1. (14 points) Determine whether each of the following is or is not a subspace of the vector space $M_{2\times 2}$ of 2×2 matrices. That is, if it is a subspace, check that the required conditions hold. If it is not give an example to show that some condition fails.

(a) The set $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = c\}$.

(b) Nonnegative matrices. That is, the set $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \geq 0\}$

(c) Non-invertible matrices. That is, the set $\{A : A^{-1}$ does not exist $\}$.

(d) Symmetric matrices. That is, $\{A : A^T = A\}$.

(a) Subspace. $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$
$= \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$ and
$k \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{pmatrix}$.
If $a_1 + d_1 = c_1$ and $a_2 + d_2 = c_2$, then $(a_1 + a_2) + (d_1 + d_2) = (c_1 + c_2)$ and $ka_1 + kd_1 = kc_1$.

(b) Not a subspace. $(-1)\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ is not in the set.

(c) Not a subspace. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not in the set.

(d) Subspace. If $A^T = A$ and $B^T = B$, then $(A + B)^T = A^T + B^T = A + B$ and $(cA)^T = c(A^T) = cA$.

2. (14 points) We use the matrix $A = \begin{pmatrix} 1 & 5 & 0 & 1 & 0 \\ 2 & 9 & 4 & 1 & 1 \\ 1 & 4 & 4 & 1 & 1 \\ 1 & 5 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$.

(a) Determine whether $B = \begin{pmatrix} 3 & 4 & 1 & 1 & -5 \end{pmatrix}^T$ is a linear combination of the columns of A. If so provide coefficients which express it as a linear combination. If it is not a linear combination, explain why not.

(b) Find a basis for the column space of A.

(c) Find a basis for the row space of A and determine the rank of A.

(a) To obtain B are a linear combination of the columns of A, we put the augmented matrix $(A|B)$ is Reduced Row Echelon Form.

$$\begin{pmatrix} 1 & 5 & 0 & 1 & 0 & 3 \\ 2 & 9 & 4 & 1 & 1 & 4 \\ 1 & 4 & 4 & 1 & 1 & 1 \\ 1 & 5 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -4 \end{pmatrix}.$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$.

$$\begin{pmatrix} 1 & 5 & 0 & 1 & 0 & 3 \\ 0 & -1 & 4 & -1 & 1 & -2 \\ 0 & -1 & 4 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 & -4 \end{pmatrix}.$$

$R_2 \to -R_2$, then $R_3 \to R_3 + R_2$.

$$\begin{pmatrix} 1 & 5 & 0 & 1 & 0 & 3 \\ 0 & 1 & -4 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 & -4 \end{pmatrix}.$$

$R_5 \to R_5 - R_3$, then $R_5 \to R_5 - 2R_4$

$$\begin{pmatrix} 1 & 5 & 0 & 1 & 0 & 3 \\ 0 & 1 & -4 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Echelon form. Now for Reduced Echelon Form
$R_2 \to R_2 + R_4$.

$$\begin{pmatrix} 1 & 5 & 0 & 1 & 0 & 3 \\ 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$R_2 \to R_2 - R_3, R_1 \to R_1 - R_3$.

$$\begin{pmatrix} 1 & 5 & 0 & 0 & 0 & 3 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$R_1 \to R_1 - 5R_2$.

$$\begin{pmatrix} 1 & 0 & 20 & 0 & 0 & 3 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The solution is $x_5 = -2, x_4 = 0, x_3 = r, x_2 = r, x_1 = 3 - 20r$. We may use $r = 0$ and so express B as $3A_1 - 2A_5$ where A_1 and A_5 are the first and fifth columns of A.

(b) A basis for the column space is $\{A_1, A_2, A_4, A_5\}$, that is the first, second, fourth and fifth columns of A.

(c) With Q the matrix in reduced echelon form row equivalent to A, a basis for the row space of A is the four nonzero rows of Q. The rank is 4.

3.(16 points) Given that $S = \{v_1, \ldots, v_n\}$ is a list of vectors in a vector space V and that $T = \{w\}$ lists a vector in V. Assume that w lies in $Span(S)$.

(a) Prove that $Span(S \cup T) = Span(S)$.

(b) Prove that $S \cup T$ is ld (i.e. linearly dependent).

(a) Since S is contained in $S \cup T$ and $S \cup T$ is contained in $Span(S \cup T)$. That is, all the vectors of $S \cup T$ are linear combinations of the vectors of $S \cup T$. Similarly, S is contained in $Span(S)$. By assumption T is contained in $Span(S)$ and so $S \cup T$ is contained in $Span(S)$.

Because $Span(S)$ and $Span(S \cup T)$ are subspaces, $S \subset Span(S \cup T)$ implies $Span(S) \subset Span(S \cup T)$ and $S \cup T \subset Span(S)$ implies $Span(S \cup T) \subset Span(S)$.

