)]^{3/2}. \end{cases}$$

Any of these with $C \geq 0$ satisfy the initial condition $y(0) = 0$.

Variables Separable, Section 2.2

If $\frac{dy}{dx} = A(x)B(y)$, we can solve by separating the variables:

$$\frac{dy}{B(y)} = A(x)dx$$

and integrating. We combine the constants of integration and usually put the single constant on the x side. If we have initial conditions, we determine the constant.
Section 2.2/13: $y' = 2x/(y + x^2y), \quad y(0) = -2.$
Factor and separate: $\int ydy = \int (2x/(1 + x^2)dx.$
$\frac{y^2}{2} = \ln(1 + x^2) + C$ and $y(0) = -2$ implies $C = 2.$
On a test stop here, But taking the square root, we get:

$$y = \pm\sqrt{4 + 2\ln(1 + x^2)}$$

and the initial condition requires the $-$ sign for the unique solution.

Homogeneous Equations, Section 2.2/34

A function $F(x, y)$ is called *homogeneous of degree n* if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$. We will only be considering functions homogeneous of degree 0 so that $F(\lambda x, \lambda y) = F(x, y)$ and so with $\lambda = 1/x$:

$$F(x, y) = F(1, y/x).$$

So we use the change-of-variable:

$$z = \frac{y}{x}, xz = y, \quad \text{and so} \quad \frac{dy}{dx} = x\frac{dz}{dx} + z.$$

So when F is homogeneous of degree zero the equation $\frac{dy}{dx} = F(x, y)$ becomes

$$x\frac{dz}{dx} + z = F(1, z), \quad \text{or} \quad x\frac{dz}{dx} = F(1, z) - z,$$

which is variables separable. Usually the integration uses partial fractions.

Section 2.2/BD34; BDM29 : $\frac{dy}{dx} = -\frac{4x+3y}{2x+y}$.

$$x\frac{dz}{dx} = -z - \frac{4+3z}{2+z} = -\frac{z^2+5z+4}{2+z}.$$

$$\frac{2+z}{(z+4)(z+1)}dz = -\frac{dx}{x}.$$

Partial fractions: $\frac{2+z}{(z+4)(z+1)} = \frac{A}{(z+4)} + \frac{B}{(z+1)}$.

$2+z = A(z+1) + B(z+4)$ and so (substituting $z = -1, -4$) $B = 1/3, A = 2/3$.

Integrate to get $\frac{2}{3}\ln(z+4) + \frac{1}{3}\ln(z+1) = -\ln(x) + C$. Multiply by 3 and exponentiate:

$$2\ln(z+4)+\ln(z+1) = -3\ln(x)+C \qquad \text{new } C.$$

$$(z+4)^2(z+1)^1 = Cx^{-3}, \qquad \text{new } C.$$

Multiply by x^3. Note that $x^2(z+4)^2 = (y+4x)^2$ and $x(z+1) = (y+x)$. So

$$(y+4x)^2(y+x) = C.$$

Exercises BD 5, 35, 37; BDM 3, 307

2.2/BD5; BDM3 : $y' = (\cos^2(x))(\cos^2(2y))$. So $\int \sec^2(2y)dy = \int \cos^2(x)dx$.

We use that $\sec^2$ is the derivative of tan and that $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1$.

$$\frac{1}{2}\tan(2y) = \frac{x}{2} + \frac{\sin(2x)}{4} + C.$$

2.2/BD35 : $\frac{dy}{dx} = \frac{x+3y}{x-y}$. and so

$x\frac{dz}{dx} = -z + \frac{1+3z}{1-z} = \frac{1+2z+z^2}{1-z}$.

$\frac{1-z}{(z+1)^2} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2}$.

With $1 - z = A(z+1) + B$, set $z = -1$ to get $B = 2$

and $z = 0$ to get $A + B = 1$ and so $A = -1$.

Integrate to get $-\ln(z+1) - 2(z+1)^{-1} = \ln(x) + C$.

So $-\ln(z+1) - \ln(x) - 2(z+1)^{-1} = C$.

$-\ln(z+1) - \ln(x) =$
$-\ln((z+1)x) =$
$-\ln(y+x)$.

$(z+1) = \frac{y+x}{x}$ and so $2(z+1)^{-1} = \frac{2x}{y+x}$. So,

$$-\ln(x+y) - \frac{2x}{y+x} = C.$$

2.2/BD37;BDM30 : $\frac{dy}{dx} = \frac{x^2-3y^2}{2xy}$. and so

$x\frac{dz}{dx} = -z + \frac{1-3z^2}{2z} = \frac{1-5z^2}{2z}$.

$$\ln(x) + C = \int \frac{dx}{x} = \int \frac{2z}{1-5z^2}dz$$

$$= -\frac{1}{5}\ln(1-5z^2). \qquad \text{Multiply by } \; 5$$

Since $\ln(x^5) + \ln(1-5z^2) = \ln(x^3 \cdot x^2(1-5z^2)$,

$$x^3(x^2 - 5y^2) = C.$$

Linear Equations, Section 2.1

A first order equation is *linear* if it can be written in the form $\frac{dy}{dt} + p(t)y = g(t)$. If $g(t) = 0$ then the equation is separable with $\frac{dy}{y} = -p(t)dt$. Notice that if $p(t)$ is a constant then this is exponential growth. Integrating and exponentiating we get $y = Ce^{-\int p(t)dt}$. When $g(t)$ is not zero, the equation is not separable and we use the *integrating factor* method.

