1. (12 points) If $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 3 & 0 \\ -1 & 1 \end{pmatrix}$, compute each of the following if it can be done. If it cannot be done, explain why not.

$$(i) \;\; AB \qquad (ii) \;\; A^{-1}B^T \qquad (iii) \;\; A + B^T$$

(i) can't multiply a 2×2 by a 3×2.

(ii) $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$ and so

$$A^{-1}B^T = \begin{pmatrix} 6/5 & 3/5 & 0 \\ -7/5 & -6/5 & 1 \end{pmatrix}.$$

(iii) can't add with different shapes.

2. (16 points) (a) If A is an $n \times n$ matrix define what it means for B to be *inverse* of A.

(b) If A is an $n \times n$ matrix with a row of zeroes, show - without using determinants - that A is not *invertible*. (That is, it does not have an inverse).

(c) If A and B are invertible $n \times n$ matrices, show that the product AB is invertible and compute its inverse.

(a) B is the inverse of A when it cancels A. That is, $AB = I_n = BA$.

(b) If A is an $n \times n$ matrix with a row of zeroes, show that A is not invertible.

If row i of A is a row of zeroes then for any $n \times n$ matrix B, row i of AB is a row of zeroes and so $AB \neq I_n$.

(c) If A and B are invertible $n \times n$ matrices, show that the product AB is invertible and compute its inverse.

$(AB)(B^{-1}A^{-1}) = I_n$ and $(B^{-1}A^{-1})(AB) = I_n$. Since $(B^{-1}A^{-1})$ cancels AB it is the inverse of AB.

3. (12 points) (a) For the row operation: To Row 3 add a copy of -5 times Row 2, write down the associated 4×4 elementary matrix E.

What is the reverse row operation and what is the inverse E^{-1}.

(b) Assume that A and B are 4×4 matrices with $det(A) = 5$ and $det(B) = 3$. Compute:

$$(i) \;\; det(3A), \;\; (ii) \;\; det(B^{-1}), \;\; (iii) \;\; det(AB^T).$$

(a) $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

The reverse row operation is: To Row 3 add a copy of 5 times a copy of Row 2.

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) (i) $det(3A) = 3^4 det(A) = 1 \cdot 5 = 405$.

(ii) $det(B^{-1}) = 1/det(B) = 1/3$.

(iii) $det(AB^T) = det(A) \cdot det(B^T) = det(A) \cdot det(B) = 15$.

4. (20 points)Assume that A is a 6×6 matrix and that Q is the 6×6 matrix in reduced echelon form row equivalent to A

(a) If A has rank 6, explain why A has an inverse.

(b) If A has rank 5, explain why A is not invertible.
You may use the fact that elementary matrices are invertible.

Q is obtained from A by a sequence of row operations $Q = Op_k(...(Op_1(A)...)$ and this is the same as multiplying by the corresponding elementary matrices. $Q = (E_k...E_1)A$. So $U = E_k...E_1$ is invertible and $Q = UA$.

If A has rank 6, then $Q = I_6$ and so U cancels A and $U^{-1} = A$. So A is invertible with inverse U.

If A has rank 5, then Q has a row of zeroes and so is not invertible. If A were invertible, then the product $UA = Q$ would be invertible which it is not.

5. (20 points)(a) Put the matrix

$$A = \begin{pmatrix} 0 & 2 & 10 & 3 & 7 & 2 \\ 0 & 0 & 0 & 3 & 9 & 6 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 1 & 5 & -1 & -4 & -4 \end{pmatrix}$$

in *reduced echelon form* and compute its *rank*.

$R_1 \leftrightarrow R_4 :$
$$\begin{pmatrix} 0 & 1 & 5 & -1 & -4 & -4 \\ 0 & 0 & 0 & 3 & 9 & 6 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 2 & 10 & 3 & 7 & 2 \end{pmatrix}$$

$R_4 \to R_4 - 2R1 :$
$$\begin{pmatrix} 0 & 1 & 5 & -1 & -4 & -4 \\ 0 & 0 & 0 & 3 & 9 & 6 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 5 & 15 & 10 \end{pmatrix}$$

$R_2 \leftrightarrow R_3 :$
$$\begin{pmatrix} 0 & 1 & 5 & -1 & -4 & -4 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 3 & 9 & 6 \\ 0 & 0 & 0 & 5 & 15 & 10 \end{pmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2, R_4 \rightarrow R_4 - 5R_2 : \begin{pmatrix} 0 & 1 & 5 & -1 & -4 & -4 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_1 \rightarrow R_1 + R_2 : \begin{pmatrix} 0 & 1 & 5 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank $= 2$.

(b) Compute the general solution for the system of equations which has A as its augmented matrix.

$$x_5 = r, \quad x_4 = 2 - 3r, \quad x_3 = s, \quad x_2 = -2 + r - 5s \quad x_1 = t.$$

6. (20 points)(a) Compute the inverse and the determinant of the following matrix

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(A|I_4) = \begin{pmatrix} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 \rightarrow R_3 - 2R_1, \quad \mu = 1$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -2 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 \rightarrow -R_3 \quad \mu = -1$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -2 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 \rightarrow R_3 - 2R_2 \quad \mu = 1$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 \rightarrow -\frac{1}{2}R_3 \quad \mu = -\frac{1}{2}$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_2 \to R_2 + R_4, R_1 - R_4 \quad \mu = 1$

$$\begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_1 \to R_1 - 2R_3 \quad \mu = 1$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_1 \to R_1 + R_2 \quad \mu = 1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

So

$$A^{-1} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $(\frac{1}{2})det(A) = det(I) = 1$. So $det(A) = 2$.

(b) Use the inverse to solve the system (no credit for any other method):

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 3 \\ & - & x_2 & & & & x_4 & = & 0 \\ 2x_1 & & & + & 2x_3 & & & = & 4 \\ & & & & & & x_4 & = & -1 \end{array}$$

With $B = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix}$ the system is $AX = B$ with solution $X = A^{-1}B = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.