Alternatively,

If u is a vector of $Span(S)$ then $u = c_1 v_1 + ... + c_n v_n$ for some list of coefficients.

So $u = c_1 v_1 + ... + c_n v_n + 0w$ which means u is a vector of $Span(S \cup T)$.

If u is a vector of $Span(S \cup T)$ then $u = c_1 v_1 + ... + c_n v_n + cw$ for some list of coefficients. But w in $Span(S)$ implies $w = x_1 v_1 + ... + x_n v_n$ for some coefficients. So

$$u = c_1 v_1 + ... + c_n v_n + c(x_1 v_1 + ... + x_n v_n) = (c_1 + cx_1)v_1 + ...(c_n + cx_n)v_n$$

Therefore, u is in $Span(S)$.

4. (14 points) For an $m \times n$ matrix A, the null space, $Null(A) = \{X \in \mathbb{R}^n : AX = \theta\}$ where θ is the 0 matrix. That is, $Null(A)$ is the solution space of the associated homogeneous system.

(a) Show that $Null(A)$ is a subspace of R^n.

(b) For the matrix (b) For the matrix $A = \begin{pmatrix} 1 & 5 & 0 & 1 & 0 \\ 2 & 9 & 4 & 1 & 1 \\ 1 & 4 & 4 & 1 & 1 \\ 1 & 5 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$ given in problem 2, find a basis for $Null(A)$. (You may use your work from problem 2.)

(a) If $AX = \theta$ and $AY = \theta$ then $A(X+Y) = AX + AY = \theta + \theta = \theta$ and $A(cX) = c(AX) = c\theta = \theta$. So the nullspace is a subspace.

(b) Using our work from (2) We put A is Reduced Row Echelon Form to obtain

$$Q = \begin{pmatrix} 1 & 0 & 20 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the general solution of the homogeneous system is : $x_5 = 0, x_4 = 0, x_3 = r, x_2 = 4r, x_1 = -20r$. The nullspace is one dimensional with a basis $\{\begin{pmatrix} -20 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$.

5. (14 points) Assume that A is an $m \times n$ matrix.
(a) Assume that the columns of A form a linearly independent list. What is the rank of A? (Explain)
If, in addition, $m = n$ so that the matrix is square, does this imply that A has an inverse? (Explain)
(b) If $m < n$, can the columns form a linearly independent list? Can the columns span $\mathbb{R}^m$? (Explain each answer)

(a) If Q is in Reduced Row Echelon Form row equivalent to A then the columns are linearly independent exactly when every column of Q has a leading 1 and so when the rank equals n.
If $m = n$, then $Q = I$ and A is invertible.

(b) If $m < n$ then the largest the rank can be is m which is less than n and so the columns cannot be linearly independent. However, if the rank is m, then there is a leading 1 in every row of Q and so the columns do span in that case.

6. (14 points) Consider the following list of polynomials in $\mathcal{P}_2$, the vector space of polynomials of degree at most two.

$$S = \{t^2 + 2t + 1, 5t^2 + 9t + 4, 4t + 4\}.$$

(error, should be)

(a) Show that the list S is linearly dependent.

(b) Obtain a basis for the $Span(S)$ and compute the dimension of $Span(S)$.

(c) Does the list span $\mathcal{P}_2$? (Explain)

6. Translating the polynomials into columns we represent the list by the matrix:

$$\begin{pmatrix} 1 & 5 & 0 \\ 2 & 9 & 4 \\ 1 & 4 & 4 \end{pmatrix}.$$

This is row equivalent to the following matrix in reduced echelon form:

$$\begin{pmatrix} 1 & 0 & 20 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because the third column has no leading 1 and so represents a free variable, the columns form a linearly dependent list. The solution of the homogeneous system is given by $x_3 = r, x_2 = 4r, x_1 = -20r$. With $r = 1$ we see that $-20(t^2 + 2t + 1) + 4(5t^2 + 9t + 4) + (4t + 4) = 0$.

Because the first two columns form a basis for the column space, a basis for $Span(S)$ is $\{t^2 + 2t + 1, 5t^2 + 9t + 4\}$.

Because $\mathcal{P}_2$ is three dimensional, $Span(S)$ is a proper subspace and S does not span $\mathcal{P}_2$.

7. (14 points)(a) For a vector space V, define "the dimension of V".

(b) Using the definition of dimension, compute the dimension of $M_{2\times 2}$ the vector space of 2×2 matrices.

(a) The dimension of V is the number of vectors in any basis for V.

(b)$M_{2\times 2} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}\}$

The associated standard basis is:

$$\{E_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$E_c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

Therefore the dimension is 4.