- Set the equation in the form $\frac{dy}{dt} + p(t)y = g(t)$, dividing (both sides!) by the coefficient of $\frac{dy}{dt}$ if it is not 1.
- Compute the *integrating factor* $\mu(t) = e^{\int p(t)dt}$ and multiply through (both sides!).
- **Check** that the left side is $[\mu(t)y]'$.
- Integrate and divide by μ to solve for y:

$$y = [\int \mu(t)g(t)dt + C]/\mu(t).$$

It is sometimes not obvious that an equation is linear.

Example: $x\frac{dy}{dx} + xy \quad = \quad x^3 + 2y.$

Collect the y' terms and the y terms and divide to get a coefficient of 1 on y'. $x\frac{dy}{dx} + (x-2)y \quad = \quad x^3$ and so:

$$\frac{dy}{dx} + (1 - \frac{2}{x})y = x^2.$$

$$\mu = exp[\int (1 - \frac{2}{x})dx] = e^{-2\ln(x)+x} = x^{-2}e^x,$$

$$x^{-2}e^x y' + (x^{-2} - 2x^{-3})e^x y = [x^{-2}e^x y]' = e^x.$$

$$x^{-2}e^x y \;=\; e^x + C, \qquad \text{or} \quad y = x^2 + Cx^2e^{-x}.$$

Exercises 2.1/BD 2, 8, 19; BDM 2,

2.1/BD2;BDM2 : $y' - 2y = t^2e^{2t}$. Multiply by
$\mu = e^{\int -2dt} = e^{-2t}$.

$$[e^{-2t}y]' = e^{-2t}y' - 2e^{-2t}y = t^2.$$

$$e^{-2t}y = \frac{1}{3}t^3 + C, \quad \text{and so} \quad y = (\frac{1}{3}t^3 + C)e^{2t}.$$

2.1/BD8 : $(1+t^2)y' + 4ty = (1+t^2)^{-2}$. Divide by $(1+t^2)$ to get
$y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3}$, with
$\mu = exp(2\ln(1+t^2)) = (1+t^2)^2$.

$$(1+t^2)^2y' + 4t(1+t^2)y = [(1+t^2)^2y]' = (1+t^2)^{-1},$$

with $(1+t^2)^2y = \arctan(t) + C$.

$$y = (\arctan(t) + C)(1+t^2)^{-2}.$$

$2.1/\text{BD}19: \; t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0, t < 0$. Divide by t^3 to get

$y' + 4t^{-1}y = t^{-3}e^{-t}$, with $\mu = e^{\int 4t^{-1}dt} = e^{4\ln(t)} = t^4$.

$$t^4y' + 4t^3y = [t^4y]' = te^{-t}.$$

Integrate by parts to get $t^4y = -te^{-t} - e^{-t} + C$.

When $t = -1, y = 0$, and so $C = 0$.
Hence,

$$y = -(t^{-3} + t^{-4})e^{-t}.$$

Exact Equations, Section 2.6 through Example 3

We begin with a review of the chain rule in several variables.If F is a function of x and y and each of these is a function of t, then for the composed function given by $t \mapsto F(x(t), y(t))$, we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}.$$

This is the dot product of the *gradient* of F $(=< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} >)$ with the velocity vector $(=< \frac{dx}{dt}, \frac{dy}{dt} >)$.

About the gradient vector field, there are two things to remember.

- The gradient of F points in the direction of greatest increase of F.
- The gradient of F is perpendicular to the *contour curves* defined by $F(x, y) = C$.

The contour curve $F(x, y) = C$ defines y *implicitly* as a function of x, with:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0.$$

This is a differential equation whose solution is the family of contour curves $F = C$. Multiplying by dx we see that it is of the form

$$M(x, y)dx + N(x, y)dy = 0.$$

The total differential of F is $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$. Think of it as the effect on F of a little change in x and y. Given a differential $M(x, y)dx + N(x, y)dy$ we would like to find an F so that it equals dF.

Such an F might not exist. When it does, the differential is called *exact*.

Think of the vectorfield $< M(x, y), N(x, y) >$ as a force field. If x and y move along a path as a function of time, the integral $W = \int_{t=a}^{t=b} M(x(t), y(t))\frac{dx}{dt} + N(x(t), y(t))\frac{dy}{dt}\; dt$ is called the *line integral* and its value is the *work* along the path between the times a and b. The result usually depends upon the choice of path.

