MATH 39100 K Test 1 - Solutions Spring, 2026

(1) (12 points each)Compute the general solution of each of the following differential equations:

(a) $y' - x\cos(3y) = 0.$

Variables Separable: $\int \sec(3y)dy = \int x dx$.

$$\frac{1}{3}\ln|\sec(3y) + tan(3y)| = C + \frac{1}{2}x^2.$$

(b) $t^2\frac{dy}{dt} + 3ty = 2e^{-t} - t^2y$

Linear: $\frac{dy}{dt} + (\frac{3}{t} + 1)y = 2t^{-2}e^{-t}$

$$\mu = exp(3\ln(t) + t) = t^3e^t.$$

$$t^3e^t\frac{dy}{dt} + (3t^2 + t^3)e^ty = [t^3e^ty]' = 2t.$$

So $t^3e^ty = t^2 + C$ or $y = (t^2 + C)/(t^3e^t)$.

(c) $y'' - 9y' = 0$ (y is a function of x).

Second Order Linear: Characteristic Equation $r^2 - 9r = 0$ with roots $r = 0, 9$.

$$y = C_1\,e^{0x} + C_2e^{9x} = C_1 + C_2e^{9x}.$$

(d) $$xydy \quad - \quad (2x^2 - 3xy + 2y^2)dx \quad = \quad 0.$$

Homogeneous with $\frac{dy}{dx} = \frac{2y^2 - 3xy + 2x^2}{xy}$. Let $z = y/x$.

$$x\frac{dz}{dx} = \frac{2z^2 - 3z + 2}{z} - z = -\frac{z^2 - 3z + 2}{z}.$$

$$-\int \frac{z}{(z-1)(z-2)} dz = \int \frac{dx}{x}.$$

$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

So $z = A(z-2) + B(z-1)$.
With $z = 1, A = -1$ and with $z = 2, B = 2$.

$$C + \ln(x) = -\ln(\frac{y-x}{x}) + 2\ln(\frac{y-2x}{x}).$$

$$3\ln(y-x) - 2\ln(\frac{y-2x}{x}) = C, \quad \text{or} \quad \frac{(y-x)x^2}{(y-2x)^2} = C.$$

(2) (12 points each) Solve the following initial value problems:

(a)(a) $\quad y'y'' = 2, \qquad$ with $\quad y(0) = 1, y'(0) = 2.$

Reduction of Order: Let $y' = v$ so that $y'' = \frac{dv}{dt}$.

$$v\frac{dv}{dt} = 2.$$

$$v^2/2 = \int vdv = \int 2dt = 2t + C_1.$$

When $t = 0$, $v = 2$ and so $C_1 = 2$, $v^2 = 4(t+1)$.
$\frac{dy}{dt} = v = 2\sqrt{t+1}$ (positive square root because $v = 2$ when $t = 0$.So

$$y = 2\int (1+t)^{1/2} dt = \frac{4}{3}(t+1)^{3/2} + C_2.$$

When $t = 0$, $y = 1$ and so $C_2 = -\frac{1}{3}$.

$$y = \frac{4(t+1)^{3/2} - 1}{3}.$$

(b) $e^{xy}(1+xy)\ dx + (x^2e^{xy} - 2y)\ dy \quad = \quad 0 \quad \text{with} \quad y(1) = 0.$

$$\frac{\partial}{\partial y}(e^{xy}(1+xy)) = e^{xy}(2x + x^2y) = \frac{\partial}{\partial x}(x^2e^{xy} - 2y)$$

Exact Equation: $F = \int(x^2e^{xy} - 2y)\ dy = xe^{xy} - y^2 + H(x)$.

$$\frac{\partial F}{\partial x} = e^{xy}(1+xy) + H'(x) = e^{xy}(1+xy).$$

$H'(x) = 0$ and so $H(x) = 0$.
$xe^{xy} - y^2 = C$. Since $y(1) = 0$, $1 - 0 = C$ and so $C = 1$.

(3)(9 points) Assume that y_1, y_2 are solutions of the equation $y''+py'+qy = 0$ where p and q are functions of t.

(a) Prove Abel's Theorem. That is, show that the Wronskian satisfies a first order differential equation.
(b) Explain why the Wronskian is either identically zero or never zero.

$0 = y_1y_2 - y_2y_1$.
$W = y_1y_2' - y_2y_1'$.
$W' = y_1y_2" - y_2y_1"$.
Multiply the first row by q, the second row by p, the third row by 1 and add.
Because y_1 and y_2 are solutions we obtain:
$0 + pW + W' \ = \ 0 + 0$. That is, $\frac{dW}{dt} + pW = 0$.

The solution is $W = C \times exp(-\int p(t)dt)$. Since the exponential is always positive, W is identically zero if $C = 0$ and is otherwise never zero.

(4) (4) (10 points)A 400 gallon tank contains 120 gallons of water in which is dissolved 20 pounds of salt. Starting at time t = 0, pure water is pumped into the tank at a rate of 6 gallons per minute. At the same time, the well-stirred mixture is pumped out at the rate of 4 gallons per minute.

Set up an initial value problem (differential equation and initial conditions) for the amount $Q(t)$ of salt (in pounds) in the tank at time t until the tank is full. Use it to determine the amount of salt in the tank the moment that the tank is filled up.

$\frac{dV}{dt} = 6 - 4$ in gal/min with $V_0 = 120$. So $V = 120 + 2t$.
In $pounds/min$

$$\frac{dQ}{dt} = 6 \cdot 0 - 5\frac{Q}{V} = -\frac{4Q}{120 + 2t}$$

with $Q_0 = 20$.

$$\frac{dQ}{Q} = -\frac{4}{120 + 2t}dt$$

So $\ln Q = -2\ln(120 + 2t) + C$ and so $Q = \frac{C}{(120+2t)^2}$.

Since $(Q(0) = 20$, $C = 20 \cdot (120)^2$ and so $Q = 20 \cdot (\frac{120}{120+2t})^2$.

The tank is full when $120 + 2t = V = 400$ and so when $t = 140$. Then $Q = 20 \cdot (\frac{120}{400})^2$ (which equals 1.8.)

(5) (9 points) I borrow $\$8,000$ at an interest rate of 7% per year, compounded continuously. I pay off the loan continuously at a rate of \$1200 per year. Set up an initial value problem (differential equation and initial conditions) whose solution is the quantity $S(t)$ of dollars that I owe at time t, until the loan is paid off. You need not solve the equation.

$$\frac{dP}{dt} = .07P - 1200,$$

with $P_0 = 8,000$.

Remember to show your work. Good luck.