(1)(20 points)(a) For the differential equation $y'' + 4y' = 0$ obtain a fundamental pair of solutions and compute the Wronskian of the pair.

(b) Solve the initial value problem: $y'' + 4y' = 2e^{4t} + 5$ with $y(0) = 0$ and $y'(0) = 0$.

(a) The characteristic polynomial for the homogeneous equation is $r^2 + 4r = r(r+4)$ with roots $0, -4$. The fundamental pair is $\{1 = e^{0t}, e^{-4t}\}$ and so the Wronskian $= y_1 y_2' - y_2 y_1' = 1 \cdot (1 \cdot -4e^{-4t}) - (e^{-4t}) \cdot 0 = -4e^{-4t}$.

(b) The general solution of the homogeneous system is $y_h = C_1 + C_2 e^{-4t}$.

The associated root for $12e^{4t}$ is 3 and the associated root for 5 is 0. So the first version of the test function is $Y_p^1 = Ae^{4t} + B$. Because 0 is a root of the homogeneous equation, the test function is $Y_p = Ae^{4t} + Bt$.

$$\begin{array}{rrcl} 0 \times & Y_p & = & 0Ae^{4t} + 0Bt \\ 4 \times & Y_p' & = & 16Ae^{4t} + 4B \\ 1 \times & Y_p'' & = & 16Ae^{4t} \end{array}$$

$$2e^{3t} + 5 = 32Ae^{4t} + 4B$$

So $A = 1/16, \ B = 5/4$.

$$\begin{aligned} y &= C_1 + C_2 e^{-4t} + (\frac{1}{16})e^{4t} + (\frac{5}{4})t \\ y' &= +(-4C_2)e^{-4t} + \frac{1}{4}e^{4t} + (\frac{5}{4}). \end{aligned}$$

Since $y(0) = y'(0) = 0.$, $0 = C_1 + C_2 + \frac{1}{16}$ and $0 = -4C_2 + \frac{3}{2}$.
So $C_2 = \frac{3}{8}, C_1 = -\frac{7}{16}$.

$$y = -\frac{7}{16} + (\frac{3}{8})e^{-4t} + (\frac{1}{16})e^{4t} + (\frac{5}{4})t.$$

(2)(20 points) (a) Compute the general solution of the differential equation $2y'' - 6y' + 5y = 0$.

(b)For the equation $2y'' - 6y' + 5y = 5e^{3t} + t\cos(t) + 7 + 2e^{3t}\sin(t) - t^3$, write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients. Do not solve for the constants.

(a) The characteristic polynomial for the homogeneous equation is $2r^2 - 6r + 5$ with roots $(3 \pm \mathbf{i})/2$.

So $y_h = e^{3t/2}(C_1\cos(t/2) + C_2\sin(t/2))$.

(b) The associated roots are as follows

For $5e^{3t}$, $\quad r = 3$.

For 7 and for $-t^3$, $\quad r = 0$.

For $t\cos(t)$, $\quad r = \pm i$.

For $2e^{3t}\sin(t)$, $\quad r = 3 \pm \mathbf{i}$.

So the first version of the test function is:

$$Y_p^1 = [Ae^{3t}] + [Bt^3 + Ct^2 + Dt + E] + [(Ft + G)\cos(t) + (Ht + I)\sin(t)] + [Je^{3t}\cos(t) + Ke^{3t}\sin(t)].$$

Since none is a root of the homogeneous equation, $Y_p = Y_p^1$.

(3)(20 points) Consider the fifth order differential equation $y^{(5)} + y^{(4)} - 2y^{(3)} - y'' - y' + 2y = g(t)$.

(a) Compute the general solution of the homogeneous equation with $g(t) = 0$. Notice that $r^5 + r^4 - 2r^3 - r^2 - r + 2 = r^3(r^2 + r - 2) - (r^2 + r - 2)$.