But if the force is the gradient of a function F, then it is called a *conservative force* with *potential function* F. In that case

$$W = \int_{t=a}^{t=b} \frac{dF}{dt}\; dt = F(x(b), y(b)) - F(x(a), y(a)).$$

The work is the potential difference between the two locations. Happily there is a simple test for exactness. If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Consider the diagram

$$\begin{array}{ccccc} & & F & & \\ & \partial_x \swarrow & & \searrow \partial_y & \\ M\ dx & & + & & N\ dy \\ & \partial_y \searrow & & \swarrow \partial_x & \\ & & =? & & \end{array}$$

Example: $(2xy^2 + 2y)dx + (2x^2y + 2x + 2y)dy = 0.$
$\frac{\partial(2xy^2+2y)}{\partial y} = 4xy + 2 = \frac{\partial(2x^2y+2x+2y)}{\partial x}.$

$$F = \int 2xy^2 + 2y\ dx = x^2y^2 + 2xy + H(y).$$

$\frac{\partial F}{\partial y} = 2x^2y + 2x + H'(y) = 2x^2y + 2x + 2y.$

$H'(y) = 2y$ and so $H(y) = y^2$. Thus, $F = x^2y^2 + 2xy + y^2$.

Solution : $x^2y^2 + 2xy + y^2 = C.$

Example :
$(e^{xy}(y\cos(x)-\sin(x))+1)dx+(e^{xy}x\cos(x)+\cos(y)+2y)dy$
$=0, \quad y(0)=0.$

Exact. Go up the y side :
$F=\int(e^{xy}x\cos(x)+\cos(y)+2y)\ dy=$
$e^{xy}\cos(x)+\sin(y)+y^2+H(x).$
$\frac{\partial F}{\partial x}=e^{xy}(y\cos(x)-\sin(x))+H'(x)=e^{xy}(y\cos(x)-\sin(x))+1.$
$H'(x)=1$, and so $F=e^{xy}\cos(x)+\sin(y)+y^2+x$. The general solution is $F=C$.

Since $y=0$ when $x=0$, $C=1$.

Example : $(2x+y)dx=(x+2y)dy$. Rewrite as $(2x+y)dx-(x+2y)dy=0$. Not exact. (It is homogeneous). Let's try anyway.

$F=\int(2x+y)\ dx=x^2+xy+H(y).$
$\frac{\partial F}{\partial y}=x+H'(y)=-x-2y$ and so $H'(y)=-2x-2y$.
ERROR.

Example: $(y\ln(x) + xy)dx + (x\ln(y) + xy)dy = 0$. Not exact (It is variables separable).

$\int \frac{\ln(x)+x}{x}\, dx = -\int \frac{\ln(y)+y}{y}\, dy$.

Split each integral: $\int \frac{\ln(x)}{x}\, dx = \frac{1}{2}(\ln(x))^2 + C$
[u substitution with $u = \ln(x)$].

2.6/BD17;BDM13 : With
$\psi(x,y) = \int_{x_0}^{x} M(s,y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt$,
the first term doesn't depend on y. So

$$\frac{\partial \psi}{\partial y} = N(x,y).$$

$$\frac{\partial \psi}{\partial x} = M(x,y_0) + \int_{y_0}^{y} \frac{\partial N}{\partial x}(x,t)\,dt.$$

By assumption $\frac{\partial N}{\partial x}(x,t) = \frac{\partial M}{\partial t}(x,t)$

and so the integral is $\int_{y_0}^{y} \frac{\partial M}{\partial t}(x,t)\,dt = M(x,y) - M(x,y_0)$.

$$\frac{\partial \psi}{\partial x} = M(x,y_0) + M(x,y) - M(x,y_0) = M(x,y).$$

Units

Numbers in science are rarely naked. They are usually attached to units. These come in two varieties.

Counting units measure amounts like gallons, pounds, centimeters, and seconds. These are usually additive.

Put a heap of 5 pounds together with a heap of 3 pounds. The combined heap weighs 8 pounds. A trip of 5 miles following a trip of 3 miles accounts for 8 miles on the odometer.

Ratio units compare amounts. You recognize these by the word *per*, which is literally the Latin word for "through" and in mathematical and scientific use means *divide*. Thus, the units for speed, miles per hour, means miles divided by hours and so is written mi/hr. The unit of price, dollars per pound, written $/lb, means dollars divided by pounds. Finally, percent, written %, means divide by one hundred.

When multiplied, units can be cancelled as in algebraic expressions. Thus, when you buy 2 pounds of hamburger at \$4 dollars per pound, you multiply 2 pounds $\times \frac{4\$}{\text{pound}}$ to get a cost of 8 \$. Notice that what is added up at the checkout is the total cost. The prices don't add.

Suppose you want to convert 60 miles per hour to the units feet per second. You cancel miles to get feet and cancel hours to get seconds. You do this by multiplying by conversion factors, version of the number 1, but with different units.

$$\frac{60\text{miles}}{\text{hour}} \cdot \frac{\text{feet}}{\text{miles}} \cdot \frac{\text{hours}}{\text{seconds}} = \frac{??\ \text{feet}}{\text{second}}.$$

$$\frac{60\text{miles}}{\text{hour}} \cdot \frac{5280\text{feet}}{1\text{miles}} \cdot \frac{1\text{hours}}{3600\text{seconds}} = \frac{88\ \text{feet}}{\text{second}}.$$

Tank and Interest Problems, Section 2.3

Example: A 200 gallon tank initially contains 50 gallons of water in which is dissolved 5 pounds of salt. At a rate of 5 gallons per minute a solution with a concentration of $1/2$ pound of salt per gallon is poured into the tank. A well-stirred solution is pumped out at a rate of 2 gallons per minute. Compute the equation for the quantity $S(t)$ of the salt in the tank up to the time t^* the tank is filled. Then compute the formula $S(t)$ for the time t beyond t^* as the tank continues to overflow.

Note the units. The volume $V(t)$ in gallons and the quantity $S(t)$ in pounds. The net changes $\frac{dV}{dt}$ and $\frac{dS}{dt}$ are in *gallons per minute* and in *pounds per minute*, respectively.

$\frac{dV}{dt} = Input - Output = 5 - 2 = 3$. (Linear Growth)
So $V = V_0 + 3t = 50 + 3t$, and the tank is filled when $V(t^*) = 200$ and so with $t^* = 50$ minutes.

$\frac{dS}{dt} = 5[gal/min] \cdot \frac{1}{2}[lb/gal] - 2[gal/min] \cdot \frac{S}{50+3t}$ [lb/gal].
This is the linear equation

$$\frac{dS}{dt} + \frac{2}{50+3t}S = \frac{5}{2}.$$

$\mu = exp(\int \frac{2}{50+3t} dt) = exp(\frac{2}{3} \ln(50+3t)) = (50+3t)^{2/3}$.

$[(50+3t)^{2/3} S]' = \frac{5}{2}(50+3t)^{2/3}$.

$(50+3t)^{2/3} S = \frac{1}{2}(50+3t)^{5/3} + C$

When $t = 0, S = 5$. So $C = 50^{2/3}5 - \frac{1}{2}50^{5/3} = -20(50)^{2/3}$.

$$S = \frac{1}{2}(50+3t) - 20(\frac{50}{50+3t})^{2/3}.$$

with $S(50) = 100 - 20(4^{-2/3})$.

When the tank is overflowing the outflow is the same as the inflow and the volume V is constant at 200.
$\frac{dS}{dt} = 5 \cdot \frac{1}{2} - 5 \cdot \frac{S}{200}$ with $t \geq t^*$.

This equation is linear and variables separable
$\frac{dS}{S-100} = -\frac{dt}{40}$. and so $\ln(S - 100) = -\frac{t}{40} + C$.

$$S = 100 + Ce^{-t/40}$$

with $S(50) = 100 - 20(4^{-2/3})$ determining C.

Instead we can restart the clock so that
$S == 100 - 20(4^{-2/3})$ at $t = 0$.

Interest

For annual interest with rate r the interest on P dollars left for one year is rP. Thus, $P_1 = P_0 + rP_0$.

For the next year the interest is rP_1 and so
$P_2 = P_1 + rP_1 = P_0 + rP_0 + rP_0 + r^2 P_0$.

Let us pause here to look at a little puzzle.
Suppose you buy something with price P, but there is a 6% tax. You must pay $P + .06P$. Suppose you are given a 10% discount. Then you would pay $P - .1P$. Now the question:

Which way would you do better?

- Take the discount first and then add the tax on the smaller amount, or
- Add the tax first and then take the discount on the whole thing, tax and all.

The answer is that the results are the same.

For the tax, instead of thinking of adding $.06P$ to P,think that you are adding 6% to 100% for a total of 106%. That is, you are multiplying by 1.06.
For the discount, you are subtracting 10% from 100% and so are paying 90% of the original price. That is, you are multiplying by .9.
So the two options are: First multiply P by .9 and then multiply by 1.06, or, alternatively first multiply by 1.06 and then by .9.

The results are the same.

Back to the interest computation:
We had $P_2 = P_1 + rP_1 = P_0 + rP_0 + rP_0 + r^2 P_0$.

But notice that $P + rP = (1 + r)P$. So each year we multiply by $(1 + r)$.

Beginning with P_0 we have $P_t = (1 + r)^t P_0$.

The units of the interest rate r are *dollars of interest per dollar of principal per year*. So in a fraction $1/n$ of a year, the interest on P is $(r/n)P$.