(b) For each of the following forcing functions $g(t)$, write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients. Do not solve for the constants.

$$(i) \quad g(t) = t^2 + te^{-t} + 2e^{2t}. \qquad (ii) \quad g(t) = 2te^t + t^2e^{-t/2}.$$

(a) The characteristic polynomial is $r^5 + r^4 - 2r^3 - r^2 - r + 2 = (r^3 - 1)(r^2 + r - 2) = (r-1)(r^2 + r + 1)(r-1)(r+2)$ The roots are $1, 1, -2, \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$.

$$y_h = C_1e^t + C_2te^t + C_3e^{-2t} + C_4e^{t/2}\cos(\sqrt{3}t/2) + C_5e^{t/2}\sin(\sqrt{3}t/2).$$

(b) (i) Associated roots are as follows:
For t^2, $r = 0$.
For te^{-t} $r = -1$.
For $2e^{2t}$ $r = 2$
None of these are roots of the homogeneous equation and so

$$Y_p = Y_p^1 = [At^2 + Bt + C] + [(Dt + E)e^{-t}] + [Fe^{2t}].$$

(ii) Associated roots are as follows:
For $2te^t$, $r = 1$.
For $t^2e^{-t/2}$ $r = -\frac{1}{2}$
$Y_p^1 = [(At+B)e^t]+[(Ct^2+Dt+E)e^{-t/2}]$ and as 1 is a root of the homogeneous repeated twice

$$Y_p = t^2[(At + B)e^t] + [(Ct^2 + Dt + E)e^{-t/2}].$$

(4)(15 points) Compute the general solution of $y'' + 9y = 12\sec(3t)$.

The characteristic polynomial for the homogeneous equation is $r^2 + 9$ with roots $\pm 3\mathbf{i}$. The solution of the homogeneous is $y_h = C_1 \cos(3t) + C_2 \sin(3t)$.
We look for a particular solution $y_p = u_1 \cos(3t) + u_2 \sin(3t)$.

The equations are:

$$\begin{aligned} u_1' \cos(3t) + u_2' \sin(3t) &= 0, \\ u_1'(-3\sin(3t)) + u_2'(3\cos(3t)) &= 12\sec(3t) \end{aligned}$$

The Wronskian is 3.
$u_1' = -12\sec(3t)\sin(3t)/3 = -4\tan(3t)$, $u_2' = 12\sec(3t)\cos(3t)/3 = 4$.
So $u_1 = -\frac{4}{3}\ln(\sec(3t)), u_2 = 4t$ and $y_p = -\frac{4}{3}\ln(\sec(3t))\cos(3t) + 4t\sin(3t)$.

$$y_g = C_1 \cos(3t) + C_2 \sin(3t) + -\frac{4}{3}\ln(\sec(3t))\cos(3t) + 4t\sin(3t).$$

(5)(5)(15 points) For the differential equation, $t^2y'' + 7ty' + 9y = 0$, $y_1 = t^{-3}$ is a solution. Use the method of Reduction of Order to compute a second solution y_2 which is independent of the first one.

Look for $y_2 = ut^{-3}$ So that

$$\begin{aligned} y_2 &= ut^{-3} \\ y_2' &= -3ut^{-4} \;+\; u't{-}3 \\ y_2'' &= 12ut^{-5} \;+\; -6u't^{-4} \;+\; u''t^{-3} \end{aligned}$$

Multiply by the coefficients and add.

$$\begin{array}{rrcl} 9\times & y_2 & = & 9ut^{-3} \\ 7t\times & y_2' & = & -21ut^{-3} \ + \ 7u't{-}2 \\ t^2\times & y_2'' & = & 12ut^{-3} \ + \ -6u't^{-2} \ + \ u''t^{-1} \\ & & & \\ & 0 & = & 0 \quad + \ u't^{-2} \ + \ u''t^{-1}. \end{array}$$

With $v = u'$, $\frac{dv}{v} = -\frac{dt}{t}$
Therefore, $\ln v = -\ln(t)$ and so $\frac{du}{dt} = t^{-1}$ and $u = \ln(t)$

$$y_2 = ut^{-3} = t^{-3}\ln(t).$$

(6)(10 points) (a) Assume that y_1, y_2 are solutions of the equation $y'' + py' + qy = 1$ where p and q are functions of t. Show that $y_1 + y_2$ is NOT a solution of this equation.

Because y_1 and y_2 are solutions we have
$y_1'' + py_1' + qy_1 = 1$ and $y_2'' + py_2' + qy_2 = 1$. Adding, we get that

$$(y_1 + y_2)'' + p(y_1 + y_2)' + q(y_1 + y_2) = 2 \neq 1.$$

(b) Compute the *Wronskian* W of the pair $y_1 = e^{3x}, y_2 = xe^x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^x \\ 3e^{3x} & (1+x)e^x \end{vmatrix}$$

$$= (1+x)e^{4x} - 3xe^{4x} = (1-2x)e^{4x}.$$

So $W = 0$ when $x = \frac{1}{2}$. Notice that this does not violate Abel's Theorem as these two functions are not solutions of a second order linear, homogeneous equation defined around $x = \frac{1}{2}$.