So if you compound semi-annually, you multiply by $(1 + \frac{r}{2})$ every half year.

The principal after one year is $(1 + \frac{r}{2})^2 P = (1 + r + \frac{r^2}{4})P$. The extra bit is the interest on the previous six month's interest. So if you compound every month, you multiply by $(1 + \frac{r}{12})$ every month. So the principal after one year is $(1 + \frac{r}{12})^{12} P$.

So compound interest with n periods per year yields P_t, after t years, given by

$$P_t = (1 + \frac{r}{n})^{nt} P_0 = ((1 + h)^{\frac{1}{h}})^{rt} P_0$$

where $h = \frac{r}{n}$ so that $nt = \frac{rt}{h}$.

So compound interest with n periods per year yields P_t, after t years, given by

$$P_t = (1 + \frac{r}{n})^{nt} = ((1 + h)^{\frac{1}{h}})^{rt}$$

where $h = \frac{r}{n}$ so that $nt = \frac{rt}{h}$.

For *continuous compounding* we take the limit as n tends to infinity or, equivalently, as h tends to zero. To recall what happens to the limit, write

$\ln((1 + h)^{\frac{1}{h}}) = \frac{\ln(1+h) - \ln(1)}{h}$ whose limit is the derivative of $\ln(x)$ at $x = 1$ and so is 1.

Hence, $\lim_{h \to 0}(1 + h)^{\frac{1}{h}} = e^1 = e$, and with continuous compounding

$$P_t = P_0 e^{rt}.$$

The equation $P_t = P_0 e^{rt}$ is exactly exponential growth with rate r. This is also derived as follows

The change in the principal due to interest is $dP = rPdt$, which is exponential growth with rate r.

The problems also feature a constant flow of k dollars per year with $k > 0$, eg for money put into a bank account, or with $k < 0$ for money paying off a loan. So the combined effect is

$$dP = (rP + k)dt.$$

2.3/ BD9; BDM7 : $P_0 = 8000$ dollars, $r = .1$ per year and $P_3 = 0$.

$\frac{dP}{dt} = .1P - k$, and so $\frac{dP}{P-10k} = .1dt$.

$P = 10k + Ce^{.1t}$, with $8000 = 10k + C, 0 = 10k + Ce^{.3}$.

So $C = -8000/(e^{.3} - 1)$ and $k = 800(e^{.3}/(e^{.3} - 1)$. So $3k - 8000 = 1257$ dollars of interest.

Miscellaneous Problems, Section 2.9/ BD 1-6, 8-11, 15, 29; BDM 1-10, 12, 22

1. Linear, 2. Separable, 3. Exact,

4. Linear (factor y), 5. Exact, 6. Linear,

8. Linear, 9. Exact,10. Separable (factor),

11. Exact, 15. Linear, 29. Homogeneous.

Reduction of Order Problems

In Calculus 201 falling body problems were introduced. With y the height, the derivative $\frac{dy}{dt}$ is the velocity v and the second derivative $\frac{d^2y}{dt^2} = \frac{dv}{dt}$ is the acceleration a. Near the earth's surface, the acceleration due to gravity is a constant labeled $-g$ (Why $-$?).

The motion due to gravity alone is given by the simple *Second Order Differential Equation* $\frac{d^2y}{dt^2} = -g$. The solution will require two integrations yielding two arbitrary constants.

We convert the second order equation in y to a first order equation in v, writing $\frac{d^2y}{dt^2} = \frac{dv}{dt} = -g$ and so $v = -gt + C$ with C the initial velocity v_0. That is,

$\frac{dy}{dt} = v = v_0 - gt$. Integrating again, and observing that the second constant of integration is the initial height, y_0, we get

$$v = v_0 - gt \qquad \text{and} \qquad y = y_0 + v_0 t - \frac{1}{2} g t^2.$$

In using these formulae, we observe that *ground* is when $y = 0$ and the *maximum height* occurs when $v = 0$.

There is an alternative procedure which uses the chain rule

$$\frac{d^2 y}{dt^2} = \frac{dv}{dt} = \frac{dy}{dt}\frac{dv}{dy} = v\frac{dv}{dy}.$$

The falling body equation becomes $\frac{d^2 y}{dt^2} = v\frac{dv}{dy} = -g$.

Separating variables and integrating we obtain $\frac{1}{2}v^2 = -gy + C$. Labeling the constant E we get:

$$\frac{1}{2}v^2 + gy = E.$$

This is *Conservation of Energy*.

Reduction of Order Problems, Section 2.9/ BD 41, 49. 48; BDM 36

2.9/BD41 : $t^2 y'' = (y')^2, t > 0$ and so $t^2 \frac{dv}{dt} = v^2$.

$\frac{dv}{v^2} = \frac{dt}{t^2}$ and so $v^{-1} = t^{-1} + C_1$

$$\frac{dy}{dt} = v = \frac{t}{C_1 t + 1}$$

$$y = \int \frac{t}{C_1 t + 1} dt = \frac{1}{C_1^2} \int \frac{u-1}{u} du.$$

and so $y = \frac{1}{C_1} t - \frac{1}{C_1^2} \ln(C_1 t + 1) + C_2$ if $C_1 \neq 0$.

Otherwise, $y = \frac{1}{2} t^2 + C_2$.

2.9/BD49 : $y'' - 3y^2 = 0, y(0) = 2, y'(0) = 4.$

$$\frac{dv}{dt} - 3y^2 = 0.$$

WON'T WORK.

$$v\frac{dv}{dy} = 3y^2, \quad \text{and so} \quad \frac{1}{2}v^2 = y^3 + C_1.$$

With $y = 2, v = 4,\ C_1 = 0$ and $\frac{dy}{dt} = v = \sqrt{2}y^{3/2}$.

$-2y^{-1/2} = \sqrt{2}t + C_2$. With $t = 0, y = 2$ and so $C_2 = -\sqrt{2}$.

$$y = \frac{2}{(t-1)^2}.$$

Notice that for $C_1 \neq 0,\ t = \int \frac{\sqrt{2}}{\sqrt{y^3+C_1}}dy$.

2.9/BD48, BDM36 : $y'y'' = 2, y(0) = 1, y'(0) = 2.$

$$\int v dv = \int 2 dt, \quad \text{and so} \quad \frac{dy}{dt} = v = 2\sqrt{t+1}.$$

So $y = \frac{4}{3}(t+1)^{3/2} - \frac{1}{3}$.

Alternatively, $v^2 \frac{dv}{dy} = 2$.

$$\frac{dy}{dt} = v = (6y+2)^{1/3} \quad \text{and so} \quad \int (6y+2)^{-1/3} dy = t + C_2.$$

So $\frac{1}{4}(6y+2)^{2/3} = t + 1$.

Second Order Differential Equations

A general second order ode (= ordinary differential equation) is of the form $y'' = F(x, y, y')$ or $\frac{d^2y}{dt^2} = F(t, y, \frac{dy}{dt})$.

For second order equations, we require two integrations to get back to the original function. So the *general solution* has two separate, arbitrary constants. An IVP in this case is of the form:

$$y'' = F(t, y, y'), \quad \text{with} \quad y(t_0) = y_0,\ y'(t_0) = y_0'.$$

Interpreting y' as the velocity and y'' as the acceleration, we thus determine the constants by specifying the initial position and the initial velocity with t_0 regarded as the initial time (usually $= 0$).

The Fundamental Existence and Uniqueness Theorem says that an IVP has a unique solution.

The General Solution

The *general solution* of the equation $y'' = F(t, y, y')$ is of the form $y(t, C_1, C_2)$ with two arbitrary constants. Supposing that these are solutions for every choice of C_1, C_2, how do we now that we have all the solutions? The answer is that we have all solutions if we can solve every IVP. To be precise:

THEOREM: If $y'' = F(t, y, y')$ is an equation to which the Fundamental Theorem applies and $y(t, C_1, C_2)$ is a solution for every C_1, C_2, then every solution is one of these provided that for every pair of real numbers A, B, we can find C_1, C_2 so that $y(t_0, C_1, C_2) = A$ and $y'(t_0, C_1, C_2) = B$.

PROOF: If $z(t)$ is any solution then let $A = z(t_0)$ and $B = z'(t_0)$, and choose C_1, C_2 so that $y(t, C_1, C_2)$ solves this IVP. Then $y(t, C_1, C_2)$ and $z(t)$ solve the same IVP. This means that $z(t) = y(t, C_1, C_2)$.

Linear Equations and Linear Operators

We will be studying exclusively second order, *linear* equations. These are of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t), \quad \text{or} \quad y'' + py' + qy = f.$$

Just as in the first order linear case, if we see $Ay'' + By' + Cy = D$ with A, B, C, D functions of t we can obtain the above form by dividing by A. When A, B and C are constants we often won't bother.

The equation is called *homogeneous* when $f = 0$. It has *constant coefficients* when p and q are constant functions of t.

These are studied by using *linear operators*. An *operator* is a function $\mathcal{L}$ whose input and output are functions. For example, $\mathcal{L}(y) = yy'$. If $y = e^x$ then $\mathcal{L}(y) = e^{2x}$ and $\mathcal{L}(\sin)(x) = \sin(x)\cos(x)$.

An operator $\mathcal{L}$ is a linear operator when it satisfies *linearity* or *The Principle of Superposition*.

$$\mathcal{L}(Cy_1 + y_2) = C\mathcal{L}(y_1) + \mathcal{L}(y_2).$$

Given functions p, q we define $\mathcal{L}(y) = y'' + py' + qy$. Observe that:

$$\begin{array}{rcccccc} C \times (y_1'' & + & py_1' & + & qy_1) \\ + \quad y_2'' & + & py_2' & + & qy_2 \\ \hline \multicolumn{5}{c}{(Cy_1 + y_2)'' + p(Cy_1 + y_2)' + q(Cy_1 + y_2)} \end{array}$$

So the homogeneous equation $y'' + py' + qy = 0$ can be written as $\mathcal{L}(y) = 0$.

General Solution of the Homogeneous Equation, Section 3.2

If y_1 and y_2 are solutions of the homogeneous equation $\mathcal{L}(y) = 0$ then by linearity $C_1 y_1 + C_2 y_2$ are solutions for every choice of constants C_1 and C_2, because
$\mathcal{L}(C_1 y_1 + C_2 y_2) = C_1 \mathcal{L}(y_1) + C_2 \mathcal{L}(y_2) = C_1 0 + C_2 0 = 0.$
Notice there are two arbitrary constants. This is the general solution provided we can solve every IVP. So given numbers A, B we want to find C_1, C_2 so that

$$C_1 y_1(t_0) + C_2 y_2(t_0) = A,$$
$$C_1 y_1'(t_0) + C_2 y_2'(t_0) = B.$$

Cramer's Rule says that this has a solution provided the coefficient determinant

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \quad \text{is nonzero.}$$

Let us review Cramer's Rule.

Start with the system of two linear equations in two unknowns x and y.

$$\begin{aligned} ax + by &= E, \\ cx + dy &= F. \end{aligned}$$

So a, b, c, d, E, F are given constants and the graph of each equation, the *locus* of the equation, is a line.
When the two lines cross (the usual case), the coordinate pair of the intersection point is the unique solution pair (x, y).

Cramer's Rule says that the solution is given by.

$$x = \begin{vmatrix} E & b \\ F & d \end{vmatrix} \div \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

$$y = \begin{vmatrix} a & E \\ c & F \end{vmatrix} \div \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

This requires that the *coefficient determinant*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

be nonzero.

If the two lines are parallel, then there are no solutions and if the two lines have the same graph then all of the points on the common line are solutions.

These are the cases when the coefficient determinant is zero.

The Wronskian

Given two functions y_1, y_2 the *Wronskian* is defined to be the function:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

Two functions are called *linearly dependent* when there are constants C_1, C_2 not both zero such that $C_1 y_1 + C_2 y_2 \equiv 0$ and so when one of the two functions is a constant multiple of the other. If $y_2 = Cy_1$ then $W(y_1, y_2) \equiv 0$. The converse is almost true:

$W/y_1^2 = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = (\frac{y_2}{y_1})'$ and so if W is identically zero the ratio $\frac{y_2}{y_1}$ is some constant C and so $y_2 = Cy_1$.

On the other hand, $y_1(t) = t^3$ and $y_2(t) = |t^3|$ have $W \equiv 0$ but are not linearly dependent. What is the problem? Hint: look for a division by zero.

Now suppose the Wronskian vanishes at a single point t_0. If $y_1'(t_0) = y_2'(t_0) = 0$ then we can pick C_1, C_2, not both zero, such that with $z = C_1y_1 + C_2y_2$ is zero at t_0 and of course $z'(t_0) = C_1y_1'(t_0) + C_2y_2'(t_0) = 0$. Otherwise, choose $C_1 = y_2'(t_0)$ and $C_1 = -y_1'(t_0)$. Check that with $z = C_1y_1 + C_2y_2$ we have

$$z(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = W(y_1, y_2)(t_0) = 0,$$
$$z'(t_0) = y_1'(t_0)y_2'(t_0) - y_2'(t_0)y_1'(t_0) = 0.$$

If y_1, y_2 are solutions of the homogeneous equation $y'' + py' + qy = 0$ then $z = C_1y_1 + C_2y_2$ is a solution with zero initial conditions. This means that $z \equiv 0$ and so y_1 and y_2 are linearly dependent.

A pair y_1, y_2 of solutions of the homogeneous equation $y'' + py' + qy = 0$ is called a *fundamental pair* if the Wronskian does not vanish. The general solution is then $C_1y_1 + C_2y_2$ because we can solve every IVP.

Abel's Theorem

Abel's Theorem shows that the Wronskian of two solutions y_1, y_2 of $y'' + py' + qy = 0$ satisfies a first order linear ode:

$$W' = y_1 y_2'' - y_2 y_1'' + y_1' y_2' - y_2' y_1' = y_1 y_2'' - y_2 y_1''$$

$$\begin{array}{rcccl} q \times & y_1 y_2 & - & y_2 y_1 & = 0 \\ p \times & y_1 y_2' & - & y_2 y_1' & = W \\ 1 \times & y_1 y_2'' & - & y_2 y_1'' & = W' \end{array}$$

The left side adds up to $y_1 0 - y_2 0 = 0$ and so $0 = W' + pW$ or $\frac{dW}{dt} + pW = 0$.

This is variables separable with solution $W = C \times e^{-\int p(t)dt}$.

Since the exponential is always positive, $W \equiv 0$ if $C = 0$ and otherwise W is never zero.

Wronskian Problems

You should be able to use the Wronskian.
Suppose you are told that y_1, y_2 is a pair of solutions of the second order, linear, homogeneous equation $y'' + py' + qy = 0$. Is this a fundamental pair ? That is, are they independent (not dependent)? This is when the Wronskian is not zero. In that case, the general solution, using which you can solve every IVP, is $C_1 y_1 + C_2 y_2$.

Notice that from Abel's Theorem, the Wronskian of these two solutions is either never zero or is always zero.

We found from Abel's Theorem that the Wronskian is a constant times an exponential function.

Example 3.2/ BD 38, BDM 29: Assume y_1, y_2 is a pair of solutions of $y'' + py' + qy = 0$ on an interval which includes t_0. If y_1 and y_2 both vanish at the point t_0, then they are not a fundamental pair. That is, one of them is a constant times the other. This is because the Wronskian at t_0 is

$$\begin{vmatrix} 0 & 0 \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0.$$

Example 3.2/ BD 39, BDM 30: Assume y_1, y_2 is a pair of solutions of $y'' + py' + qy = 0$ on an interval which includes t_0. If y_1 and y_2 each have a local maximum at the point t_0, then they are not a fundamental pair. That is, one of them is a constant times the other. This is because the Wronskian at t_0 is

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ 0 & 0 \end{vmatrix} = 0.$$

(Recall that at a local maximum, the derivative vanishes).

Example 3.2/ BD 16, BDM 13: The function $y_1 = \sin(t^2)$ cannot be a solution of $y'' + py' + qy = 0$ on an interval which includes 0. This is because $y_1(0) = 0$ and $y_1'(0) = 0$. If it were a solution, then it would solve the IVP

$$y'' + py' + qy = 0, \quad y(0) = 0, \;\; y'(0) = 0.$$

But the constant function $y = 0$ is a solution of this IVP and so is the only solution.

Notice that for any y_2 the Wronskian at 0 of y_1, y_2 is

$$\begin{vmatrix} 0 & y_2(0) \\ 0 & y_2'(0) \end{vmatrix} = 0.$$

Homogeneous, Constant Coefficients, Sections 3.1, 3.3

The derivative itself is an example of a linear operator: $D(y) = y'$. For a linear operator $\mathcal{L}$ a function y is an *eigenvector* with *eigenvalue* r when $\mathcal{L}(y) = ry$. For the operator D, an eigenvector y satisfies $y' = D(y) = ry$. This is exponential growth with growth rate r and so the solutions are constant multiples of $y = e^{rt}$. Thus, such exponential functions have a special role to play.

A second order, linear, homogeneous equation with constant coefficients is of the form $Ay'' + By' + Cy = 0$ with A, B, C constants and $A \neq 0$.

We look for a solution of the form $y = e^{rt}$. Substituting and factoring we get

$$(Ar^2 + Br + C)e^{rt} = 0.$$

Since the exponential is always positive, e^{rt} is a solution precisely when r satisfies the characteristic equation $Ar^2 + Br + C = 0$.

The quadratic equation $Ar^2 + Br + C = 0$ has roots

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

The nature of the two roots depends on the discriminant $B^2 - 4AC$.

CASE 1: The simplest case is when $B^2 - 4AC > 0$. There are then two distinct real roots r_1 and r_2. This gives us two special solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$

Observe that the Wronskian of these two is:

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0.$$

The general solution is then $C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

We will see later that it is important to be able to go backwards as well. That is, if e^{rt} is a solution of $Ay'' + By' + Cy = 0$ with A, B, C constants, then r is a root of the characteristic equation $Ar^2 + Br + C = 0$.

Contrast: $y'' - 3y = 0$ with $y'' - 3y' = 0$.

The characteristic equation of the first is $r^2 - 3 = 0$ with roots $\pm\sqrt{3}$ and general solution

$$y = C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t}.$$

The characteristic equation of the second is $r^2 - 3r = 0$ with roots $0, 3$ and general solution

$$y = C_1 e^{0t} + C_2 e^{3t} = C_1 + C_2 e^{3t}.$$

Example: IVP $y'' - 5y' - 6y = 0, \quad y(0) = 6, y'(0) = 1.$

$0 = r^2 - 5r - 6 = (r-6)(r+1)$
with roots $6, -1$

$$y = C_1 e^{6t} + C_2 e^{-t}.$$

$$y' = 6C_1 e^{6t} - C_2 e^{-t}.$$

At $t = 0$ we get

$$\begin{aligned} 6 &= C_1 + C_2. \\ 1 &= 6C_1 - C_2. \end{aligned}$$

So $C_1 = 1$, $C_2 = 5$. and the solution is $Y = e^{6t} + 5e^{-t}$.